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ON ZAGIER-HOFFMAN'S CONJECTURES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We study Todd-Thakur's analogues of Zagier-Hoffman's conjectures in positive characteristic. These conjectures predict the dimension and an explicit basis $\mathcal{T}_w$ of the span of characteristic p multiple zeta values of fixed weight w which were introduced by Thakur as analogues of classical multiple zeta values of Euler.

In the present paper we first establish the algebraic part of these conjectures which states that the span of characteristic p multiple zeta values of weight w is generated by the set $\mathcal{T}_w$. As a consequence, we obtain upper bounds for the dimension. This is the analogue of Brown's theorem and also those of Deligne-Goncharov and Terasoma.

We then prove two results towards the transcendental part of these conjectures. First, we establish the linear independence for a large subset of $\mathcal{T}_w$ and yield lower bounds for the dimension. Second, for small weights we prove the linear independence for the whole set $\mathcal{T}_w$ and completely solve Zagier-Hoffman's conjectures in positive characteristic. Our key tool is the Anderson-Brownawell-Papanikolas criterion for linear independence in positive characteristic.

1. INTRODUCTION

1.1. Classical multiple zeta values. Multiple zeta values of Euler (MZV's for short) are real numbers of the form

$$\zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}, \quad \text{where } n_i \geq 1, n_r \geq 2.$$

Here r is called the depth and $w = n_1 + \dots + n_r$ is called the weight of the presentation $\zeta(n_1, \dots, n_r)$ (see [5, Remark 1.28] for more details). For $r = 1$ we recover the special values $\zeta(n)$ for $n \geq 2$ of the Riemann zeta function. These values have been studied in different contexts, for example Witten's zeta functions, Vassiliev knot invariants or mixed Tate motives. We refer the reader to the excellent survey of Zagier [31] and the recent book of Burgos Gil and Fresan [5] for more details and more complete references.

As mentioned in [5], one can argue that the main goal of the theory of MZV's is to understand all $\mathbb{Q}$ -linear relations among MZV's. Zagier [31] and Hoffman [16] proposed some conjectures for the $\mathbb{Q}$ -vector space spanned by MZV's. More precisely, let $\mathcal{Z}_k$ be the $\mathbb{Q}$ -vector space spanned by MZV's of weight k . We define a Fibonacci-like sequence of integers d_k as follows. Letting $d_0 = 1, d_1 = 0$ and $d_2 = 1$

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we define $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$. The following conjecture was stated by Zagier [31].

Conjecture 1.1 (Zagier's conjecture). *For $k \in \mathbb{N}$ we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

Hoffman [16] went further and suggested a refinement of Zagier's conjecture.

Conjecture 1.2 (Hoffman's conjecture). *The $\mathbb{Q}$ -vector space $\mathcal{Z}_k$ is generated by the basis consisting of MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

Roughly speaking, we could break Zagier's and Hoffman's conjectures into two parts of different nature.

- *The algebraic part* of Zagier's conjecture concerns the upper bound for the dimension, i.e. $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$. For Hoffman's conjecture, it states that $\mathcal{Z}_k$ is generated by MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.
- *The transcendental part* of Zagier's conjecture concerns the lower bound for the dimension, i.e. $\dim_{\mathbb{Q}} \mathcal{Z}_k \geq d_k$. For Hoffman's conjecture, it states that MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$ are $\mathbb{Q}$ -linearly independent.

In the last two decades the algebraic part of these conjectures was completely solved by the seminal works of Brown [6], Deligne-Goncharov [13] and Terasoma [21]. Although Zagier-Hoffman's conjectures are easily stated, the proofs of Brown, Deligne-Goncharov and Terasoma use the theory of mixed Tate motives.

Theorem 1.3 (Deligne-Goncharov, Terasoma). *For $k \in \mathbb{N}$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.*

Theorem 1.4 (Brown). *The $\mathbb{Q}$ -vector space $\mathcal{Z}_k$ is generated by MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

Consequently, every MZV can be written as a $\mathbb{Q}$ -linear combination of $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$. Unfortunately, in the words of Deligne [12], the proof of Brown does not provide a usable algorithm to find out such a linear combination.

Despite major progress, the transcendental part of Zagier's and Hoffman's conjectures is completely out of reach. To our knowledge, we do not even know any single $k \in \mathbb{N}$ for which $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is bigger than 1!

1.2. Characteristic p multiple zeta values. By a well-known analogy between the arithmetic of number fields and that of function fields, conceived of in the 1930s by Carlitz, we now switch to the function field setting.

Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational place ∞ . Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞ .

In [8] Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ ($n \in \mathbb{N}$) given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. Here A_+ denotes the set of monic polynomials in A . For any tuple of positive integers

$\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [22] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. We call $\text{depth}(\mathfrak{s}) = r$ the depth and $w(\mathfrak{s}) = s_1 + \dots + s_r$ the weight of $\mathfrak{s}$ and $\zeta_A(\mathfrak{s})$. We extend the definition of $\zeta_A(\mathfrak{s})$ to the empty tuple by defining the associated zeta value to be 1. We note that each MZV does not vanish (see [24]), and that Carlitz zeta values are exactly depth one MZV's.

Many works have revealed the importance of these zeta values for both their independent interest and for their applications to a wide variety of arithmetic applications. We refer the reader to the excellent surveys of Thakur [26, 27] for more details and more exhaustive references.

Similar to the classical setting, the main goal of this theory is to determine all linear relations over K among MZV's. Of particular interest we now state analogues of Zagier-Hoffman's conjectures in positive characteristic formulated by Thakur in [26, §8] and by Todd in [28].

Conjecture 1.5 (Zagier's conjecture in positive characteristic). *Letting*

$$d(w) = \begin{cases} 1 & \text{if } w = 0, \\ 2^{w-1} & \text{if } 1 \leq w \leq q-1, \\ 2^{w-1} - 1 & \text{if } w = q, \end{cases}$$

we put $d(w) = \sum_{i=1}^q d(w-i)$ for $w > q$. For any $w \in \mathbb{N}$, if $\mathcal{Z}_w$ denotes the K -span of MZV's of weight w , then

$$\dim_K \mathcal{Z}_w = d(w).$$

We note that if we set $d(w) = 0$ for $w < 0$, then the equality $d(w) = \sum_{i=1}^q d(w-i)$ holds for every integer $w \neq 0, q$.

Thakur [26, §8.2] formulated a refinement of the above conjecture which could be considered as an analogue of Hoffman's conjecture in positive characteristic.

Conjecture 1.6 (Hoffman's conjecture in positive characteristic). *A K -basis $\mathcal{T}_w$ for $\mathcal{Z}_w$ is given by $\zeta_A(s_1, \dots, s_r)$ of weight w with $s_i \leq q$ for $1 \leq i < r$, and $s_r < q$.*

1.3. Main results. In this paper we first establish the algebraic part of Conjectures 1.5 and 1.6. This is the analogue of Brown's theorem [6] and also those of Deligne-Goncharov [13] and Terasoma [21] in positive characteristic.

Theorem A (Brown's theorem in positive characteristic). *Let $w \in \mathbb{N}$. Then every MZV of weight w can be written as a K -linear combination of MZV's in the set $\mathcal{T}_w$. In particular, $\dim_K \mathcal{Z}_w \leq d(w)$.*

Contrary to the work of Brown [6], our proof gives an algorithm to express every MZV of weight w as a K -linear combination of MZV's in the set $\mathcal{T}_w$.

We next prove two results towards the transcendental part of Conjectures 1.5 and 1.6. First, we obtain the linear independence over K for a large subset of $\mathcal{T}_w$ which is also conjectured by Thakur [26, §8.2].

Theorem B. *Let $w \in \mathbb{N}$. We define $\mathcal{T}_w^0$ to be the subset of $\mathcal{T}_w$ given by $\zeta_A(s_1, \dots, s_r)$ of weight w with $s_i < q$ for $1 \leq i \leq r$. Then MZV's in $\mathcal{T}_w^0$ are all linearly independent over K . In particular, $\dim_K \mathcal{Z}_w \geq d_0(w) := |\mathcal{T}_w^0|$.*

We observe that if $w \leq q$, then $\mathcal{T}_w^0$ and $\mathcal{T}_w$ are the same. As an immediate consequence of Theorem B we deduce

Corollary C. *Let $w \in \mathbb{N}$ with $w \leq q$. Then $\mathcal{T}_w$ is a K -basis for $\mathcal{Z}_w$. In particular,*

$$\dim_K \mathcal{Z}_w = d(w).$$

Corollary C is purely a transcendental result since the algebraic part (i.e. Theorem A) is trivial when $w \leq q$.

Finally, we improve Corollary C and completely solve Zagier-Hoffman's conjectures in positive characteristic for $w \leq 2q - 2$.

Theorem D. *Let $w \in \mathbb{N}$ with $w \leq 2q - 2$. Then $\mathcal{T}_w$ is a K -basis for $\mathcal{Z}_w$. In particular, $\dim_K \mathcal{Z}_w = d(w)$.*

Note that when $q = 2$, Corollary C and Theorem D are the same. When $q > 2$, contrary to Corollary C, Theorem D needs both the algebraic part and the transcendental part. These are the first known dimensions for MZV's of fixed weight w in positive characteristic with $w > q > 2$.

We present below the list of all known cases where one can determine completely dimensions for characteristic p MZV's of fixed weight. We are grateful to one of the referees for pointing out the last case.

- $w = 1$: we have $d(w) = 1$, and this case follows from the fact that $\zeta_A(1) \neq 0$.
- $w = 2$ and $q = 2$: we have $d(w) = 1$, and this case is known by Thakur [22, Theorem 5.10.13] and follows from the fact that $\zeta_A(2) \neq 0$ and the fundamental relation R_1 given below (see also (2.8)).
- $w = 2$ and $q > 2$: we have $d(w) = 2$, and this case was proved by Mishiba [19, Corollary 1.5].
- $w = 3$ and $q = 2$: we have $d(w) = 2$, and this case is already known. The inequality $d(w) \leq 2$ (i.e. Theorem A in this case) can be easily proved. Further, we have $d(w) \geq 2$ by [22, Theorem 5.10.2].

The first three cases are covered by Corollary C. However, the last one when $w = 3$ and $q = 2$ is not covered by Theorem D.

1.4. Ingredients of the proofs. To prove Theorem A we use the previous works of Thakur and Todd on linear relations among MZV's (see §2). Roughly speaking, all linear relations among MZV's are expected to be generated by two operations $\mathcal{B}^*$ and $\mathcal{C}$ and one fundamental relation called R_1 given by

$$\zeta_A(q) + D_1 \zeta_A(1, q-1) = 0, \quad \text{with } D_1 = \theta^q - \theta \in K^*.$$

More precisely, starting with $\zeta_A(s_1, \dots, s_r)$ satisfying $s_1 > q$ we find a way to express $\zeta_A(s_1, \dots, s_r)$ as a K -linear combination of $\zeta_A(t_1, \dots, t_k)$ with $t_1 \leq q$. Once we have $s_1 \leq q$, we continue to lower the second entry s_2 until $s_2 \leq q$. Repeating this process plus a little extra work we obtain a proof of Theorem A (see §3).

The proofs of Theorems B and D are completely of different flavor. They are based on two key ingredients: the theory of Anderson t -motives introduced by Anderson [1] and developed further in [2, 15], and a very powerful transcendental tool called the Anderson-Brownawell-Papanikolas criterion devised in [2] (see §4 for details). This criterion has turned out to be very fruitful in function field arithmetic, see for example [2, 9, 10, 20]. In particular, the proof of Theorems B is inspired by [10] (see §5). We first construct t -motives which lift linear relations

among MZV's. Next we apply the Anderson-Brownawell-Papanikolas criterion to deduce rationality results which allow us to conclude.

To prove Theorem D we construct another set of MZV's $\mathcal{T}'_w$ having the same cardinality as $\mathcal{T}_w$ and succeed in extending Theorem B to this set. Thus we obtain a lower bound $\dim_K \mathcal{Z}_w \geq d(w)$. Combining this lower bound with the upper bound of Theorem A yields $\dim_K \mathcal{Z}_w = d(w)$, and Theorem D follows (see §6).

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2. ALGEBRAIC TOOLS

In this section we investigate linear relations for MZV's. Our approach is based on techniques dealing with power sums studied by Thakur [25] and crucial operations introduced by Todd [28].

We continue with the notation in the introduction. Let $\mathbb{F}_q$ be a finite field having q elements of characteristic $p > 0$. Recall that $A = \mathbb{F}_q[\theta]$, A_+ denotes the set of monic polynomials in A , $K = \mathbb{F}_q(\theta)$ equipped with the rational place ∞ , K_∞ denotes the completion of K at ∞ , and $\mathbb{C}_\infty$ denotes the completion of a fixed algebraic closure $\overline{K}$ of K at ∞ . We denote by v_∞ the discrete valuation on K_∞ corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$ and $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K . The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞ .

In what follows, we use upper-case letters or Fraktur characters (e.g. $W, V, \mathfrak{s}, \mathfrak{t}$) for tuples of positive integers and normal characters (e.g. s_1, t_1) for scalars.

Let $\mathbb{N}$ be the set of positive integers. Letting $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ we set $s_i = 0$ for $i > \text{depth}(\mathfrak{s}) = r$. For any tuple $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, recall that

$$\zeta_A(\mathfrak{s}) = \zeta_A(s_1, \dots, s_r) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum is over $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$; $\text{depth}(\mathfrak{s}) := r$ is called the depth, and $w(\mathfrak{s}) := s_1 + \dots + s_r$ the weight of $\mathfrak{s}$ and $\zeta_A(\mathfrak{s})$. We have extended the definition of $\zeta_A(\mathfrak{s})$ to the empty tuple by defining the associated zeta value to be 1. Note that each MZV does not vanish (see [24]).

2.1. Power sums. Let $\mathfrak{s} = (s_1, \dots, s_r)$ and $\mathfrak{t} = (t_1, \dots, t_k)$ be tuples of positive integers. We say that $\mathfrak{s} \leq \mathfrak{t}$ if the following assertions hold:

- For all $i \in \mathbb{N}$ we have $s_1 + \dots + s_i \leq t_1 + \dots + t_i$ where we recall $s_i = 0$ (resp. $t_i = 0$) for i bigger than the depth of $\mathfrak{s}$ (resp. $\mathfrak{t}$).
- $\mathfrak{s}$ and $\mathfrak{t}$ have the same weight.

Letting $\mathfrak{s} = (s_1, s_2, \dots, s_r) \in \mathbb{N}^r$ we set $\mathfrak{s}_- := (s_2, \dots, s_r)$. For $i \in \mathbb{N}$ we define $T_i(\mathfrak{s})$ to be the tuple $(s_1 + \dots + s_i, s_{i+1}, \dots, s_r)$. Note that $T_1(\mathfrak{s}) = \mathfrak{s}$. Further, for tuples of positive integers $\mathfrak{s}, \mathfrak{t}$ and for $i \in \mathbb{N}$, if $T_i(\mathfrak{s}) \leq T_i(\mathfrak{t})$, then $T_k(\mathfrak{s}) \leq T_k(\mathfrak{t})$ for all $k \geq i$.

For $d \in \mathbb{Z}$ we introduce

$$S_d(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $d = \deg a_1 > \dots > \deg a_r$. Further, we define

$$S_{<d}(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty$$

where the sum is over $(a_1, \dots, a_r) \in A_+^r$ with $d > \deg a_1 > \dots > \deg a_r$. Thus

$$S_{<d}(\mathfrak{s}) = \sum_{i=0}^{d-1} S_i(\mathfrak{s}), \quad S_d(\mathfrak{s}) = S_d(s_1)S_{<d}(\mathfrak{s}_-) = S_d(s_1)S_{<d}(s_2, \dots, s_r).$$

Here by convention we define empty sums to be 0 and empty products to be 1. In particular, $S_{<d}$ of the empty tuple is equal to 1.

We briefly recall some results of Thakur concerning power sums in [25] (see also [26, §5.2]). Thakur first proved (see [25, Theorems 1 and 2]) that for all $a, b \in \mathbb{N}$, there exist $\Delta_{a,b}^i \in \mathbb{F}_p$ for $0 < i < a + b$ such that for all $d \in \mathbb{Z}$,

$$(2.1) \quad S_d(a)S_d(b) = S_d(a+b) + \sum_{0 < i < a+b} \Delta_{a,b}^i S_d(a+b-i, i).$$

Shortly after, Chen [11] gave explicit formulas for the coefficients $\Delta_{a,b}^i$ and proved

$$\Delta_{a,b}^i = \begin{cases} (-1)^{a-1} \binom{i-1}{a-1} + (-1)^{b-1} \binom{i-1}{b-1} & \text{if } (q-1) \mid i \text{ and } 0 < i < a+b, \\ 0 & \text{otherwise.} \end{cases}$$

Using this product for the product of two power sums of depth 1, one can define the product of two power sums of arbitrary depth as follows. For two tuples of positive integers $\mathfrak{a} = (a_1, \dots, a_r)$ and $\mathfrak{b} = (b_1, \dots, b_k)$, we recall $\mathfrak{a}_- = (a_2, \dots, a_r)$ and $\mathfrak{b}_- = (b_2, \dots, b_k)$. Then we use the following formulas

$$(2.2) \quad \begin{aligned} S_d(\mathfrak{a})S_d(\mathfrak{b}) &:= (S_d(a_1)S_d(b_1))(S_{<d}(\mathfrak{a}_-)S_{<d}(\mathfrak{b}_-)) \\ &= S_d(a_1 + b_1)(S_{<d}(\mathfrak{a}_-)S_{<d}(\mathfrak{b}_-)) \\ &\quad + \sum_{0 < i < a_1+b_1} \Delta_{a_1,b_1}^i S_d(a_1 + b_1 - i)(S_{<d}(i)(S_{<d}(\mathfrak{a}_-)S_{<d}(\mathfrak{b}_-))), \end{aligned}$$

and

$$(2.3) \quad S_d(\mathfrak{a})S_{<d}(\mathfrak{b}) = S_{<d}(\mathfrak{b})S_d(\mathfrak{a}) := S_d(a_1)(S_{<d}(\mathfrak{a}_-)S_{<d}(\mathfrak{b})),$$

$$(2.4) \quad \begin{aligned} S_{<d}(\mathfrak{a})S_{<d}(\mathfrak{b}) &:= \sum_{i < d} S_i(a_1)(S_{<i}(\mathfrak{a}_-)S_{<i}(\mathfrak{b})) + \sum_{i < d} S_i(b_1)(S_{<i}(\mathfrak{a})S_{<i}(\mathfrak{b}_-)) \\ &\quad + \sum_{i < d} S_i(\mathfrak{a})S_i(\mathfrak{b}). \end{aligned}$$

From the above formulas we deduce (see [25, Theorem 3])

Proposition 2.1 (Thakur). *Let $\mathfrak{a} = (a_1, \dots, a_r)$ and $\mathfrak{b} = (b_1, \dots, b_k)$ be two tuples of positive integers.*

1) *There exist constants $f_i \in \mathbb{F}_p$ and tuples of positive integers $\mathfrak{c}_i$ with $\mathfrak{c}_i \leq \mathfrak{a} + \mathfrak{b}$ and $\text{depth}(\mathfrak{c}_i) \leq \text{depth}(\mathfrak{a}) + \text{depth}(\mathfrak{b})$ for all i , such that for all $d \in \mathbb{Z}$,*

$$(2.5) \quad S_d(\mathfrak{a})S_d(\mathfrak{b}) = \sum_i f_i S_d(\mathfrak{c}_i).$$

2) There exist constants $f'_i \in \mathbb{F}_p$ and tuples of positive integers $\mathbf{c}'_i$ with $\mathbf{c}'_i \leq \mathbf{a} + \mathbf{b}$ and $\text{depth}(\mathbf{c}'_i) \leq \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b})$ for all i , such that for all $d \in \mathbb{Z}$,

$$(2.6) \quad S_{<d}(\mathbf{a})S_{<d}(\mathbf{b}) = \sum_i f'_i S_{<d}(\mathbf{c}'_i).$$

Proof. We write down a complete proof for the convenience of the reader.

The proof is by induction on $\text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b})$. We start with $\text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}) = 2$. Thus $\text{depth}(\mathbf{a}) = \text{depth}(\mathbf{b}) = 1$. Then Proposition 2.1 follows from the explicit formulas (2.1) and (2.4).

Suppose that Proposition 2.1 holds for tuples $\mathbf{a}, \mathbf{b}$ such that $\text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}) < d$ where $d \in \mathbb{N}$ and $d \geq 3$. We claim that Proposition 2.1 holds for tuples $\mathbf{a}, \mathbf{b}$ such that $\text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}) = d$.

We first prove Part 1 by using the formula (2.2). By the induction hypothesis there exist constants $f'_j, g'_{ij} \in \mathbb{F}_p$ and

- tuples $\mathbf{c}'_j$ with $\mathbf{c}'_j \leq \mathbf{a}_- + \mathbf{b}_-$ and $\text{depth}(\mathbf{c}'_j) \leq \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b}_-)$,
- tuples $\mathbf{d}'_{ij}$ with $\mathbf{d}'_{ij} \leq (i) + \mathbf{a}_- + \mathbf{b}_-$ and $\text{depth}(\mathbf{d}'_{ij}) \leq 1 + \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b}_-)$,

such that for all $d \in \mathbb{Z}$ and all $0 < i < a_1 + b_1$,

$$S_{<d}(\mathbf{a}_-)S_{<d}(\mathbf{b}_-) = \sum_j f'_j S_{<d}(\mathbf{c}'_j), \quad S_{<d}(i)(S_{<d}(\mathbf{a}_-)S_{<d}(\mathbf{b}_-)) = \sum_j g'_{ij} S_{<d}(\mathbf{d}'_{ij}).$$

Note that $\text{depth}(\mathbf{a}_-) = \text{depth}(\mathbf{a}) - 1$ and $\text{depth}(\mathbf{b}_-) = \text{depth}(\mathbf{b}) - 1$.

The formula (2.2) gives

$$\begin{aligned} S_d(\mathbf{a})S_d(\mathbf{b}) &= S_d(a_1 + b_1)(S_{<d}(\mathbf{a}_-)S_{<d}(\mathbf{b}_-)) \\ &\quad + \sum_{0 < i < a_1 + b_1} \Delta_{a_1, b_1}^i S_d(a_1 + b_1 - i)(S_{<d}(i)(S_{<d}(\mathbf{a}_-)S_{<d}(\mathbf{b}_-))) \\ &= \sum_j f'_j S_d(a_1 + b_1)S_{<d}(\mathbf{c}'_j) + \sum_{0 < i < a_1 + b_1} \Delta_{a_1, b_1}^i S_d(a_1 + b_1 - i) \sum_j g'_{ij} S_{<d}(\mathbf{d}'_{ij}) \\ &= \sum_j f'_j S_d(a_1 + b_1, \mathbf{c}'_j) + \sum_{0 < i < a_1 + b_1} \Delta_{a_1, b_1}^i \sum_j g'_{ij} S_d(a_1 + b_1 - i, \mathbf{d}'_{ij}). \end{aligned}$$

We see that

$$(a_1 + b_1, \mathbf{c}'_j) \leq (a_1 + b_1, \mathbf{a}_- + \mathbf{b}_-) = \mathbf{a} + \mathbf{b},$$

$$\text{depth}(a_1 + b_1, \mathbf{c}'_j) \leq 1 + \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b}_-) < \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}),$$

and for all $0 < i < a_1 + b_1$,

$$(a_1 + b_1 - i, \mathbf{d}'_{ij}) \leq (a_1 + b_1 - i, (i) + \mathbf{a}_- + \mathbf{b}_-) \leq \mathbf{a} + \mathbf{b},$$

$$\text{depth}(a_1 + b_1 - i, \mathbf{d}'_{ij}) \leq 2 + \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b}_-) = \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}).$$

The proof of Part 1 is complete.

We now prove Part 2 by using the formula (2.4). By the induction hypothesis there exist constants $f'_j, g'_{ij} \in \mathbb{F}_p$ and

- tuples $\mathbf{c}''_j$ with $\mathbf{c}''_j \leq \mathbf{a}_- + \mathbf{b}$ and $\text{depth}(\mathbf{c}''_j) \leq \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b})$,
- tuples $\mathbf{d}''_j$ with $\mathbf{d}''_j \leq \mathbf{a} + \mathbf{b}_-$ and $\text{depth}(\mathbf{d}''_j) \leq \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}_-)$,

such that for all $d \in \mathbb{Z}$,

$$S_{<d}(\mathbf{a}_-)S_{<d}(\mathbf{b}) = \sum_j f''_j S_{<d}(\mathbf{c}''_j), \quad S_{<d}(\mathbf{a})S_{<d}(\mathbf{b}_-) = \sum_j g''_j S_{<d}(\mathbf{d}''_j).$$

By Part 1 there exist constants $f_j \in \mathbb{F}_p$ and tuples of positive integers $\mathbf{c}_j$ with $\mathbf{c}_j \leq \mathbf{a} + \mathbf{b}$ and $\text{depth}(\mathbf{c}_j) \leq \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b})$ for all j , such that for all $d \in \mathbb{Z}$,

$$S_d(\mathbf{a}) S_d(\mathbf{b}) = \sum_j f_j S_d(\mathbf{c}_j).$$

Putting all together into (2.4) yields

$$\begin{aligned} S_{<d}(\mathbf{a}) S_{<d}(\mathbf{b}) &= \sum_{i < d} S_i(a_1) (S_{<i}(\mathbf{a}_-) S_{<i}(\mathbf{b})) + \sum_{i < d} S_i(b_1) (S_{<i}(\mathbf{a}) S_{<i}(\mathbf{b}_-)) + \sum_{i < d} S_i(\mathbf{a}) S_i(\mathbf{b}) \\ &= \sum_j f_j'' S_{<d}(a_1, \mathbf{c}_j'') + \sum_j g_j'' S_{<d}(b_1, \mathbf{d}_j'') + \sum_j f_j S_{<d}(\mathbf{c}_j). \end{aligned}$$

Moreover,

$$\begin{aligned} (a_1, \mathbf{c}_j'') &\leq (a_1, \mathbf{a}_- + \mathbf{b}) \leq \mathbf{a} + \mathbf{b}, \\ \text{depth}(a_1, \mathbf{c}_j'') &\leq 1 + \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b}) = \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b}). \end{aligned}$$

Similarly, $(b_1, \mathbf{d}_j'') \leq \mathbf{a} + \mathbf{b}$ and $\text{depth}(b_1, \mathbf{d}_j'') \leq \text{depth}(\mathbf{a}) + \text{depth}(\mathbf{b})$. Thus we get Part 2. $\square$

As a direct consequence, we deduce the following result for the product $S_d(\mathbf{a}) S_{<d}(\mathbf{b})$. In fact, by Proposition 2.1, Part 2, there exist constants $f_i' \in \mathbb{F}_p$ and tuples of positive integers $\mathbf{c}_i'$ with $\mathbf{c}_i' \leq \mathbf{a}_- + \mathbf{b}$ and $\text{depth}(\mathbf{c}_i') \leq \text{depth}(\mathbf{a}_-) + \text{depth}(\mathbf{b})$ for all i , such that for all $d \in \mathbb{Z}$, we have $S_{<d}(\mathbf{a}_-) S_{<d}(\mathbf{b}) = \sum_i f_i' S_{<d}(\mathbf{c}_i')$. Thus

$$(2.7) \quad S_d(\mathbf{a}) S_{<d}(\mathbf{b}) = S_d(a_1) (S_{<d}(\mathbf{a}_-) S_{<d}(\mathbf{b})) = \sum_i f_i' S_d(a_1, \mathbf{c}_i').$$

For the rest of this paper, when we wish to express $S_d(\mathbf{a}) S_d(\mathbf{b})$ (resp. $S_{<d}(\mathbf{a}) S_{<d}(\mathbf{b})$, $S_d(\mathbf{a}) S_{<d}(\mathbf{b})$) as an $\mathbb{F}_p$ -linear combination of power sums $S_d(\mathbf{c})$ (resp. $S_{<d}(\mathbf{c})$, $S_d(\mathbf{c})$), the product (2.5) (resp. (2.6), (2.7)) will be used.

Remark 2.2. 1) We do not know whether the expression of $S_d(\mathbf{a}) S_d(\mathbf{b})$ (resp. $S_{<d}(\mathbf{a}) S_{<d}(\mathbf{b})$) as an $\mathbb{F}_p$ -linear combination of $S_d(\mathbf{c}_i)$ (resp. $S_{<d}(\mathbf{c}_i)$) as in Proposition 2.1 is unique. Note that it follows immediately from a conjecture of Thakur: the multiple zeta values $\zeta_A(\mathbf{s})$ are linearly independent over $\mathbb{F}_p$ (see [26, §6.3]).

2) We also mention the elementary fact which is useful in the sequel: for $a, b \in \mathbb{N}$ with $a + b \leq q$ we have $S_d(a) S_d(b) = S_d(a + b)$ for all $d \in \mathbb{Z}$. This equality follows immediately from (2.1). In fact, it was known from explicit formulas for power sums $S_d(a)$ with $a \leq q$ and $d \in \mathbb{Z}$ (see for example [28, Equation (3.3)]).

2.2. Binary relations and Todd's operations.

Definition 2.3. 1) A *binary relation* R of weight w is given by a collection of elements a_i, b_i of K such that for all $d \in \mathbb{Z}$,

$$\sum_i a_i S_d(\mathbf{s}_i) + \sum_i b_i S_{d+1}(\mathbf{t}_i) = 0$$

where the sum runs through tuples $\mathbf{s}_i$ and $\mathbf{t}_i$ of weight w . We denote the above equality by $R(d)$.

2) A binary relation is called a *fixed relation* if $b_i = 0$ for all i .

We denote by $\mathfrak{BR}_w$ the set of binary relations of weight w . An important example is the fundamental relation called R_1 (see [23, §3.4.6]) and given by

$$(2.8) \quad S_d(q) + D_1 S_{d+1}(1, q-1) = 0, \quad \text{with } D_1 = \theta^q - \theta \in K^*.$$

Remark 2.4. Note that binary relations give rise to linear relations among MZV's by taking the sum over $d \in \mathbb{Z}$:

$$\sum_i a_i \zeta_A(\mathfrak{s}_i) + \sum_i b_i \zeta_A(\mathfrak{t}_i) = 0.$$

In what follows we fix a binary relation R of weight w given by

$$R(d) : \sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) = 0, \quad a_i, b_i \in K.$$

Then Todd [28] defined the $\mathcal{B}^*$ maps and the $\mathcal{C}$ maps between the previous sets of binary relations as follows.

We first introduce the map $\mathcal{B}_v^* : \mathfrak{B}\mathfrak{R}_w \rightarrow \mathfrak{B}\mathfrak{R}_{w+v}$ attached to a singleton tuple (v) . We express $S_d(v) \sum_{j < d} R(d)$ as

$$\begin{aligned} 0 &= S_d(v) \left(\sum_{j < d} \sum_i a_i S_j(\mathfrak{s}_i) + \sum_{j < d} \sum_i b_i S_{j+1}(\mathfrak{t}_i) \right) \\ &= S_d(v) \left(\sum_i a_i S_{<d}(\mathfrak{s}_i) + \sum_i b_i (S_{<d}(\mathfrak{t}_i) + S_d(\mathfrak{t}_i)) \right) \\ &= \sum_i a_i S_d(v, \mathfrak{s}_i) + \sum_i b_i S_d(v, \mathfrak{t}_i) + \sum_i b_i \sum_j f_{ij} S_d(\mathfrak{c}_{ij}). \end{aligned}$$

Here the last equality holds since by (2.5) we have $S_d(v) S_d(\mathfrak{t}_i) = \sum_j f_{ij} S_d(\mathfrak{c}_{ij})$ for some $f_{ij} \in \mathbb{F}_p$ and some tuples $\mathfrak{c}_{ij}$ of weight $w + v$.

Proposition 2.5. *The map $\mathcal{B}_v^*$ sends R to the fixed relation*

$$\sum_i a_i S_d(v, \mathfrak{s}_i) + \sum_i b_i S_d(v, \mathfrak{t}_i) + \sum_i b_i S_d(v) S_d(\mathfrak{t}_i) = 0.$$

We point out two particular cases that will be used later (see [28, Theorems 3.22 and 3.24]).

Corollary 2.6. *Suppose that $b_i = 0$ for all i . Then $\mathcal{B}_v^*(R)$ equals*

$$\sum_i a_i S_d(v, \mathfrak{s}_i) = 0.$$

Proof. This corollary is an immediate consequence of Proposition 2.5. □

Corollary 2.7. *Suppose that for all i we have $v + t_{i1} \leq q$ where $\mathfrak{t}_i = (t_{i1}, \mathfrak{t}_{i-})$. Then $\mathcal{B}_v^*(R)$ equals*

$$\sum_i a_i S_d(v, \mathfrak{s}_i) + \sum_i b_i S_d(v, \mathfrak{t}_i) + \sum_i b_i S_d(v + t_{i1}, \mathfrak{t}_{i-}) = 0.$$

Proof. By Proposition 2.5 the map $\mathcal{B}_v^*$ sends R to the fixed relation

$$\sum_i a_i S_d(v, \mathfrak{s}_i) + \sum_i b_i S_d(v, \mathfrak{t}_i) + \sum_i b_i S_d(v) S_d(\mathfrak{t}_i) = 0.$$

For all i we have

$$S_d(v) S_d(\mathfrak{t}_i) = (S_d(v) S_d(t_{i1})) S_{<d}(\mathfrak{t}_{i-}) = S_d(v + t_{i1}) S_{<d}(\mathfrak{t}_{i-}) = S_d(v + t_{i1}, \mathfrak{t}_{i-}).$$

Here the first equality follows from (2.2), and the second equality holds by Remark 2.2, Part 2 and the fact that $v + t_{i1} \leq q$. Hence the corollary follows. □

Let $W = (w_1, \dots, w_r)$ be a tuple of positive integers. We define $\mathcal{B}_W^* : \mathfrak{BR}_w \rightarrow \mathfrak{BR}_{w+w(W)}$ by $\mathcal{B}_W^* := \mathcal{B}_{w_1}^* \circ \dots \circ \mathcal{B}_{w_r}^*$. Next we define $\mathcal{C}_W : \mathfrak{BR}_w \rightarrow \mathfrak{BR}_{w+w(W)}$ as follows. We express $S_{<d+1}(W)R(d)$ as

$$\begin{aligned} 0 &= S_{<d+1}(W) \left(\sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) \right) \\ &= \sum_i a_i S_d(\mathfrak{s}_i) (S_{<d}(W) + S_d(W)) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) S_{<d+1}(W) \\ &= \sum_i a_i S_d(\mathfrak{s}_i) S_{<d}(W) + \sum_i a_i S_d(\mathfrak{s}_i) S_d(W) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) S_{<d+1}(W). \end{aligned}$$

By (2.5) and (2.7) we then deduce

$$0 = S_{<d+1}(W) \left(\sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) \right) = \sum_j f_j S_d(\mathfrak{c}_j) + \sum_\ell g_\ell S_{d+1}(\mathfrak{d}_\ell)$$

for some $f_j, g_\ell \in \mathbb{F}_p$ and some tuples $\mathfrak{c}_j, \mathfrak{d}_\ell$ of weight $w + w(W)$.

Proposition 2.8. *The map $\mathcal{C}_W$ sends R to the binary relation*

$$\sum_i a_i S_d(\mathfrak{s}_i) S_{<d}(W) + \sum_i a_i S_d(\mathfrak{s}_i) S_d(W) + \sum_i b_i S_{d+1}(\mathfrak{t}_i) S_{<d+1}(W) = 0.$$

The following lemma is a particular case of Proposition 2.8.

Lemma 2.9. *Let $W = (w_1, \dots, w_r) = (w_1, W_-)$ be a tuple of positive integers. Recall that R_1 is defined as in (2.8). Then $\mathcal{C}_W(R_1)$ can be written as*

$$S_d(q + w_1, W_-) + \sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(1, \mathfrak{t}_i) = 0$$

for some $a_i, b_i \in K$ and some tuples of positive integers $\mathfrak{s}_i, \mathfrak{t}_i$ satisfying

- for all i , $\mathfrak{s}_i \leq (q) + W$ and $s_{i1} < q + w_1$,
- for all i , $\mathfrak{t}_i \leq (q - 1) + W$.

Proof. By Proposition 2.8 $\mathcal{C}_W(R_1)$ equals

$$S_d(q, W) + S_d(q) S_d(W) + D_1 S_{d+1}(1) (S_{<d+1}(q - 1) S_{<d+1}(W)) = 0.$$

By (2.2), (2.6) and Proposition 2.1 we have

$$S_d(q, W) + S_d(q) S_d(W) = S_d(q + w_1, W_-) + \sum_i a_i S_d(\mathfrak{s}_i),$$

$$D_1 S_{<d+1}(q - 1) S_{<d+1}(W) = \sum_i b_i S_{<d+1}(\mathfrak{t}_i),$$

for some $a_i, b_i \in K$ and some tuples $\mathfrak{s}_i, \mathfrak{t}_i$ satisfying

- for all i , $\mathfrak{s}_i \leq (q) + W$ and $s_{i1} < q + w_1$,
- for all i , $\mathfrak{t}_i \leq (q - 1) + W$.

Thus $\mathcal{C}_W(R_1)$ equals

$$S_d(q + w_1, W_-) + \sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(1, \mathfrak{t}_i) = 0$$

with the desired properties. $\square$

To end this section we introduce a new operation $\mathcal{BC}_q : \mathfrak{BR}_w \rightarrow \mathfrak{BR}_{w+q}$. By Proposition 2.5, $\mathcal{B}_q^*(R)$ is the following fixed relation

$$\sum_i a_i S_d(q, \mathfrak{s}_i) + \sum_i b_i S_d(q, \mathfrak{t}_i) + \sum_i b_i S_d(q) S_d(\mathfrak{t}_i) = 0.$$

By Proposition 2.8 the binary relation $\mathcal{C}_{\mathfrak{t}_i}(R_1)$ with R_1 as in (2.8) equals

$$S_d(q, \mathfrak{t}_i) + S_d(q) S_d(\mathfrak{t}_i) + D_1 S_{d+1}(1) (S_{<d+1}(q-1) S_{<d+1}(\mathfrak{t}_i)) = 0.$$

It follows that the combination $\mathcal{BC}_q(R) := \mathcal{B}_q^*(R) - \sum_i b_i \mathcal{C}_{\mathfrak{t}_i}(R_1)$ equals

$$\sum_i a_i S_d(q, \mathfrak{s}_i) - \sum_i b_i D_1 S_{d+1}(1) (S_{<d+1}(q-1) S_{<d+1}(\mathfrak{t}_i)) = 0.$$

Note that by (2.6) we can write

$$- \sum_i b_i D_1 S_{d+1}(1) (S_{<d+1}(q-1) S_{<d+1}(\mathfrak{t}_i)) = \sum_{i,j} b_{ij} S_{d+1}(1, \mathfrak{t}_{ij})$$

for some $b_{ij} \in K$ and some tuples $\mathfrak{t}_{ij}$ with $\mathfrak{t}_{ij} \leq (q-1) + \mathfrak{t}_i$. In particular, the binary relation $\mathcal{BC}_q(R)$ equals

$$\sum_i a_i S_d(q, \mathfrak{s}_i) + \sum_{i,j} b_{ij} S_{d+1}(1, \mathfrak{t}_{ij}) = 0.$$

Proposition 2.10. *The relation $\mathcal{BC}_q(R) := \mathcal{B}_q^*(R) - \sum_i b_i \mathcal{C}_{\mathfrak{t}_i}(R_1)$ equals*

$$\sum_i a_i S_d(q, \mathfrak{s}_i) - \sum_i b_i D_1 S_{d+1}(1) (S_{<d+1}(q-1) S_{<d+1}(\mathfrak{t}_i)) = 0.$$

2.3. Relations generated by the fundamental relation. We mention that the relation conjecture formulated by Todd (see [28, §5]) states that a spanning set for linear relations of weight $w + q$ with $w > 0$ is given by

$$\bigcup \mathcal{B}_U^* \circ \mathcal{C}_V(R_1)$$

where the union runs through the set of tuples of positive integers U and V (possibly empty) such that (U, V) is a tuple of positive integers of weight w .

Definition 2.11. 1) Let $\mathfrak{s} = (s_1, \dots, s_r)$ be a tuple of positive integers. We denote by $0 \leq i \leq r$ the biggest integer such that $s_j \leq q$ for all $1 \leq j \leq i$ and define the *initial tuple* $\text{Init}(\mathfrak{s})$ of $\mathfrak{s}$ to be the tuple

$$\text{Init}(\mathfrak{s}) := (s_1, \dots, s_i).$$

In particular, if $s_1 > q$, then $i = 0$ and $\text{Init}(\mathfrak{s})$ is the empty tuple.

2) For two different tuples $\mathfrak{s}$ and $\mathfrak{t}$, we consider the lexicographical order for initial tuples and write $\text{Init}(\mathfrak{t}) \preceq \text{Init}(\mathfrak{s})$ (resp. $\text{Init}(\mathfrak{t}) \prec \text{Init}(\mathfrak{s})$, $\text{Init}(\mathfrak{t}) \succeq \text{Init}(\mathfrak{s})$ and $\text{Init}(\mathfrak{t}) \succ \text{Init}(\mathfrak{s})$).

Proposition 2.12. *Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ with $s_1, \dots, s_{k-1} \leq q$ and $s_k > q$ for some $1 \leq k \leq r$. Recall that $\text{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$ as defined in Definition 2.11. Then $\zeta_A(\mathfrak{s})$ is equal to a K -linear combination of MZV's which can be decomposed into three types*

$$(2.9) \quad \zeta_A(\mathfrak{s}) = \underbrace{\sum_i a_i \zeta_A(\mathfrak{s}'_i)}_{\text{type 1}} + \underbrace{\sum_i b_i \zeta_A(\mathfrak{t}'_i)}_{\text{type 2}} + \underbrace{\sum_i c_i \zeta_A(\mathfrak{u}_i)}_{\text{type 3}},$$

with $a_i, b_i, c_i \in K$ such that the following properties hold:

- For all tuples $\mathbf{t}$ appearing on the right-hand side,

$$T_k(\mathbf{t}) \leq T_k(\mathbf{s}), \text{ and } \text{depth}(\mathbf{t}) \geq \text{depth}(\mathbf{s}).$$

- For tuples $\mathbf{s}'$ of type 1 with respect to $\mathbf{s}$,
 - either $\text{Init}(\mathbf{s}') \succ \text{Init}(\mathbf{s})$,
 - or $\text{Init}(\mathbf{s}') = \text{Init}(\mathbf{s})$ and $s'_k < s_k$.
- For tuples $\mathbf{t}'$ of type 2 with respect to $\mathbf{s}$, for all ℓ such that $k \leq \ell \leq \text{depth}(\mathbf{s})$, we have

$$t'_1 + \cdots + t'_\ell < s_1 + \cdots + s_\ell.$$

- For tuples $\mathbf{u}$ of type 3 with respect to $\mathbf{s}$, we have $\text{Init}(\mathbf{u}) \succ \text{Init}(\mathbf{s})$.

Proof. We set $W := (s_k - q, s_{k+1}, \dots, s_r)$. By Lemma 2.9 the binary relation $\mathcal{C}_W(R_1)$ equals

$$(2.10) \quad S_d(s_k, \dots, s_r) + \sum_i a_i S_d(\mathbf{s}_i) + \sum_i b_i S_{d+1}(1, \mathbf{t}_i) = 0$$

where $a_i, b_i \in K$, and for all i , we have

- $\mathbf{s}_i \leq (q) + W = (s_k, s_{k+1}, \dots, s_r)$ and $s_{i1} < s_k$,
- $\mathbf{t}_i \leq (q-1) + W = (s_k - 1, s_{k+1}, \dots, s_r)$.

Letting $s_0 := 0$ we can suppose that there exists $0 \leq j \leq k-1$ such that $s_j < q$ and $s_{j+1} = \cdots = s_{k-1} = q$. In what follows, for $m \in \mathbb{N}$, $q^{\{m\}}$ denotes the sequence of m consecutive q 's, and $q^{\{0\}}$ is just the empty tuple.

Proposition 2.10 applied $k-j-1$ times to (2.10) gives

$$(2.11) \quad S_d(q^{\{k-j-1\}}, s_k, \dots, s_r) + \sum_i a_i S_d(q^{\{k-j-1\}}, \mathbf{s}_i) \\ + \sum_{i_1 \dots i_{k-j}} b_{i_1 \dots i_{k-j}} S_{d+1}(1, \mathbf{t}_{i_1 \dots i_{k-j}}) = 0.$$

Here we define by induction $b_{i_1 \dots i_\ell} = -b_{i_1 \dots i_{\ell-1}} c_{i_1 \dots i_\ell} D_1$ where

$$(2.12) \quad S_{<d+1}(q-1) S_{<d+1}(1, \mathbf{t}_{i_1 \dots i_{\ell-1}}) = \sum_{i_\ell} c_{i_1 \dots i_\ell} S_{<d+1}(\mathbf{t}_{i_1 \dots i_\ell}), \quad c_{i_1 \dots i_\ell} \in \mathbb{F}_p.$$

By (2.12) and Proposition 2.1, for all $\ell \geq 2$,

$$\mathbf{t}_{i_1 \dots i_\ell} \leq (q-1) + (1, \mathbf{t}_{i_1 \dots i_{\ell-1}}) = (q, \mathbf{t}_{i_1 \dots i_{\ell-1}}).$$

Thus

$$(2.13) \quad \mathbf{t}_{i_1 \dots i_{k-j}} \leq (q^{\{k-j-1\}}, \mathbf{t}_{i_1}) \leq (q^{\{k-j-1\}}, s_k - 1, s_{k+1}, \dots, s_r).$$

Since $s_j < q$, by Corollary 2.7 we apply $\mathcal{B}_{s_j}^*$ to (2.11) to get the fixed relation

$$S_d(s_j, \dots, s_r) + \sum_i a_i S_d(s_j, q^{\{k-j-1\}}, \mathbf{s}_i) \\ + \sum_{i_1 \dots i_{k-j}} b_{i_1 \dots i_{k-j}} S_d(s_j, 1, \mathbf{t}_{i_1 \dots i_{k-j}}) + \sum_{i_1 \dots i_{k-j}} b_{i_1 \dots i_{k-j}} S_d(s_j + 1, \mathbf{t}_{i_1 \dots i_{k-j}}) = 0.$$

Next by Corollary 2.6 we apply $\mathcal{B}_{s_1}^* \circ \cdots \circ \mathcal{B}_{s_{j-1}}^*$ to the above relation to obtain

$$(2.14) \quad \begin{aligned} & S_d(s_1, \dots, s_r) + \sum_i a_i S_d(s_1, \dots, s_j, q^{\{k-j-1\}}, \mathfrak{s}_i) \\ & + \sum_{i_1 \dots i_{k-j}} b_{i_1 \dots i_{k-j}} S_d(s_1, \dots, s_j, 1, \mathfrak{t}_{i_1 \dots i_{k-j}}) \\ & + \sum_{i_1 \dots i_{k-j}} b_{i_1 \dots i_{k-j}} S_d(s_1, \dots, s_{j-1}, s_j + 1, \mathfrak{t}_{i_1 \dots i_{k-j}}) = 0. \end{aligned}$$

We analyze each tuple except the first one appearing in the above expression. Recall that $\mathfrak{s} = (s_1, \dots, s_j, q^{\{k-j-1\}}, s_k, \dots, s_r)$. Thus $\text{depth}(\mathfrak{s}) = r$, $T_k(\mathfrak{s}) = (s_1 + \cdots + s_k, s_{k+1}, \dots, s_r)$, and $\text{Init}(\mathfrak{s}) = (s_1, \dots, s_j, q^{\{k-j-1\}})$.

Type 1: tuples $\mathfrak{s}' = (s_1, \dots, s_j, q^{\{k-j-1\}}, \mathfrak{s}_i)$ in the second sum of (2.14).

By the above discussion we know that $\mathfrak{s}_i \leq (s_k, s_{k+1}, \dots, s_r)$ and $s_{i1} < s_k$. It follows that $\mathfrak{s}' \leq \mathfrak{s}$. We then deduce

- $\text{depth}(\mathfrak{s}') \geq \text{depth}(\mathfrak{s})$, and $T_k(\mathfrak{s}') \leq T_k(\mathfrak{s})$,
- $\text{Init}(\mathfrak{s}') \succeq (s_1, \dots, s_j, q^{\{k-j-1\}}) = \text{Init}(\mathfrak{s})$.

Further, $s'_k = s_{i1} < s_k$.

Type 2: tuples $\mathfrak{t}' = (s_1, \dots, s_j, 1, \mathfrak{t}_{i_1 \dots i_{k-j}})$ in the third sum of (2.14).

By (2.13) we know that $\mathfrak{t}_{i_1 \dots i_{k-j}} \leq (q^{\{k-j-1\}}, s_k - 1, s_{k+1}, \dots, s_r)$. Thus

$$\text{depth}(\mathfrak{t}') \geq j + 1 + (r - j) = r + 1 > \text{depth}(\mathfrak{s}).$$

Further, for all ℓ with $k \leq \ell \leq r$, we claim that

$$t'_1 + \cdots + t'_\ell < s_1 + \cdots + s_\ell.$$

In fact, for $\ell = k$,

$$\begin{aligned} t'_1 + \cdots + t'_k & \leq s_1 + \cdots + s_j + 1 + (k - j - 1)q \\ & < s_1 + \cdots + s_j + (k - j - 1)q + s_k \\ & = s_1 + \cdots + s_k. \end{aligned}$$

Here the second inequality follows from the fact that $s_k > q > 1$.

For $k + 1 \leq \ell \leq r$, we have

$$\begin{aligned} t'_1 + \cdots + t'_\ell & \leq s_1 + \cdots + s_j + 1 + (k - j - 1)q + (s_k - 1) + s_{k+1} + \cdots + s_{\ell-1} \\ & < s_1 + \cdots + s_j + (k - j - 1)q + s_k + s_{k+1} + \cdots + s_\ell \\ & = s_1 + \cdots + s_\ell. \end{aligned}$$

Here the second inequality holds since $s_\ell > 0$.

Type 3: tuples $\mathfrak{u} = (s_1, \dots, s_j + 1, \mathfrak{t}_{i_1 \dots i_{k-j}})$ in the last sum of (2.14).

We stress that these terms appear only in the case where $j \geq 1$. As before, by (2.13) we know that $\mathfrak{t}_{i_1 \dots i_{k-j}} \leq (q^{\{k-j-1\}}, s_k - 1, s_{k+1}, \dots, s_r)$. It implies that

- $\text{depth}(\mathfrak{u}) \geq j + (r - j) = r = \text{depth}(\mathfrak{s})$,
- $T_k(\mathfrak{u}) \leq T_k(\mathfrak{s})$,
- $\text{Init}(\mathfrak{u}) \succeq (s_1, \dots, s_j + 1) \succ \text{Init}(\mathfrak{s})$.

The proof is finished. $\square$

Proposition 2.13. *Let $\mathfrak{s} = (s_1, \dots, s_k) \in \mathbb{N}^k$ with $s_1, \dots, s_{k-1} \leq q$ and $s_k = q$. Then $\zeta_A(\mathfrak{s})$ is equal to a K -linear combination of MZV 's which can be decomposed into two types*

$$(2.15) \quad \zeta_A(\mathfrak{s}) = \underbrace{\sum_i b_i \zeta_A(\mathfrak{t}'_i)}_{\text{type 2}} + \underbrace{\sum_i c_i \zeta_A(\mathfrak{u}_i)}_{\text{type 3}},$$

with $b_i, c_i \in K$ such that the following properties hold:

- For all tuples $\mathfrak{t}$ appearing on the right-hand side,

$$T_k(\mathfrak{t}) \leq T_k(\mathfrak{s}), \text{ and } \text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s}).$$

- For tuples $\mathfrak{t}'$ of type 2 with respect to $\mathfrak{s}$, we have

$$t'_1 + \dots + t'_k < s_1 + \dots + s_k.$$

- For tuples $\mathfrak{u}$ of type 3 with respect to $\mathfrak{s}$, we have $\text{Init}(\mathfrak{u}) \succ \text{Init}(\mathfrak{s})$.

Proof. Letting W be the empty tuple, we note that $\mathcal{C}_W(R_1)$ is R_1 . Then the proof follows the same lines as that of Proposition 2.12, and we omit the details. $\square$

3. PROOF OF THEOREM A

This section is devoted to a proof of Theorem A.

Definition 3.1. Let $k \in \mathbb{N}$ and $\mathfrak{s}$ be a tuple of positive integers. We say that $\mathfrak{s}$ is k -admissible if it satisfies the following two conditions:

- 1) $s_1, \dots, s_k \leq q$.
- 2) $\mathfrak{s}$ is not of the form $(s_1, \dots, s_r)$ with $r \leq k$, $s_1, \dots, s_{r-1} \leq q$, and $s_r = q$.

Here we recall $s_i = 0$ for $i > \text{depth}(\mathfrak{s})$.

Proposition 3.2. *For $k \in \mathbb{N}$ and for all tuples $\mathfrak{s}$ we can express $\zeta_A(\mathfrak{s})$ as a K -linear combination of $\zeta_A(\mathfrak{t})$ such that $\mathfrak{t}$ is k -admissible, $T_k(\mathfrak{t}) \leq T_k(\mathfrak{s})$, and $\text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s})$.*

Proof. The proof is by induction on k . We denote by H_k the corresponding hypothesis for k :

For all tuples $\mathfrak{s}$ we can express $\zeta_A(\mathfrak{s})$ as a K -linear combination of $\zeta_A(\mathfrak{t})$ such that $\mathfrak{t}$ is k -admissible, $T_k(\mathfrak{t}) \leq T_k(\mathfrak{s})$, and $\text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s})$.

We start with $k = 1$ and note that $T_1(\mathfrak{t}) = \mathfrak{t}$ for all tuples $\mathfrak{t}$. We observe that a tuple $\mathfrak{t}$ is 1-admissible if $\mathfrak{t} \neq (q)$ and $t_1 \leq q$. We now prove H_1 by induction on the first term s_1 of $\mathfrak{s}$.

In fact, if $s_1 \leq q$, then either $\mathfrak{s} = (q)$ or $\mathfrak{s}$ is 1-admissible. If $\mathfrak{s} = (q)$, we are done by using R_1 as in (2.8). Otherwise, $\mathfrak{s}$ is 1-admissible, and we are also done.

Suppose that for $s > q$, H_1 holds for all tuples $\mathfrak{s}$ with $s_1 \leq s - 1$. We claim that H_1 holds for all tuples $\mathfrak{s} = (s_1, \dots, s_r)$ with $s_1 = s$. In fact, we set $W := (s_1 - q, s_2, \dots, s_r)$. By Lemma 2.9 the binary relation $\mathcal{C}_W(R_1)$ equals

$$S_d(s_1, \dots, s_r) + \sum_i a_i S_d(\mathfrak{s}_i) + \sum_i b_i S_{d+1}(1, \mathfrak{t}_i) = 0$$

where $a_i, b_i \in K$, and we have

- for all i , $\mathfrak{s}_i \leq (q) + W = (s_1, \dots, s_r)$ and $s_{i1} < s_1$,
- for all i , $\mathfrak{t}_i \leq (q - 1) + W = (s_1 - 1, s_2, \dots, s_r)$.

By Remark 2.4 we get

$$\zeta_A(\mathfrak{s}) + \sum_i a_i \zeta_A(\mathfrak{s}_i) + \sum_i b_i \zeta_A(1, \mathfrak{t}_i) = 0.$$

The induction hypothesis implies that H_1 holds for all tuples $\mathfrak{s}_i$. All tuples $(1, \mathfrak{t}_i)$ are 1-admissible. We then conclude that H_1 holds for $\mathfrak{s}$. Thus we have proved H_1 .

Let $k \in \mathbb{N}$ with $k \geq 2$. We suppose that H_{k-1} holds. We now prove H_k . The proof is again by induction on the sum $s := s_1 + \cdots + s_k$.

If $s = 1$, then it is clear that H_k holds. Let $s \in \mathbb{N}$ with $s \geq 2$. Suppose that H_k holds for $\mathfrak{s}$ with $s_1 + \cdots + s_k < s$. We now prove that H_k holds for all tuples $\mathfrak{s}$ with $s_1 + \cdots + s_k = s$.

In fact, let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple with $s_1 + \cdots + s_k = s$. We suppose further that $\mathfrak{s}$ is not k -admissible and satisfies $\text{depth}(\mathfrak{s}) = r \geq k$. In fact, by H_{k-1} we can suppose that $\mathfrak{s}$ is $(k-1)$ -admissible, but not k -admissible. In particular, we get $\text{depth}(\mathfrak{s}) \geq k$.

An algorithm. Starting with $\mathfrak{s}$ which is not k -admissible and satisfies $\text{depth}(\mathfrak{s}) \geq k$ and $s_1 + \cdots + s_k = s$, we give an algorithm to show that H_k holds for $\mathfrak{s}$.

Step 1. Since $\mathfrak{s}$ is not k -admissible and satisfies $\text{depth}(\mathfrak{s}) \geq k$, there are two possibilities for $\mathfrak{s}$:

- **Case 1:** $\mathfrak{s} = (s_1, \dots, s_r)$ with $s_1, \dots, s_{k_1-1} \leq q$, $s_{k_1} > q$ for some $k_1 \leq k$.
- **Case 2:** $\mathfrak{s} = (s_1, \dots, s_k)$ with $s_1, \dots, s_{k-1} \leq q$ and $s_k = q$.

For Case 1 we apply Proposition 2.12 to $\mathfrak{s}$ to obtain an expression (2.9) for $\zeta_A(\mathfrak{s})$. Otherwise, for Case 2 we apply Proposition 2.13 to $\mathfrak{s}$ to obtain an expression (2.15) for $\zeta_A(\mathfrak{s})$. Thus we always have

$$\zeta_A(\mathfrak{s}) = \underbrace{\sum_i a_i \zeta_A(\mathfrak{s}'_i)}_{\text{type 1}} + \underbrace{\sum_i b_i \zeta_A(\mathfrak{t}'_i)}_{\text{type 2}} + \underbrace{\sum_i c_i \zeta_A(\mathfrak{u}_i)}_{\text{type 3}}, \quad \text{with } a_i, b_i, c_i \in K.$$

We denote by $S(\mathfrak{s})$ the set of all the tuples $\mathfrak{t}$ appearing in this expression.

We claim that for all tuples $\mathfrak{t} \in S(\mathfrak{s})$, we have $T_k(\mathfrak{t}) \leq T_k(\mathfrak{s})$ and $\text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s})$. In fact, if we are in Case 1, then by Proposition 2.12, $T_{k_1}(\mathfrak{t}) \leq T_{k_1}(\mathfrak{s})$ and $\text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s})$. Since $k_1 \leq k$, it follows that $T_k(\mathfrak{t}) \leq T_k(\mathfrak{s})$. Otherwise, we are in Case 2, and the claim follows from Proposition 2.13.

Consequently, for all tuples $\mathfrak{t} \in S(\mathfrak{s})$ we get $t_1 + \cdots + t_k \leq s_1 + \cdots + s_k = s$. We divide the set $S(\mathfrak{s})$ into two disjoint subsets

$$S(\mathfrak{s}) = S(\mathfrak{s})_0 \sqcup S(\mathfrak{s})_1$$

where $S(\mathfrak{s})_0$ (resp. $S(\mathfrak{s})_1$) consists of all the tuples $\mathfrak{t} \in S(\mathfrak{s})$ such that $t_1 + \cdots + t_k < s$ (resp. $t_1 + \cdots + t_k = s$).

We claim that if $\mathfrak{t}' \in S(\mathfrak{s})$ is of type 2 with respect to $\mathfrak{s}$, then $\mathfrak{t}' \in S(\mathfrak{s})_0$. In fact, if we are in Case 1, the claim results from Proposition 2.12 and the fact that $k_1 \leq k \leq \text{depth}(\mathfrak{s})$. Otherwise, in Case 2 the claim follows immediately from Proposition 2.13.

Step 2. For tuples $\mathfrak{t} \in S(\mathfrak{s})_0$, since $t_1 + \cdots + t_k < s$, we apply the induction hypothesis H_k for $\mathfrak{t}$, and we are done with $\zeta_A(\mathfrak{t})$. Thus we are reduced to deal with tuples belonging to $S(\mathfrak{s})_1$. Let $\mathfrak{s}_1$ be such a tuple. Then by the above discussion $\mathfrak{s}_1$ is of type 1 or type 3 with respect to $\mathfrak{s}$. Further, we have seen that $T_k(\mathfrak{s}_1) \leq T_k(\mathfrak{s})$, $\text{depth}(\mathfrak{s}_1) \geq \text{depth}(\mathfrak{s}) \geq k$, and $s_{11} + \cdots + s_{1k} = s$. If $\mathfrak{s}_1$ is k -admissible, then we

are done with $\zeta_A(\mathfrak{s}_1)$. Otherwise, $\mathfrak{s}_1$ is not k -admissible and $\text{depth}(\mathfrak{s}_1) \geq k$. We then repeat Steps 1 and 2 for $\mathfrak{s}_1$, and so on.

Effectiveness of the algorithm. We claim that the algorithm must end after a finite number of steps. Suppose that it is not the case. Then we obtain an infinite sequence of tuples $\mathfrak{s}_0 := \mathfrak{s}, \mathfrak{s}_1, \mathfrak{s}_2, \dots$ such that for all i the following properties hold:

- $\mathfrak{s}_i$ is not k -admissible, and $\text{depth}(\mathfrak{s}_i) \geq k$,
- $T_k(\mathfrak{s}_{i+1}) \leq T_k(\mathfrak{s}_i)$, and $\text{depth}(\mathfrak{s}_{i+1}) \geq \text{depth}(\mathfrak{s}_i)$,
- $s_{i1} + \dots + s_{ik} = s$,
- $\mathfrak{s}_{i+1}$ is of type 1 or type 3 with respect to $\mathfrak{s}_i$.

By Proposition 2.12 and Proposition 2.13,

- If $\mathfrak{s}_{i+1}$ is of type 1 with respect to $\mathfrak{s}_i$, then $\text{Init}(\mathfrak{s}_{i+1}) \succeq \text{Init}(\mathfrak{s}_i)$.
- If $\mathfrak{s}_{i+1}$ is of type 3 with respect to $\mathfrak{s}_i$, then $\text{Init}(\mathfrak{s}_{i+1}) \succ \text{Init}(\mathfrak{s}_i)$.

Since $\mathfrak{s}_i$ is not k -admissible, it follows that $\text{depth}(\text{Init}(\mathfrak{s}_i)) \leq k$. Thus $\text{Init}(\mathfrak{s}_i) \preceq q^{\{k\}}$ where $q^{\{k\}}$ is the sequence of k consecutive q 's. It implies that for i sufficiently large, we always have $\text{Init}(\mathfrak{s}_{i+1}) = \text{Init}(\mathfrak{s}_i)$. This forces that for i sufficiently large, $\mathfrak{s}_{i+1}$ is of type 1 with respect to $\mathfrak{s}_i$. Since $\text{Init}(\mathfrak{s}_{i+1}) = \text{Init}(\mathfrak{s}_i)$, it follows from Proposition 2.12 that $s_{(i+1)\ell} < s_{i\ell}$ where $\ell := \text{depth}(\text{Init}(\mathfrak{s}_i)) + 1$. Therefore, we get a contradiction.

We conclude that our algorithm must end after a finite number of steps. The proof is finished. $\square$

Proof of Theorem A. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple of weight w . Proposition 3.2 applied to $k = w$ implies that we can express $\zeta_A(\mathfrak{s})$ as a K -linear combination of $\zeta_A(\mathfrak{t})$ such that $\mathfrak{t}$ is w -admissible, $T_w(\mathfrak{t}) \leq T_w(\mathfrak{s})$, and $\text{depth}(\mathfrak{t}) \geq \text{depth}(\mathfrak{s})$. Since $w(\mathfrak{t}) = w$ and $\mathfrak{t}$ is w -admissible, we get $\mathfrak{t} \in \mathcal{T}_w$. Then Theorem A follows. $\square$

4. TRANSCENDENTAL TOOLS

In this section we briefly review the basic theory of Anderson dual t -motives. We refer the reader to [7, §1.5] and [15, §2.4] for more details.

4.1. Dual t -motives. Letting t be another variable we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ equipped with the Gauss norm $\|\cdot\|$.

For $k \in \mathbb{Z}$ we consider the k -fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\begin{aligned} \mathbb{C}_\infty((t)) &\rightarrow \mathbb{C}_\infty((t)) \\ f = \sum_i a_i t^i &\mapsto f^{(k)} := \sum_i a_i^{q^k} t^i \end{aligned}$$

We extend k -fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entry-wise. We denote by $\overline{K}[t, \sigma]$ be the non-commutative $\overline{K}[t]$ -algebra generated by a new variable σ with the rules $\sigma f = f^{(-1)} \sigma$ for all $f \in \overline{K}[t]$.

Definition 4.1. An *effective dual t -motive* is a left $\overline{K}[t, \sigma]$ -module $\mathcal{M}$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t - \theta)^\ell (\mathcal{M} / \sigma \mathcal{M}) = \{0\}.$$

We mention that effective dual t -motives are called Frobenius modules in [10, §2.2]. Note that Hartl and Juschka [15, §2.4] introduced a more general notion of dual t -motives. In particular, effective dual t -motives are always dual t -motives.

Throughout this paper we will always work with effective dual t -motives. Therefore, we will sometimes drop the word "effective" where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t -motives. Then a morphism of effective dual t -motives $\mathcal{M} \rightarrow \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t, \sigma]$ -modules. We denote by $\mathcal{F}$ the category of effective dual t -motives equipped with the trivial object $\mathbf{1}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \text{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$ -module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$ -basis for $\mathcal{M}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ given by a matrix $\Phi \in \text{Mat}_r(\overline{K}[t])$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \text{GL}_r(\mathbb{T})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$.

4.2. Dual t -motives connected to MZV's. We briefly review Anderson-Thakur polynomials introduced in [3]. For $k \geq 0$ we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^k [\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$ we write $n - 1 = \sum_{j \geq 0} n_j q^j$ with $0 \leq n_j \leq q - 1$ and define $\Gamma_n := \prod_{j \geq 0} D_j^{n_j}$. We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^j (\theta^{q^\ell} - t^{q^\ell})$ for $j \geq 1$. Then Anderson-Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n \geq 1} \frac{\alpha_n(t)}{\Gamma_n} x^n := x \left(1 - \sum_{j \geq 0} \frac{\gamma_j(t)}{D_j} x^{q^j} \right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t : $H_n(t) = \alpha_n(t)|_{t=\theta, \theta=t}$. By [3,

Equation (3.7.3)] we get that $\|H_n\| < |\theta|_{\infty}^{\frac{nq}{q-1}}$. We mention that H_n here is denoted by H_{n-1} in [3, 10, 17].

Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple. Following [4] we consider the dual t -motives $\mathcal{M}_{\mathfrak{s}}$ and $\mathcal{M}'_{\mathfrak{s}}$ attached to $\mathfrak{s}$ given by

$$\Phi_{\mathfrak{s}} = \begin{pmatrix} (t - \theta)^{s_1 + \dots + s_r} & 0 & 0 & \dots & 0 \\ H_{s_1}^{(-1)}(t - \theta)^{s_1 + \dots + s_r} & (t - \theta)^{s_2 + \dots + s_r} & 0 & \dots & 0 \\ 0 & H_{s_2}^{(-1)}(t - \theta)^{s_2 + \dots + s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t - \theta)^{s_r} & 0 \\ 0 & \dots & 0 & H_{s_r}^{(-1)}(t - \theta)^{s_r} & 1 \end{pmatrix} \in \text{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s}} \in \text{Mat}_r(\overline{K}[t])$ which is the upper left $r \times r$ submatrix of $\Phi_{\mathfrak{s}}$.

Throughout this paper, we work with the Carlitz period $\tilde{\pi}$ which is a fundamental period of the Carlitz module (see [14, 22]). We fix a choice of $(q-1)$ st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta^{q^i}} \right) \in \mathbb{T}^{\times}$$

so that $\Omega^{(-1)} = (t - \theta)\Omega$ and $\frac{1}{\Omega(\theta)} = \tilde{\pi}$. Given $\mathfrak{s}$ as above, Chang introduced the following series (see [9, Lemma 5.3.1] and also [10, Equation (2.3.2)]):

$$(4.1) \quad \mathfrak{L}(\mathfrak{s}) = \mathfrak{L}(s_1, \dots, s_r) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} H_{s_r})^{(i_r)} \dots (\Omega^{s_1} H_{s_1})^{(i_1)}.$$

Letting $\Gamma_{\mathfrak{s}} = \Gamma_{s_1} \dots \Gamma_{s_r}$, by [9, Equation (5.5.3)] we have

$$(4.2) \quad \mathfrak{L}(\mathfrak{s})(\theta) = \Gamma_{\mathfrak{s}} \zeta_A(\mathfrak{s}) / \tilde{\pi}^{w(\mathfrak{s})}.$$

If we denote $\mathcal{E}$ the ring of series $\sum_{n \geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n \rightarrow +\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$, then any $f \in \mathcal{E}$ is an entire function. It is proved that $\mathfrak{L}(\mathfrak{s}) \in \mathcal{E}$ (see [9, Lemma 5.3.1]).

Then the matrix given by

$$\Psi_{\mathfrak{s}} = \begin{pmatrix} \Omega^{s_1 + \dots + s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1)\Omega^{s_2 + \dots + s_r} & \Omega^{s_2 + \dots + s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2)\Omega^{s_3 + \dots + s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1, \dots, s_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2, \dots, s_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1, \dots, s_r) & \mathfrak{L}(s_2, \dots, s_r) & \dots & \mathfrak{L}(s_r) & 1 \end{pmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi_{\mathfrak{s}}^{(-1)} = \Phi_{\mathfrak{s}} \Psi_{\mathfrak{s}}.$$

Thus $\Psi_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t -motive $\mathcal{M}_{\mathfrak{s}}$.

We also denote by $\Psi'_{\mathfrak{s}}$ the upper $r \times r$ submatrix of $\Psi_{\mathfrak{s}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t -motive $\mathcal{M}'_{\mathfrak{s}}$.

4.3. The Anderson-Brownawell-Papanikolas criterion. We close this section by recalling the Anderson-Brownawell-Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

Theorem 4.2 (Anderson-Brownawell-Papanikolas). *Let $\Phi \in \mathrm{Mat}_{\ell}(\overline{K}[t])$ be a matrix such that $\det \Phi = c(t - \theta)^s$ for some $c \in \overline{K}$ and $s \in \mathbb{Z}^{\geq 0}$. Let $\psi \in \mathrm{Mat}_{\ell \times 1}(\mathcal{E})$ be a vector satisfying $\psi^{(-1)} = \Phi \psi$ and $\rho \in \mathrm{Mat}_{1 \times \ell}(\overline{K})$ such that $\rho \psi(\theta) = 0$. Then there exists a vector $P \in \mathrm{Mat}_{1 \times \ell}(\overline{K}[t])$ such that*

$$P\psi = 0 \quad \text{and} \quad P(\theta) = \rho.$$

5. PROOF OF THEOREM B

This section is devoted to a proof of Theorem B. The proof is by induction on the weight $w \in \mathbb{N}$. For $w = 1$ we are done since $\zeta_A(1)$ is nonzero. One can show that $\zeta_A(1)$ is even transcendental over K by Wade [29] (see also [30]). Suppose that for $w' < w$ MZV's in $\mathcal{T}_{w'}^0$ are all linearly independent over K , hence over $\overline{K}$ by [9, Proposition 4.3.1].

We claim that MZV's in $\mathcal{T}_w^0$ are all linearly independent over K . Suppose that there exist $\zeta_A(\mathfrak{s}_i) \in \mathcal{T}_w^0$ and $a_i \in A \setminus \{0\}$ such that

$$\sum_i a_i \Gamma_{\mathfrak{s}_i} \zeta_A(\mathfrak{s}_i) = 0.$$

We show that this relation leads to a contradiction. The proof of this fact is divided into several steps.

5.1. Step 1. We first construct a dual t -motive to which we will apply the Anderson-Brownawell-Papanikolas criterion. In what follows we set $a_i(t) := a_i|_{\theta=t} \in \mathbb{F}_q[t] \setminus \{0\}$.

For each $\mathfrak{s}_i$ we have attached to it a matrix $\Phi_{\mathfrak{s}_i}$. For each $\mathfrak{s}_i$ we write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and define the set of tuples

$$I(\mathfrak{s}_i) = \{\emptyset, (s_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)})\}.$$

Recall that $\mathfrak{s}_i$ is of weight w , that means $s_{i1} + \dots + s_{i\ell_i} = w$.

We construct a new matrix Φ' by merging the same rows of $\Phi'_{s_1}, \dots, \Phi'_{s_n}$ as follows. We set $I := \cup_i I(s_i)$. Then the matrix Φ' will be a matrix indexed by elements of I , write $\Phi' = (\Phi'_{\mathbf{t}, \mathbf{t}'})_{\mathbf{t}, \mathbf{t}' \in I} \in \text{Mat}_{|I|}(\overline{K}[t])$. For the row which corresponds to the empty tuple $\emptyset$ we define

$$\Phi'_{\emptyset, \mathbf{t}'} = \begin{cases} (t - \theta)^w & \text{if } \mathbf{t}' = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by $\mathbf{t} = (s_{i1}, \dots, s_{ij})$ for some i and $1 \leq j \leq \ell_i - 1$ we put

$$\Phi'_{\mathbf{t}, \mathbf{t}'} = \begin{cases} (t - \theta)^{w-w(\mathbf{t}')} & \text{if } \mathbf{t}' = \mathbf{t}, \\ H_{s_{ij}}^{(-1)}(t - \theta)^{w-w(\mathbf{t}')} & \text{if } \mathbf{t}' = (s_{i1}, \dots, s_{i(j-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Phi'_{s_i} = (\Phi'_{\mathbf{t}, \mathbf{t}'})_{\mathbf{t}, \mathbf{t}' \in I(s_i)}$ for all i .

We define $\Phi \in \text{Mat}_{|I|+1}(\overline{K}[t])$ by

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \text{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathbf{t}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t]),$$

where

$$v_{\mathbf{t}} = \begin{cases} a_i(t) H_{s_{i\ell_i}}^{(-1)}(t - \theta)^{w-w(\mathbf{t})} & \text{if } \mathbf{t} = (s_{i1}, \dots, s_{i(\ell_i-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

We now introduce a rigid analytic trivialization matrix Ψ for Φ . We define $\Psi' = (\Psi'_{\mathbf{t}, \mathbf{t}'})_{\mathbf{t}, \mathbf{t}' \in I} \in \text{GL}_{|I|}(\mathbb{T})$ as follows. For the row which corresponds to the empty tuple $\emptyset$ we define

$$\Psi'_{\emptyset, \mathbf{t}'} = \begin{cases} \Omega^w & \text{if } \mathbf{t}' = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by $\mathbf{t} = (s_{i1}, \dots, s_{ij})$ for some i and $1 \leq j \leq \ell_i - 1$ we put

$$\Psi'_{\mathbf{t}, \mathbf{t}'} = \begin{cases} \mathfrak{L}(\mathbf{t}) \Omega^{w-w(\mathbf{t})} & \text{if } \mathbf{t}' = \emptyset, \\ \mathfrak{L}(s_{i(k+1)}, \dots, s_{ij}) \Omega^{w-w(\mathbf{t})} & \text{if } \mathbf{t}' = (s_{i1}, \dots, s_{ik}) \text{ for some } 1 \leq k \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Psi'_{s_i} = (\Psi'_{\mathbf{t}, \mathbf{t}'})_{\mathbf{t}, \mathbf{t}' \in I(s_i)}$ for all i .

We define $\Psi \in \text{GL}_{|I|+1}(\mathbb{T})$ by

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \text{GL}_{|I|+1}(\mathbb{T}), \quad \mathbf{f} = (f_{\mathbf{t}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\mathbb{T})$$

with

$$(5.1) \quad f_{\mathbf{t}} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i})$$

where the sum runs through the set of indices i such that $\mathbf{t} = (s_{i1}, \dots, s_{ik})$ for some $0 \leq k \leq \ell_i - 1$. In particular, $f_{\emptyset} = \sum_i a_i(t) \mathfrak{L}(s_i)$.

By construction and by §4.2, we get $\Psi^{(-1)} = \Phi \Psi$, that means Ψ is a rigid analytic trivialization for Φ .

5.2. Step 2. Next we apply the Anderson-Brownawell-Papanikolas criterion and combine with the induction hypothesis to get some rationality results. More precisely, we claim that for all $\mathfrak{t} \in I$, $f_{\mathfrak{t}}(\theta)$ belongs to K where $f_{\mathfrak{t}}$ is given as in (5.1).

In fact, we consider the first column vector of Ψ

$$\psi = \left(\begin{array}{c} \Psi'_{\mathfrak{t}, \emptyset} \\ f_{\emptyset} \end{array} \right)_{\mathfrak{t} \in I}.$$

Then we have $\psi^{(-1)} = \Phi\psi$.

We also observe that for all $\mathfrak{t} \in I$ we have $\Psi'_{\mathfrak{t}, \emptyset} = \mathfrak{L}(\mathfrak{t})\Omega^{w-w(\mathfrak{t})}$. Further, by (4.2),

$$f_{\emptyset}(\theta) = \sum_i a_i \mathfrak{L}(\mathfrak{s}_i)(\theta) = \sum_i a_i \Gamma_{\mathfrak{s}_i} \zeta_A(\mathfrak{s}_i) / \tilde{\pi}^w = 0.$$

By Theorem 4.2 with $\rho = (0, \dots, 0, 1)$ we deduce that there exists $\mathbf{h} = (g_{\mathfrak{t}}, g) \in \text{Mat}_{1 \times (|I|+1)}(\overline{K}[t])$ such that $\mathbf{h}\psi = 0$, and that $g_{\mathfrak{t}}(\theta) = 0$ for $\mathfrak{t} \in I$ and $g(\theta) \neq 0$. If we put $\mathbf{g} := (1/g)\mathbf{h} \in \text{Mat}_{1 \times (|I|+1)}(\overline{K}(t))$, then all the entries of $\mathbf{g}$ are regular at $t = \theta$.

Now we have

$$(5.2) \quad (\mathbf{g} - \mathbf{g}^{(-1)}\Phi)\psi = \mathbf{g}\psi - (\mathbf{g}\psi)^{(-1)} = 0.$$

We write $\mathbf{g} - \mathbf{g}^{(-1)}\Phi = (B_{\mathfrak{t}}, 0)_{\mathfrak{t} \in I}$. We claim that $B_{\mathfrak{t}} = 0$ for all $\mathfrak{t} \in I$. In fact, expanding (5.2) we obtain

$$(5.3) \quad \sum_{\mathfrak{t} \in I} B_{\mathfrak{t}} \mathfrak{L}(\mathfrak{t}) \Omega^{w-w(\mathfrak{t})} = 0.$$

By [9, Lemma 5.3.5] (see also [10, Proposition 2.3.3]) we see that for $\mathfrak{t} \in I$ and $j \in \mathbb{N}$,

$$(5.4) \quad \mathfrak{L}(\mathfrak{t})(\theta^{q^j}) = (\mathfrak{L}(\mathfrak{t})(\theta))^{q^j}$$

which is also nonzero by (4.2) and [24, Theorem 4].

We put $w_0 := \max_{\mathfrak{t} \in I} w(\mathfrak{t})$ and denote by $I(w_0)$ the set of $\mathfrak{t} \in I$ such that $w(\mathfrak{t}) = w_0$. Then dividing (5.3) by Ω^{w-w_0} yields

$$(5.5) \quad \sum_{\mathfrak{t} \in I} B_{\mathfrak{t}} \mathfrak{L}(\mathfrak{t}) \Omega^{w_0-w(\mathfrak{t})} = \sum_{\mathfrak{t} \in I(w_0)} B_{\mathfrak{t}} \mathfrak{L}(\mathfrak{t}) + \sum_{\mathfrak{t} \in I \setminus I(w_0)} B_{\mathfrak{t}} \mathfrak{L}(\mathfrak{t}) \Omega^{w_0-w(\mathfrak{t})} = 0.$$

Since each $B_{\mathfrak{t}}$ belongs to $\overline{K}(t)$, they are defined at $t = \theta^{q^j}$ for $j \gg 1$. Note that the function Ω has a simple zero at $t = \theta^{q^k}$ for $k \in \mathbb{N}$. Specializing (5.5) at $t = \theta^{q^j}$ and using (5.4) yields

$$\sum_{\mathfrak{t} \in I(w_0)} B_{\mathfrak{t}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t})(\theta))^{q^j} = 0$$

for $j \gg 1$. Since $w_0 < w$, by the induction hypothesis we deduce that $B_{\mathfrak{t}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $\mathfrak{t} \in I(w_0)$. Since each $B_{\mathfrak{t}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathfrak{t}} = 0$ for all $\mathfrak{t} \in I(w_0)$.

Next, we put $w_1 := \max_{\mathfrak{t} \in I \setminus I(w_0)} w(\mathfrak{t})$ and denote by $I(w_1)$ the set of $\mathfrak{t} \in I$ such that $w(\mathfrak{t}) = w_1$. Dividing (5.3) by Ω^{w-w_1} and specializing at $t = \theta^{q^j}$ yields

$$\sum_{\mathfrak{t} \in I(w_1)} B_{\mathfrak{t}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t})(\theta))^{q^j} = 0$$

for $j \gg 1$. Since $w_1 < w$, by the induction hypothesis we deduce that $B_{\mathbf{t}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $\mathbf{t} \in I(w_1)$. Since each $B_{\mathbf{t}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathbf{t}} = 0$ for all $\mathbf{t} \in I(w_1)$. Repeating the previous arguments we deduce that $B_{\mathbf{t}} = 0$ for all $\mathbf{t} \in I$ as required.

We have proved that $\mathbf{g} - \mathbf{g}^{(-1)}\Phi = 0$. Thus

$$\begin{pmatrix} \text{Id} & 0 \\ (g_{\mathbf{t}}/g)_{\mathbf{t} \in I} & 1 \end{pmatrix}^{(-1)} \Phi = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ (g_{\mathbf{t}}/g)_{\mathbf{t} \in I} & 1 \end{pmatrix}.$$

By [10, Proposition 2.2.1] we see that the common denominator b of $g_{\mathbf{t}}/g$ for $\mathbf{t} \in I$ belongs to $\mathbb{F}_q[t] \setminus \{0\}$. If we put $\delta_{\mathbf{t}} = bg_{\mathbf{t}}/g$ for $\mathbf{t} \in I$ which belong to $\overline{K}[t]$ and $\delta := (\delta_{\mathbf{t}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\overline{K}[t])$, then

$$(5.6) \quad \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix}.$$

If we put $X := \begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix}$, then $X^{(-1)} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} X$. By [20, §4.1.6] there exist $\nu_{\mathbf{t}} \in \mathbb{F}_q(t)$ for $\mathbf{t} \in I$ such that if we set $\nu = (\nu_{\mathbf{t}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\mathbb{F}_q(t))$,

$$X = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ \nu & 1 \end{pmatrix}.$$

Thus the equation $\begin{pmatrix} \text{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix} = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ \nu & 1 \end{pmatrix}$ implies

$$(5.7) \quad \delta\Psi' + b\mathbf{f} = \nu.$$

The left-hand side belongs to $\mathbb{T}$, so does the right-hand side. Thus $\nu = (\nu_{\mathbf{t}})_{\mathbf{t} \in I} \in \text{Mat}_{1 \times |I|}(\mathbb{F}_q[t])$. For any $j \in \mathbb{N}$, by specializing (5.7) at $t = \theta^{q^j}$ and using (4.2) and the fact that Ω has a simple zero at $t = \theta^{q^j}$ we deduce that

$$\mathbf{f}(\theta) = \nu(\theta)/b(\theta).$$

Thus for all $\mathbf{t} \in I$, $f_{\mathbf{t}}(\theta)$ given as in (5.1) belongs to K . The claim is proved.

5.3. Step 3. In this final step we deduce a contradiction from the previous rationality results.

For $\mathfrak{s}_i$ we let $m_i \in \mathbb{Z}^{\geq 0}$ be the biggest index such that $(s_{i1}, \dots, s_{im_i})$ belongs to $I(\mathfrak{s}_j)$ for some $j \neq i$. Note that $s_{i(m_i+1)} \neq 0$ since the weights of s_i and s_j ($j \neq i$) are the same. For $m_i + 1 \leq k < \ell_i$ we consider $\mathbf{t} = (s_{i1}, \dots, s_{ik})$. Thus (5.1) gives $f_{\mathbf{t}} = a_i(t)\mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i})$. By Step 2 we know that $f_{\mathbf{t}}(\theta)$ belongs to K . By (4.2) we get

$$a_i \Gamma_{(s_{i(k+1)}, \dots, s_{i\ell_i})} \zeta_A(s_{i(k+1)}, \dots, s_{i\ell_i}) / \widetilde{\pi}^{s_{i(k+1)} + \dots + s_{i\ell_i}} \in K.$$

Since $a_i \in K^*$, $s_{i(k+1)} + \dots + s_{i\ell_i}$ is divisible by $q-1$. Since this condition holds for all $m_i + 1 \leq k < \ell_i$, it follows that $s_{i(k+1)}$ is divisible by $q-1$ for all $m_i + 1 \leq k < \ell_i$. Since $1 \leq s_{i(k+1)} \leq q-1$, we conclude that

$$s_{i(k+1)} = q-1, \quad \text{for all } m_i + 1 \leq k < \ell_i.$$

Let $m := \max_i m_i$. Then we can find two different tuples, says $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be such that $m_1 = m_2 = m$. Thus $(s_{11}, \dots, s_{1m}) = (s_{21}, \dots, s_{2m})$ and $s_{1(m+1)} \neq s_{2(m+1)}$. The previous discussion shows that $s_{1(m+2)} = \dots = q-1$ and $s_{2(m+2)} = \dots = q-1$. Combining with the fact that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same weight yields

$s_{1(m+1)} \equiv s_{2(m+1)} \pmod{q-1}$. Since $1 \leq s_{1(m+1)}, s_{2(m+1)} \leq q-1$, we deduce $s_{1(m+1)} = s_{2(m+1)}$, which is a contradiction.

The proof of Theorem B is finished.

Remark 5.1. In Step 2 of the proof of Theorem B we use the crucial fact that if $\zeta_A(s_1, \dots, s_r)$ belongs to $\mathcal{T}_w^0$, then $\zeta_A(s_1, \dots, s_{r-1})$ belongs to $\mathcal{T}_{w-s_r}^0$. However, this property does not hold for $\mathcal{T}_w$.

6. PROOF OF THEOREM D

As noticed in Remark 5.1 we cannot extend directly Theorem B to the set $\mathcal{T}_w$. In this section we provide a tricky way to bypass this problem. To do so, for $w \in \mathbb{N}$ we consider another set $\mathcal{T}'_w$ consisting of MZV's of weight w of the form $\zeta_A(s_1, \dots, s_r)$ where q does not divide s_i for all i .

Lemma 6.1. 1) The sets $\mathcal{T}_w$ and $\mathcal{T}'_w$ have the same cardinality.

2) If $\zeta_A(s_1, \dots, s_r)$ belongs to $\mathcal{T}'_w$, then $\zeta_A(s_1, \dots, s_{r-1})$ belongs to $\mathcal{T}'_{w-s_r}$.

Proof. Part 2 is clear from the definition of $\mathcal{T}'_w$. For Part 1 we put $d'(w) := |\mathcal{T}'_w|$. If $w \leq q$, then it is clear that $|\mathcal{T}_w| = |\mathcal{T}'_w|$ since $\mathcal{T}_w = \mathcal{T}'_w$. It suffices to show that $|\mathcal{T}'_w|$ satisfies the same Fibonacci-like relations as that of $|\mathcal{T}_w|$. In fact, let $(s_1, \dots, s_r) \in \mathbb{N}^r$ such that q does not divide s_i for all i . If $1 \leq s_1 \leq q-1$, then $(s_2, \dots, s_r) \in \mathcal{T}'_{w-s_1}$. Otherwise, $s_1 > q$ and it is clear that $(s_1 - q, s_2, \dots, s_r) \in \mathcal{T}'_{w-q}$. Thus we get $d'(w) = \sum_{i=1}^q d'(w-i)$ as desired. The proof of Part 1 is finished. $\square$

In what follows we suppose that $w \leq 2q-2$ and extend Theorem B to $\mathcal{T}'_w$. As mentioned in the introduction, when $q = 2$ Corollary C and Theorem D are the same. Thus from now on we will assume that $q > 2$.

Theorem 6.2. Suppose that $q > 2$ and $w \leq 2q-2$. Then MZV's in $\mathcal{T}'_w$ are all linearly independent over K .

Proof. The proof follows the same lines as that of Theorem B. We skip Steps 1 and 2 and give full details for Step 3.

The proof is by induction on the weight $w \in \mathbb{N}$. For $w < q$ we have $\mathcal{T}_w = \mathcal{T}'_w$, and we are done by Theorem B. Letting $w \in \mathbb{N}$ with $w \leq 2q-2$, we suppose that for $w' < w$ MZV's in $\mathcal{T}'_{w'}$ are all linearly independent over K .

We claim that MZV's in $\mathcal{T}'_w$ are all linearly independent over K . Suppose that there exist $\zeta_A(\mathfrak{s}_i) \in \mathcal{T}'_w$ and $a_i \in A \setminus \{0\}$ such that $\sum_i a_i \Gamma_{\mathfrak{s}_i} \zeta_A(\mathfrak{s}_i) = 0$. We show that this relation leads to a contradiction and divide the proof into three steps.

Steps 1 and 2. We set $a_i(t) := a_i|_{\theta=t} \in \mathbb{F}_q[t] \setminus \{0\}$. For each $\mathfrak{s}_i$ we write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i})$ and define the set of tuples $I(\mathfrak{s}_i) = \{\emptyset, (s_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)})\}$. Recall that $\mathfrak{s}_i$ is of weight w , that means $s_{i1} + \dots + s_{i\ell_i} = w$. We set $I := \cup_i I(\mathfrak{s}_i)$. By Lemma 6.1 Steps 1 and 2 carry over without modification as the size restriction on $\mathfrak{s}_i$ of Theorem B was never used for them. Therefore, if for $\mathfrak{t} \in I$ we set

$$f_{\mathfrak{t}} = \sum_i a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i})$$

where the sum is over indices i such that $\mathfrak{t} = (s_{i1}, \dots, s_{ik})$ for some $0 \leq k \leq \ell_i - 1$, then $f_{\mathfrak{t}}(\theta)$ belongs to K .

Step 3. Let $\mathbf{t} \in I$ and $\mathbf{t} \neq \emptyset$. Then $\mathbf{t} = (s_{i1}, \dots, s_{ik})$ for some i and $1 \leq k \leq \ell_i - 1$. We denote by $J(\mathbf{t})$ the set of all such i . It follows that $f_{\mathbf{t}} = \sum_{i \in J(\mathbf{t})} a_i(t) \mathfrak{L}(s_{i(k+1)}, \dots, s_{i\ell_i})$ and by (4.2),

$$(6.1) \quad f_{\mathbf{t}}(\theta) = \sum_{i \in J(\mathbf{t})} a_i \Gamma_{(s_{i(k+1)}, \dots, s_{i\ell_i})} \zeta_A(s_{i(k+1)}, \dots, s_{i\ell_i}) / \tilde{\pi}^{w-w(\mathbf{t})} \in K.$$

We claim that $w - w(\mathbf{t})$ is divisible by $q - 1$. In fact, suppose that $w - w(\mathbf{t})$ is not divisible by $q - 1$, then $\tilde{\pi}^{w-w(\mathbf{t})} \notin K_{\infty}$. Combining with (6.1) yields

$$\sum_{i \in J(\mathbf{t})} a_i \Gamma_{(s_{i(k+1)}, \dots, s_{i\ell_i})} \zeta_A(s_{i(k+1)}, \dots, s_{i\ell_i}) = 0.$$

The MZV's appearing in the above equality belong to $\mathcal{T}'_{w-w(\mathbf{t})}$. Recalling that $a_i \neq 0$ for $i \in J(\mathbf{t})$ we then obtain a contradiction by the induction hypothesis.

We have proved that $w - w(\mathbf{t})$ is divisible by $q - 1$ for all $\mathbf{t} \in I \setminus \emptyset$. Consequently, letting $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i})$ we conclude that $s_{i2}, \dots, s_{i\ell_i}$ are all divisible by $q - 1$. We now use the hypothesis $w \leq 2q - 2$ to conclude that either $\mathfrak{s}_i = (w)$ or $\mathfrak{s}_i = (k, q - 1)$ where $k = w - (q - 1)$ satisfying $1 < k \leq q - 1$. In other words, we have a linear relation

$$(6.2) \quad a_1 \Gamma_w \zeta_A(w) + a_2 \zeta_A(k, q - 1) = 0, \quad \text{for some } a_1, a_2 \in K^*.$$

We can suppose that $a_1, a_2 \in A \setminus \{0\}$. We have reduced to a very simple case which was already studied in [10] and also in [17].

We claim that (6.2) leads to a contradiction¹. In fact, we put $s_1 = k$ and $s_2 = q - 1$. By (5.6) it follows that there exist $\delta_1, \delta_2 \in \overline{K}[t]$ such that

$$(6.3) \quad \delta_1 = \delta_1^{(-1)}(t - \theta)^w + \delta_2^{(-1)} H_{s_1}^{(-1)}(t - \theta)^w + a_1(t) H_w^{(-1)}(t - \theta)^w,$$

$$(6.4) \quad \delta_2 = \delta_2^{(-1)}(t - \theta)^{s_2} + a_2(t) H_{s_2}^{(-1)}(t - \theta)^{s_2}.$$

Kuan and Lin showed (see [17, proof of Theorem 2]) that δ_1, δ_2 belong to $K[t]$, and

$$\deg_{\theta} \delta_i \leq \frac{q(s_i + \dots + s_2)}{q - 1}, \quad i = 1, 2.$$

Observe that if $(\delta_1, \delta_2, a_1(t), a_2(t)) \in K[t]^2 \times \mathbb{F}_q[t]^2$ is a solution of the above system, then $(f\delta_1, f\delta_2, fa_1(t), fa_2(t)) \in K[t]^2 \times \mathbb{F}_q[t]^2$ is also a solution for all $f \in \mathbb{F}_q[t]$.

We proceed by direct calculations. We first solve (6.4)

$$\delta_2 = \delta_2^{(-1)}(t - \theta)^{s_2} + a_2(t) H_{s_2}^{(-1)}(t - \theta)^{s_2}.$$

We know $H_{s_2} = 1$. It is easy to see that $a'_2(t) = t^q - t$ and $\delta'_2 = \theta^q - t^q$ is a solution. If we set $\delta := a'_2(t)\delta_2 - a_2(t)\delta'_2 \in K[t]$, then

$$\begin{aligned} \delta &= a'_2(t)\delta_2 - a_2(t)\delta'_2 \\ &= a'_2(t)(\delta_2^{(-1)}(t - \theta)^{s_2} + a_2(t)H_{s_2}^{(-1)}(t - \theta)^{s_2}) \\ &\quad - a_2(t)(\delta_2'^{(-1)}(t - \theta)^{s_2} + a'_2(t)H_{s_2}^{(-1)}(t - \theta)^{s_2}) \\ &= a'_2(t)\delta_2^{(-1)}(t - \theta)^{s_2} - a_2(t)\delta_2'^{(-1)}(t - \theta)^{s_2} \\ &= \delta^{(-1)}(t - \theta)^{s_2}. \end{aligned}$$

¹F. Pellarin pointed out that there was an elementary proof using Mahler's method (private communication, August 2020).

Here the last equality holds since $a_2(t), a'_2(t)$ belong to $\mathbb{F}_q[t]$. Comparing the degree of t we deduce that $\delta = 0$. Thus $a_2(t) = fa'_2(t) = f(t^q - t)$, $\delta_2 = f\delta'_2 = f(\theta^q - t^q)$ for some $f \in \mathbb{F}_q[t]$ and $f \neq 0$.

We know that $H_{s_1} = 1$ and $H_w = H_{q+k-1} = (t^q - t) + (k-1)(t^q - \theta^q)$. Replacing these in (6.3) we have to solve

$$(6.5) \quad \delta_1 = \delta_1^{(-1)}(t - \theta)^w + \delta_2^{(-1)}(t - \theta)^w + a_1(t)H_w^{(-1)}(t - \theta)^w$$

where $\delta_1 \in K[t]$ and $\deg_\theta \delta_1 \leq \frac{qw}{q-1}$. We will treat separately two cases: $1 < k < q-1$ and $k = q-1$.

If $1 < k < q-1$, then $w < 2q-2$. It follows that $\deg_\theta \delta_1 \leq w+1$ since $\deg_\theta \delta_1 \leq \frac{qw}{q-1}$. From (6.5) we see that δ_1 is divisible by $(t - \theta)^w$. Then we write

$$\delta_1 = (a\theta + b)(t - \theta)^w, \quad a, b \in \mathbb{F}_q[t].$$

Replacing this expression in (6.5) and twisting once yields

$$a\theta^q + b = (a\theta + b)(t^q - \theta^q)(t - \theta)^{k-1} + (\theta^q - t^q)f + [(t^q - t) + (k-1)(t^q - \theta^q)]a_1(t).$$

Comparing the coefficients for θ^k and θ^{k-1} we get $a = 0$ and $b = 0$. We then compare the coefficients for θ^q and get $f = (k-1)a_1(t)$. Substituting $f = (k-1)a_1(t)$ yields a contradiction $t^q - t = 0$, as $a_1(t) \neq 0$.

Otherwise, $k = q-1$, then $w = 2q-2$. As before, δ_1 is divisible by $(t - \theta)^w$ and satisfies $\deg_\theta \delta_1 \leq \frac{qw}{q-1} = w+2$. Thus we write

$$\delta_1 = (a\theta^2 + b\theta + c)(t - \theta)^w, \quad a, b, c \in \mathbb{F}_q[t].$$

Replacing this expression in (6.5) and twisting once yields

$$a\theta^{2q} + b\theta^q + c = (a\theta^2 + b\theta + c)(t - \theta)^{2q-2} + (\theta^q - t^q)f + [(t^q - t) - 2(t^q - \theta^q)]a_1(t).$$

Comparing the coefficients for θ^{2q-1} and θ^{2q-2} , we get $a\theta^2 + b\theta + c = a(t - \theta)^2$. Multiplying both sides by $t^q - t$ and using the equality

$$(t^q - t)[(t^q - t) - 2(t^q - \theta^q)] = (t - \theta^q)^2 - (t^q - \theta^q)^2,$$

yields

$$[a(t^q - t) - a_1(t)](t - \theta^q)^2 = [a(t^q - t) - a_1(t)](t - \theta)^{2q} + (\theta^q - t^q)f(t^q - t).$$

Thus

$$[a(t^q - t) - a_1(t)][(t - \theta^q)^2 - (t - \theta)^{2q}] = (\theta^q - t^q)f(t^q - t).$$

Since $f \neq 0$, we also obtain a contradiction since the right-hand side is divisible by $\theta - t$ but not the left-hand side.

The proof of Theorem 6.2 is finished. $\square$

Remark 6.3. Our method could not extend to the case $w = 2q-1$. This is because of the following fact: $\zeta_A(2q-1)$ and $\zeta_A(1, 2q-2)$ are K -linearly dependent, which could be verified by writing down $\mathcal{C}_{(q-1)}(R_1)$ (see also [18]).

Proof of Theorem D. Theorem D follows from Theorem A, Lemma 6.1 and Theorem 6.2. $\square$

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