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# THE S-MEASURE FOR ALGEBRAIC INTEGERS HAVING ALL THEIR CONJUGATES IN A SECTOR.

V. FLAMMANG

## Abstract

Let  $\alpha$  be a nonzero algebraic integer of degree  $d$  whose all conjugates  $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_d$  lie in a sector  $|\arg z| \leq \theta$ ,  $0 < \theta \leq 90^\circ$ . We define the *S-measure* of  $\alpha$  by  $S(\alpha) = \sum_{i=1}^d |\alpha_i|$  and the *absolute S-measure* of  $\alpha$  by  $s(\alpha) = S(\alpha)/d$ . We compute the greatest lower bound  $c(\theta)$  of  $s(\alpha)$  for  $\alpha$  belonging to twelve subintervals of  $(0, \theta)$ . Among these subintervals, three are complete. These computations use the principle of explicit auxiliary functions and our recursive algorithm.

## 1 Introduction

Let  $\alpha$  be a nonzero algebraic integer of degree  $d$ . We define the *S-measure* of  $\alpha$  by:

$$S(\alpha) = \sum_{i=1}^d |\alpha_i|.$$

The *absolute S-measure* of  $\alpha$  is the quantity  $s(\alpha) = S(\alpha)/d$ .

If  $P$  is the minimal polynomial of  $\alpha$ , then  $S(P) = S(\alpha)$  and  $s(P) = s(\alpha)$ .

The arithmetic-geometric inequality shows that  $s(\alpha) \geq 1$ . Since  $\alpha$  is a nonzero algebraic integer the quantity  $|\alpha_1 \dots \alpha_d|$  is a positive integer, and therefore  $|\alpha_1 \dots \alpha_d| \geq 1$ . The equality holds if and only if  $\alpha$  is a root of unity. Indeed, if  $s(\alpha) = 1$ , then  $\frac{\sum_{i=1}^d |\alpha_i|}{d} = \prod_{i=1}^d |\alpha_i|^{1/d}$ . Thus, for  $i = 1, \dots, d$ , all the  $|\alpha_i|$  are equal to each other and then equal to 1. The theorem of Kronecker [K] implies that  $\alpha$  is a root of unity.

In this paper, we consider an algebraic integer  $\alpha$  whose all conjugates lie in a sector  $|\arg z| \leq \theta$ ,  $0 < \theta \leq 90^\circ$ .

Note that the case  $\theta$  equals to 0 is well known.  $\alpha$  is then a *totally positive* algebraic integer i.e., all its conjugates are positive real numbers. Thus, the S-measure of  $\alpha$  is then the usual trace of  $\alpha$  and was already studied by a lot of authors. For more details, see for example [S], [ABP], [AP1], [AP2], [LW], [DW], [F1].

Hence, we focus here our attention on  $\theta > 0$ . We follow a work of M. Langevin [La] on the absolute Mahler measure  $\Omega(\alpha) = (\prod_{i=1}^d \max(1, |\alpha_i|))^{1/d}$  of algebraic integers  $\alpha$  having all their conjugates in a sector. He proved that there exists a function  $c(\theta)$  on  $[0, 180^\circ)$ , always  $> 1$ , such that if  $\alpha \neq 0$  is not a root of unity, whose conjugates all lie in  $S_\theta = \{z \mid |\arg z| \leq \theta, 0 \leq \theta \leq 180^\circ\}$ , then  $\Omega(\alpha) \geq c(\theta)$ . But his result was not numerical. G. Rhin and C. Smyth [RS] were the first to succeed in finding the exact value of  $c(\theta)$  for  $\theta$  in nine subintervals of  $[0, 120^\circ]$  and conjectured that  $c(\theta)$  is a “staircase” function of  $\theta$ , which is constant except for finitely many left discontinuities in any closed subinterval of  $[0, 180^\circ)$ . Their method used explicit auxiliary polynomials found heuristically. In 2004, thanks to Wu’s algorithm [Wu], G. Rhin and Q. Wu [RW] gave the exact value of  $c(\theta)$  for four new subintervals of  $[0, 140^\circ]$  and extended four existing subintervals. In 2013, the author and G. Rhin [FR] found for the first time a complete subinterval and a fourteenth subinterval. A complete subinterval is an interval on which the function  $c(\theta)$  is constant, with jump discontinuities at each end. These improvements were due to our recursive algorithm. From there, we studied most of the usual measures for  $\alpha$  having all its conjugates in a sector, in particular the usual trace [F2].

Here, we prove the following result

**Theorem 1.** *There exist a left discontinuous, strictly positive, staircase function  $g$  on  $[0, 90^\circ)$  and a positive, continuous, monotonically decreasing function  $f$  on  $[0, 90^\circ)$  such that:*

*for  $\theta \in [0, 90^\circ)$ , if  $\alpha$  is a nonzero algebraic integer of degree  $\geq 2$  whose conjugates all lie in  $S_\theta = \{z \mid |\arg z| \leq \theta, 0 \leq \theta \leq 90^\circ\}$  then*

$$s(\alpha) \geq \min(f(\theta), g(\theta))$$

*Moreover, the exact value of  $c(\theta) = \inf_{|\arg \alpha| \leq \theta} s(\alpha)$  is known on twelve subintervals of  $[0, 90^\circ)$ .*

These intervals are given in Table 1.

The function  $g(\theta)$  is a decreasing staircase function having left discontinuities. The function  $g(\theta)$  is the smallest value of  $s(\alpha)$  that could be found for  $\alpha$  having all its conjugates in  $|\arg z| \leq \theta$ .

The function  $f(\theta)$  is given by  $f(\theta) = \max_{1 \leq i \leq 12} (f_i(\theta))$ , and the functions  $f_i(\theta)$  are of the type:

$$f_i(\theta) = \min_{z \in S_\theta} \left( |z| - \sum_{1 \leq j \leq J} c_{ij} \log |Q_{ij}(z)| \right)$$

where the polynomials  $Q_{ij}$  with integer coefficients and the positive numbers  $c_{ij}$  are read off from Table 2.

**Table 1** Intervals  $[\theta_i, \theta_i']$  where  $c(\theta)$  is known exactly. Here  $c(\theta) = c(\theta_i) = s(P)$  for  $\theta \in [\theta_i, \theta_i']$ .

$P$  denotes the polynomial whose all conjugates lie in  $S_{\theta_i}$ . The angles are given in degrees.

$i$	$c(\theta)$	$\theta_i$	$\theta_i'$	$P$
1	1.5	0	20.1	$x^2 - 3x + 1$
2	1.428241	21.640384	26.408740	$x^3 - 4x^2 + 5x - 1$
3	1.395643	26.408740	29.86	$x^4 - 5x^3 + 9x^2 - 5x + 1$
4	1.365660	32.569940	33.14	$x^6 - 7x^5 + 21x^4 - 32x^3 + 25x^2 - 8x + 1$
5	1.280776	38.668282	40.65	$x^4 - 4x^3 + 7x^2 - 4x + 1$
6	1.221499	47.941432	50.882970	$x^6 - 5x^5 + 13x^4 - 17x^3 + 13x^2 - 5x + 1$
7	1.214392	50.882970	53.32	$x^6 - 5x^5 + 13x^4 - 18x^3 + 15x^2 - 6x + 1$
8	1.196310	56.004692	57.66	$x^8 - 6x^7 + 20x^6 - 40x^5 + 53x^4 - 45x^3 + 24x^2 - 7x + 1$
9	1.190591	59.066144	63.91	$x^6 - 4x^5 + 10x^4 - 13x^3 + 11x^2 - 4x + 1$
10	1.144122	64.086354	72.86	$x^4 - 2x^3 + 4x^2 - 2x + 1$
11	1.094450	76.795630	80.656154	$x^4 - x^3 + 3x^2 - x + 1$
12	1.073092	80.656154	87.5	$x^3 - x^2 + 2x - 1$

## 2 The explicit auxiliary functions

Let  $\alpha$  be an algebraic integer of degree  $d$  all of which conjugates  $\alpha_1 = \alpha, \dots, \alpha_d$  lie in  $S_\theta = \{z \mid |\arg z| \leq \theta\}$ , with minimal polynomial  $P$ . The auxiliary functions  $f_i$ ,  $1 \leq i \leq 12$ , are of the following type:

$$\forall z \in S_\theta, f(z) = |z| - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|, \quad (2.1)$$

where the coefficients  $c_j$  are positive real numbers and the polynomials  $Q_j$  are nonzero in  $\mathbb{Z}[z]$ , not necessarily irreducible, but not divisible by  $P$ . The main point is to choose the numbers  $c_j$  and the polynomials  $Q_j$  in order to maximize the minimum  $m$  of  $f$  on  $S_\theta$ .

If we have  $\sum_{i=1}^d f(\alpha_i) \geq md$  then

$$S(\alpha) = \sum_{i=1}^d |\alpha_i| \geq md + \sum_{j=1}^J c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i) \right|.$$

Since  $P$  does not divide any  $Q_j$ ,  $\prod_{i=1}^d Q_j(\alpha_i)$  is a nonzero integer because it is the resultant of  $P$  and  $Q_j$ . Therefore, we have

$$s(\alpha) \geq m.$$

The main difficulty in this procedure is to find a good list of polynomials  $Q_j$  which gives a value of  $m$  as large as possible. For this purpose, we link the auxiliary function to the integer transfinite diameter in order to find our polynomials by the recursive algorithm.

### 3 Auxiliary functions and integer transfinite diameter

In this section, we will need the following definition:

Let  $K$  be a compact subset of  $\mathbb{C}$ . We define *the integer transfinite diameter of  $K$*  by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = n}} |P|_{\infty, K}^{\frac{1}{n}},$$

where  $|P|_{\infty, K} = \sup_{z \in K} |P(z)|$ . If  $\varphi$  is a positive function defined on  $K$ , the  $\varphi$ -*integer transfinite diameter of  $K$*  is defined as

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = n}} \sup_{z \in K} \left( |P(z)|^{\frac{1}{n}} \varphi(z) \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A]. It is an important tool in the study of rational approximations of logarithms of rational numbers. Inside the auxiliary function (2.1), we replace the numbers  $c_j$  by rational numbers  $a_j/q$  where  $q$  is a common denominator of the  $c_j$  for  $1 \leq j \leq J$ . Then we can write:

$$\text{for } z \in S_{\theta}, f(z) = |z| - \frac{t}{r} \log |Q(z)| \geq m \quad (3.1)$$

where  $Q = \prod_{j=1}^J Q_j^{a_j} \in \mathbb{Z}[z]$  is of degree  $r = \sum_{i=1}^J a_j \deg Q_j$  and  $t = \sum_{j=1}^J c_j \deg Q_j$ . We want to get a function whose minimum  $m$  in the sector  $S_{\theta}$  is as large as possible. Thus we search for a polynomial  $Q \in \mathbb{Z}[z]$  such that

$$\sup_{z \in S_{\theta}} |Q(z)|^{t/r} (e^{|z|})^{-1} \leq e^{-m}.$$

If we suppose that  $t$  is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z}, \varphi}(S_{\theta}) = \liminf_{\substack{r \geq 1 \\ r \rightarrow +\infty}} \inf_{\substack{P \in \mathbb{Z}[z] \\ \deg(P) = r}} \sup_{z \in S_{\theta}} \left( |P(z)|^{\frac{t}{r}} \varphi(z) \right).$$

where we use the weight  $\varphi(z) = (e^{|z|})^{-1}$ .

## 4 Construction of the auxiliary functions

The polynomials involved in the auxiliary functions are found by our recursive algorithm developed in [F3] from Wu's algorithm [Wu]. It replaces a heuristic search for suitable polynomials by a systematic inductive search. Suppose that we have already found a list  $Q_1, Q_2, \dots, Q_J$  of suitable polynomials. Then we use semi-infinite linear programming (introduced into number theory by C.J. Smyth [S]) to optimize  $f$  for this set of polynomials (i.e., to get the greatest possible  $m$ ). We obtain the real positive numbers  $c_1, c_2, \dots, c_J$  and then  $f$  in the form (3.1) as above. The function  $f$  is invariant under complex conjugation so we can limit ourselves to the sector  $S'_\theta = \{z \in \mathbb{C} \text{ such that } 0 \leq \arg z \leq \theta\}$ . Since the function  $f$  is harmonic in  $S'_\theta$  outside the union of arbitrarily small disks around the roots of the polynomials  $Q_j$ , the minimum is taken on the boundary of  $S'_\theta$ . Thus, it is sufficient to consider the minimum of  $f$  on the half line  $R_\theta = \{z = se^{i\theta}, s \geq 0\}$ .

The auxiliary function on the half line is:

$$f(z) = s - \sum_{1 \leq j \leq J} c_j \log |Q_j(z)|.$$

Thus, by our algorithm, we seek a polynomial  $R(z) = \sum_{l=0}^k a_l z^l \in \mathbb{Z}[z]$ , where  $k$  is varying from 4 to 15 successively, such that

$$\sup_{s \geq 0} |Q(se^{i\theta})R(se^{i\theta})| \frac{t}{r+k} (e^s)^{-1} \leq e^{-m},$$

i.e., such that

$$\sup_{s \geq 0} |Q(se^{i\theta})R(se^{i\theta})| (e^s)^{-\frac{r+k}{t}}$$

is as small as possible.

But, here,  $R(se^{i\theta})$  is not a linear forms in the unknown coefficients  $a_i$  of  $R$ . We replace them by their real parts and their imaginary parts. Then, we get the following real linear forms

$$|Q(s_n e^{i\theta})| \operatorname{Re}(R(s_n e^{i\theta})) (e^{s_n})^{-\frac{r+k}{t}} \quad \text{and} \quad |Q(s_n e^{i\theta})| \operatorname{Im}(R(s_n e^{i\theta})) (e^{s_n})^{-\frac{r+k}{t}}.$$

The  $s_n$  are suitable points in  $(0, \infty)$ , including the points where  $f$  has its least local minima. All these linear forms define a real lattice on  $\mathbb{Z}^{k+1}$ . We use algorithm LLL to obtain a small vector in this lattice.

Then, we get a polynomial  $R$  whose factors  $R_j$  are good candidates to enlarge the set of polynomials  $(Q_1, Q_2, \dots, Q_J)$ . We only keep the polynomials  $R_j$  which have a nonzero coefficient  $c_j$  in the newly optimized auxiliary function  $f$ . After optimization, some previous polynomials  $Q_j$  may have a zero coefficient  $c_j$  and so are removed.

**Table 2: The auxiliary functions  $f_i$ ,  $1 \leq i \leq 12$ .**

$f_1$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.54074137	0	$x$
	1.0000000	0.64261192	0	$x - 1$
	1.5000000	0.13252114	0	$x^2 - 3x + 1$
	1.5303349	0.00694451	19.533923	$x^{11} - 16x^{10} + 112x^9 - 447x^8 + 1115x^7 - 1798x^6 + 1883x^5 - 1264x^4 + 531x^3 - 133x^2 + 18x - 1$
	1.4282414	0.08169266	21.640384	$x^3 - 4x^2 + 5x - 1$
	1.4806329	0.02908250	32.317239	$x^{10} - 14x^9 + 85x^8 - 291x^7 + 613x^6 - 815x^5 + 677x^4 - 339x^3 + 98x^2 - 15x + 1$
$f_2$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.56975402	0	$x$
	1.0000000	0.77184525	0	$x - 1$
	1.4282414	0.07029371	21.640384	$x^3 - 4x^2 + 5x - 1$
	1.4782611	0.00634725	25.535459	$x^5 - 7x^4 + 17x^3 - 18x^2 + 7x - 1$
	1.3956439	0.03715149	26.408740	$x^4 - 5x^3 + 9x^2 - 5x + 1$
	1.4666102	0.00582468	27.821143	$x^6 - 8x^5 + 26x^4 - 41x^3 + 31x^2 - 9x + 1$
	1.5087080	0.00143308	28.849859	$x^8 - 11x^7 + 52x^6 - 134x^5 + 201x^4 - 174x^3 + 82x^2 - 20x + 2$
	1.4320220	0.00119483	32.281080	$x^7 - 9x^6 + 34x^5 - 67x^4 + 72x^3 - 40x^2 + 11x - 1$
	1.4331455	0.01240370	32.422861	$x^7 - 9x^6 + 34x^5 - 67x^4 + 72x^3 - 39x^2 + 10x - 1$
	1.4410315	0.00448025	32.724266	$x^7 - 9x^6 + 34x^5 - 66x^4 + 69x^3 - 37x^2 + 10x - 1$
	1.4312869	0.00316917	37.481532	$x^7 - 9x^6 + 34x^5 - 66x^4 + 68x^3 - 35x^2 + 9x - 1$
$f_3$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.58114845	0	$x$
	1.0000000	0.76150091	0	$x - 1$
	1.3956439	0.03170582	26.408740	$x^4 - 5x^3 + 9x^2 - 5x + 1$
	1.5062565	0.00371221	29.619471	$x^6 - 8x^5 + 27x^4 - 46x^3 + 40x^2 - 15x + 2$
	1.7320508	0.01765706	30.000000	$x^2 - 3x + 3$
	1.4798946	0.00283543	30.333904	$x^{10} - 13x^9 + 76x^8 - 257x^7 + 546x^6 - 743x^5 + 636x^4 - 326x^3 + 96x^2 - 15x + 1$
	1.4856233	0.00365455	31.605909	$x^7 - 9x^6 + 35x^5 - 71x^4 + 77x^3 - 40x^2 + 10x - 1$
	1.5478075	0.01357778	32.185004	$x^3 - 4x^2 + 6x - 2$
	1.4177256	0.00425036	32.536719	$x^{12} - 15x^{11} + 103x^{10} - 422x^9 + 1135x^8 - 2084x^7 + 2641x^6 - 2294x^5 + 1341x^4 - 515x^3 + 124x^2 - 17x + 1$
	1.3656607	0.02118463	32.569940	$x^6 - 7x^5 + 21x^4 - 32x^3 + 25x^2 - 8x + 1$
	1.4725753	0.00603112	32.870830	$x^7 - 9x^6 + 35x^5 - 72x^4 + 81x^3 - 45x^2 + 11x - 1$
	1.4650511	0.00585525	34.381807	$x^{11} - 14x^{10} + 89x^9 - 332x^8 + 794x^7 - 1254x^6 + 1308x^5 - 885x^4 + 382x^3 - 101x^2 + 15x - 1$
	1.3521869	0.00668035	34.985334	$x^5 - 6x^4 + 14x^3 - 15x^2 + 6x - 1$
$f_4$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.56838131	0	$x$
	1.0000000	0.77021312	0	$x - 1$
	1.5478075	0.00171754	32.185004	$x^3 - 4x^2 + 6x - 2$
	1.3656607	0.02265560	32.569940	$x^6 - 7x^5 + 21x^4 - 32x^3 + 25x^2 - 8x + 1$
	1.4910451	0.00213815	34.193790	$x^4 - 5x^3 + 10x^2 - 7x + 2$
	1.3521869	0.01440308	34.985334	$x^5 - 6x^4 + 14x^3 - 15x^2 + 6x - 1$
	1.3973223	0.00151006	35.031799	$x^{11} - 13x^{10} + 78x^9 - 278x^8 + 643x^7 - 996x^6 + 1037x^5 - 715x^4 + 320x^3 - 89x^2 + 14x - 1$
	1.4102499	0.00407700	35.274805	$x^{10} - 12x^9 + 66x^8 - 214x^7 + 446x^6 - 614x^5 + 556x^4 - 322x^3 + 115x^2 - 23x + 2$
	1.4226863	0.00818534	35.357349	$x^{10} - 12x^9 + 66x^8 - 213x^7 + 438x^6 - 585x^5 + 500x^4 - 262x^3 + 82x^2 - 14x + 1$
	1.4206206	0.00058659	36.322451	$x^{15} - 18x^{14} + 153x^{13} - 805x^{12} + 2906x^{11} - 7561x^{10} + 14526x^9 - 20798x^8 + 22166x^7 - 17417x^6 + 9923x^5 - 4004x^4 + 1108x^3 - 199x^2 + 21x - 1$
	1.4665625	0.00084278	37.323334	$x^{13} - 16x^{12} + 120x^{11} - 550x^{10} + 1700x^9 - 3702x^8 + 5778x^7 - 6461x^6 + 5105x^5 - 2780x^4 + 1013x^3 - 236x^2 + 32x - 2$
	1.3808230	0.00322863	37.650150	$x^{10} - 12x^9 + 65x^8 - 205x^7 + 410x^6 - 533x^5 + 447x^4 - 235x^3 + 75x^2 - 13x + 1$
	1.3737014	0.00116992	38.363201	$x^{11} - 13x^{10} + 77x^9 - 270x^8 + 615x^7 - 944x^6 + 984x^5 - 688x^4 + 312x^3 - 88x^2 + 14x - 1$
	1.2807764	0.02023734	38.668282	$x^4 - 4x^3 + 7x^2 - 4x + 1$
	1.3500797	0.00237483	39.090356	$x^9 - 10x^8 + 46x^7 - 123x^6 + 207x^5 - 222x^4 + 148x^3 - 57x^2 + 12x - 1$
$f_5$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.56549176	0	$x$
	1.0000000	0.82799146	0	$x - 1$
	1.2807764	0.01663368	38.668282	$x^4 - 4x^3 + 7x^2 - 4x + 1$
	1.2887120	0.04592145	40.894844	$x^3 - 3x^2 + 4x - 1$
	1.3025959	0.01073176	41.920317	$x^6 - 6x^5 + 17x^4 - 25x^3 + 20x^2 - 7x + 1$
	1.2993422	0.00373700	43.441992	$x^7 - 7x^6 + 23x^5 - 41x^4 + 43x^3 - 25x^2 + 8x - 1$
	1.2882975	0.00079616	44.070606	$x^8 - 8x^7 + 30x^6 - 64x^5 + 83x^4 - 65x^3 + 31x^2 - 8x + 1$
	1.2703626	0.00312681	45.173320	$x^6 - 6x^5 + 16x^4 - 22x^3 + 17x^2 - 6x + 1$
	1.3256092	0.00502304	46.440516	$x^{10} - 10x^9 + 48x^8 - 138x^7 + 259x^6 - 324x^5 + 270x^4 - 145x^3 + 50x^2 - 10x + 1$
	1.2971977	0.00828626	51.015623	$x^7 - 7x^6 + 23x^5 - 41x^4 + 42x^3 - 23x^2 + 7x - 1$
$f_6$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.62533984	0	$x$
	1.0000000	0.13129538	60.000000	$x^2 - x + 1$
	1.4142136	0.01666709	45.000000	$x^2 - 2x + 2$
	1.2214993	0.00012139	47.941432	$x^6 - 5x^5 + 13x^4 - 17x^3 + 13x^2 - 5x + 1$
	1.1513878	0.08048976	49.353681	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	1.2143924	0.03396658	50.882970	$x^6 - 5x^5 + 13x^4 - 18x^3 + 15x^2 - 6x + 1$
	1.2257003	0.01375343	53.350320	$x^8 - 6x^7 + 20x^6 - 38x^5 + 48x^4 - 38x^3 + 20x^2 - 6x + 1$
	1.2237982	0.02183003	54.861223	$x^8 - 6x^7 + 20x^6 - 39x^5 + 50x^4 - 40x^3 + 21x^2 - 6x + 1$

$f_7$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.64156873	0	$x$
	1.4142136	0.02560466	45.000000	$x^2 - 2x + 2$
	1.1513878	0.08849685	49.353681	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	1.2143924	0.02688682	50.882970	$x^6 - 5x^5 + 13x^4 - 18x^3 + 15x^2 - 6x + 1$
	1.2257003	0.01486371	53.350320	$x^8 - 6x^7 + 20x^6 - 38x^5 + 48x^4 - 38x^3 + 20x^2 - 6x + 1$
	1.2237982	0.01372826	54.861223	$x^8 - 6x^7 + 20x^6 - 39x^5 + 50x^4 - 40x^3 + 21x^2 - 6x + 1$
	1.0000000	0.12411259	60.000000	$x^2 - x + 1$
$f_8$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.66258051	0	$x$
	1.1513878	0.08777864	49.353681	$x^4 - 3x^3 + 5x^2 - 3x + 1$
	1.1963101	0.00140181	56.004692	$x^8 - 6x^7 + 20x^6 - 40x^5 + 53x^4 - 45x^3 + 24x^2 - 7x + 1$
	1.1598550	0.01093296	59.015770	$x^3 - 2x^2 + 3x - 1$
	1.1905913	0.02037738	59.066144	$x^6 - 4x^5 + 10x^4 - 13x^3 + 11x^2 - 4x + 1$
	1.0000000	0.19173198	60.000000	$x^2 - x + 1$
$f_9$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.69106953	0	$x$
	1.1905913	0.00047744	59.066144	$x^6 - 4x^5 + 10x^4 - 13x^3 + 11x^2 - 4x + 1$
	1.0000000	0.20669082	60.000000	$x^2 - x + 1$
	1.2329260	0.00070272	63.571914	$x^7 - 4x^6 + 12x^5 - 19x^4 + 23x^3 - 15x^2 + 7x - 1$
	1.1441228	0.06530631	64.086354	$x^4 - 2x^3 + 4x^2 - 2x + 1$
	1.2104994	0.02478898	64.985950	$x^7 - 4x^6 + 12x^5 - 19x^4 + 22x^3 - 14x^2 + 5x - 1$
$f_{10}$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.65189622	0	$x$
	1.0000000	0.20329094	60.000000	$x^2 - x + 1$
	1.1441228	0.05925279	64.086354	$x^4 - 2x^3 + 4x^2 - 2x + 1$
	1.4142136	0.01542631	69.295189	$x^2 - x + 2$
	1.1660867	0.01043488	73.001452	$x^8 - 3x^7 + 10x^6 - 14x^5 + 20x^4 - 14x^3 + 10x^2 - 3x + 1$
	1.1710692	0.02295000	74.639018	$x^8 - 3x^7 + 10x^6 - 15x^5 + 22x^4 - 16x^3 + 11x^2 - 3x + 1$
	1.1435156	0.00095069	77.579788	$x^6 - 2x^5 + 6x^4 - 5x^3 + 6x^2 - 2x + 1$
$f_{11}$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.67127297	0	$x$
	1.0944505	0.029235734	76.795630	$x^4 - x^3 + 3x^2 - x + 1$
	1.1307279	0.01817266	84.841604	$x^6 - x^5 + 5x^4 - 3x^3 + 5x^2 - x + 1$
	1.1476672	0.02486566	86.221414	$x^6 - x^5 + 5x^4 - 3x^3 + 6x^2 - x + 1$
	1.0000000	0.22968256	90.000000	$x^2 + 1$
$f_{12}$	$s(Q_j)$	$c_j$	$\theta(Q_j)$	$Q_j$
	0	0.74200412	0	$x$
	1.0944505	0.08810863	76.795630	$x^4 - x^3 + 3x^2 - x + 1$
	1.0730921	0.002505677	80.656154	$x^3 - x^2 + 2x - 1$
	1.0000000	0.14047981	90.000000	$x^2 + 1$

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