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# Application of Perron Trees to Geometric Maximal Operators

Anthony Gauvan

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## Abstract

To be able to construct a generalized Perron trees as in [11] with a family of triangles  $T$  implies that the maximal operator  $M_T$  associated to this family is unbounded on  $L^p(\mathbb{R}^2)$  for any  $1 \leq p < \infty$ . If  $T'$  is a family of triangles which is geometrically adapted to the family  $T$ , we prove that  $M_{T'}$  is also unbounded on  $L^p(\mathbb{R}^2)$  for any  $1 \leq p < \infty$ .

We work in the euclidean plane  $\mathbb{R}^2$ ; if  $U$  is a measurable subset of  $\mathbb{R}^2$  we denote by  $|U|$  its two dimensional Lebesgue measure.

## 1 Introduction

Denote by  $R$  the collection containing all rectangles of  $\mathbb{R}^2$ ; for  $r \in R$  we define its *orientation* as the angle  $\omega_r \in [0, \pi)$  that its longest side makes with the  $Ox$ -axis and its *eccentricity* as the ratio  $\kappa_r \in (0, 1]$  of its shortest side by its longest side.

If  $B$  is an arbitrary non empty family contained in  $R$ , we define the associated *derivation basis*  $B^*$  by

$$B^* = \{\vec{t} + hr : \vec{t} \in \mathbb{R}^2, h > 0, r \in B\}.$$

The derivation basis  $B^*$  is the smallest invariant by dilation and translation collection that contains any element of  $B$ . We define the *geometric maximal operator*  $M_B$  generated by  $B$  by

$$M_B f(x) := \sup_{x \in r \in B^*} \frac{1}{|r|} \int_r |f|$$

for any  $f \in L^1_{loc}(\mathbb{R}^2)$  and  $x \in \mathbb{R}^2$ . Here the supremum is taken on elements  $r$  in  $B^*$  that contain the point  $x$ . The definitions of  $B^*$  and  $M_B$  remain valid when we consider that  $B$  is an arbitrary family composed of open bounded convex sets. In this note, for technical reasons we will work with triangles instead of rectangles.

For  $p \in (1, \infty]$  we define the operator norm  $\|M_B\|_p$  of  $M_B$  by

$$\|M_B\|_p = \sup_{\|f\|_p=1} \|M_B f\|_p.$$

If  $\|M_B\|_p < \infty$  we say that  $M_B$  is bounded on  $L^p(\mathbb{R}^2)$ . The  $L^p$ -boundedness of a maximal operator  $M_B$  is related to the *geometry* that the family  $B$  exhibits. We will say that the operator  $M_B$  is a **good operator** when it is bounded on  $L^p$  for any  $p > 1$ . On the other side, we say that the operator  $M_B$  is a **bad operator** when it is unbounded on  $L^p$  for any  $1 < p < \infty$ .

## Classic examples

We first recall some well known results about classic maximal operators. Suppose that  $B$  is reduced to a single element, say  $B = \{c_0\}$  where  $c_0 = [0, 1] \times [0, 1]$ . This square yields the so-called *Hardy-Littlewood* maximal operator that we will simply denote  $M := M_{\{c_0\}}$ . The operator  $M$  is not bounded from  $L^1$  to  $L^1$ . However it has weak-type  $(1, 1)$  *i.e.* there is a constant  $C > 0$  such that

$$|\{Mf > t\}| \leq C \times \frac{\|f\|_1}{t}$$

for any  $f \in L^1$  and  $t > 0$ . We invite the reader to see [9] page 37. Actually, it is not difficult to see that if

$$\kappa(B) := \inf_{R \in B} \kappa_R > 0$$

then the operator  $M_B$  is pointwise bounded by a multiple of  $M$  and so  $M_B$  has also weak-type  $(1, 1)$ . Using Marcinkiewicz interpolation theorem and the fact that we always have  $\|M_B f\|_\infty \leq \|f\|_\infty$ , one deduces that  $M_B$  is bounded on  $L^p$  for any  $p > 1$  in this case. More generally, as soon as  $M_B$  is bounded on  $L^{p_0}$  for some  $p_0 > 1$  it is then bounded on  $L^p$  for any  $p \geq p_0$  using the same method.

Different situations can occur when  $\kappa(B) = 0$ ; that is to say when  $B$  contains arbitrary thin rectangles. Consider the family

$$B^{\{0\}} = \{r \in R : \omega_r\}$$

*i.e.* the family  $B^{\{0\}}$  contains all rectangles whose sides are parallel to the  $Ox-Oy$  axis. In this case, the operator  $M_{B^{\{0\}}}$  is commonly called the *strong* maximal operator and we denote it  $M_s$ . It is known that the operator  $M_s$  does not have weak-type  $(1, 1)$ . Yet  $M_s$  is bounded on  $L^p$  for any  $p > 1$ . We invite the reader to see [6] for a geometric proof of this fact.

Consider now the maximal operator  $M_R$  *i.e.* the maximal operator generated by all the rectangles contained in the plane. In this case, because the collection  $R$  is "too big", we have for any  $1 < p < \infty$

$$\|M_R\|_p = \infty.$$

We invite the reader to see [9] page 116. To sum up, we have the following facts.

- The Hardy-Littlewood maximal operator  $M$  has weak-type  $(1, 1)$  and is a good operator.
- More generally, for any  $B \subset R$ , if  $\kappa(B) > 0$  then  $M_B$  has weak-type  $(1, 1)$  and is a good operator.
- The strong maximal operator  $M_s$  is a good operator.
- The maximal operator  $M_R$  generated by all rectangles of the plane is a bad operator.

## Directional maximal operators

A lot of research has been done in the case where  $B$  is equal to

$$B^\Omega := \{r \in R : \omega_r \in \Omega\}$$

where  $\Omega$  is an arbitrary set of directions in  $[0, \pi)$ . In other words,  $B^\Omega$  is the set of all rectangles whose orientation belongs to  $\Omega$ . We say that  $B^\Omega$  is a *directional basis* and to alleviate the notation we denote

$$M_{B^\Omega} := M^\Omega.$$

In the literature, the operator  $M^\Omega$  is said to be a *directional maximal operator*. The study of those operators goes back at least to Cordoba and Fefferman's article [7] in which they use geometric techniques to show that if

$$\Omega = \left\{ \frac{\pi}{2^k} \right\}_{k \geq 1}$$

then  $M^\Omega$  has weak-type  $(2, 2)$ . A year later, using Fourier analysis techniques, Nagel, Stein and Wainger proved in [12] that  $M^\Omega$  is actually bounded on  $L^p$  for any  $p > 1$ . In [1], A. Alfonseca has proved that if the set of direction  $\Omega$  is a *lacunary set of finite order* then the operator  $M^\Omega$  is bounded on  $L^p$  for any  $p > 1$ . In [4], M. Bateman completely characterized the  $L^p$ -boundedness of directional operators. Precisely he proved the following theorem.

**Theorem 1.** *Fix an arbitrary set of directions  $\Omega \subset [0, \pi)$ . The directional maximal operator  $M^\Omega$  is either bounded on  $L^p$  for any  $1 < p < \infty$  or either unbounded on  $L^p$  for any  $1 < p < \infty$ . In other words, an operator  $M^\Omega$  is either good or bad.*

Hence we know that a set of directions  $\Omega$  yields an directional operator  $M^\Omega$  that is either good or bad. Merging the vocabulary, we will say that a set of directions  $\Omega$  is a *good set of directions* when  $M^\Omega$  is good and that it is a *bad set of directions* when  $M^\Omega$  is bad. The notion of good/bad is well understood for a set of directions  $\Omega$  and the associated directional operator  $M^\Omega$ . To say it bluntly,  $\Omega$  is a good set of directions if and only if it can be included in a finite union of lacunary sets of finite order. If this is not possible,  $\Omega$  is a bad set of directions. Rather than giving the definition of lacunarity of finite order (we refer to [4] for this) we provide some examples. In the following cases the set of directions  $\Omega$  is good *i.e.*  $M^\Omega$  is bounded on  $L^p$  for any  $1 < p < \infty$  :

- when  $\Omega$  is finite,
- when  $\Omega = \left\{ \frac{\pi}{2^k} \right\}_{k \geq 1}$ ,
- more generally when  $\Omega$  is included in a lacunary sequence.

On the other hand, the set of directions  $\Omega$  is bad *i.e.*  $M^\Omega$  is unbounded on  $L^p$  for any  $1 < p < \infty$  when :

- $\Omega$  contains an open interval,
- $\Omega$  is the ternary Cantor set,
- more surprisingly,  $\Omega = \left\{ \frac{\pi}{k^s} \right\}_{k \geq 1}$  for a positive fixed  $s > 0$ .

## Questions

A natural question arises from all this : can one find a family  $B$  in  $R$  such that the operator  $M_B$  is unbounded on  $L^{1+\epsilon}$  for some small  $\epsilon > 0$  and bounded on  $L^{1+A}$  for some large  $A > 1$  ? We believe that such a family does not exist and we thank P. Hagelstein and A. Stokolos for communicating us the following conjecture.

**Conjecture 1.** *Any maximal operator  $M_B$  generated by a family  $B$  included in  $R$  is either bounded on  $L^p$  for all finite  $p > 1$  or either unbounded on  $L^p$  for all finite  $p > 1$ .*

In the general setting, Hagelstein and Stokolos proved in [10] that if the operator  $M_B$  satisfies a *tauberian inequality* then it is bounded on  $L^p$  for large  $p < \infty$ . Precisely, they show that if there exist  $0 < \eta_0 < 1$  and  $C_0 > 0$  such that

$$|\{M_B \mathbb{1}_E > \eta_0\}| \leq C_0 \times |E|$$

for any bounded measurable set  $E$ , then  $M_B$  is at least bounded on  $L^{p_0}$  for a large  $p_0$ . The problem here is that  $p_0$  depends *a priori* on the basis  $B$ .

Another interesting problem can be phrased as follows : given  $B$  included in  $R$  find the optimal Orlicz space  $L^\Phi$  on which the operator  $M_B$  is bounded. This question won't be addressed here but the interested reader might consult Stokolos' article [13] or more recently D'Aniello, Moonens and Rosenblatt's article [2] or D'Aniello and Moonens' article [3].

To tackle the conjecture at least two questions must be addressed.

### Question 1 : rarefaction of a directional basis

Fix an arbitrary family  $B$  contained in  $R$  satisfying the following conditions :

- the directions contained in  $B$  *i.e.* the set  $\Omega(B) := \{\omega_r : r \in B\}$  is a bad set of directions ;
- for any  $\omega \in \Omega(B)$  we *only* have access to arbitrary thin rectangles in  $B$  in the direction  $\omega$  *i.e.* for any  $\omega \in \Omega(B)$  we have

$$\kappa(B, \omega) := \inf_{r \in B, \omega_r = \omega} \kappa_r = 0.$$

Loosely speaking, we assume that  $B$  is a *rarefied* version of the directional basis  $B^\Omega$ . Under those conditions, can one show that the operator  $M_B$  is still bad ? In a submitted work, we have proven that such an operator  $M_B$  is indeed bad.

### Question 2 : fleeing bases

The second question is more complex and it is the one that we partially address in this note. Fix an arbitrary family  $B$  included in  $R$  satisfying the following conditions :

- the directions that are accessible with arbitrary thin rectangles form a good set of directions *i.e.* the set of directions

$$\Omega(B, 0) := \{\omega \in [0, \pi) : \forall \epsilon > 0, \exists r \in B, \kappa_r < \epsilon, \omega_r = \omega\}$$

is a good set of directions ;

- yet for any  $\epsilon > 0$  the set of directions

$$\Omega(B, \epsilon) := \{\omega \in [0, \pi) : \exists r \in B, \kappa_r < \epsilon, \omega_r = \omega\}$$

is always a bad set of directions.

Can one show that such an operator  $M_B$  is good ? or bad ? In this note, we consider different families  $B$  satisfying those conditions and show that in those cases the operator  $M_B$  is indeed good or bad.

## Results

We exhibit different geometric maximal operators verifying the condition of the last question and which are either good or bad. Precisely we denote by

$$\mathbf{t} = \{t_k\}_{k \geq 1} \subset [0, 1]$$

a sequence decreasing to 0 and by

$$\mathbf{x} = \{x_k\}_{k \geq 1} \subset (0, \infty)$$

any positive sequence. For  $A, B, C \in \mathbb{R}^2$ , we denote by  $ABC$  the triangle whose vertices are the points  $A, B$  and  $C$ . If  $t$  is a point in  $\mathbb{R}^2$ , we might denote it by  $\vec{t}$  to insist on the fact that it used as a translation vector. We define two families of triangles. First, for any  $k \geq 1$ , we define the triangle  $T_k$  as

$$T_k = OA_k A_{k+1}$$

where  $O = (0, 0)$ ,  $A_k = (1, t_k)$  and  $A_{k+1} = (1, t_{k+1})$ . Observe that we have only used the sequence  $\mathbf{t}$  in order to define the triangles  $T_k$ . Then we define the triangle  $T_k(\mathbf{x})$  as

$$T_k(\mathbf{x}) = OA_{k+1}(A_{k+1} + \vec{X}_k)$$

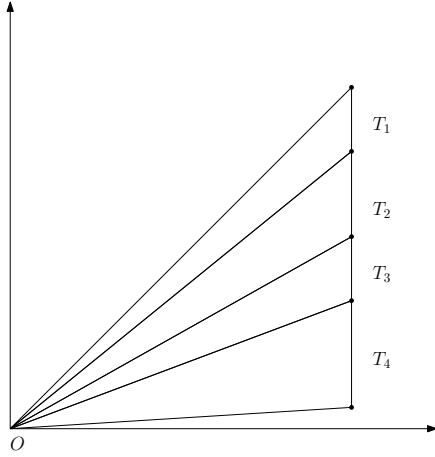


Figure 1: A representation of the triangles  $T_k$ .

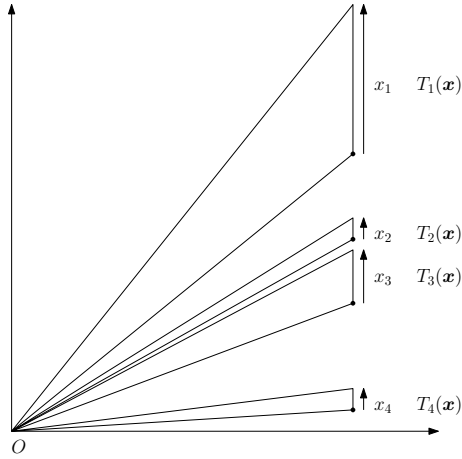


Figure 2: A representation of the triangles  $T_k(\mathbf{x})$ .

where  $\vec{X}_k := (0, x_k)$ . We will study the collection  $B$  defined as

$$B := \{T_k(\mathbf{x})\}_{k \geq 1}.$$

Now that we have defined the triangles  $T_k$  and  $T_k(\mathbf{x})$  we can interpret the sequence  $\mathbf{t}$  as a sequence of tangent of angles that tends to 0 and the sequence  $\mathbf{x}$  as an arbitrary sequence of eccentricity. Our first result reads as follow.

**Theorem 2.** *Suppose there is a constant  $C > 0$  such that for any  $k \geq 1$*

$$t_k \leq C \times x_k.$$

*In this case the operator  $M_B$  is a good operator i.e.  $M_B$  is bounded on  $L^p$  for any  $p > 1$*

Defining the quantity

$$\tau_{\mathbf{t}} := \sup_{k \geq 0, l \leq k} \left( \frac{t_{k+l+l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+l+l} - t_{k+l}} \right)$$

we will prove the following proposition.

**Theorem 3.** *Suppose that*

$$\tau_{\mathbf{t}} < \infty$$

*and that there is a constant  $\mu_0 > 0$  such that for any  $k \geq 1$*

$$x_k < \mu_0 |t_k - t_{k+1}|.$$

*In this case, the maximal operator  $M_B$  is a bad operator i.e.  $M_B$  is unbounded on  $L^p$  for any finite  $p \geq 1$ .*

## Structure and strategy

This note is structured as follow.

- In section 2 we give two geometric estimates involving maximal operators.
- In section 3 we recall known facts about the construction of so-called *Perron trees* and *generalized Perron trees*.
- Finally in the last two sections, we prove Theorems 2 and 3.

To prove that an operator  $M_B$  is a good operator we are simply going to exhibit another operator  $M'$  that is known to be good and we will check that we have

$$M_B \leq C \times M'.$$

This will immediately imply that  $M_B$  is good also. Conversely, to prove that an operator  $M_B$  is bad, it is enough to find a small fixed value  $0 < \eta_0 < 1$  such that for any  $\epsilon > 0$  there is a subset  $X$  in  $\mathbb{R}^2$  such that

$$|X| \leq \epsilon \times |\{M_B \mathbb{1}_X > \eta_0\}|.$$

Indeed if this holds then for any  $p > 1$  we have

$$\int (M_B \mathbb{1}_X)^p \geq \eta_0^p \times |\{M_B \mathbb{1}_X > \eta_0\}| \geq \eta_0^p \times \frac{\|\mathbb{1}_X\|_p^p}{\epsilon}$$

since  $|X| = \|\mathbb{1}_X\|_p^p$ . Hence we have for any  $\epsilon > 0$  the following lower bound on the  $L^p$ -norm of  $M_B$

$$\|M_B\|_p^p \geq \frac{\eta_0^p}{\epsilon}$$

*i.e.* for any  $p > 1$  we have  $\|M_B\|_p^p = \infty$  since  $\epsilon > 0$  is arbitrary low in the latter inequality. This exactly means that  $M_B$  is bad. We are going to build our set  $X$  as a *generalized Perron tree*. Before discussing about those trees, we are going to detail some geometric estimates that will be useful.

## Acknowledgments

I thank Laurent Moonens and Emmanuel Russ for their kind advices.

## 2 Geometric estimates

We fix an arbitrary open triangle  $T_1 = ABC$ . Consider the triangle  $T_2$  defined as

$$T_2 := \vec{B} + \frac{1}{2}(T_1 - \vec{A}),$$

see figure 3.

**Lemma 1** (Geometric estimate I). *The following inclusion holds*

$$T_2 \subset \left\{ M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4} \right\}.$$

*In other words, the level set  $\{M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4}\}$  contains an affine copy of  $T_1$ .*

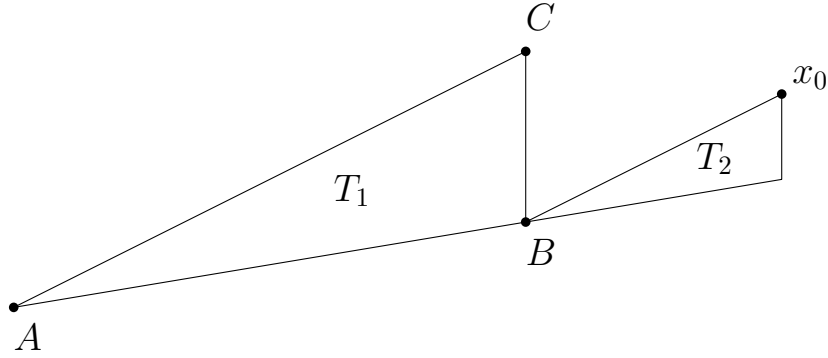


Figure 3: A representation of the triangles  $T_1$  and  $T_2$ .

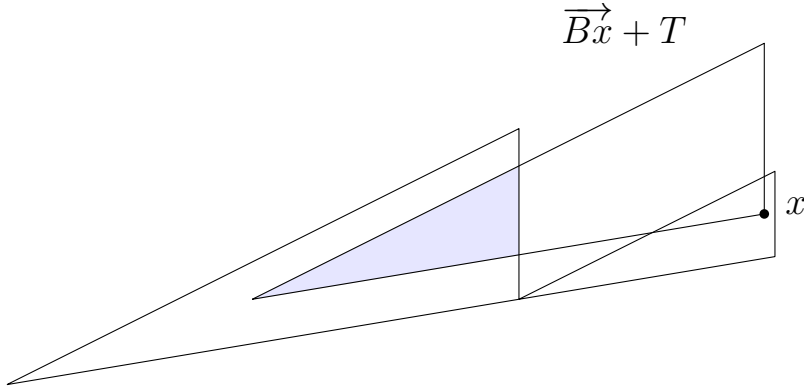


Figure 4: The proof of lemma 1 consists in observing that the blue shaded area represents a quarter of the triangle  $\overrightarrow{Bx} + T$ .

*Proof.* Fix  $x \in \overrightarrow{B} + \frac{1}{2}(T_1 - \overrightarrow{A})$ . It suffices to observe (see figure 4) that we have

$$x \in \overrightarrow{Bx} + T_1$$

and that

$$|T_1 \cap (\overrightarrow{Bx} + T_1)| \geq \frac{1}{4} |\overrightarrow{Bx} + T_1|.$$

Hence  $x \in \{M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4}\}$ . □

We now fix an arbitrary positive constant  $\mu_0 > 0$  and for all triangles  $T$  with vertices  $(A, B, C)$  we define the triangle  $T_{\mu_0}$  as

$$T_{\mu_0} := AB(B + \mu_0 \times \overrightarrow{BC}).$$

For any  $0 < x < \mu_0$  we define similarly the triangle

$$T(x) = AB(B + x \times \overrightarrow{BC}).$$



**Lemma 2** (Geometric estimate II). *For any couple  $(T_1, T(x))$  as defined above, the following inclusion holds*

$$T_2 \subset \{M_{\{T(x)\}}\mathbb{1}_{T_1} \geq \eta(\mu_0)\}$$

where  $\eta(\mu_0) = \inf \left\{ \frac{1}{4}, \frac{1}{4\mu_0} \right\}$ .

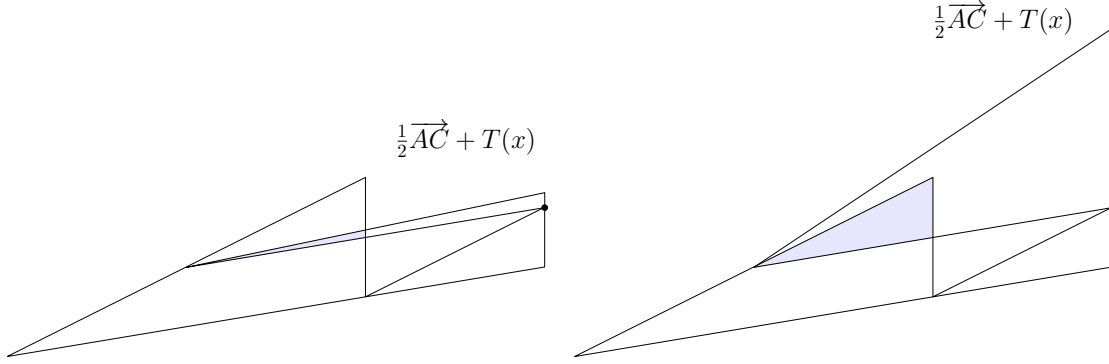


Figure 5: In the proof of lemma 2, we distinguish two cases : either  $x$  is very small as on the left side of the figure, either  $x$  is close to the parameter  $\mu_0$  as on the right of the figure.

*Proof.* The proof is akin to the proof of lemma 1, see figure 5 for a geometric representation. It is enough to check

$$x_0 \in \{\mathcal{M}_{T(x)}\mathbb{1}_{T_1} > \eta(\mu_0)\}$$

where  $x_0 = B + \frac{1}{2}\overrightarrow{AC}$  because this is the "worst case". To begin with, observe that we have

$$x_0 \in \frac{1}{2}\overrightarrow{AC} + T(x).$$

Now either we have

$$\left| T_1 \cap \left( \frac{1}{2}\overrightarrow{AC} + T(x) \right) \right| \geq \frac{1}{4}|T(x)|$$

and in this case we have  $x_0 \in \{M_{\{T(x)\}}\mathbb{1}_{T_1} > \frac{1}{4}\}$ ; or we have

$$\left| T_1 \cap \left( \frac{1}{2}\overrightarrow{AC} + T(x) \right) \right| = \frac{1}{4}|T_1|$$

and in this case we have  $x_0 \in \{M_{\{T(x)\}}\mathbb{1}_{T_1} \geq \frac{1}{4\mu_0}\}$  since

$$\frac{|T_1|}{|T(x)|} \geq \frac{|T_1|}{|T_{\mu_0}|} = \frac{1}{\mu_0}.$$

□

### 3 Generalized Perron trees

We are going to recall how one can construct a *Perron tree* as in [9] and more generally how one can construct a *generalized Perron tree* as in [11].

Before stating any proposition, we would like to say that loosely speaking the construction of a Perron tree is a procedure that specifies how one can

- (1) cut a triangle  $T$  in approximatively  $2^n$  thin triangles  $T_k$
- (2) stack up those thin triangles in a new set  $X$  that is usually called a Perron tree
- (3) the Perron tree  $X$  has different properties with respect to the thin triangles  $T_k$ . Among others we have  $|X| \ll |T|$ .

We formulate this more precisely in the following propositions. Denote by  $T$  the triangle whose vertices are the points  $O$ ,  $A = (1, 0)$  and  $B = (1, 1)$ . For any  $n \geq 1$  and  $1 \leq k \leq 2^n$  denote by  $T_k$  the triangle whose vertices are the points  $O$ ,  $A_k = (1, \frac{k}{2^n})$  and  $A_{k-1} = (1, \frac{k-1}{2^n})$ .

**Proposition 1** (see [9], page 112). *Fix any positive ratio  $\frac{1}{2} < \alpha < 1$  and any integer  $n \geq 1$ . There exist  $2^n$  vectors*

$$\vec{s}_k := (0, s_k)$$

*satisfying the following properties*

- *the set  $X$  defined as*

$$X = \bigcup_{k \leq 2^n} (\vec{s}_k + T_k)$$

*satisfies*

$$|X| \leq (\alpha^{2^n} + 2(1 - \alpha)) \times |T|;$$

- *for any  $k \neq l$  the triangles  $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$  and  $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$  are disjoint.*

*We say that the set  $X$  is a Perron tree of scale  $(\alpha, n)$  and we denote it by  $X_{\alpha, n}$ .*

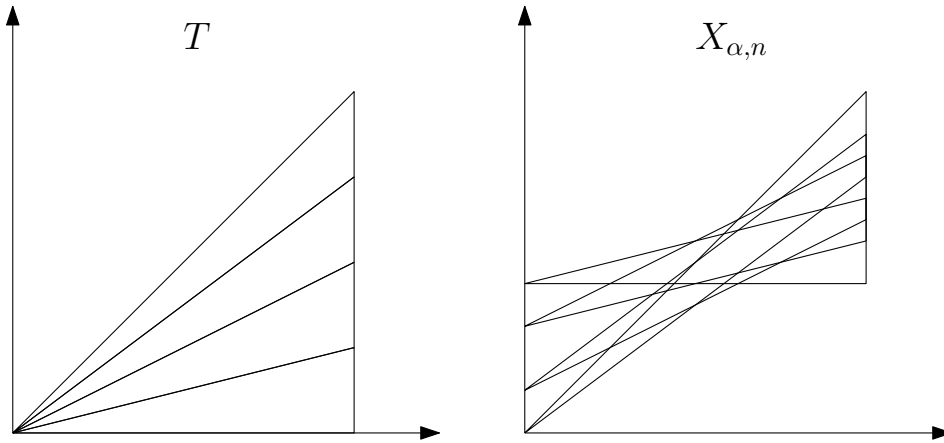


Figure 6: A representation of a Perron tree  $X_{\alpha, n}$  for  $n = 2$ .

Observe that for any  $\epsilon > 0$ , one can first choose  $\alpha$  close to one and then  $n$  large enough in order to have

$$|X_{\alpha, n}| \leq \epsilon \times |T|.$$

To obtain such an inequality, it is sufficient to have a condition on the thin triangles  $T_k$  that ensures in some sense that they have comparable area.

Indeed, suppose now that we define for any  $k \geq 1$  the triangle  $T_k$  as the one whose vertices are the points  $O$ ,  $G_k = (1, \frac{1}{2^k})$  and  $G_{k+1} = (1, \frac{1}{2^{k+1}})$ . In this situation, for any  $I \subset \mathbb{N}$  and any sequence of vectors  $\{\vec{s}_i\}_{i \in I} \subset \mathbb{R}^2$  the set  $X_I$  defined as

$$X_I = \bigcup_{i \in I} (\vec{s}_i + T_i)$$

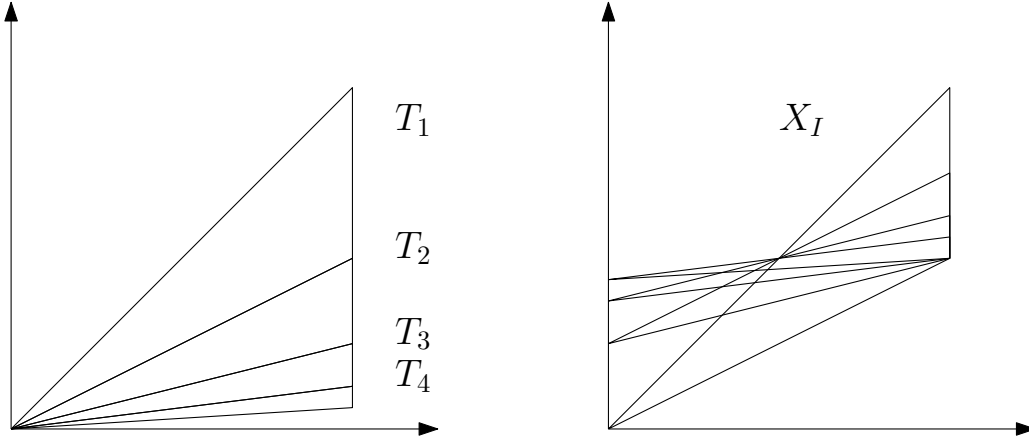


Figure 7: If the areas of the triangles  $T_k$  decrease too fast then we cannot stack them up efficiently.

satisfies the following inequality

$$|X_I| \geq |T_{i_0}| \geq \frac{1}{2} \left| \bigcup_{i \in I} T_i \right|$$

where  $i_0 := \min I$ . We hence cannot, in this case, hope to stack up the triangles  $T_k$  into a set  $X$  that has a small area compared to the sum of the areas of the  $T_k$ , see figure 7 for an illustration of this phenomenon.

We consider a sequence  $\mathbf{t} = \{t_k\}_{k \geq 1}$  which is decreasing to 0 and such that  $t_1 = 1$ . We define the following quantity

$$\tau_{\mathbf{t}} := \sup_{k \geq 0, l \leq k} \left( \frac{t_{k+l+l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+l+l} - t_{k+l}} \right).$$

For any  $k \geq 1$  consider the triangle  $T_k$  whose vertices are the points  $O, A_k = (1, t_k)$  and  $A_{k+1} = (1, t_{k+1})$ . The following proposition is a generalisation of the previous one.

**Proposition 2.** [see [11]] Suppose that  $\tau_{\mathbf{t}} < \infty$ . In this case for any positive ratio  $\alpha$  close to 1 and any integer  $n \geq 1$ , there exists an integer  $N \gg 1$  and  $2^n$  vectors

$$\vec{s}_k := (0, s_k)$$

satisfying the following properties

- the set  $X$  defined as

$$X = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + T_k)$$

satisfies

$$|X| \leq (\alpha^{2^n} + \tau_{\mathbf{t}}(1 - \alpha)) \times |T_{N+1} \sqcup \dots \sqcup T_{N+2^n}|;$$

- for any  $k \neq l$  the triangles  $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$  and  $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$  are disjoint.

We say that the set  $X$  is a generalized Perron tree of scale  $(\alpha, n)$  and we denote it by  $X_{\alpha, n, \mathbf{t}}$ .

The fact that the triangles  $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$  and  $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$  are disjoint is not proved in [11]; this can be proved easily by geometry inspection.

## 4 Proof of Theorem 2

For any  $k \geq 1$  consider the rectangle  $R_k = [0, 1] \times [0, x_k + t_k]$ ; we have

$$T_k(\mathbf{x}) \subset R_k,$$

$|T_k(\mathbf{x})| = \frac{1}{2}x_k$  and  $|R_k| = x_k + t_k$ . Since we have supposed that  $t_k \leq C \times x_k$  for all  $k \geq 1$ , we have

$$|R_k| \leq (1 + C)x_k = 2(1 + C)|T_k(\mathbf{x})|.$$

Those estimations allow us to use the strong maximal operator  $M_s$  to dominate pointwise  $M_B$ . Fix any  $f \in L^1_{loc}(\mathbb{R}^2)$  and any  $T \in B^*$ . Using the fact that  $B^*$  is generated by  $B$ , there exists a rectangle  $R \in \mathcal{I}$  such that

$$T \subset R$$

and

$$|R| \leq 2(1 + C)|T|.$$

Hence we have

$$\frac{1}{|T|} \int_T |f| \leq \frac{2(1 + C)}{|R|} \int_R |f|$$

and this shows that for any  $x \in \mathbb{R}^2$

$$M_B f(x) \leq 2(1 + C)M_s f(x).$$

The conclusion comes from the fact that the strong maximal operator  $M_s$  is a good operator.

## 5 Proof of Theorem 3

Recall that we suppose there is a constant  $\mu_0 > 0$  such that for any  $k \geq 1$

$$x_k < \mu_0 \times |t_k - t_{k+1}|.$$

To begin with, we are going to construct a Perron tree  $X_{\alpha, n, t}$  with the triangles

$$\{T_k\}_{k \geq 1}.$$

Then we will exploit this Perron tree  $X_{\alpha, n, t}$  and the sequence of triangles

$$B = \{T_k(\mathbf{x})\}_{k \geq 1}$$

to show that  $M_B$  is a bad operator. Precisely we prove the following claim.

**Claim 1.** *For any  $\alpha$  close to 1 and any  $n \in \mathbb{N}$ , the Perron tree  $X := X_{\alpha, n, t}$  satisfies the following inequality*

$$|X| \leq \epsilon \times |\{M_B \mathbb{1}_X > \eta(\mu_0)\}|$$

where  $\epsilon = \alpha^{2n} + \tau_t(1 - \alpha)$  and  $B = \{T_k(\mathbf{x})\}_{k \geq 1}$ .

*Proof.* Fix  $\alpha$  close to 1 and  $n \in \mathbb{N}$  and consider a Perron tree of scale  $(\alpha, n)$

$$X := X_{\alpha, n, t} = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + T_k)$$

where  $N$  is given by proposition 2. Fix any  $k \in \{N + 1, \dots, N + 2^n\}$  and consider the couple of triangles

$$(\vec{s}_k + T_k, \vec{s}_k + T_k(\mathbf{x}))$$

or more simply the couple  $(T_k, T_k(\mathbf{x}))$  which is the same up to a translation. We can apply lemma 2 to this couple

$$(\vec{s}_k + T_k, \vec{s}_k + T_k(\mathbf{x})) = (T_1, T(x))$$

which yields the following inclusion

$$\left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_{\{T_k(\mathbf{x})\}} \mathbb{1}_{\vec{s}_k + T_k} > \eta(\mu_0)\}.$$

Using the fact that  $\{T_k(\mathbf{x})\} \subset B$  we have

$$M_{T_k(\mathbf{x})} \leq M_B$$

and so we can replace  $M_{\{T_k(\mathbf{x})\}}$  by  $M_B$  *i.e*

$$\left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_B \mathbb{1}_{\vec{s}_k + T_k} > \eta(\mu_0)\}.$$

We form now the union on  $k \in \{N+1, \dots, N+2^n\}$  of the precedent inclusion which by definition yields

$$\bigsqcup_{k=1}^{2^n} \left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_B \mathbb{1}_X > \eta(\mu_0)\}.$$

In the latter inclusion, the fact that the union is disjoint comes from proposition 2. Hence this gives in terms of Lebesgue measure

$$\sum_{N+1 \leq k \leq N+2^n} \frac{1}{4}|T_k| \leq |\{M_B \mathbb{1}_X > \eta(\mu_0)\}|.$$

Using the fact that  $X$  is a Perron tree constructed with the triangles  $T_k$  we have

$$|X| \leq (\alpha^{2^n} + \tau_{\mathbf{t}}(1 - \alpha)) \times |T_{N+1} \sqcup \dots \sqcup T_{N+2^n}|.$$

In other words we have

$$|X| \leq 4 \times (\alpha^{2^n} + \tau_{\mathbf{t}}(1 - \alpha)) \times |\{M_B \mathbb{1}_X > \eta(\mu_0)\}|.$$

□

As detailed in section 1, the claim implies that for any  $p > 1$  we have

$$\|M_B\|_p \geq \eta(\mu_0)(4\alpha^{2^n} + 4\tau_{\mathbf{t}}(1 - \alpha))^{-\frac{1}{p}}$$

for any  $\alpha$  close to 1 and any  $n \in \mathbb{N}$ . The fact that constant  $\eta(\mu_0)$  is independant of the scale  $(\alpha, n)$  concludes : we have  $\|M_B\|_p = \infty$  for any  $p > 1$  *i.e.*  $M_B$  is a bad operator.

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