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THE V-MEASURE FOR ALGEBRAIC INTEGERS HAVING ALL THEIR CONJUGATES IN A SECTOR.

V. FLAMMANG

Abstract

Let α be a nonzero algebraic integer of degree d whose all conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$ lie in a sector $|\arg z| \leq \theta, 0 \leq \theta \leq 90^\circ$. We define the *V-measure* of α by $V(\alpha) = \prod_{i=1}^d (1 + |\alpha_i|)(1 + 1/|\alpha_i|)$ and the *absolute V-measure* of α by $v(\alpha) = V(\alpha)^{1/\deg(\alpha)}$. Firstly, we consider the case $\theta = 0$. We study the set \mathcal{V} of the quantities $v(\alpha)$. We prove that there exists a number l such that \mathcal{V} is dense in (l, ∞) . Then we find the seven smallest points of \mathcal{V} in $(4, l)$. In case of $0 < \theta \leq 90^\circ$, we compute the greatest lower bound $c(\theta)$ of the absolute V-measure of α , for α belonging to ten subintervals of $]0, 90[$. Moreover, among these subintervals, one is complete and six are complete and consecutive. It is only the second time that such a phenomenon appears. The first time was when we studied the *house* of such α , i.e., the largest modulus of its conjugates. All computations are done by using the method of explicit auxiliary functions. The polynomials involved in these functions are found by our recursive algorithm.

1 Introduction

Let α be a nonzero algebraic integer of degree d with conjugates $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$. We define the *V-measure* of α by:

$$V(\alpha) = \prod_{i=1}^d (1 + |\alpha_i|)(1 + 1/|\alpha_i|).$$

The *absolute V-measure* of α is the quantity $v(\alpha) = V(\alpha)^{1/d}$.

If P is the minimal polynomial of α , then $V(P) = V(\alpha)$ and $v(P) = v(\alpha)$.

It is obvious that $v(\alpha) \geq 4$ and the equality holds if and only if α is a root of unity. Indeed, if for $i = 1, \dots, d$, we have $(1 + |\alpha_i|)(1 + 1/|\alpha_i|) = 4$ then for all $i = 1, \dots, d$, $|\alpha_i| = 1$. By the theorem of Kronecker, we deduce that α is a root of unity.

Remark: The scheme of this paper follows the scheme of our previous work [F2].

1.1 The case $\theta = 0$

We suppose first that $\theta = 0$. Then α is a *totally positive algebraic integer* i.e., all its conjugates are positive real numbers.

Theorem 1. *For all nonzero totally positive algebraic integer α whose minimal polynomial P satisfies $|P(0)| = 1$,*

$$V(\alpha) \text{ is a square.}$$

This is because, in this case, $V(\alpha) = P(-1)^2$.

Now, we want to study the structure of the set \mathcal{V} of the quantities $v(\alpha)$ and prove the following

Theorem 2. \mathcal{V} is dense in (l, ∞) where $l = \lim_{n \rightarrow \infty} v(\beta_n^2) = 5.649376\dots$

The β_n^2 were defined by C.J. Smyth [Sm1] as follows:

$$\begin{cases} \beta_0^2 &= 1 \\ \beta_n^2 &= \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2 \end{cases}$$

β_n^2 is a totally positive algebraic integer of degree 2^n .

In order to determine the structure of the set \mathcal{V} in the gap $(4, l)$, we prove the following

Theorem 3. *If α is a nonzero totally positive algebraic integer whose minimal polynomial is different from $x - 1$, $x - 2$, $x^2 - 5x + 5$, $x^3 - 7x^2 + 14x - 7$, $x^4 - 9x^3 + 27x^2 - 31x + 11$, $x^4 - 9x^3 + 26x^2 - 28x + 9$ and $x^6 - 13x^5 + 64x^4 - 151x^3 + 177x^2 - 96x + 19$ then we have:*

$$v(\alpha) \geq 4.935240.$$

Therefore, the seven smallest points of \mathcal{V} in $(4, l)$ are:

$$\begin{aligned} 4 &= v(x - 1), \\ 4.5 &= v(x - 2), \\ 4.880511 &= v(x^4 - 9x^3 + 27x^2 - 31x + 11), \\ 4.919349 &= v(x^2 - 5x + 5), \\ 4.925915 &= v(x^6 - 13x^5 + 64x^4 - 151x^3 + 177x^2 - 96x + 19), \\ 4.932882 &= v(x^4 - 9x^3 + 26x^2 - 28x + 9), \\ 4.934380 &= v(x^3 - 7x^2 + 14x - 7). \end{aligned}$$

1.2 The case $0 < \theta < 90^\circ$

Now, we consider algebraic integers α with all conjugates in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$ with $0 < \theta < 90^\circ$. We follow the work of M. Langevin [La] on the absolute Mahler measure $\Omega(\alpha) = (\prod_{i=1}^d \max(1, |\alpha_i|))^{1/d}$ of algebraic integers α having all their conjugates in a sector. He proved that there exists a function $c(\theta)$ on $[0, 180^\circ)$, always > 1 , such that if $\alpha \neq 0$ is not a root of unity, whose conjugates all lie in S_θ , then $\Omega(\alpha) \geq c(\theta)$. His result is not numerical. G. Rhin and C. Smyth [RS] succeeded in finding the exact value of $c(\theta)$ for θ in nine subintervals of $[0, 120^\circ]$ and conjectured that $c(\theta)$ is a "staircase" decreasing function of θ , which is constant except for finitely many left discontinuities in any closed subinterval of $[0, 180^\circ)$. Their method uses explicit auxiliary polynomials $A(X) = X^a R(X)$ where R is a reciprocal polynomial in $\mathbb{Z}[X]$, found by heuristic methods. In 2004, thanks to Wu's algorithm [Wu], G. Rhin and Q. Wu [RW] gave the exact value of $c(\theta)$ for four new subintervals of $[0, 140^\circ]$ and extended four existing subintervals. In 2013, the author and G. Rhin [FR] found for the first time a complete subinterval and a fourteenth subinterval. A complete subinterval is an interval on which the function $c(\theta)$ describing the minimum on the sector $|\arg z| \leq \theta$ is constant, with jump discontinuities at each end. These improvements are due to our recursive algorithm.

We give in Table 3 a list of polynomials Q_i with $\theta_i = \varphi(Q_i) = \max\{|\arg z| \text{ such that } Q_i(z) = 0\}$. Now we define two functions f and g on $[0, 90)$. The function $g(\theta)$ is the decreasing staircase function having left discontinuities at the angles θ_i given in Table 3 and such that $g(\theta_i) = v(Q_i)$. It gives the smallest known value of $v(\alpha)$ for $\alpha \in S_\theta$ then $c(\theta) \leq g(\theta)$. For $1 \leq i \leq 10$, we define 10 non-increasing functions f_i for $\theta \in [\theta_i, \theta_{i+1}]$ as follows:

$$f_i(\theta) = \min_{z \in S_\theta} \left(\log(1 + |z|)(1 + 1/|z|) - \sum_{1 \leq j \leq J} c_{ij} \log |Q_{ij}(z)| \right).$$

The polynomials Q_{ij} and the coefficients c_{ij} can be read off from Table 3. The function f is such that $f(\theta) = f_i(\theta)$ when $\theta \in [\theta_i, \theta_{i+1})$ for $1 \leq i \leq 10$. Since the functions f_i are continuous we have $f(\theta) \rightarrow f_i(\theta_{i+1})$ when $\theta \rightarrow \theta_{i+1}^-$. We do not find any function f_i such that $f_i(\theta_i) > g(\theta_i)$ for the other intervals $[\theta_i, \theta_{i+1})$, by Kronecker's theorem we may define $f(\theta) = 4$ for $\theta \geq \theta_{11}$. Then the function f is non-increasing on $[0, 90)$.

Theorem 4. *The non-increasing functions f, g, c satisfy the following inequalities:*

$$\min(f(\theta), g(\theta)) \leq c(\theta) \leq g(\theta) \quad (0 \leq \theta < 90).$$

Moreover, the exact value of $c(\theta)$ is known on ten subintervals of $[0, 90)$.

These intervals are given in Table 1. One can read off the six intervals $[\theta_i, \theta'_i)$ for $1 \leq i \leq 6$, the interval $[\theta_8, \theta'_8)$ and the three intervals $[\theta_7, \theta'_7)$, $[\theta_9, \theta'_9)$ and $[\theta_{10}, \theta'_{10}]$ where $c(\theta)$ is known exactly. For θ in each of these intervals we have $f(\theta) > g(\theta)$ so $c(\theta) = c(\theta_i)$. Outside these intervals, $c(\theta) \leq g(\theta)$.

Table 1: The 10 intervals where $c(\theta)$ is known. The polynomials in the last column are the minimal polynomial of an algebraic integer belonging to S_{θ_i} and they are also listed in **Table 3**.

i	$c(\theta)$	θ_i	θ'_i	Q
1	4.5	0	20.557484	$z - 1$
2	4.493902	20.557484	22.386177	$x^8 - 13z^7 + 74z^6 - 239z^5 + 476z^4 - 595z^3 + 454z^2 - 194z + 37$
3	4.470758	22.386177	27.431718	$z^4 - 7z^3 + 19z^2 - 23z + 11$
4	4.447255	27.431718	27.828202	$x^8 - 13z^7 + 76z^6 - 258z^5 + 554z^4 - 769z^3 + 675z^2 - 344z + 79$
5	4.430500	27.828202	30	$x^{12} - 19z^{11} + 168z^{10} - 910z^9 + 3354z^8 - 8847z^7 + 17112z^6 - 24455z^5 + 25648z^4 - 19281z^3 + 9886z^2 - 3116z + 460$
6	4.309401	30	36.243112	$z^2 - 3z + 3$
7	4.273004	36.243112	43.15	$z^4 - 5z^3 + 11z^2 - 11z + 5$
8	4.121320	45	61.188506	$x^2 - 2z + 2$
9	4.114907	61.188506	69.28	$z^4 - z^3 + 2z^2 + 1$
10	4.073390	73.631614	85.33	$z^3 + z + 1$

2 Denseness of the set \mathcal{V}

2.1 Study of the sequence $(v(\beta_n^2))_{n \geq 1}$

As above the sequence $(\beta_n^2)_{n \geq 0}$ are defined as follows:

$$\begin{cases} \beta_0^2 &= 1 \\ \beta_n^2 &= \beta_{n+1}^2 + \beta_{n+1}^{-2} - 2 \end{cases}$$

We first prove :

Lemma 5.

$$v(\beta_n^2) = 4 \prod_{i=0}^{n-1} (1 + \lambda_i)^{1/2^i} \quad \text{where } \lambda_0 = \frac{1}{4} \text{ and } \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \quad \text{for } i \geq 0.$$

Proof

For $n \geq 0$, we put $\gamma_n = \beta_n^2$, so $\gamma_n = \gamma_{n+1} + \gamma_{n+1}^{-1} - 2$.

Therefore, we have: $V(\beta_n^2) = V(\gamma_n) = \prod_{i=1}^{2^n} (1 + \gamma_{n,i})(1 + \gamma_{n,i}^{-1}) = \prod_{i=1}^{2^{n-1}} ((1 + \gamma_{n,i})(1 + \gamma_{n,i}^{-1}))^2$ because $\gamma_{n,i} = \gamma_{2^{n-1}+i}$ for $1 \leq i \leq 2^{n-1}$. (For $1 \leq i \leq 2^n$, $\gamma_{n,i}$ denotes the conjugates of γ_n).

Thus: $V(\beta_n^2) = \prod_{i=1}^{2^{n-1}} (2 + \gamma_{n,i} + \gamma_{n,i}^{-1})^2 = \prod_{i=1}^{2^{n-1}} (4 + \gamma_{n-1,i})^2$ Finally, we have $V(\beta_n^2) = \left[4^{2^{n-1}} \prod_{i=1}^{2^{n-1}} (1 + 1/4\gamma_{n-1,i}) \right]^2$.

We need here a more general lemma that we proved in [F1]:

Lemma 6. *The notations are the same as previously.*

$$\prod_{i=1}^{2^n} (1 + \lambda_0 \gamma_{n,i}) = \left(\prod_{i=0}^n (1 + \lambda_i)^{1/2^i} \right)^{2^n} \quad \text{where } \lambda_0 = \frac{1}{4} \text{ and } \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2} \quad \text{for } i \geq 0.$$

Now, we can conclude that $v(\beta_n^2) = 4 \prod_{i=0}^{n-1} (1 + \lambda_i)^{1/2^i}$.

A short program with Pari (see Annex below) computes the limit l of the sequence $v(\beta_n^2)$ and we get $l = 5.649376 \dots$. The limit l also gives an upper bound for the first accumulation point of \mathcal{V} .

2.2 Proof of Theorem 2

We follow the notations and proof of C.J. Smyth in [Sm1]. For a given function $g : [0, \infty) \rightarrow \mathbb{R}$, put $\mathcal{M}(g)$ the set of all means

$$M_g(\alpha) = \frac{1}{d} \sum_{i=1}^d g(|\alpha_i|)$$

for α a totally real algebraic integer, i.e., all its conjugates $\alpha_1 = \alpha, \dots, \alpha_d$ are real numbers. When the limits exist, put $a(g) = \lim_{n \rightarrow \infty} M_g(\beta_n)$ and $c(g) = \lim_{n \rightarrow \infty} M_g(2 \cos(2\pi/n))$.

Our choice here for g is

$$g : x \rightarrow \log(1 + x^2)(1 + 1/x^2) \quad (2.1)$$

because then $M_g(\alpha) = \log v(\alpha^2)$.

The proof consists in two parts.

2.2.1 First step of the proof

C.J. Smyth [Sm1] proved the following

Theorem 7. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a monotonic increasing function, zero on $[0, 1]$ such that*

$$\lim_{x \rightarrow \infty} g(x+1)/g(x) = 1$$

and the values of $\log_2 g(2k+1) \bmod 1$ ($k=0,1,2,\dots$) are everywhere dense in $(0, 1)$.

Then the limit $a(g)$ exists and $\mathcal{M}(g)$ is dense in $(a(g), \infty)$.

We replace the function g defined by (3.1) by the function g^* which satisfies the hypothesis of Theorem 7:

$$g^*(x) = \begin{cases} g(x) & \text{if } x > 1 \\ 0 & \text{if } 0 \leq x \leq 1 \end{cases} \quad (2.2)$$

Since $M_g(\beta_n) = M_{g^*}(\beta_n)$, the existence of $a(g^*)$ implies those of $a(g)$ and $a(g^*) = a(g)$.

It is easy to see that g^* satisfies the first hypothesis of Theorem 7. So, it is sufficient to study the denseness of the set $\mathcal{F} = \{\log_2 g(2k+1) \bmod 1, k \in \mathbb{N}\}$.

Let $t \in [0, 1]$ and $\epsilon > 0$. Does there exist $f \in \mathcal{F}$ such that $|f(k) - t| < \epsilon$? We search for n and k satisfying:

$$|\log_2 g^*(2k+1) - t - n| < \epsilon$$

i.e.,

$$|\log g^*(2k+1) - t' - n \log 2| < \epsilon' \quad (2.3)$$

The uniform continuity of the function \log on $[1, \infty)$ gives :

$$\forall \epsilon' > 0, \exists \eta(\epsilon') \text{ such that } \forall x, y > 0, |x - y| < \eta(\epsilon') \Rightarrow |\log x - \log y| < \epsilon'.$$

We choose n such that $2^{-n} < \eta(\epsilon')$ and k such that $|(2k+1) - (g^*)^{-1}(2^n e^{t'})| \leq 1$. As $(g^*)'$ is bounded by 1, the mean value Theorem for g^* on $(1, \infty)$ gives:

$$|g^*(2k+1) - 2^n e^{t'}| \leq 1,$$

i.e.,

$$|2^{-n} g^*(2k+1) - e^{t'}| \leq 2^{-n} < \eta(\epsilon')$$

and the inequality (3.3) follows immediately. Thus, we have proved that $\mathcal{M}(g)$ is dense in $(a(g^*), \infty) = (a(g), \infty) = (\log l, \infty) = (1.731545\dots, \infty)$.

2.2.2 Second step of the proof

C.J. Smyth [Sm1] established the following

Theorem 8. *Let $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a function such that $\lim_{x \rightarrow \infty} g(x) = \infty$ and which satisfies a Lipschitz condition*

$$|g(x) - g(y)| < B(\lambda)|x - y|$$

for $x, y \in [0, \lambda]$, for each $\lambda > 0$. Then $\mathcal{M}(g)$ is dense on $(c(g), \infty)$, where

$$c(g) = \frac{2}{\pi} \int_0^{\pi/2} g(2 \cos \theta) d\theta.$$

For our function g defined by (3.1), the Lipschitz condition is satisfied for $B(\lambda) = 2\lambda$. Thus, $\mathcal{M}(g)$ is dense on $(c(g), \infty) = (1.924847\dots, \infty)$.

2.2.3 Conclusion

We have proved that $\mathcal{M}(g)$ is dense on $(\min(a(g), c(g)), \infty) = (\log l, \infty)$ which means that \mathcal{V} is dense on (l, ∞) , where $l = \lim_{n \rightarrow \infty} v(\beta_n^2) = 5.649376\dots$

3 Proof of Theorem 3

3.1 The explicit auxiliary function

The auxiliary function involved in Theorem 3 is of the following type:

$$\text{for } x > 0, f(x) = \log(1+x)\left(1 + \frac{1}{x}\right) - \sum_{0 \leq j \leq J} c_j \log |Q_j(x)|$$

where the c_j are positive real numbers and the polynomials Q_j are non zero polynomials in $\mathbb{Z}[x]$.

If α is a totally positive algebraic integer with conjugates $\alpha_1 = \alpha, \dots, \alpha_d$, we have

$$\sum_{i=1}^d f(\alpha_i) \geq md,$$

where m denotes the minimum of the function f . Thus,

$$\log V(\alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

Now assume that P does not divide any Q_j , then $\prod_{i=1}^d Q_j(\alpha_i)$ is a nonzero integer because it is the resultant of P and Q_j .

Therefore, if α is not a root of Q_j , we have

$$v(\alpha) \geq e^m.$$

The problem becomes to find a convenient set of polynomials Q_j which gives a value of m as large as possible. This leads us to link the auxiliary function with the integer transfinite diameter in order to find the polynomials with our recursive algorithm.

3.2 Auxiliary functions and integer transfinite diameter

Let K be a compact subset of \mathbb{C} .

If φ is a positive function defined on K , the φ -integer transfinite diameter of K is defined as

$$t_{\mathbb{Z},\varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = n}} \sup_{x \in K} \left(|P(x)|^{\frac{1}{n}} \varphi(x) \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A].

Inside the auxiliary function (1), we replace the numbers c_j by rational numbers a_j/q where q is an integer such that $q \cdot c_j$ is an integer for all $0 \leq j \leq J$. Then we can write:

$$\text{for } x > 0, \quad f(x) = \log(1+x) \left(1 + \frac{1}{x}\right) - \frac{t}{r} \log |Q(x)| \quad (2)$$

where $Q = \prod_{i=0}^J Q_j^{a_j} \in \mathbb{Z}[X]$ is of degree $r = \sum_{i=0}^J a_i \deg(Q_i)$ and $t = \sum_{j=0}^J c_j \deg(Q_j)$ is a positive real number. We want to get a function whose minimum m is as large as possible. It means to search a polynomial $Q \in \mathbb{Z}[X]$ such that

$$\sup_{x>0} |Q(x)|^{t/r} ((1+x)(1+1/x))^{-1} \leq e^{-m}.$$

If we suppose that t is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z},\varphi}((0, \infty)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow +\infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = r}} \sup_{x > 0} \left(|P(x)|^{\frac{t}{r}} \varphi(x) \right)$$

where we use the weight $\varphi(x) = ((1+x)(1+1/x))^{-1}$.

We replace the compact subset K by the infinite interval $(0, \infty)$, but the weight φ ensures that the quantity $t_{\mathbb{Z},\varphi}((0, \infty))$ is finite.

3.2.1 Construction of the auxiliary function

We proceed as follows: suppose that we know Q_1, Q_2, \dots, Q_J . We use the semi-infinite linear programming (introduced in number theory by C. J. Smyth [Sm2]) to optimize f for this set of polynomials (i.e., to get the greatest possible m). We obtain the numbers c_1, c_2, \dots, c_J and f in

the form (2) with $t = \sum_{i=1}^J c_i \deg(Q_i)$.

Then for several value of k , we seek a polynomial $R(x) = \sum_{l=0}^k a_l x^l \in \mathbb{Z}[x]$ such that

$$\sup_{x>0} |Q(x)R(x)|^{\frac{t}{r+k}} ((1+x)(1+1/x))^{-1} \leq e^{-m},$$

i.e., such that

$$\sup_{x>0} |Q(x)R(x)| ((1+x)(1+1/x))^{-(r+k)/t}$$

is as small as possible.

We apply the LLL algorithm to the linear forms in a_0, \dots, a_k

$$Q(x_i)R(x_i)((1+x_i)(1+1/x_i))^{-(r+k)/t}.$$

The x_i are control points uniformly distributed in the interval $[0,70]$, including the points where f has its least local minima. We get a polynomial R whose factors R_j are good candidates to extend the list of polynomials (Q_1, Q_2, \dots, Q_J) . We only keep the polynomials R_j which have a nonzero coefficient c_j in the new optimized auxiliary function f . After optimization, some previous polynomials Q_j may have a zero coefficient and are removed.

In order to get the constant of Theorem 3, we take k from 4 to 20 successively.

The polynomials and the coefficients c_j involved in the auxiliary function of Theorem 3 are listed in the Table 2 below.

Remark: The main change with Wu's algorithm is that our polynomials are obtained by induction.

4 Proof of Theorem 4

Now, α is a nonzero algebraic integer with conjugates lie in a sector $S_\theta = \{z \in \mathbb{C} \text{ such that } |\arg z| \leq \theta\}$ with $0 < \theta \leq 90^\circ$. The auxiliary functions f_i , $1 \leq i \leq 10$ are of the following type:

$$\forall z \in S_\theta, f(z) = \log(1 + |z|)(1 + 1/|z|) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|,$$

where the c_j are positive real numbers and the polynomials Q_j are non zero polynomials in $\mathbb{Z}[z]$.

Since the function f is harmonic outside a finite set containing the roots of the polynomials Q_j , it takes its minimum on the boundary of S_θ . It is clear that $f(z) \rightarrow \infty$ as $z \rightarrow \infty$. Thus, as we have $f(z) = f(\bar{z})$, it is sufficient to search the minimum of f on the upper edge of the sector S_θ where $z = xe^{i\theta}$ with $x > 0$.

The auxiliary function on the half line $R_\theta = \{z \in \mathbb{C}, z = xe^{i\theta}, x > 0\}$ is

$$f(z) = \log(1 + x)(1 + \frac{1}{x}) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|.$$

Concerning the integer transfinite diameter, we replace here the compact K by the half line R_θ . But, again, the weight $\varphi(z) = (|z| + 1/|z|)^{-1}$ ensures that the quantity $t_{\mathbb{Z},\varphi}(R_\theta)$ is a finite number.

Then, for several values of k , we search a polynomial $R(z) = \sum_{l=0}^k a_l z^l \in \mathbb{Z}[z]$ such that

$$\sup_{x>0} |Q(z)R(z)|^{\frac{t}{r+k}} |(1+x)(1+1/x)|^{-1} \leq e^{-m},$$

where m is the minimum of the function f . We search

$$\sup_{x>0} |Q(z)R(z)| |(1+x)(1+1/x)|^{-(r+k)/t}$$

is as small as possible.

Since $R(z)$ is no more a real linear form in the unknown coefficients a_i , we replace it by its real part and its imaginary part. Then, we apply LLL to the linear forms

$$|Q(z_n) \cdot \text{Re}(R(z_n))| \cdot |(1+x_n)(1+1/x_n)|^{-(r+k)/t} \quad \text{and} \quad |Q(z_n) \cdot \text{Im}(R(z_n))| \cdot |(1+x_n)(1+1/x_n)|^{-(r+k)/t}.$$

The x_n are suitable control points in $[0, 50]$, including the points where f has its least local minima. Then we apply our recursive algorithm as described in Section 3.

The polynomials and the coefficients c_j involved in the auxiliary function of Theorem 4 are listed in the Table 3 below.

Annex: The program which computes $\lim_{n \rightarrow \infty} v(\beta_n^2)$

```
N=25;  
lambda=vector(N,j,0);  
lambda[1]=1/4; for(k=2,N,lambda[k]=lambda[k-1]/(1+lambda[k-1])^2);p=4;for(k=1,N-1,p=p*(1+lambda[k])^(1/2)^(k-1);print(p*1.))
```

The computation gives:

```
5.00000000000000  
5.3851648071345  
5.5385668995107  
5.6017408320507  
5.6284946631685  
5.6400801272366  
5.6451863873055  
5.6474688546314  
5.6485007769183  
5.6489716924285  
5.6491882700948  
5.6492885311707  
5.6493352061849  
5.6493570404592  
5.6493672976162  
5.6493721340930  
5.6493744221264  
5.6493755077355  
5.6493760241907  
5.6493762704706  
5.6493763881680  
5.6493764445270  
5.6493764715632  
5.6493764845544
```

Table 2: The polynomials involved in Theorem 3.

j	c_j	Q_j
1	0.134139923394118	$x - 1$
2	0.061749635022796	$x - 2$
3	0.009744089471627	$x - 3$
4	0.001848231653815	$x^2 - 3 + 1$
5	0.008718157080652	$x^2 - 4 + 2$
6	0.004623942020398	$x^2 - 5 + 5$
7	0.001677775683442	$x^3 - 6x^2 + 9 - 3$
8	0.000770087575792	$x^3 - 7x^2 + 14 - 7$
9	0.003025585967283	$x^4 - 9x^3 + 26x^2 - 28 + 9$
10	0.008560749688132	$x^4 - 9x^3 + 27x^2 - 31 + 11$
11	0.004698394045964	$x^6 - 13x^5 + 64x^4 - 151x^3 + 177x^2 - 96 + 19$
12	0.001126497871938	$x^6 - 13x^5 + 65x^4 - 158x^3 + 193x^2 - 110 + 23$
13	0.000001874362705	$x^6 - 13x^5 + 65x^4 - 158x^3 + 194x^2 - 113 + 25$
14	0.000403173239206	$x^6 - 14x^5 + 76x^4 - 203x^3 + 278x^2 - 182 + 43$
15	0.000383084810882	$x^7 - 16x^6 + 102x^5 - 333x^4 + 596x^3 - 580x^2 + 282 - 53$
16	0.000260704467608	$x^8 - 18x^7 + 134x^6 - 536x^5 + 1252x^4 - 1736x^3 + 1385x^2 - 578 + 97$
17	0.000745912646983	$x^8 - 18x^7 + 134x^6 - 537x^5 + 1260x^4 - 1760x^3 + 1418x^2 - 598 + 101$
18	0.000652672384412	$x^8 - 18x^7 + 135x^6 - 547x^5 + 1299x^4 - 1835x^3 + 1492x^2 - 633 + 107$
19	0.000352222775710	$x^8 - 19x^7 + 149x^6 - 627x^5 + 1539x^4 - 2239x^3 + 1868x^2 - 809 + 139$
20	0.000068815531942	$x^8 - 19x^7 + 149x^6 - 627x^5 + 1541x^4 - 2254x^3 + 1907x^2 - 850 + 153$
21	0.000014696986402	$x^{12} - 28x^{11} + 344x^{10} - 2451x^9 + 11268x^8 - 35159x^7 + 76200x^6 - 115315x^5 + 120628x^4 - 84845x^3 + 38001x^2 - 9718 + 1073$

Table 3: The auxiliary functions $f_i, 1 \leq i \leq 10$

f_1	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.50000	0.109569	0	$x - 2$
	4.00000	0.159943	0	$x - 1$
	4.49390	0.001016	20.5575	$x^8 - 13x^7 + 74x^6 - 239x^5 + 476x^4 - 595x^3 + 454x^2 - 194x + 37$
	4.47076	0.001195	22.3862	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	4.53330	0.000250	22.7179	$x^{10} - 17x^9 + 131x^8 - 599x^7 + 1793x^6 - 3659x^5 + 5140x^4 - 4894x^3 + 3016x^2 - 1086x + 175$
	4.58284	9.89994 E-5	23.1681	$x^5 - 9x^4 + 33x^3 - 60x^2 + 53x - 17$
	4.64998	0.002097	23.7496	$x^4 - 8x^3 + 25x^2 - 35x + 19$
	4.53248	0.000469	24.0255	$x^{10} - 17x^9 + 131x^8 - 599x^7 + 1792x^6 - 3651x^5 + 5112x^4 - 4840x^3 + 2956x^2 - 1050x + 166$
	4.52195	0.000883	24.4961	$x^{10} - 17x^9 + 131x^8 - 599x^7 + 1793x^6 - 3660x^5 + 5146x^4 - 4909x^3 + 3035x^2 - 1098x + 178$
f_2	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.140130	0	$x - 1$
	4.49390	0.004873	20.5575	$x^8 - 13x^7 + 74x^6 - 239x^5 + 476x^4 - 595x^3 + 454x^2 - 194x + 37$
	4.47076	0.008004	22.3862	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	4.52195	0.000301	24.4961	$x^{10} - 17x^9 + 131x^8 - 599x^7 + 1793x^6 - 3660x^5 + 5146x^4 - 4909x^3 + 3035x^2 - 1098x + 178$
	4.30940	0.029232	30.0000	$x^2 - 3x + 3$
f_3	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.156836	0	$x - 1$
	4.30940	0.023194	30.0000	$x^2 - 3x + 3$
	4.47076	0.004332	22.3862	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	4.53161	5.86076 E-5	25.5367	$x^4 - 7x^3 + 19x^2 - 22x + 10$
	4.44065	6.45845 E-5	26.2404	$x^{12} - 19x^{11} + 168x^{10} - 910x^9 + 3354x^8 - 8846x^7 + 17102x^6 - 24411x^5 + 25540x^4 - 19123x^3 + 9750x^2 - 3053x + 448$
	4.48215	0.000362	26.4803	$x^{12} - 20x^{11} + 186x^{10} - 1059x^9 + 4100x^8 - 11350x^7 + 23008x^6 - 34386x^5 + 37591x^4 - 29318x^3 + 15494x^2 - 4989x + 743$
	4.68328	0.006789	26.5651	$x^2 - 4x + 5$
	4.44726	0.000805	27.4317	$x^8 - 13x^7 + 76x^6 - 258x^5 + 554x^4 - 769x^3 + 675x^2 - 344x + 79$
	4.47572	0.000172	27.5096	$x^{12} - 20x^{11} + 186x^{10} - 1059x^9 + 4099x^8 - 11338x^7 + 22944x^6 - 34191x^5 + 37224x^4 - 28887x^3 + 15191x^2 - 4876x + 727$
	4.45763	0.000147	28.2826	$x^{12} - 19x^{11} + 168x^{10} - 909x^9 + 3340x^8 - 8758x^7 + 16778x^6 - 23643x^5 + 24325x^4 - 17837x^3 + 8867x^2 - 2693x + 381$
	4.44764	0.001217	28.3244	$x^6 - 10x^5 + 43x^4 - 100x^3 + 132x^2 - 94x + 29$
	4.46165	0.000199	28.3538	$x^{12} - 19x^{11} + 168x^{10} - 909x^9 + 3340x^8 - 8757x^7 + 16767x^6 - 23592x^5 + 24196x^4 - 17646x^3 + 8703x^2 - 2619x + 368$
	4.48237	0.000966	28.9238	$x^7 - 12x^6 + 64x^5 - 194x^4 + 359x^3 - 404x^2 + 256x - 71$
	4.49944	0.000320	29.9530	$x^{13} - 22x^{12} + 225x^{11} - 1414x^{10} + 6082x^9 - 18880x^8 + 43463x^7 - 75057x^6 + 97177x^5 - 93148x^4 + 64276x^3 - 30269x^2 + 8738x - 1173$
	4.50347	0.000233	30.9258	$x^{10} - 17x^9 + 133x^8 - 626x^7 + 1954x^6 - 4212x^5 + 6334x^4 - 6550x^3 + 4454x^2 - 1799x + 329$
f_4	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.160737	0	$x - 1$
	4.47076	0.004736	22.3862	$x^4 - 7x^3 + 19x^2 - 23x + 11$
	4.68328	0.007227	26.5651	$x^2 - 4x + 5$
	4.44726	0.001341	27.4317	$x^8 - 13x^7 + 76x^6 - 258x^5 + 554x^4 - 769x^3 + 675x^2 - 344x + 79$
	4.43050	0.000275	27.8282	$x^{12} - 19x^{11} + 168x^{10} - 910x^9 + 3354x^8 - 8847x^7 + 17112x^6 - 24455x^5 + 25648x^4 - 19281x^3 + 9886x^2 - 3116x + 460$
	4.44764	0.001702	28.3244	$x^6 - 10x^5 + 43x^4 - 100x^3 + 132x^2 - 94x + 29$
	4.46165	0.000953	28.3538	$x^{12} - 19x^{11} + 168x^{10} - 909x^9 + 3340x^8 - 8757x^7 + 16767x^6 - 23592x^5 + 24196x^4 - 17646x^3 + 8703x^2 - 2619x + 368$
	4.48237	0.001132	28.9238	$x^7 - 12x^6 + 64x^5 - 194x^4 + 359x^3 - 404x^2 + 256x - 71$
	4.30940	0.020855	30.0000	$x^2 - 3x + 3$
f_5	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.149253	0	$x - 1$
	4.44726	0.000224	27.4317	$x^8 - 13x^7 + 76x^6 - 258x^5 + 554x^4 - 769x^3 + 675x^2 - 344x + 79$
	4.43050	0.000276	27.8282	$x^{12} - 19x^{11} + 168x^{10} - 910x^9 + 3354x^8 - 8847x^7 + 17112x^6 - 24455x^5 + 25648x^4 - 19281x^3 + 9886x^2 - 3116x + 460$
	4.44764	0.000758	28.3244	$x^6 - 10x^5 + 43x^4 - 100x^3 + 132x^2 - 94x + 29$
	4.30940	0.024490	30.0000	$x^2 - 3x + 3$
	4.37789	0.007228	32.0950	$x^4 - 6x^3 + 15x^2 - 17x + 8$
	4.39727	0.001220	34.0406	$x^8 - 12x^7 + 65x^6 - 205x^5 + 410x^4 - 531x^3 + 436x^2 - 209x + 46$
	4.40730	0.001488	34.0856	$x^8 - 12x^7 + 66x^6 - 213x^5 + 438x^4 - 585x^3 + 496x^2 - 245x + 55$

f_6	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.120028	0	$x - 1$
	4.30940	0.012943	30.0000	$x^2 - 3x + 3$
	4.32814	0.001299	34.5817	$x^6 - 8x^5 + 29x^4 - 58x^3 + 68x^2 - 44x + 13$
	4.31264	0.004190	35.6858	$x^8 - 10x^7 + 47x^6 - 131x^5 + 236x^4 - 280x^3 + 215x^2 - 98x + 21$
	4.27300	0.008190	36.2431	$x^4 - 5x^3 + 11x^2 - 11x + 5$
	4.35739	0.001903	37.7612	$x^4 - 5x^3 + 11x^2 - 10x + 4$
	4.12132	0.019883	45.0000	$x^2 - 2x + 2$
f_7	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.06036	0	$x - 1$
	4.27300	0.008216	36.2431	$x^4 - 5x^3 + 11x^2 - 11x + 5$
	4.15075	0.003199	39.0700	$2x^6 - 11x^5 + 29x^4 - 43x^3 + 39x^2 - 20x + 5$
	4.34442	0.001774	39.4782	$x^6 - 7x^5 + 23x^4 - 40x^3 + 41x^2 - 23x + 6$
	4.36481	0.000167	41.8833	$x^7 - 9x^6 + 33x^5 - 69x^4 + 88x^3 - 70x^2 + 32x - 7$
	4.12132	0.02604	45.0000	$x^2 - 2x + 2$
	4.21656	0.01509	45.3100	$x^4 - 4x^3 + 8x^2 - 7x + 3$
	4.20509	0.000723	63.2729	$x^6 - 5x^5 + 15x^4 - 26x^3 + 28x^2 - 17x + 5$
f_8	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.12132	0.05118	45.0000	$x^2 - 2x + 2$
	4.00000	0.03096	60.0000	$x^2 - x + 1$
f_9	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.00000	0.036463	60.0000	$x^2 - x + 1$
	4.11491	0.012541	61.1985	$x^4 - x^3 + 2x^2 + 1$
f_{10}	$v(Q_j)$	c_j	$\theta(Q_j)$	Q_j
	4.07339	0.007302	73.6316	$x^3 + x + 1$
	4.20906	0.008674	79.0744	$x^8 - 2x^7 + 10x^6 - 12x^5 + 28x^4 - 19x^3 + 26x^2 - 8x + 7$
	4.21560	0.009987	80.2809	$x^4 - x^3 + 5x^2 - 2x + 4$
	4.20719	0.000880	83.1995	$x^6 - x^5 + 7x^4 - 4x^3 + 13x^2 - 3x + 6$
	4.11803	0.013675	84.2506	$x^7 - 2x^6 + 7x^5 - 10x^4 + 14x^3 - 14x^2 + 8x - 5$
	4.12132	0.010461	90.0000	$x^2 + 2$

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