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COMPARISON OF THE MAHLER MEASURE AND THE LENGTH FOR ALGEBRAIC INTEGERS HAVING ALL THEIR CONJUGATES IN A SECTOR.

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Abstract

In this paper, explicit auxiliary functions are used to get lower and upper bounds for the Mahler measure involving the length for an algebraic integer having all its conjugates in a sector. Mostly, these bounds improve the usual ones. At our knowledge, it is the first time that such a work is done.

1 Introduction

Let α be a nonzero algebraic number of degree d with minimal polynomial $P = a_0x^d + a_1x^{d-1} + \dots + a_d = a_0(x - \alpha_1)\dots(x - \alpha_d)$. We define:

- the *Mahler measure* of α as $M(\alpha) = \prod_{i=1}^d \max(1, |\alpha_i|)$,
- the *length* of α as $L(\alpha) = \sum_{i=0}^d |a_i|$.

We have the well-known inequalities : $2^{-d}L(\alpha) \leq M(\alpha) \leq L(\alpha)$.

Now, we consider that α is an algebraic integer whose all conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$, with $0 \leq \theta \leq 90^\circ$. Since the algebraic integers $\alpha_1 = \alpha, \dots, \alpha_d$ have positive real parts, we can show by induction that then the polynomial P has a sequence of

coefficients with alternate signs so $L(\alpha) = |P(-1)| = \prod_{i=1}^d |1 + \alpha_i|$.

In 2014, we studied the case of $\theta = 0$ [F2]. It means that the conjugates of α are all positive real numbers. Using the method of explicit auxiliary functions and our recursive algorithm, we proved:

Theorem 1. *If the minimal polynomial P of α is different from $x, x - 1, x^2 - 3x + 1, x^4 - 7x^3 + 13x^2 - 7x + 1, x^2 - 4x + 1, x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$ and $x^8 - 15x^7 + 83x^6 - 220x^5 + 303x^4 - 220x^3 + 83x^2 - 15x + 1$ then*

$$1.058358^d L(\alpha)^{0.562454} \leq M(\alpha) \leq 0.379128^d L(\alpha)^{1.803995}.$$

These inequalities are generally better than the classical ones. For instance, if α has minimal polynomial $x^3 - 6x^2 + 8x - 1$, we get:

$$1.058358^d L(\alpha)^{0.562454} = 5.63845 \leq M(\alpha) = 7.18421 \leq 8.10185 = 0.379128^d L(\alpha)^{1.803995}$$

instead of

$$2^{-d}L(\alpha) = 2 \leq M(\alpha) = 7.18421 \leq L(\alpha) = 16.$$

The purpose of this paper is to establish such inequalities in the case of $0 < \theta \leq 90$. It is the first time that this kind of work is done at our knowledge. We need to recall a theoretical result of M. Langevin [La] on the Mahler measure. He proved that there exists a function $c(\theta)$ on $[0, 180^\circ)$, always > 1 , such that if $\alpha \neq 0$ is not a root of unity, whose conjugates all lie in S_θ , then $M(\alpha)^{1/d} \geq c(\theta)$ where d denotes the degree of α . In 1995, G. Rhin and C. Smyth [RS] were the first to succeed in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 120^\circ]$ and conjectured that $c(\theta)$ is a "staircase" decreasing function of θ , which is constant except for finitely many left discontinuities in any closed subinterval of $[0, 180^\circ)$. They used the method of explicit auxiliary functions with polynomials found by heuristic search. Later, in 2004, G. Rhin and Q. Wu [RW] gave the exact value of $c(\theta)$ for four new subintervals of $[0, 140^\circ]$ and extended four existing subintervals. These improvements were allowed thanks to Wu's algorithm [Wu]. Based on this algorithm we have developed our recursive algorithm where the polynomials are found by induction. It allowed the author and G. Rhin [FR] in 2013 to find for the first time a complete subinterval and a fourteenth subinterval. A complete subinterval is an interval on which the function $c(\theta)$ describing the minimum on the sector $|\arg z| \leq \theta$ is constant, with jump discontinuities at each end. In 2014 [F3], we applied this reasoning to the length. It means that we computed the greatest lower bound $c(\theta)$ of the quantities $L(\alpha)^{1/d}$ for θ belonging to eight subintervals of $[0, 90)$. Here again, one subinterval is complete. The following table summarizes our results:

Table 1 Intervals $[\theta_i, \theta'_i]$ for $i \neq 6$ and $[\theta_6, \theta'_6)$ where $c(\theta)$ is known exactly. Here $c(\theta) = c(\theta_i) = L(P)^{1/\deg P}$ for $\theta \in [\theta_i, \theta'_i]$ when $i \neq 6$ and for $\theta \in [\theta_6, \theta'_6)$.

i	$c(\theta)$	θ_i	θ'_i	P
1	2.236068	0	18.26	$x^2 - 3x + 1$
2	2.140695	26.408740	29.78	$x^4 - 5x^3 + 9x^2 - 5x + 1$
3	2.030543	38.668282	40.42	$x^4 - 4x^3 + 7x^2 - 4x + 1$
4	1.950116	47.941432	48.68	$x^8 - 7x^7 + 25x^6 - 50x^5 + 63x^4 - 50x^3 + 25x^2 - 7x + 1$
5	1.898828	49.353681	54.93	$x^4 - 3x^3 + 5x^2 - 3x + 1$
6	1.853006	59.360240	60	$x^8 - 5x^7 + 16x^6 - 29x^5 + 37x^4 - 29x^3 + 16x^2 - 5x + 1$
7	1.732050	60	71.19	$x^2 - x + 1$
8	1.626576	76.795630	81.02	$x^4 - x^3 + 3x^2 - x + 1$

From now on, we suppose that the conjecture of G. Rhin and C. Smyth on the function $c(\theta)$ is true. We prove the following results:

Theorem 2. *If α is an algebraic integer whose conjugates all lie in:*

$$\begin{aligned}
[0, 18.26) \quad & \text{then} \quad 1.141148^d \cdot L(\alpha)^{0.427453} \leq M(\alpha) \leq 0.218243^d \cdot L(\alpha)^{2.490525} \\
[18.26, 29.78) \quad & \text{then} \quad 0.681486^d \cdot L(\alpha)^{0.814756} \leq M(\alpha) \leq 0.343833^d \cdot L(\alpha)^{1.990903} \\
[29.78, 40.42) \quad & \text{then} \quad 0.951716^d \cdot L(\alpha)^{0.447824} \leq M(\alpha) \leq 0.215121^d \cdot L(\alpha)^{2.356599} \\
[40.42, 48.68) \quad & \text{then} \quad 0.826667^d \cdot L(\alpha)^{0.599317} \leq M(\alpha) \leq 0.128230^d \cdot L(\alpha)^{3.544544}
\end{aligned}$$

$$\begin{aligned}
[48.68, 54.93) \quad & \text{then} \quad 0.820391^d \cdot L(\alpha)^{0.524156} \leq M(\alpha) \leq 0.197247^d \cdot L(\alpha)^{3.016706} \\
[54.93, 60) \quad & \text{then} \quad 0.528810^d \cdot L(\alpha)^{1.159859} \leq M(\alpha) \leq 0.152149^d \cdot L(\alpha)^{3.545630} \\
[60, 71.19) \quad & \text{then} \quad 0.952391^d \cdot L(\alpha)^{0.434332} \leq M(\alpha) \leq 0.547165^d \cdot L(\alpha)^{1.677066} \\
[71.19, 81.02) \quad & \text{then} \quad 0.903804^d \cdot L(\alpha)^{0.457938} \leq M(\alpha) \leq 0.582441^d \cdot L(\alpha)^{1.719218}
\end{aligned}$$

We give in Table 2 below an example for each sector.

The proof of these different inequalities uses the principle of explicit auxiliary function introduced into Number Theory by C. J. Smyth [Sm1]. The method is based on the fact that the resultant of two polynomials in $\mathbb{Z}[X]$ with no common roots is a nonzero integer.

To get the lower bounds, the auxiliary function is of the following type:

$$\forall z \in S_\theta, f(z) = \log \max(1, |z|) - c_0 \log |z + 1| - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)| \quad (1.1)$$

where the c_j 's are positive real numbers and the polynomials Q_j are nonzero elements of $\mathbb{Z}[X]$. Then

$$\sum_{i=1}^d f(\alpha_i) \geq md,$$

where m denotes the minimum of the function f on the sector S_θ . Thus

$$\log M(\alpha) \geq md + c_0 \log L(\alpha) + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

If the minimal polynomial P of α does not divide any Q_j then $\prod_{i=1}^d Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j . In this case, we have

$$M(\alpha) \geq e^{md} L(\alpha)^{c_0}.$$

To get the upper bounds, the auxiliary function is of the following type:

$$\forall z \in S_\theta, f(z) = -\log \max(1, |z|) + c_0 \log |z + 1| - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|.$$

In each case, f is invariant under complex conjugation. We can then limit ourselves to $0 \leq \arg z \leq \theta$. Since the function f is harmonic outside the union of arbitrary small disks around the roots of the polynomials Q_j , the minimum is taken on the upper edge of S_θ where $z = xe^{i\theta}$ with $x > 0$.

The auxiliary function on the half line $R_\theta = \{z \in \mathbb{C}, z = xe^{i\theta}, x > 0\}$ is

$$f(z) = \log \max(1, x) - c_0 \log |z + 1| - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|. \quad (1.2)$$

In the following sections, we explain how to construct an auxiliary function, especially how to find a good set of polynomials Q_j . All the computations are done on a MacBook Pro with the languages Pascal and Pari.

2 Construction of an explicit auxiliary function

2.1 Rewriting the auxiliary function (1.2)

Inside this auxiliary function, we replace the numbers c_j by rational numbers a_j/q where q is an integer such that $q \cdot c_j$ is an integer for all $1 \leq j \leq J$. Then we have

$$\forall x > 0, f(z) = \log \max(1, x) - c_0 \log |z + 1| - \frac{t}{r} \log |Q(z)| \geq m \quad (2.1)$$

where $Q = \prod_{i=0}^J Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{i=1}^J a_j \deg(Q_j)$ and $t = \sum_{j=1}^J c_j \deg(Q_j)$ is a positive real number. We want to get a function whose minimum m is as large as possible. It means to search a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(z)|^{t/r} \frac{|z+1|^{c_0}}{\max(1, x)} \leq e^{-m}.$$

If we suppose that t is fixed, we need to get an effective upper bound for the quantity

$$t_{\mathbb{Z}, \varphi}((0, \infty)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow +\infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = r}} \sup_{x > 0} \left(|P(z)|^{\frac{t}{r}} \varphi(z) \right).$$

where we use the weight $\varphi(z) = \frac{|z+1|^{c_0}}{\max(1, x)}$.

It is clear that this quantity is closely related to the weighted version of the integer transfinite diameter of a compact subset K of \mathbb{C}

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = n}} \sup_{z \in K} \left(|P(z)|^{\frac{1}{n}} \varphi(z) \right).$$

where φ is a positive function defined on K . We replace K by the infinite interval $(0, \infty)$, but the weight φ ensures that the quantity $t_{\mathbb{Z}, \varphi}((0, \infty))$ is finite.

2.2 How to find the polynomials Q_j

We proceed as follows: suppose that we know Q_1, Q_2, \dots, Q_J . We use the semi-infinite linear programming method (introduced into number theory by C. J. Smyth [Sm2]) to optimize f for this set of polynomials (i.e., to get the greatest possible m). We obtain the numbers $c_0, c_1, c_2, \dots, c_J$ and f in the form (1.2) with $t = \sum_{j=1}^J c_j \deg(Q_j)$.

Then, for several values of k , we search a polynomial $R(z) = \sum_{l=0}^k a_l z^l \in \mathbb{Z}[z]$ such that

$$\sup_{x>0} |Q(z)R(z)|^{\frac{t}{r+k}} \frac{|z+1|^{c_0}}{\max(1, x)} \leq e^{-m},$$

where m is the minimum of the function f . We search

$$\sup_{x>0} |Q(z)R(z)| \left(\frac{|z+1|^{c_0}}{\max(1, x)} \right)^{-(r+k)/t}$$

is as small as possible.

Since $R(z)$ is not a real linear form in the unknown coefficients a_i , we replace it by its real part and its imaginary part. Then, we apply LLL to the linear forms

$$|Q(z_n) \cdot \operatorname{Re}(R(z_n))| \cdot \left(\frac{|z_n+1|^{c_0}}{\max(1, x_n)} \right)^{-(r+k)/t} \quad \text{and} \quad |Q(z_n) \cdot \operatorname{Im}(R(z_n))| \cdot \left(\frac{|z_n+1|^{c_0}}{\max(1, x_n)} \right)^{-(r+k)/t}.$$

The x_n are suitable control points in $[0, 50]$, including the points where f has its least local minima. Remember that $z_n = x_n e^{i\theta}$. We get a polynomial R whose factors R_j are good candidates to extend the list of polynomials (Q_1, Q_2, \dots, Q_J) . We only keep the polynomials R_j which have a nonzero coefficient c_j in the new optimized auxiliary function f . After optimization, some previous polynomials Q_j may have a zero coefficient and are removed.

The polynomials Q_j and their coefficients c_j can be read off from Table 3 and Table 4 below.

2.3 Where the Rhin-Smyth conjecture is needed

Within our semi-infinite linear programming program, we need a lower bound for the quantities $L(\alpha)^{1/d}$ where α is an algebraic integer having all its conjugates in a sector. Remember that the function $c(\theta)$ is exactly the greatest lower bound of these quantities. That is why the eight subintervals of $[0, 90)$ appear in Theorem 2. Because on each of them, either we know the exact value of $c(\theta)$, or, under the hypothesis that the conjecture of G. Rhin and C. Smyth is true, we have a lower bound for it thus for the quantities $L(\alpha)^{1/d}$. We give an example of how it works. For $\theta \in [0, 18.26)$, we have [F3] $c(\theta) = 2.236068$. But we do not know the exact value of $c(\theta)$ on $[18.26, 26.408740)$. Nevertheless, if we suppose that $c(\theta)$ is a staircase decreasing function of θ , then we can say that $c(\theta) \geq 2.140695$ which is the exact value of $c(\theta)$ on the following subinterval $[26.408740, 29.78)$. We iterate this reasoning on each subinterval where the exact value of $c(\theta)$ is not known.

Table 2: P is the minimal polynomial of the algebraic integer α whose all conjugates lie in the sector $[\theta_i, \theta'_i]$.

$[\theta_i, \theta'_i]$	P	$2^{-d}L(\alpha)$	Our lower bound	$M(\alpha)$	Our upper bound	$L(\alpha)$
[0, 18.26)	$x^3 - 5x^2 + 7x - 1$	1.75000	4.59136	6.22226	7.43501	14.0000
[18.26, 29.78)	$x^3 - 5x^2 + 8x - 3$	2.12500	3.18337	5.61347	11.4485	17.0000
[29.78, 40.42)	$x^4 - 4x^3 + 7x^2 - 4x + 1$	1.06250	2.91779	4.33064	4.55268	17.0000
[40.42, 48.68)	$x^3 - 3x^2 + 4x - 1$	1.12500	2.10807	3.14790	5.08538	9.00000
[48.68, 54.93)	$x^4 - 3x^3 + 5x^2 - 3x + 1$	0.81250	1.73766	2.96557	3.47122	13.0000
[54.93, 60)	$x^3 - 2x^2 + 3x - 1$	0.875000	1.41287	2.32472	3.4931	7.00000
[60, 71.19)	$x^4 - 2x^3 + 4x^2 - 2x + 1$	0.62500	2.23901	2.89005	4.26129	10.0000
[71.19, 81.02)	$x^3 - x^2 + 2x - 1$	0.625000	1.54279	1.75488	3.14369	5.00000

Table 3: The polynomials Q_j and their coefficients c_j used for the lower bounds in Theorem 2.

[0, 18.26)	c_j	Q_j
	0.069302390379	x
	0.371118419047	$x - 1$
	0.031411836366	$x^2 - 3x + 1$
[18.26, 29.78)	c_j	Q_j
	0.002437560806	$5x^4 - 18x^3 + 29x^2 - 23x + 8$
	0.001061948615	$5x^4 - 19x^3 + 32x^2 - 26x + 9$
	0.009415541713	$10x^4 - 34x^3 + 49x^2 - 34x + 10$
	0.000307427745	$13x^4 - 45x^3 + 65x^2 - 45x + 13$
	0.008285897341	$2x^7 + 9x^6 - 81x^5 + 224x^4 - 322x^3 + 265x^2 - 120x + 24$
	0.004756616202	$x^9 + 71x^8 - 509x^7 + 1633x^6 - 3106x^5 + 3835x^4 - 3148x^3 + 1681x^2 - 536x + 79$
	0.000727997673	$2x^9 - 111x^8 + 741x^7 - 2383x^6 + 4623x^5 - 5858x^4 + 4948x^3 - 2721x^2 + 893x - 135$
	0.001049319352	$x^{11} - 200x^{10} + 1715x^9 - 7034x^8 + 17843x^7 - 30825x^6 + 37797x^5 - 33283x^4 + 20788x^3 - 8827x^2 + 2308x - 284$
	0.001949219424	$2x^{11} - 195x^{10} + 1618x^9 - 6542x^8 + 16471x^7 - 28360x^6 + 34767x^5 - 30686x^4 + 19251x^3 - 8225x^2 + 2167x - 269$
	0.001033849131	$3x^{11} - 205x^{10} + 1703x^9 - 6988x^8 + 17893x^7 - 31307x^6 + 38924x^5 - 34753x^4 + 21991x^3 - 9447x^2 + 2494x - 309$
	0.000521463318	$5x^{11} - 363x^{10} + 3023x^9 - 12303x^8 + 31050x^7 - 53315x^6 + 64844x^5 - 56497x^4 + 34818x^3 - 14542x^2 + 3726x - 447$
	0.003314331646	$11x^{13} - 775x^{12} + 7505x^{11} - 36392x^{10} + 111965x^9 - 240826x^8 + 379836x^7 - 449505x^6 + 401780x^5 - 269162x^4 + 131873x^3 - 44910x^2 + 9567x - 968$
	0.000216099044	$12x^{13} - 817x^{12} + 7920x^{11} - 38487x^{10} + 118612x^9 - 255348x^8 + 402718x^7 - 476099x^6 + 424719x^5 - 283724x^4 + 138504x^3 - 46965x^2 + 9956x - 1002$
	0.000861280624	$13x^{13} - 797x^{12} + 7747x^{11} - 38024x^{10} + 118599x^9 - 258563x^8 + 412985x^7 - 494313x^6 + 446216x^5 - 301407x^4 + 148633x^3 - 50851x^2 + 10860x - 1099$
[29.78, 40.42)	c_j	Q_j
	0.215858078320	$x^2 - x + 1$
	0.022246718914	$2x^2 - 3x + 2$
	0.001855528030	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	0.001442212353	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	0.001307301497	$3x^4 - 7x^3 + 8x^2 - 4x + 1$
	0.000415058230	$3x^4 - 9x^3 + 13x^2 - 9x + 3$
	0.001852500229	$x^6 - 5x^5 + 13x^4 - 17x^3 + 13x^2 - 5x + 1$
	0.000492561336	$x^8 - 9x^7 + 39x^6 - 98x^5 + 159x^4 - 171x^3 + 121x^2 - 52x + 11$
	0.000120050649	$x^8 - 9x^7 + 39x^6 - 100x^5 + 168x^4 - 188x^3 + 138x^2 - 61x + 13$
	0.000342003517	$2x^8 - 12x^7 + 38x^6 - 72x^5 + 89x^4 - 72x^3 + 38x^2 - 12x + 2$
	0.001681915713	$x^{10} - 3x^9 + 20x^7 - 52x^6 + 69x^5 - 52x^4 + 20x^3 - 3x + 1$
	0.000125294152	$x^{10} - 11x^9 + 60x^8 - 200x^7 + 450x^6 - 711x^5 + 801x^4 - 637x^3 + 345x^2 - 116x + 19$
	0.001910181120	$5x^{12} - 50x^{11} + 248x^{10} - 773x^9 + 1668x^8 - 2607x^7 + 3019x^6 - 2607x^5 + 1668x^4 - 773x^3 + 248x^2 - 50x + 5$

[40.42, 48.68)	c_j	Q_j
	0.02590718645	$x^2 - 2x + 2$
	0.01941817601	$2x^2 - 2x + 1$
	0.11827937410	$2x^2 - 3x + 2$
	0.00038026400	$4x^4 - 9x^3 + 10x^2 - 5x + 1$
	0.01060190670	$7x^6 - 29x^5 + 57x^4 - 65x^3 + 46x^2 - 19x + 4$
[48.68, 54.93)	c_j	Q_j
	0.155106991371	$2x^2 - 3x + 2$
	0.008668166571	$2x^5 - 8x^4 + 14x^3 - 11x^2 + 4x + 1$
	0.003912861880	$6x^7 - 44x^6 + 127x^5 - 215x^4 + 230x^3 - 161x^2 + 68x - 15$
	0.004034414973	$9x^7 - 35x^6 + 78x^5 - 98x^4 + 82x^3 - 40x^2 + 12x - 1$
	0.004310120971	$4x^8 - 33x^7 + 124x^6 - 273x^5 + 395x^4 - 384x^3 + 252x^2 - 102x + 22$
	0.003806593612	$8x^8 - 49x^7 + 148x^6 - 269x^5 + 324x^4 - 261x^3 + 139x^2 - 44x + 7$
	0.001615690909	$6x^9 - 62x^8 + 235x^7 - 539x^6 + 809x^5 - 851x^4 + 620x^3 - 311x^2 + 96x - 16$
[54.93, 60)	c_j	Q_j
	0.00000010000	$x^4 - 2x^3 + 4x^2 - 2x + 1$
	0.00000010000	$x^4 - 3x^3 + 6x^2 - 4x + 2$
	0.00000010000	$2x^4 - 4x^3 + 6x^2 - 3x + 1$
	0.00000010000	$2x^4 - 4x^3 + 7x^2 - 4x + 2$
	0.00000010000	$x^6 - x^5 + 4x^3 - 5x^2 + 3x - 1$
	0.00000010000	$x^6 - 9x^5 + 25x^4 - 41x^3 + 37x^2 - 19x + 5$
	0.00000010000	$x^7 - 29x^6 + 412x^5 - 1091x^4 + 1844x^3 - 1589x^2 + 813x - 198$
	0.00000010000	$x^8 - 48x^7 + 2076x^6 - 8817x^5 + 21913x^4 - 30384x^3 + 27630x^2 - 13250x + 3929$
[60, 71.19)	c_j	Q_j
	0.177813299876	$x^2 + 1$
	0.022335665277	$2x^2 - x + 2$
	0.019836674768	$x^6 - x^5 + 5x^4 - 3x^3 + 5x^2 - x + 1$
	0.005717360870	$3x^6 - 4x^5 + 12x^4 - 9x^3 + 12x^2 - 4x + 3$
	0.001223177207	$x^{10} - 3x^9 + 18x^8 - 33x^7 + 65x^6 - 67x^5 + 70x^4 - 40x^3 + 24x^2 - 6x + 2$
[71.19, 81.02)	c_j	Q_j
	0.004602700691	$x^6 - x^5 + 5x^4 - 3x^3 + 5x^2 - x + 1$
	0.085737005206	$3x^6 - 4x^5 + 12x^4 - 9x^3 + 12x^2 - 4x + 3$

Table 4: The polynomials Q_j and their coefficients c_j used for the upper bounds in Theorem 2.

[0, 18.26)	c_j	Q_j
	0.647286246321	x
	0.097976553424	$x^2 - 3x + 1$
[18.26, 29.78)	c_j	Q_j
	0.474121906237	x
	0.010151342146	$x^4 - 6x^3 + 12x^2 - 6x + 1$
	0.000222468351	$2x^7 - 23x^6 + 94x^5 - 177x^4 + 144x^3 - 59x^2 + 12x - 1$
[29.78, 40.42)	c_j	Q_j
	0.678499578979	x
[40.42, 48.68)	c_j	Q_j
	1.13905764949	x
	0.04916744121	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	0.01162652604	$x^6 - 5x^5 + 13x^4 - 17x^3 + 13x^2 - 5x + 1$
[48.68, 54.93)	c_j	Q_j
	0.878492951062	x
	0.029955915957	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	0.017156080155	$x^4 - 3x^3 + 6x^2 - 3x + 1$
	0.003933620090	$x^4 - 3x^3 + 7x^2 - 3x + 1$
	0.006942288287	$x^8 - 6x^7 + 21x^6 - 41x^5 + 53x^4 - 41x^3 + 21x^2 - 6x + 1$
[54.93, 60)	c_j	Q_j
	1.092450513855	x
	0.045279700135	$x^4 - 2x^3 + 4x^2 - 2x + 1$
	0.022451405283	$x^8 - 5x^7 + 16x^6 - 29x^5 + 37x^4 - 29x^3 + 16x^2 - 5x + 1$
[60, 71.19)	c_j	Q_j
	0.338533155338	x
[71.19, 81.02)	c_j	Q_j
	0.359609490561	x

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