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Application of Perron Trees to Geometric Maximal Operators

Anthony Gauvan

February 9, 2022

Abstract

We characterize the $L^p(\mathbb{R}^2)$ boundedness of the geometric maximal operator $M_{a,b}$ associated to the basis $\mathcal{B}_{a,b}$ ($a, b > 0$) which is composed of rectangles R whose eccentricity and orientation is of the form

$$(e_R, \omega_R) = \left(\frac{1}{n^a}, \frac{\pi}{4n^b} \right)$$

for some $n \in \mathbb{N}^*$. The proof relies on the exploitation of exceptional geometric sets as constructed in [12], namely *generalized Perron trees*.

We work in the euclidean plane \mathbb{R}^2 ; if A is a measurable subset of \mathbb{R}^2 we denote by $|A|$ its two dimensional Lebesgue measure. We denote by $A \sqcup B$ the union of A and B when $|A \cap B| = 0$.

1 Introduction

Denote by \mathcal{R} the collection containing all rectangles of \mathbb{R}^2 ; for $R \in \mathcal{R}$ we define its *orientation* as the angle $\omega_R \in [0, \pi)$ that its longest side makes with the Ox -axis and its *eccentricity* as the ratio $e_R \in (0, 1]$ of its shortest side and its longest side.

For an arbitrary non empty family \mathcal{B} contained in \mathcal{R} , we define the associated *derivation basis* \mathcal{B}^* by

$$\mathcal{B}^* = \{ \vec{t} + hR : \vec{t} \in \mathbb{R}^2, h > 0, R \in \mathcal{B} \}.$$

The derivation basis \mathcal{B}^* is simply the smallest collection which is invariant by dilation and translation and that contains any element of \mathcal{B} . Without loss of generality, we will constantly confuse the derivation basis \mathcal{B}^* and one of its generator \mathcal{B} .

Our object of interest will be the *geometric maximal operator* $M_{\mathcal{B}}$ generated by \mathcal{B} which is defined as

$$M_{\mathcal{B}}f(x) := \sup_{x \in R \in \mathcal{B}^*} \frac{1}{|R|} \int_R |f|$$

for any $f \in L^1_{loc}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Observe that the supremum is taken on elements of \mathcal{B}^* that contain the point x . The definitions of \mathcal{B}^* and $M_{\mathcal{B}}$ remain valid when we consider that \mathcal{B} is an arbitrary family composed of open bounded convex sets. For example in this note, for technical reasons and without loss of generality, we will work at some point with triangles instead of rectangles.

For $p \in (1, \infty]$ we define as usual the operator norm $\|M_{\mathcal{B}}\|_p$ of $M_{\mathcal{B}}$ by

$$\|M_{\mathcal{B}}\|_p = \sup_{\|f\|_p=1} \|M_{\mathcal{B}}f\|_p.$$

If $\|M_{\mathcal{B}}\|_p < \infty$ we say that $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$. The boundedness of a maximal operator $M_{\mathcal{B}}$ is related to the geometry that the family \mathcal{B} exhibits.

Definition 1. We will say that the operator $M_{\mathcal{B}}$ is a **good operator** when it is bounded on L^p for any $p > 1$. On the other hand, we say that the operator $M_{\mathcal{B}}$ is a **bad operator** when it is unbounded on L^p for any $1 < p < \infty$.

On the $L^p(\mathbb{R}^2)$ scale, to be able to say that a operator $M_{\mathcal{B}}$ is good or bad is an optimal result. Of course, one can be interested by the behavior near endpoint ($p = 1$ and $p = \infty$) but we won't consider this question here ; the reader might consult [2], [3], [14] or [4].

Directional maximal operators

A lot of research has been done in the case where \mathcal{B} is equal to $\mathcal{R}_{\Omega} := \{R \in \mathcal{R} : \omega_R \in \Omega\}$ where Ω is an arbitrary set of directions in $[0, \pi)$. In other words, \mathcal{R}_{Ω} is the set of all rectangles whose orientation belongs to Ω . We say that \mathcal{R}_{Ω} is a *directional basis* and to alleviate the notation we denote

$$M_{\mathcal{R}_{\Omega}} := M_{\Omega}.$$

In the literature, the operator M_{Ω} is said to be a *directional maximal operator*. The study of those operators goes back at least to Cordoba and Fefferman's article [8] in which they use geometric techniques to show that if $\Omega = \{\frac{\pi}{2^k}\}_{k \geq 1}$ then M_{Ω} has weak-type $(2, 2)$. A year later, using Fourier analysis techniques, Nagel, Stein and Wainger proved in [13] that M_{Ω} is actually bounded on L^p for any $p > 1$. In [1], A. Alfonseca has proved that if the set of direction Ω is a *lacunary set of finite order* then the operator M_{Ω} is bounded on L^p for any $p > 1$. Finally in [5], M. Bateman proved the converse and so characterized the L^p -boundedness of directional operators. Precisely he proved the following theorem.

Theorem 2 (Bateman's Theorem). *Fix an arbitrary set of directions $\Omega \subset [0, \pi)$. The directional maximal operator M_{Ω} is either good or bad.*

Hence we know that a set of directions Ω always yields a directional operator M_{Ω} that is either good or bad. Merging the vocabulary, we use the following definition.

Definition 3. *We say that a set of directions Ω is a good set of directions when M_{Ω} is good and that it is a bad set of directions when M_{Ω} is bad.*

The notion of good/bad is perfectly understood for a set of directions Ω and the associated directional operator M_{Ω} . To say it bluntly, Ω is a good set of directions if and only if it can be included in a finite union of lacunary sets of finite order. If this is not possible, then Ω is a bad set of directions ; see [5].

Geometric maximal operators

We recall two results in the direction of Bateman's Theorem for an arbitrary basis \mathcal{B} included in \mathcal{R} . The first one is a result in [10] where P. Hagelstein and A. Stokolos proved the following theorem.

Theorem 4. *Fix an arbitrary basis \mathcal{B} in \mathcal{R} and suppose that there exist constants $t_0 \in (0, 1)$ and $C_0 > 1$ such that for any bounded measurable set $E \subset \mathbb{R}^2$ one has*

$$|\{M_{\mathcal{B}}\mathbb{1}_E > t_0\}| \leq C_0 |E|.$$

In this case there exists p_0 depending on (t_0, C_0) such that for any $p > p_0$ we have $\|M_{\mathcal{B}}\|_p < \infty$.

In [9], we have shown that one can associate to any basis \mathcal{B} include in \mathcal{R} a geometric quantity denoted by $\lambda_{[\mathcal{B}]} \in \mathbb{N} \cup \{\infty\}$ that we call the *analytic split* of the family \mathcal{B} . We insist on the fact that the analytic split *is not* defined by abstract means but really concrete ; in a certain setting one can easily compute it.

Theorem 5. *For any basis \mathcal{B} in \mathcal{R} and any $1 < p < \infty$ we have*

$$\log(\lambda_{[\mathcal{B}]}) \lesssim_p \|M_{\mathcal{B}}\|_p^p.$$

On one hand, observe that this Theorem implies that basis \mathcal{B} whose analytic split is infinite yields bad maximal operators $M_{\mathcal{B}}$. On the other hand, it is easy (in a certain setting) to exhibit a lot of basis \mathcal{B} whose analytic split is infinite.

Results

In this note we consider a family of geometric maximal operators which are not directional maximal operators. Moreover we will always work with bases \mathcal{B} such that its associated set of directions

$$\Omega_{\mathcal{B}} := \{\omega_R : R \in \mathcal{B}\}$$

is a bad set of directions. Indeed if $\Omega_{\mathcal{B}}$ is a good set of direction using the trivial estimate $M_{\mathcal{B}} \leq M_{\Omega_{\mathcal{B}}}$ we know that $M_{\mathcal{B}}$ is also a good operator.

Fix two real positive numbers $a, b > 0$ and denote by $\mathcal{B}_{a,b}$ the basis of rectangles R whose eccentricity and orientation is of the form

$$(e_R, \omega_R) = \left(\frac{1}{n^a}, \frac{\pi}{4n^b} \right)$$

for some $n \in \mathbb{N}^*$. We denote by $M_{a,b}$ the operator associated to the basis $\mathcal{B}_{a,b}$. We prove the following theorem.

Theorem 6. *If $a < b$ then $M_{a,b}$ is a good operator. If $a \geq b$ then $M_{a,b}$ is a bad operator.*

We prove Theorem 6 thanks to Theorems 7 and 8. Denote by $\mathbf{t} = \{t_k\}_{k \geq 1} \subset [0, \frac{\pi}{4}]$ a sequence decreasing to 0 and by $\mathbf{e} = \{e_k\}_{k \geq 1} \subset (0, 1]$ any positive sequence. One should consider the sequence \mathbf{t} as a sequence of angles (or tangent of angles) that forms a bad set of directions whereas the sequence \mathbf{e} stands for an arbitrary sequence of eccentricity. For $k \geq 1$ consider a rectangle

$$R_k := R_k(\mathbf{e}, \mathbf{t})$$

whose orientation and eccentricity are defined by $(e_{R_k}, \omega_{R_k}) = (e_k, t_k)$. Define then the basis

$$\mathcal{B} = \mathcal{B}(\mathbf{t}, \mathbf{e})$$

as the one generated by the rectangles $\{R_k\}_{k \geq 1}$. Our first result reads as follow.

Theorem 7. *Suppose there is a constant $C > 0$ such that for any $k \geq 1$, $t_k \leq Ce_k$. In this case the operator $M_{\mathcal{B}}$ is a good operator.*

We define now the following quantity associated to the sequence \mathbf{t}

$$\tau_{\mathbf{t}} := \sup_{k \geq 0, l \leq k} \left(\frac{t_{k+2l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+2l} - t_{k+l}} \right) \in (0, \infty].$$

This quantity yields information on the goodness/badness of the set $\{t_k\}_{k \geq 1}$ seen as a set of directions. Indeed if $\tau_{\mathbf{t}}$ is finite then the set of directions $\Omega = \{t_k\}_{k \geq 1}$ forms a bad set of directions. In some sense, this quantity indicates to which point the sequence \mathbf{t} is *uniformly distributed near 0*. For example, the sequence $\mathbf{t} = \{\frac{1}{k}\}$ look likes a uniform distribution near 0 and we have $\tau_{\mathbf{t}} < \infty$. On the other hand the sequence $\mathbf{t} = \{\frac{1}{2^k}\}_{k \geq 1}$ converges rapidly to 0 and we have $\tau_{\mathbf{t}} = \infty$. The second result reads as follow.

Theorem 8. *Suppose that $\tau_{\mathbf{t}} < \infty$ and also that there is a constant $\mu_0 > 0$ such that for any $k \geq 1$ we have $e_k < \mu_0 |t_k - t_{k+1}|$. In this case, the maximal operator $M_{\mathcal{B}}$ is a bad operator.*

How can we prove that $M_{\mathcal{B}}$ is bad ?

To prove that an operator $M_{\mathcal{B}}$ is bad, the idea is to create an exceptional geometric set adapted to the basis \mathcal{B} ; precisely, one can try to find a small fixed value $0 < \eta_0 < 1$ such that for any $\epsilon > 0$ there is a subset X in \mathbb{R}^2 satisfying

$$|X| \leq \epsilon |\{M_{\mathcal{B}}\mathbb{1}_X > \eta_0\}|.$$

If this holds then for any $p > 1$ we have

$$\int (M_{\mathcal{B}}\mathbb{1}_X)^p \geq \eta_0^p |\{M_{\mathcal{B}}\mathbb{1}_X > \eta_0\}| \geq \eta_0^p \frac{\|\mathbb{1}_X\|_p^p}{\epsilon}$$

since $|X|^{\frac{1}{p}} = \|\mathbb{1}_X\|_p$. Hence for any $\epsilon > 0$ we have $\|M_{\mathcal{B}}\|_p \geq \eta_0^p \epsilon^{-\frac{1}{p}}$ and $\|M_{\mathcal{B}}\|_p^p = \infty$ for any $1 < p < \infty$. The question remains to understand how one can find/construct such a set X ? Of course this possibility depends on the basis \mathcal{B} . For example consider the case where $\mathcal{B} := \mathcal{R}$ is as big as possible. The following property is true (it is a consequence of proposition 1) : for any large constant $A > 1$ there exists a finite family of rectangles $\{R_i\}_{i \leq m}$ in \mathcal{R} satisfying

$$\left| \bigcup_{i \leq m} 2R_i \right| \geq A \left| \bigcup_{i \leq m} R_i \right|.$$

Considering then the set $X = \bigcup_{i \leq m} R_i$ it is easy to see that one has

$$|X| \leq \frac{1}{A} |\{M_{\mathcal{R}}\mathbb{1}_X > \frac{1}{4}\}|$$

which implies that the maximal operator $M_{\mathcal{R}}$ is a bad operator. A *Perron tree* (or *generalized Perron tree*) formed with a basis \mathcal{B} of rectangles is a concrete construction of such a set X (or more precisely a sequence of sets) for any $\epsilon > 0$ and a fixed value η_0 .

From rectangles to triangles

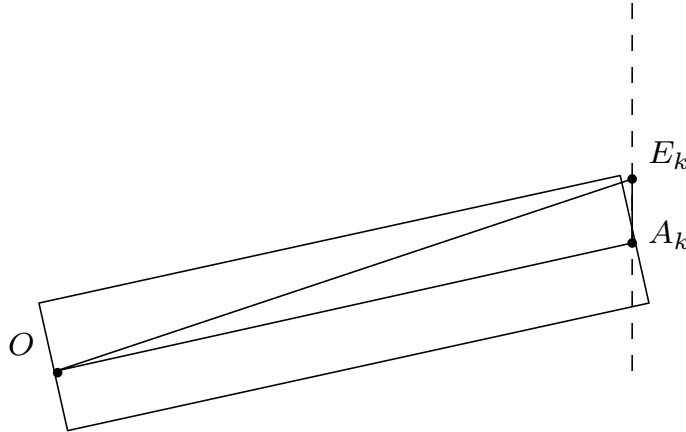


Figure 1: A rectangle R_k and a triangle T_k , both object are oriented along $\simeq t_k$ and have an eccentricity $\simeq e_k$.

Without loss of generality, we will work at some point with triangles instead of rectangles. For any $k \geq 1$ define the triangle T_k as

$$T_k := T_k(\mathbf{e}, \mathbf{t}) = OA_kE_k$$

where $O = (0, 0)$, $A_k = (1, t_k)$ and $E_k = (1, t_k + e_k)$. Loosely speaking, the triangle T_k is a triangle which is *oriented* along the direction t_k and of *eccentricity* e_k . Denoting by \mathcal{B}' the basis generated by the triangles T_k one can observe that we have the following property. For any $R \in \mathcal{B}$ there exists $T \in \mathcal{B}'$ satisfying for some vector $\vec{t} \in \mathbb{R}^2$

$$\vec{t} + \frac{1}{16}T \subset R \subset T$$

and conversely for any $T \in \mathcal{B}'$ there exists $R \in \mathcal{B}$ satisfying for some vector $\vec{t} \in \mathbb{R}^2$

$$\vec{t} + \frac{1}{16}R \subset T \subset R.$$

This implies that for any $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$ we have

$$M_{\mathcal{B}}f(x) \simeq M_{\mathcal{B}'}f(x).$$

Hence it is equivalent to work with \mathcal{B} or with \mathcal{B}' and we will denote both basis by \mathcal{B} .

Acknowledgments

I warmly thank Laurent Moonens and Emmanuel Russ for their kind advices.

2 Proof of Theorem 7

It is well know that the operator $M_{\{0\}}$ associated to the basis $\mathcal{R}_{\{0\}} = \{R \in \mathcal{R} : \omega_R = 0\}$ is a good operator. Now by geometric inspection and using the property that $t_k < Ce_k$ one can prove that for any $R \in \mathcal{B}$ there exists a rectangle $P \in \mathcal{R}_{\{0\}}$ such that

$$R \subset P$$

and also

$$|P| \leq 8(1 + C)|R|.$$

This property allow us to use the operator $M_{\{0\}}$ to dominate pointwise $M_{\mathcal{B}}$. Fix any $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and any $R \in \mathcal{B}$ and the associated rectangle $P \in \mathcal{R}_{\{0\}}$; we have

$$\frac{1}{|R|} \int_R |f| \leq \frac{8(1 + C)}{|P|} \int_P |f|$$

and this shows that for any $x \in \mathbb{R}^2$ we have

$$M_{\mathcal{B}}f(x) \leq 8(1 + C)M_{\{0\}}f(x).$$

The conclusion comes from the fact that the strong maximal operator $M_{\{0\}}$ is a good operator.

3 Geometric estimates

We establish two geometric estimates that will be useful. Fix an arbitrary open triangle $\Delta = ABC$ and consider the triangle Δ_2 defined as $\Delta_2 := \vec{B} + \frac{1}{2}(\Delta - \vec{A})$.

Lemma 1 (Geometric estimate I). *The following inclusion holds*

$$\Delta_2 \subset \left\{ M_{\{\Delta\}} \mathbb{1}_{\Delta} \geq \frac{1}{4} \right\}.$$

In other words, the level set $\{M_{\{\Delta\}} \mathbb{1}_{\Delta} \geq \frac{1}{4}\}$ contains Δ_2 .

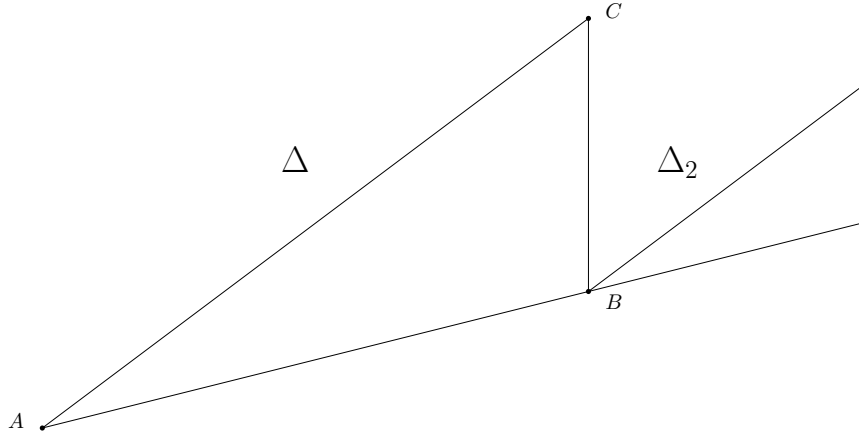


Figure 2: The triangles Δ and Δ_2 will usually be in this position.

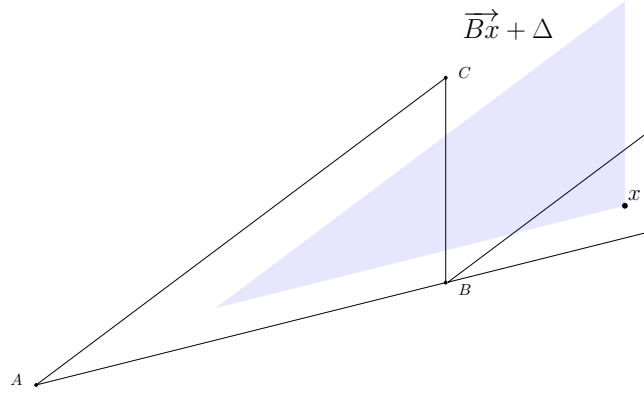


Figure 3: The proof of lemma 1 relies on the fact that $|\Delta \cap (\overrightarrow{Bx} + \Delta)| = \frac{1}{4}|\Delta|$.

Proof. Fix $x \in \overrightarrow{B} + \frac{1}{2}(\Delta - \overrightarrow{A})$. It suffices to observe that we have $x \in \overrightarrow{Bx} + \Delta$ and that $|\Delta \cap (\overrightarrow{Bx} + \Delta)| \geq \frac{1}{4}|\overrightarrow{Bx} + \Delta|$. Hence $x \in \{M_{\{\Delta\}}\mathbb{1}_\Delta \geq \frac{1}{4}\}$. \square

We will need a more general version of the previous estimate. For $e \in \mathbb{R}_+$ and $\Delta = ABC$ as before, define the triangle T as

$$T := T(e, \Delta) = AB(B + e\overrightarrow{BC}).$$

Lemma 2 (Geometric estimate II). *For any couple (Δ, T) as defined above, the following inclusion holds $\Delta_2 \subset \{M_{\{T\}}\mathbb{1}_\Delta \geq \eta(e)\}$ where $\eta(e) = \inf \{\frac{1}{4}, \frac{1}{4e}\}$.*

Proof. The proof is akin to the proof of lemma 1 and we invite the reader to look at figure ?? for a geometric representation. It is enough to check

$$x_0 \in \{M_{\{T\}}\mathbb{1}_\Delta > \eta(e)\}$$

where $x_0 = B + \frac{1}{2}\overrightarrow{AC}$ because this is the worst case. To begin with, observe that we have $x_0 \in \frac{1}{2}\overrightarrow{AC} + T$. We distinguish then two situations ; if we have

$$0 < e \leq 1$$

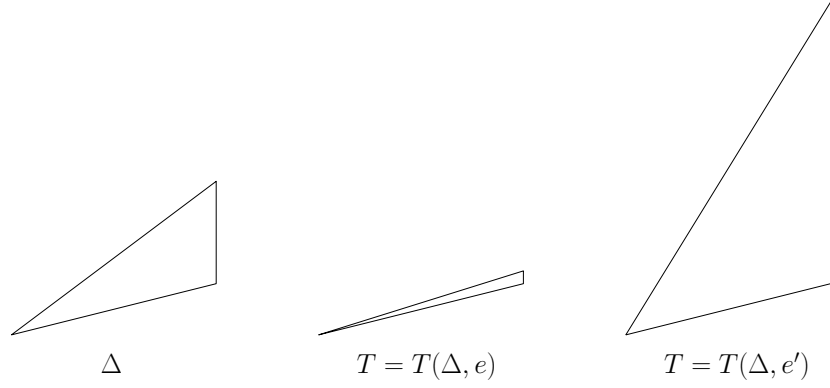


Figure 4: A representation of Δ and $T = T(\Delta, e)$ for $e \ll 1$ and $e' > 1$.

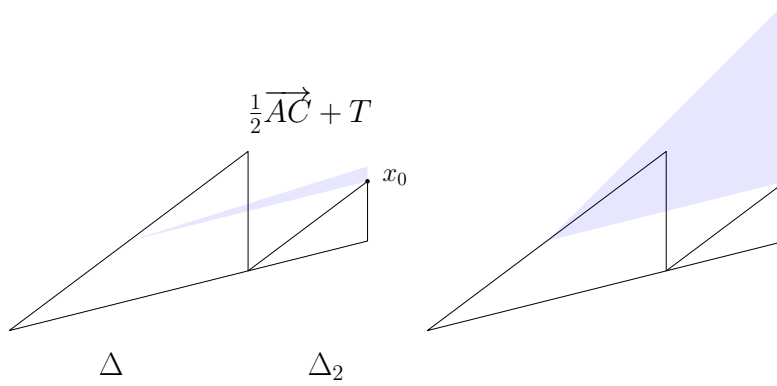


Figure 5: An illustration of the argument of lemma 2 ; the left side represents the case $0 < e \leq 1$ and the right side the case $e > 1$; the triangle in shaded blue represents $\frac{1}{2}\overrightarrow{AC} + T$.

we claim that we are in the situation corresponding to the left situation in figure ?? that is to say we have

$$\left| \Delta \cap \left(\frac{1}{2}\overrightarrow{AC} + T \right) \right| = \frac{1}{4}|T|$$

and so

$$x_0 \in \left\{ M_{\{T\}} \mathbb{1}_\Delta \geq \frac{1}{4} \right\}.$$

The second situation corresponds to the case where $1 < e$; in this case, we have (see figure ??)

$$\left| \Delta \cap \left(\frac{1}{2}\overrightarrow{AC} + T \right) \right| \geq \frac{1}{4}|\Delta| \geq \frac{1}{4e}|T|.$$

This shows that we have $x_0 \in \left\{ M_{\{T\}} \mathbb{1}_\Delta > \frac{1}{4e} \right\}$ which concludes. \square

4 Generalized Perron trees

Denote by Δ_k the triangle whose vertices are the points $O, A_k = (1, t_k)$ and $A_{k+1} = (1, t_{k+1})$. Recall that we have supposed $\tau_t < \infty$.

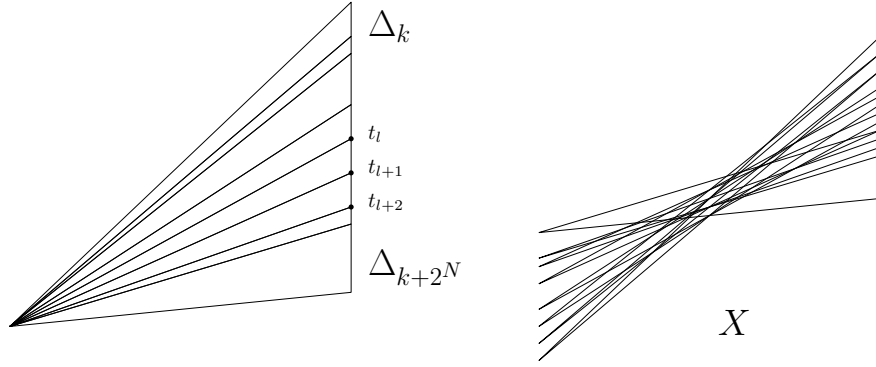


Figure 6: A representation of some Δ_k and on the left side a Perron tree X generated with those triangles. The idea is that for large n one has $|X| \ll |\Delta_k \sqcup \dots \sqcup \Delta_{k+2^N}|$; plus the second property of proposition 1.

Proposition 1 (Generalized Perron Tree). *For any positive ratio α close to 1 and any integer $n \geq 1$, there exists an integer $N \gg 1$ and 2^n vectors $\vec{s}_k := (0, s_k)$ such that defining the set*

$$X = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + \Delta_k)$$

we have the following properties

- $|X| \leq (\alpha^{2^n} + \tau_t(1 - \alpha)) |\Delta_{N+1} \sqcup \dots \sqcup \Delta_{N+2^n}|$;
- for any $k \neq l$ the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}\Delta_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}\Delta_l$ are disjoint.

We say that the set X is a generalized Perron tree of scale (α, n) and we denote it by $X_{\alpha, n}(\mathbf{t})$.

The fact that the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}\Delta_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}\Delta_l$ are disjoint is not proved in [12] but this can be proved easily by geometry inspection. Observe that for any $\epsilon > 0$, one can first choose α close to one and then n large enough in order to have

$$|X_{\alpha, n}(\mathbf{t})| \leq \epsilon |\Delta_{N+1} \sqcup \dots \sqcup \Delta_{N+2^n}|$$

for some large N . To obtain such an inequality, we need a sufficient condition on the thin triangles T_k that ensures in some sense that they are comparable.

Indeed, suppose that we had defined for any $k \geq 1$ the triangle Δ_k as the one whose vertices are the points $O, G_k = (1, \frac{1}{2^k})$ and $G_{k+1} = (1, \frac{1}{2^{k+1}})$. In this situation, for any $I \subset \mathbb{N}$ and any sequence of vectors $\{\vec{s}_i\}_{i \in I} \subset \mathbb{R}^2$ the set X_I defined as

$$X_I = \bigcup_{i \in I} (\vec{s}_i + \Delta_i)$$

satisfies the following inequality

$$|X_I| \geq |\Delta_{i_0}| \geq \frac{1}{2} \left| \bigcup_{i \in I} \Delta_i \right|$$

where $i_0 := \min I$. Hence we cannot hope to stack up the triangles Δ_k into a set X that has a small area compared to the sum of the areas of the Δ_k .

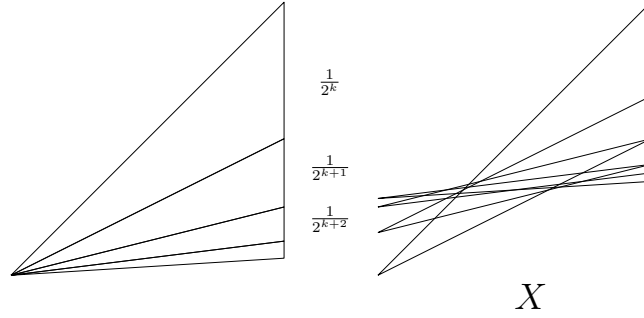


Figure 7: It quite difficult to construct a Perron ; one needs a condition to assure that the triangles Δ_k are *comparable* in some sense. On this figure, the Δ_k differs to much and one will always have $|X| \simeq |\cup \Delta_i|$ as explained.

Hopefully this example shed light on the condition imposed on \mathbf{t} which is

$$\tau_{\mathbf{t}} := \sup_{k \geq 0, l \leq k} \left(\frac{t_{k+2l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+2l} - t_{k+l}} \right) < \infty.$$

This assure that the triangles Δ_k are comparable in some sense and that we can construct *generalized Perron trees* with them.

5 Proof of Theorem 8

Recall that we suppose there is a constant $\mu_0 > 0$ such that for any $k \geq 1$, $e_k < \mu_0 |t_k - t_{k+1}|$. To begin with, we are going to construct a Perron tree $X_{\alpha, n}(\mathbf{t})$ with the triangles $\{\Delta_k\}_{k \geq 1}$. Then we will exploit this Perron tree $X_{\alpha, n}(\mathbf{t})$ with the triangles $\mathcal{B} = \{T_k\}_{k \geq 1}$ to show that $M_{\mathcal{B}}$ is a bad operator. Precisely we prove the following claim.

Claim 1. *For any α close to 1 and any $n \in \mathbb{N}$, the Perron tree $X := X_{\alpha, n}(\mathbf{t})$ satisfies the following inequality*

$$|X| \leq \epsilon |\{M_{\mathcal{B}} \mathbb{1}_X > \eta(\mu_0)\}|$$

where $\epsilon = \alpha^{2n} + \tau_{\mathbf{t}}(1 - \alpha)$.

Proof. Fix α close to 1 and $n \in \mathbb{N}$ and consider a Perron tree of scale (α, n)

$$X := X_{\alpha, n}(\mathbf{t}) = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + \Delta_k)$$

where N is given by proposition 1. Fix any $k \in \{N+1, \dots, N+2^n\}$ and consider the couple of triangles

$$(\vec{s}_k + \Delta_k, \vec{s}_k + T_k)$$

or more simply the couple (Δ_k, T_k) which is the same up to a translation. We can apply lemma 2 to this couple which yields the following inclusion

$$\left(\vec{A}_{k+1} + \vec{s}_k \right) + \frac{1}{2} \Delta_k \subset \{M_{\{T_k\}} \mathbb{1}_{\vec{s}_k + \Delta_k} > \eta(\mu_0)\}.$$

Since we have $M_{T_k} \leq M_{\mathcal{B}}$ we also have

$$\left(\vec{A}_{k+1} + \vec{s}_k \right) + \frac{1}{2} \Delta_k \subset \{M_{\mathcal{B}} \mathbb{1}_{\vec{s}_k + \Delta_k} > \eta(\mu_0)\}.$$

We form now the union on $k \in \{N + 1, \dots, N + 2^n\}$ of the precedent inclusion which by definition yields

$$\bigsqcup_{k=1}^{2^n} \left(\vec{A}_{k+1} + \vec{s}_k \right) + \frac{1}{2} \Delta_k \subset \{M_{\mathcal{B}} \mathbb{1}_X > \eta(\mu_0)\}.$$

In the latter inclusion, the fact that the union is disjoint comes from proposition 1. Hence this gives in terms of Lebesgue measure

$$\sum_{N+1 \leq k \leq N+2^n} \frac{1}{4} |\Delta_k| \leq |\{M_{\mathcal{B}} \mathbb{1}_X > \eta(\mu_0)\}|.$$

Using the fact that X is a Perron tree constructed with the triangles Δ_k we have

$$|X| \leq (\alpha^{2^n} + \tau_t(1 - \alpha)) |\Delta_{N+1} \sqcup \dots \sqcup \Delta_{N+2^n}|.$$

In other words we have

$$|X| \leq 4(\alpha^{2^n} + \tau_t(1 - \alpha)) |\{M_{\mathcal{B}} \mathbb{1}_X > \eta(\mu_0)\}|.$$

□

The claim implies that for any $p > 1$ we have

$$\|M_{\mathcal{B}}\|_p \geq \eta(\mu_0)(4\alpha^{2^n} + 4\tau_t(1 - \alpha))^{-\frac{1}{p}}$$

for any α close to 1 and any $n \in \mathbb{N}$. The fact that constant $\eta(\mu_0)$ is independant of the scale (α, n) concludes : we have $\|M_{\mathcal{B}}\|_p = \infty$ for any $p > 1$ *i.e.* $M_{\mathcal{B}}$ is a bad operator.

6 Proof of Theorem 6

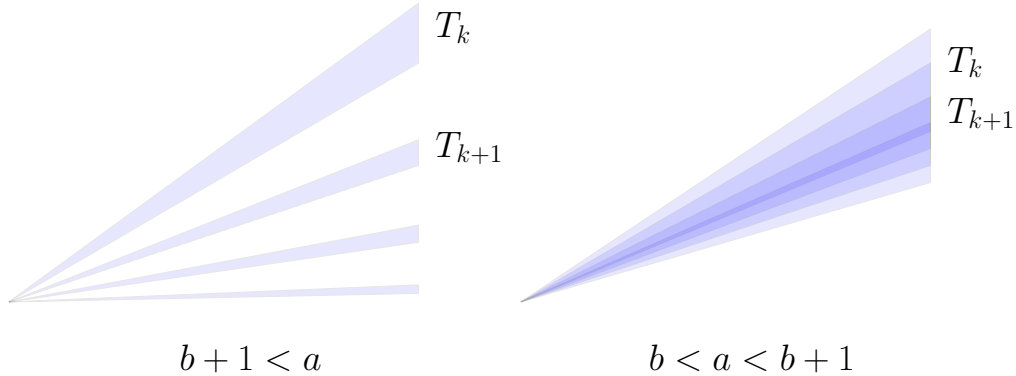


Figure 8: On the left side a representation of the regime $a > b + 1$. In this situation, the triangles T_k do not overlap at all for large k (actually the gap gets bigger with k). On the right side a representation of the regime $b + 1 > a > b$. In this situation, the triangles T_k tend to completely overlap each other.

We are now ready to prove Theorem 6 ; the case $a \leq b$ is consequence of Theorem 7 and the case $a \geq b + 1$ is a consequence of Theorem 8. It remains to deal with the case $b < a < b + 1$. In this regime, the problem is that the triangles $\{T_k\}$ are too much overlapped. The idea is to extract some of them $\{T_{k_n}\}_{n \geq 1}$ which are almost disjoint (up to some intersection of a fixed ratio) and to construct a Perron tree with this family. The following lemma will be helpful.

Lemma 3. *If there exists a polynome $P \in \mathbb{R}[X]$ such that for any $k \geq 1$ we have $1 \leq P(k)t_k$ then $\tau_{\mathbf{t}} < \infty$.*

We consider the triangle T_k defined as $T_k = OA_kE_k$ where $O = (0, 0)$, $A_k = (1, \frac{\pi}{4n^b})$ and $E_k = (1, \frac{\pi}{4n^b} + \frac{1}{n^a})$. Since we suppose that $b < a < b + 1$ the triangles T_k and T_{k+1} tend to overlap completely and this forbid a direct construction of a Perron tree with the triangles $\{T_k\}_{k \geq 1}$. Fix an integer k and let's look for $l > k$ such that T_k and T_l are disjoint. If we consider $l = 2k$, we have

$$\frac{\pi}{4(2k)^b} + \frac{1}{(2k)^a} < \frac{\pi}{4k^b}$$

and so $T_k \cap T_{2k} = O$ and the triangles are disjoint. However we can easily verify that the extracted sequence

$$\mathbf{t}' = \left\{ \frac{\pi}{4(2^n k)^b} \right\}_{n \geq 1}$$

won't satisfy the condition

$$\tau_{\mathbf{t}'} < \infty.$$

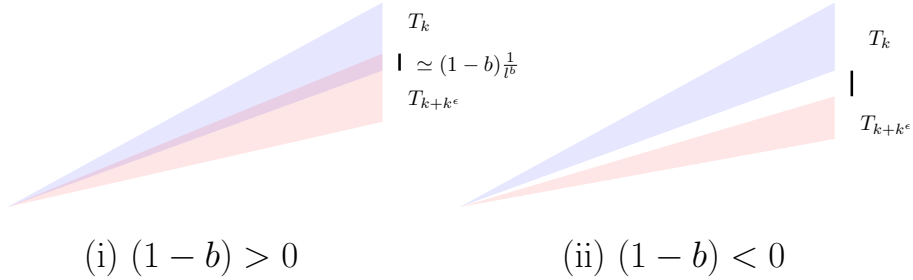


Figure 9: If we recursively set $l \simeq k + k^\epsilon$ with $\epsilon = b + 1 - a$; on long scale we obtain the following situation depending on the sign of $(1 - b)$. The triangles T_k and T_l are almost adjacent and we can apply Theorem 8.

Hence we fix a small constant $\epsilon > 0$ and look for a l of the form

$$l = k + k^\epsilon.$$

We have

$$\frac{\pi}{4l^b} + \frac{1}{l^b} \simeq \frac{1}{k^b(1 + k^{\epsilon-1})^b} + \frac{1}{l^b} \simeq \frac{1}{k^b}(1 - bk^{\epsilon-1}) + \frac{1}{l^b}$$

and so

$$\left(\frac{\pi}{4l^b} + \frac{1}{l^b} \right) - \frac{\pi}{4k^b} \simeq \frac{1}{l^b} - \frac{b}{k^{b+1-\epsilon}} \simeq \frac{1}{k^a} - \frac{b}{k^{b+1-\epsilon}}.$$

Hence, if we choose

$$\epsilon := b + 1 - a > 0$$

we have

$$\left(\frac{\pi}{4l^b} + \frac{1}{l^b} \right) - \frac{\pi}{4k^b} \simeq (1 - b) \frac{1}{l^b}.$$

This means that either (i) the triangles T_k and T_l are disjoint but not too far (ii) either a constant fraction of T_l is out of T_k , depending on the value of b . We have set

$$l := k_1 = k + k^\epsilon := k_0 + k_0^\epsilon = F(k_0).$$

We define by induction

$$k_{n+1} = F(k_n).$$

We have

$$k_{n+1} - k_n = k_n^\epsilon.$$

This leads to the following majoration : there exists a polynome P such that for every $k \geq 1$, we have

$$k_n \leq P(n).$$

Using Lemma 3 shows that we have $\tau_{\mathbf{t}'} < \infty$ where $\mathbf{t}' = \{t_{k_n}\}_{n \geq 1}$ and so we can construct a generalized Perron tree as before with the triangles $\{\Delta_{k_n}\}_{n \geq 1}$ associated to the sequence \mathbf{t}' and exploit it with the triangles $\{T_{k_n}\}_{n \geq 1}$; that is to say we apply Theorem 8. This concludes the case $b < a < b + 1$.

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