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Non Directional Geometric Maximal Operators and the All- p /No- p Conjecture

Anthony Gauvan

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Abstract

The All- p /No- p conjecture states that given a collection of rectangles in the plane which is invariant by dilation and translation, either the associated maximal operator is bounded in L^p for any $1 < p \leq \infty$ or it is unbounded on L^p for any $1 \leq p < \infty$. Focusing on the case of triangles, we exhibit *non-directional* geometric maximal operators which satisfy the All- p /No- p conjecture. To the best of our knowledge, those are among the first examples of this kind. To do so, we develop techniques relying on geometric estimates and generalized Perron trees as constructed in [11] by K. Hare and J.-O. Rönning.

We work in the euclidean plane \mathbb{R}^2 ; if U is a measurable subset of \mathbb{R}^2 we denote by $|U|$ its two dimensional Lebesgue measure.

1 Introduction

Denote by \mathcal{R} the collection containing all rectangles of \mathbb{R}^2 ; for $R \in \mathcal{R}$ we define its *orientation* as the angle $\omega_R \in [0, \pi)$ that its longest side makes with the Ox -axis and its *eccentricity* as the ratio $\kappa_R \in (0, 1]$ of its shortest side by its longest side.

If \mathcal{B} is an arbitrary non empty family contained in \mathcal{R} , we define the associated *derivation basis* \mathcal{B}^* by

$$\mathcal{B}^* = \{ \vec{t} + hR : \vec{t} \in \mathbb{R}^2, h > 0, R \in \mathcal{B} \}.$$

The derivation basis \mathcal{B}^* is the smallest invariant by dilation and translation collection that contains any element of \mathcal{B} . We define the *geometric maximal operator* $M_{\mathcal{B}}$ generated by \mathcal{B} by

$$M_{\mathcal{B}}f(x) := \sup_{x \in R \in \mathcal{B}^*} \frac{1}{|R|} \int_R |f|$$

for any $f \in L^1_{loc}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. Here the supremum is taken on elements R in \mathcal{B}^* that contain the point x . The definitions of \mathcal{B}^* and $M_{\mathcal{B}}$ remain valid when

we consider that \mathcal{B} is an arbitrary family composed of open bounded convex sets. For example in this note, we will work with triangles instead of rectangles.

For $p \in (1, \infty]$ we define the operator norm $\|M_{\mathcal{B}}\|_p$ of $M_{\mathcal{B}}$ by

$$\|M_{\mathcal{B}}\|_p = \sup_{\|f\|_p=1} \|M_{\mathcal{B}}f\|_p.$$

If $\|M_{\mathcal{B}}\|_p < \infty$ we say that $M_{\mathcal{B}}$ is bounded on $L^p(\mathbb{R}^2)$. The L^p -boundedness of a maximal operator $M_{\mathcal{B}}$ is related to the *geometry* that the family \mathcal{B} exhibits. We will say that the operator $M_{\mathcal{B}}$ is a **good operator** when it is bounded on L^p for any $p > 1$. On the other side, we say that the operator $M_{\mathcal{B}}$ is a **bad operator** when it is unbounded on L^p for any $1 < p < \infty$.

Classic examples

We first recall some well known results about classic maximal operators. Suppose that \mathcal{B} is reduced to a single element, say $\mathcal{B} = \{c_0\}$ where $c_0 = [0, 1] \times [0, 1]$. This square yields the so-called *Hardy-Littlewood* maximal operator that we will simply denote $M := M_{\{c_0\}}$. The operator M is not bounded from L^1 to L^1 . However it has weak-type $(1, 1)$ *i.e.* there is a constant $C > 0$ such that

$$|\{Mf > t\}| \leq C \times \frac{\|f\|_1}{t}$$

for any $f \in L^1$ and $t > 0$. We invite the reader to see [9] page 37. Actually, it is not difficult to see that if

$$\kappa(\mathcal{B}) := \inf_{R \in \mathcal{B}} \kappa_R > 0$$

then the operator $M_{\mathcal{B}}$ is pointwise bounded by a multiple of M and so $M_{\mathcal{B}}$ has also weak-type $(1, 1)$. Using Marcinkiewicz interpolation theorem and the fact that we always have $\|M_{\mathcal{B}}f\|_{\infty} \leq \|f\|_{\infty}$, one deduces that $M_{\mathcal{B}}$ is bounded on L^p for any $p > 1$ in this case. More generally, as soon as $M_{\mathcal{B}}$ is bounded on L^{p_0} for some $p_0 > 1$ it is then bounded on L^p for any $p \geq p_0$ using the same method.

Different situations can occur when $\kappa(\mathcal{B}) = 0$; that is to say when \mathcal{B} contains arbitrary thin rectangles. Consider the family

$$\mathcal{I} = \{[a, b] \times [c, d] : a < b, c < d\}$$

i.e. the family \mathcal{I} contains all rectangles whose sides are parallel to the *Ox-Oy* axis. In this case, the operator $M_{\mathcal{I}}$ is commonly called the *strong* maximal operator and we denote it M_s . It is known that the operator M_s does not have weak-type $(1, 1)$. Yet M_s is bounded on L^p for any $p > 1$. We invite the reader to see [6] for a geometric proof of this fact.

Consider now the maximal operator $M_{\mathcal{R}}$ *i.e.* the maximal operator generated by all the rectangles contained in the plane. In this case, because the collection \mathcal{R} is "too big", we have for any $1 < p < \infty$

$$\|M_{\mathcal{R}}\|_p = \infty.$$

We invite the reader to see [9] page 116. To sum up, we have the following facts.

- The Hardy-Littlewood maximal operator M has weak-type $(1, 1)$ and is a good operator.
- More generally, for any $\mathcal{B} \subset \mathcal{R}$, if $\kappa(\mathcal{B}) > 0$ then $M_{\mathcal{B}}$ has weak-type $(1, 1)$ and is a good operator.
- The strong maximal operator M_s is a good operator.
- The maximal operator $M_{\mathcal{R}}$ generated by all rectangles of the plane is a bad operator.

Directional maximal operators

A lot of research has been done in the case where \mathcal{B} is equal to

$$\mathcal{B}^{\Omega} := \{R \in \mathcal{R} : \omega_R \in \Omega\}$$

where Ω is an arbitrary set of directions in $[0, \pi)$. In other words, \mathcal{B}^{Ω} is the set of all rectangles whose orientation belongs to Ω . We say that \mathcal{B}^{Ω} is a *directional basis* and to alleviate the notation we denote

$$M_{\mathcal{B}^{\Omega}} := M^{\Omega}.$$

In the literature, the operator M^{Ω} is said to be a *directional maximal operator*. The study of those operators goes back at least to Cordoba and Fefferman's article [7] *On differentiation of integrals* in which they use geometric techniques to show that if

$$\Omega = \left\{ \frac{\pi}{2^k} \right\}_{k \geq 1}$$

then M^{Ω} has weak-type $(2, 2)$. A year later, using Fourier analysis techniques, Nagel, Stein and Wainger proved in [12] *Differentiation in lacunary directions* that M^{Ω} is actually bounded on L^p for any $p > 1$.

In [1] *Strong type inequalities and an almost-orthogonality principle for families of maximal operators along directions in \mathbb{R}^2* , Alfonseca has proved that if the set of direction Ω is a *lacunary set of finite order* then the operator M^{Ω} is bounded on L^p for any $p > 1$.

In [4] *Keakeya sets and directional maximal operators in the plane*, M. Bate-man completely characterized the L^p -boundedness of directional operators. Precisely he proved the following theorem.

Theorem 1. *Fix an arbitrary set of directions $\Omega \subset [0, \pi)$. The directional maximal operator M^{Ω} is either bounded on L^p for any $1 < p < \infty$ or either unbounded on L^p for any $1 < p < \infty$. In other words, an operator M^{Ω} is either good or bad.*

Hence we know that a set of directions Ω yields an directional operator M^{Ω} that is either good or bad. Merging the vocabulary, we will say that a set of directions Ω is a *good set of directions* when M^{Ω} is good and that it is a *bad*

set of directions when M^Ω is bad. The notion of good/bad is well understood for a set of directions Ω and the associated directional operator M^Ω . To say it bluntly, Ω is a good set of directions if and only if it can be included in a finite union of lacunary sets of finite order. If this is not possible, Ω is a bad set of directions. Rather than giving the definition of lacunarity of finite order (we refer to [4] for this) we provide some examples. In the following cases the set of directions Ω is good *i.e.* M^Ω is bounded on L^p for any $1 < p < \infty$:

- when Ω is finite,
- when $\Omega = \left\{ \frac{\pi}{2^k} \right\}_{k \geq 1}$,
- more generally when Ω is included in a lacunary sequence.

On the other hand, the set of directions Ω is bad *i.e.* M^Ω is unbounded on L^p for any $1 < p < \infty$ when :

- Ω contains an open interval,
- Ω is the ternary Cantor set,
- more surprisingly, $\Omega = \left\{ \frac{\pi}{k^s} \right\}_{k \geq 1}$ for a positive fixed $s > 0$.

Questions

A natural question arises from all this : can one find a family \mathcal{B} in \mathcal{R} such that the operator $M_{\mathcal{B}}$ is unbounded on $L^{1+\epsilon}$ for some small $\epsilon > 0$ and bounded on L^{1+A} for some large $A > 1$? We believe that such a family does not exist and we thank P. Hagelstein and A. Stokolos for communicating us the following conjecture.

Conjecture 1. (*All-p/No-p*) Any maximal operator $M_{\mathcal{B}}$ generated by a family \mathcal{B} included in \mathcal{R} is either bounded on L^p for all finite $p > 1$ or either unbounded on L^p for all finite $p > 1$.

In the general setting, Hagelstein and Stokolos proved in [10] *Tauberian conditions for geometric maximal operators* that if the operator $M_{\mathcal{B}}$ satisfies a *tauberian inequality* then it is bounded on L^p for large $p < \infty$. Precisely, they show that if there exist $0 < \eta_0 < 1$ and $C_0 > 0$ such that

$$|\{M_{\mathcal{B}}\mathbb{1}_E > \eta_0\}| \leq C_0 \times |E|$$

for any bounded measurable set E , then $M_{\mathcal{B}}$ is at least bounded on L^{p_0} for a large p_0 . The problem here is that p_0 depends *a priori* on the basis \mathcal{B} .

To tackle the conjecture, we formulate two questions.

Question 1 : rarefaction of a directional basis

Fix an arbitrary family \mathcal{B} contained in \mathcal{R} satisfying the following conditions :

- the directions contained in \mathcal{B} *i.e.* the set $\Omega(\mathcal{B}) := \{\omega_R : R \in \mathcal{B}\}$ is a bad set of directions ;
- for any $\omega \in \Omega(\mathcal{B})$ we *only* have access to arbitrary thin rectangles in \mathcal{B} in the direction ω *i.e.* for any $\omega \in \Omega(\mathcal{B})$ we have

$$\kappa(\mathcal{B}, \omega) := \inf_{R \in \mathcal{B}, \omega_R = \omega} = 0.$$

Loosely speaking, we assume that \mathcal{B} is a *rarefied* version of the directional basis \mathcal{B}^Ω . Under those conditions, can one show that the operator $M_{\mathcal{B}}$ is still bad ? In a submitted work, we have proven that such an operator $M_{\mathcal{B}}$ is indeed bad.

Question 2 : fleeing bases

The second question is more complex and it is the one that we partially address in this note. Fix an arbitrary family \mathcal{B} included in \mathcal{R} satisfying the following conditions :

- the directions that are accessible with arbitrary thin rectangles form a good set of directions *i.e.* the set of directions

$$\Omega(\mathcal{B}, 0) := \{\omega \in [0, \pi) : \forall \epsilon > 0, \exists R \in \mathcal{B}, \kappa_R < \epsilon, \omega_R = \omega\}$$

is a good set of directions ;

- yet for any $\epsilon > 0$ the set of directions

$$\Omega(\mathcal{B}, \epsilon) := \{\omega \in [0, \pi) : \exists R \in \mathcal{B}, \kappa_R < \epsilon, \omega_R = \omega\}$$

is always a bad set of directions.

Can one show that in this case the operator $M_{\mathcal{B}}$ is always good ? or bad ? In this note, we consider different families \mathcal{B} satisfying those conditions and show that in those cases the operator $M_{\mathcal{B}}$ is indeed good or bad.

Results

We exhibit different *non-directional* maximal operators verifying the All- p /No- p conjecture. Precisely we denote by

$$\mathbf{t} = \{t_k\}_{k \geq 1} \subset [0, 1]$$

a sequence decreasing to 0 and by

$$\mathbf{x} = \{x_k\}_{k \geq 1} \subset (0, \infty)$$

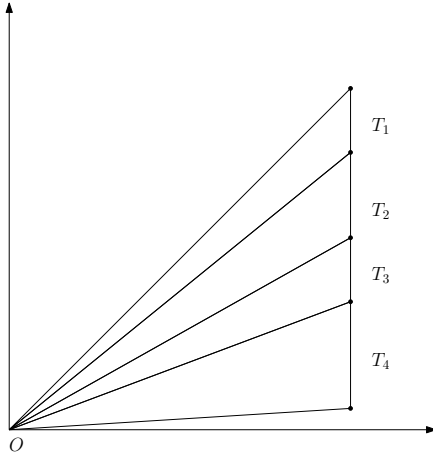


Figure 1: A representation of the triangles T_k .

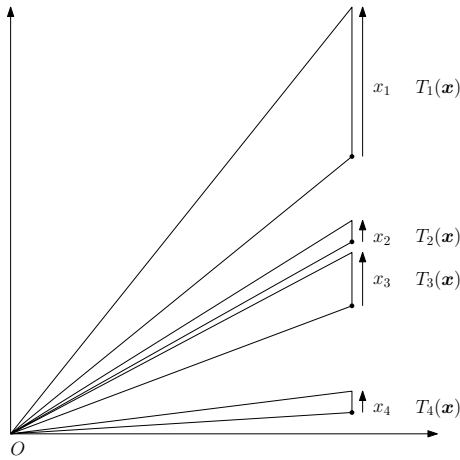


Figure 2: A representation of the triangles $T_k(\mathbf{x})$.

any positive sequence. For $A, B, C \in \mathbb{R}^2$, we denote by ABC the triangle whose vertices are the points A, B and C . If t is a point in \mathbb{R}^2 , we might denote it by \vec{t} to insist on the fact that it used as a translation vector. We define two families of triangles. First, for any $k \geq 1$, we define the triangle T_k as

$$T_k = OA_k A_{k+1}$$

where $O = (0, 0)$, $A_k = (1, t_k)$ and $A_{k+1} = (1, t_{k+1})$. Observe that we have only used the sequence \mathbf{t} in order to define the triangles T_k . Then we define the triangle $T_k(\mathbf{x})$ as

$$T_k(\mathbf{x}) = OA_{k+1}(A_{k+1} + \vec{X}_k)$$

where $\vec{X}_k := (0, x_k)$. We will study the collection \mathcal{B} defined as

$$\mathcal{B} := \{T_k(\mathbf{x})\}_{k \geq 1}.$$

Now that we have defined the triangles T_k and $T_k(\mathbf{x})$ we can interpret the sequence \mathbf{t} as a sequence of tangent of angles that tends to 0 and the sequence \mathbf{x} as an arbitrary sequence of eccentricity. Our first result reads as follow.

Theorem 2. *Suppose there is a constant $C > 0$ such that for any $k \geq 1$*

$$t_k \leq C \times x_k.$$

In this case the operator $M_{\mathcal{B}}$ is a good operator i.e. $M_{\mathcal{B}}$ is bounded on L^p for any $p > 1$

Defining the quantity

$$\tau_{\mathbf{t}} := \sup_{k \geq 0, l \leq k} \left(\frac{t_{k+l+l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+l+l} - t_{k+l}} \right)$$

we will prove the following proposition.

Theorem 3. *Suppose that*

$$\tau_{\mathbf{t}} < \infty$$

and that there is a constant $\mu_0 > 0$ such that for any $k \geq 1$

$$x_k < \mu_0 |t_k - t_{k+1}|.$$

In this case, the maximal operator $M_{\mathcal{B}}$ is a bad operator i.e. $M_{\mathcal{B}}$ is unbounded on L^p for any finite $p \geq 1$.

Structure and strategy

This note is structured as follow.

- In section 2 we give two geometric estimates involving maximal operators.
- In section 3 we recall known facts about the construction of so-called *Perron trees* and *generalized Perron trees*.
- Finally in the last two sections, we prove Theorems 2 and 3.

To prove that an operator $M_{\mathcal{B}}$ is a good operator we are simply going to exhibit another operator M' that is known to be good and we will check that we have

$$M_{\mathcal{B}} \leq C \times M'.$$

This will immediately imply that $M_{\mathcal{B}}$ is good also. Conversely, to prove that an operator $M_{\mathcal{B}}$ is bad, it is enough to find a small fixed value $0 < \eta_0 < 1$ such that for any $\epsilon > 0$ there is a subset X in \mathbb{R}^2 such that

$$|X| \leq \epsilon \times |\{M_{\mathcal{B}} \mathbb{1}_X > \eta_0\}|.$$

Indeed if this holds then for any $p > 1$ we have

$$\int (M_{\mathcal{B}} \mathbb{1}_X)^p \geq \eta_0^p \times |\{M_{\mathcal{B}} \mathbb{1}_X > \eta_0\}| \geq \eta_0^p \times \frac{\|\mathbb{1}_X\|_p^p}{\epsilon}$$

since $|X| = \|\mathbb{1}_X\|_p^p$. Hence we have for any $\epsilon > 0$ the following lower bound on the L^p -norm of $M_{\mathcal{B}}$

$$\|M_{\mathcal{B}}\|_p^p \geq \frac{\eta_0^p}{\epsilon}$$

i.e. for any $p > 1$ we have $\|M_{\mathcal{B}}\|_p^p = \infty$ since $\epsilon > 0$ is arbitrary low in the latter inequality. This exactly means that $M_{\mathcal{B}}$ is bad. We are going to build our set X as a *generalized Perron tree*. Before discussing about those trees, we are going to detail some geometric estimates that will be useful.

Acknowledgments

I would like to thank Laurent Moonens and Emmanuel Russ for their kind advices.

2 Geometric estimates

We fix an arbitrary open triangle $T_1 = ABC$. Consider the triangle T_2 defined as

$$T_2 := \vec{B} + \frac{1}{2}(T_1 - \vec{A}),$$

see figure 3.

Lemma 1 (Geometric estimate I). *The following inclusion holds*

$$T_2 \subset \left\{ M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4} \right\}.$$

In other words, the level set $\{M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4}\}$ contains an affine copy of T_1 .

Proof. Fix $x \in \vec{B} + \frac{1}{2}(T_1 - \vec{A})$. It suffices to observe (see figure 4) that we have

$$x \in \vec{Bx} + T_1$$

and that

$$|T_1 \cap (\vec{Bx} + T_1)| \geq \frac{1}{4} |\vec{Bx} + T_1|.$$

Hence $x \in \{M_{\{T_1\}} \mathbb{1}_{T_1} \geq \frac{1}{4}\}$. □

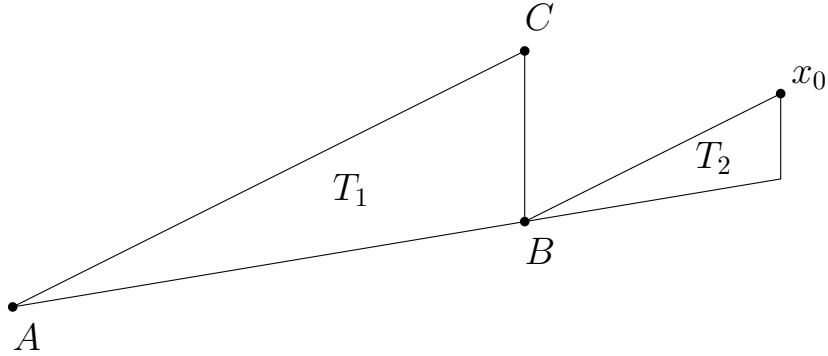


Figure 3: A representation of the triangles T_1 and T_2 .

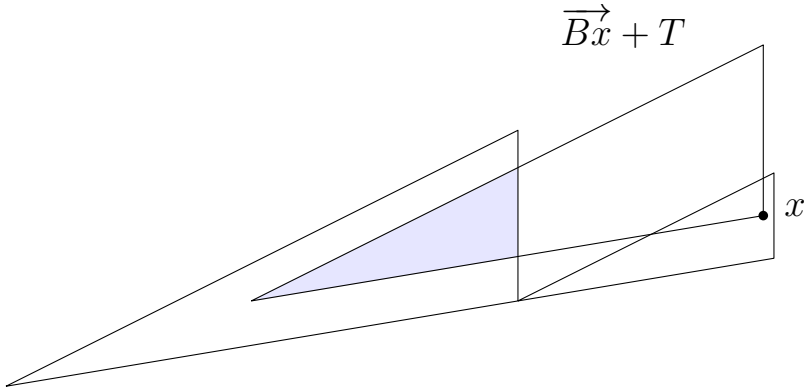


Figure 4: The proof of lemma 1 consists in observing that the blue shaded area represents a quarter of the triangle $\overrightarrow{Bx} + T$.

We now fix an arbitrary positive constant $\mu_0 > 0$ and for all triangles T with vertices (A, B, C) we define the triangle T_{μ_0} as

$$T_{\mu_0} := AB(B + \mu_0 \times \overrightarrow{BC}).$$

For any $0 < x < \mu_0$ we define similarly the triangle

$$T(x) = AB(B + x \times \overrightarrow{BC}).$$

Lemma 2 (Geometric estimate II). *For any couple $(T_1, T(x))$ as defined above, the following inclusion holds*

$$T_2 \subset \{M_{\{T(x)\}} \mathbb{1}_{T_1} \geq \eta(\mu_0)\}$$

where $\eta(\mu_0) = \inf \left\{ \frac{1}{4}, \frac{1}{4\mu_0} \right\}$.

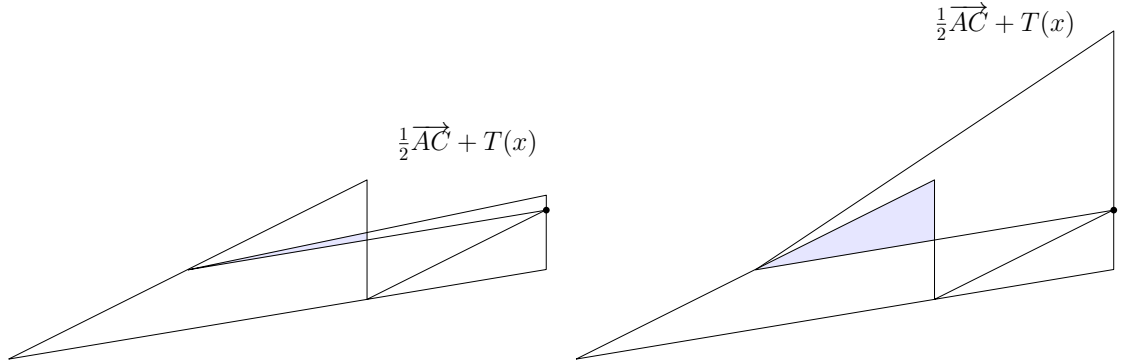


Figure 5: In the proof of lemma 2, we distinguish two cases : either x is very small as on the left side of the figure, either x is close to the parameter μ_0 as on the right of the figure.

Proof. The proof is akin to the proof of lemma 1, see figure 5 for a geometric representation. It is enough to check

$$x_0 \in \{ \mathcal{M}_{T(x)} \mathbb{1}_{T_1} > \eta(\mu_0) \}$$

where $x_0 = B + \frac{1}{2} \overrightarrow{AC}$ because this is the "worst case". To begin with, observe that we have

$$x_0 \in \frac{1}{2} \overrightarrow{AC} + T(x).$$

Now either we have

$$\left| T_1 \cap \left(\frac{1}{2} \overrightarrow{AC} + T(x) \right) \right| \geq \frac{1}{4} |T(x)|$$

and in this case we have $x_0 \in \{ M_{\{T(x)\}} \mathbb{1}_{T_1} > \frac{1}{4} \}$; or we have

$$\left| T_1 \cap \left(\frac{1}{2} \overrightarrow{AC} + T(x) \right) \right| = \frac{1}{4} |T_1|$$

and in this case we have $x_0 \in \{ M_{\{T(x)\}} \mathbb{1}_{T_1} \geq \frac{1}{4\mu_0} \}$ since

$$\frac{|T_1|}{|T(x)|} \geq \frac{|T_1|}{|T_{\mu_0}|} = \frac{1}{\mu_0}.$$

□

3 Generalized Perron trees

We are going to recall how one can construct a *Perron tree* as in [9] and more generally how one can construct a *generalized Perron tree* as in [11] *Applications of generalized Perron trees to maximal functions and density bases*.

Before stating any proposition, we would like to say that loosely speaking the construction of a Perron tree is a procedure that specifies how one can

- (1) cut a triangle T in approximately 2^n thin triangles T_k
- (2) stack up those thin triangles in a new set X that is usually called a Perron tree
- (3) the Perron tree X has different properties with respect to the thin triangles T_k . Among others we have $|X| \ll |T|$.

We formulate this more precisely in the following propositions. Denote by T the triangle whose vertices are the points O , $A = (1, 0)$ and $B = (1, 1)$. For any $n \geq 1$ and $1 \leq k \leq 2^n$ denote by T_k the triangle whose vertices are the points O , $A_k = (1, \frac{k}{2^n})$ and $A_{k-1} = (1, \frac{k-1}{2^n})$.

Proposition 1 (see [9], page 112). *Fix any positive ratio $\frac{1}{2} < \alpha < 1$ and any integer $n \geq 1$. There exist 2^n vectors*

$$\vec{s}_k := (0, s_k)$$

satisfying the following properties

- *the set X defined as*

$$X = \bigcup_{k \leq 2^n} (\vec{s}_k + T_k)$$

satisfies

$$|X| \leq (\alpha^{2^n} + 2(1 - \alpha)) \times |T|;$$

- *for any $k \neq l$ the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$ are disjoint.*

We say that the set X is a Perron tree of scale (α, n) and we denote it by $X_{\alpha, n}$.

Observe that for any $\epsilon > 0$, one can first choose α close to one and then n large enough in order to have

$$|X_{\alpha, n}| \leq \epsilon \times |T|.$$

To obtain such an inequality, it is sufficient to have a condition on the thin triangles T_k that ensures in some sense that they have comparable area.

Indeed, suppose now that we define for any $k \geq 1$ the triangle T_k as the one whose vertices are the points O , $G_k = (1, \frac{1}{2^k})$ and $G_{k+1} = (1, \frac{1}{2^{k+1}})$. In this situation, for any $I \subset \mathbb{N}$ and any sequence of vectors $\{\vec{s}_i\}_{i \in I} \subset \mathbb{R}^2$ the set X_I defined as

$$X_I = \bigcup_{i \in I} (\vec{s}_i + T_i)$$

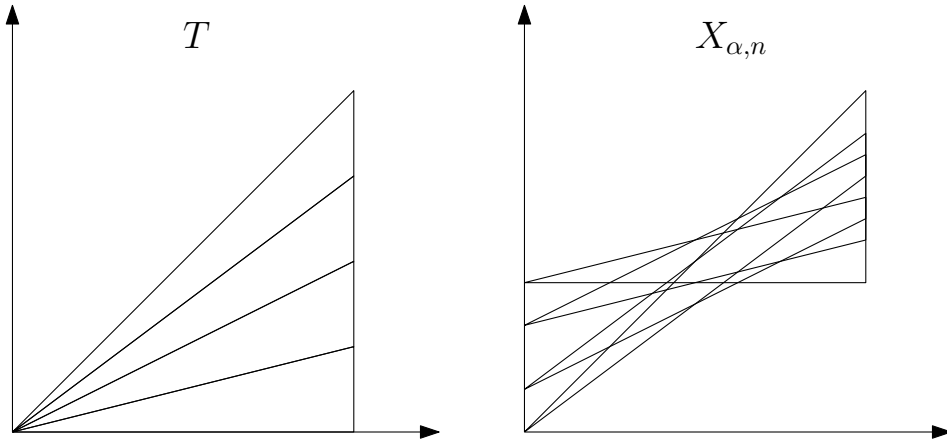


Figure 6: A representation of a Perron tree $X_{\alpha, n}$ for $n = 2$.

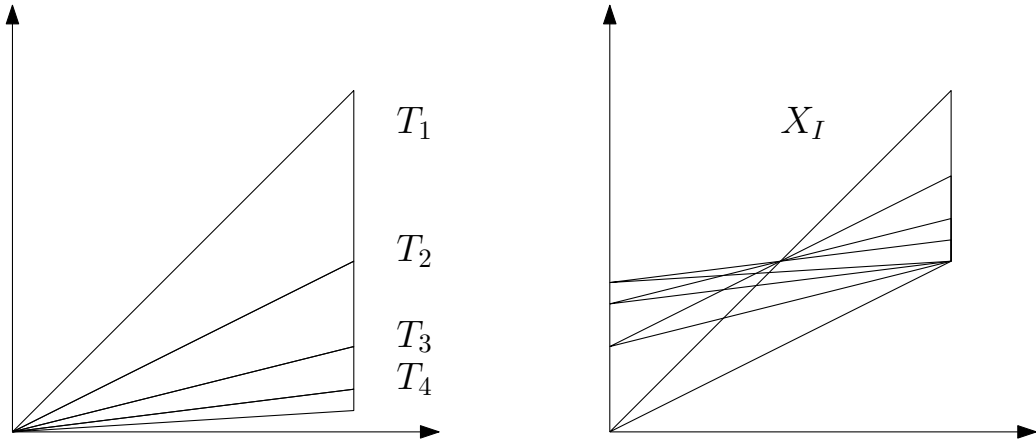


Figure 7: If the areas of the triangles T_k decrease too fast then we cannot stack them up efficiently.

satisfies the following inequality

$$|X_I| \geq |T_{i_0}| \geq \frac{1}{2} \left| \bigcup_{i \in I} T_i \right|$$

where $i_0 := \min I$. We hence cannot, in this case, hope to stack up the triangles T_k into a set X that has a small area compared to the sum of the areas of the T_k , see figure 7 for an illustration of this phenomenon.

We consider a sequence $\mathbf{t} = \{t_k\}_{k \geq 1}$ which is decreasing to 0 and such that

$t_1 = 1$. We define the following quantity

$$\tau_t := \sup_{k \geq 0, l \leq k} \left(\frac{t_{k+l+l} - t_{k+l}}{t_{k+l} - t_k} + \frac{t_{k+l} - t_k}{t_{k+l+l} - t_{k+l}} \right).$$

For any $k \geq 1$ consider the triangle T_k whose vertices are the points O , $A_k = (1, t_k)$ and $A_{k+1} = (1, t_{k+1})$. The following proposition is a generalisation of the previous one.

Proposition 2. *[see [11]] Suppose that $\tau_t < \infty$. In this case for any positive ratio α close to 1 and any integer $n \geq 1$, there exists an integer $N \gg 1$ and 2^n vectors*

$$\vec{s}_k := (0, s_k)$$

satisfying the following properties

- the set X defined as

$$X = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + T_k)$$

satisfies

$$|X| \leq (\alpha^{2^n} + \tau_t(1 - \alpha)) \times |T_{N+1} \sqcup \dots \sqcup T_{N+2^n}|;$$

- for any $k \neq l$ the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$ are disjoint.

We say that the set X is a generalized Perron tree of scale (α, n) and we denote it by $X_{\alpha, n, t}$.

The fact that the triangles $(\vec{A}_k + \vec{s}_k) + \frac{1}{2}T_k$ and $(\vec{A}_l + \vec{s}_l) + \frac{1}{2}T_l$ are disjoint is not proved in [11]; this can be proved easily by geometry inspection.

4 Proof of Theorem 2

For any $k \geq 1$ consider the rectangle $R_k = [0, 1] \times [0, x_k + t_k]$; we have

$$T_k(\mathbf{x}) \subset R_k,$$

$|T_k(\mathbf{x})| = \frac{1}{2}x_k$ and $|R_k| = x_k + t_k$. Since we have supposed that $t_k \leq C \times x_k$ for all $k \geq 1$, we have

$$|R_k| \leq (1 + C)x_k = 2(1 + C)|T_k(\mathbf{x})|.$$

Those estimations allow us to use the strong maximal operator M_s to dominate pointwise $M_{\mathcal{B}}$. Fix any $f \in L^1_{loc}(\mathbb{R}^2)$ and any $T \in \mathcal{B}^*$. Using the fact that \mathcal{B}^* is generated by \mathcal{B} , there exists a rectangle $R \in \mathcal{I}$ such that

$$T \subset R$$

and

$$|R| \leq 2(1+C)|T|.$$

Hence we have

$$\frac{1}{|T|} \int_T |f| \leq \frac{2(1+C)}{|R|} \int_R |f|$$

and this shows that for any $x \in \mathbb{R}^2$

$$M_{\mathcal{B}}f(x) \leq 2(1+C)M_s f(x).$$

The conclusion comes from the fact that the strong maximal operator M_s is a good operator.

5 Proof of Theorem 3

Recall that we suppose there is a constant $\mu_0 > 0$ such that for any $k \geq 1$

$$x_k < \mu_0 \times |t_k - t_{k+1}|.$$

To begin with, we are going to construct a Perron tree $X_{\alpha, n, \mathbf{t}}$ with the triangles

$$\{T_k\}_{k \geq 1}.$$

Then we will exploit this Perron tree $X_{\alpha, n, \mathbf{t}}$ and the sequence of triangles

$$\mathcal{B} = \{T_k(\mathbf{x})\}_{k \geq 1}$$

to show that $M_{\mathcal{B}}$ is a bad operator. Precisely we prove the following claim.

Claim 1. *For any α close to 1 and any $n \in \mathbb{N}$, the Perron tree $X := X_{\alpha, n, \mathbf{t}}$ satisfies the following inequality*

$$|X| \leq \epsilon \times |\{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}|$$

where $\epsilon = \alpha^{2n} + \tau_{\mathbf{t}}(1 - \alpha)$ and $\mathcal{B} = \{T_k(\mathbf{x})\}_{k \geq 1}$.

Proof. Fix α close to 1 and $n \in \mathbb{N}$ and consider a Perron tree of scale (α, n)

$$X := X_{\alpha, n, \mathbf{t}} = \bigcup_{N+1 \leq k \leq N+2^n} (\vec{s}_k + T_k)$$

where N is given by proposition 2. Fix any $k \in \{N+1, \dots, N+2^n\}$ and consider the couple of triangles

$$(\vec{s}_k + T_k, \vec{s}_k + T_k(\mathbf{x}))$$

or more simply the couple $(T_k, T_k(\mathbf{x}))$ which is the same up to a translation. We can apply lemma 2 to this couple

$$(\vec{s}_k + T_k, \vec{s}_k + T_k(\mathbf{x})) = (T_1, T(x))$$

which yields the following inclusion

$$\left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_{\{T_k(\mathbf{x})\}}\mathbb{1}_{\vec{s}_k+T_k} > \eta(\mu_0)\}.$$

Using the fact that $\{T_k(\mathbf{x})\} \subset \mathcal{B}$ we have

$$M_{T_k(\mathbf{x})} \leq M_{\mathcal{B}}$$

and so we can replace $M_{\{T_k(\mathbf{x})\}}$ by $M_{\mathcal{B}}$ *i.e.*

$$\left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_{\mathcal{B}}\mathbb{1}_{\vec{s}_k+T_k} > \eta(\mu_0)\}.$$

We form now the union on $k \in \{N+1, \dots, N+2^n\}$ of the precedent inclusion which by definition yields

$$\bigsqcup_{k=1}^{2^n} \left(\vec{A}_{k+1} + \vec{s}_k\right) + \frac{1}{2}T_k \subset \{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}.$$

In the latter inclusion, the fact that the union is disjoint comes from proposition 2. Hence this gives in terms of Lebesgue measure

$$\sum_{N+1 \leq k \leq N+2^n} \frac{1}{4}|T_k| \leq |\{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}|.$$

Using the fact that X is a Perron tree constructed with the triangles T_k we have

$$|X| \leq (\alpha^{2^n} + \tau_{\mathbf{t}}(1 - \alpha)) \times |T_{N+1} \sqcup \dots \sqcup T_{N+2^n}|.$$

In other words we have

$$|X| \leq 4 \times (\alpha^{2^n} + \tau_{\mathbf{t}}(1 - \alpha)) \times |\{M_{\mathcal{B}}\mathbb{1}_X > \eta(\mu_0)\}|.$$

□

As detailed in section 1, the claim implies that for any $p > 1$ we have

$$\|M_{\mathcal{B}}\|_p \geq \eta(\mu_0)(4\alpha^{2^n} + 4\tau_{\mathbf{t}}(1 - \alpha))^{-\frac{1}{p}}$$

for any α close to 1 and any $n \in \mathbb{N}$. The fact that constant $\eta(\mu_0)$ is independant of the scale (α, n) concludes : we have $\|M_{\mathcal{B}}\|_p = \infty$ for any $p > 1$ *i.e.* $M_{\mathcal{B}}$ is a bad operator.

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