Kakeya-type sets for Geometric Maximal Operators

Anthony Gauvan

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Abstract

Given any family of rectangles G, one defines a natural number $\lambda_{[G]}$ called its *analytic split* and satisfying

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p$$

for all $1 < p < \infty$, where M_G is the Hardy-Littlewood type maximal operator associated to the family G. As an application, we completely characterize the boundeness of planar *rarefied directional maximal operators* on L^p for $1 < p < \infty$. Precisely, if Ω is an arbitrary set of angles in $[0, \frac{\pi}{4})$, we prove that any *rarefied basis* G of the directional basis R^Ω yields an operator M_G that has the same L^p-behavior as the directional maximal operator M^Ω for $1 < p < \infty$.

We work in the euclidean plane $\mathbb{R}^2$; if u is a measurable subset we denote by $|u|$ its Lebesgue measure. If x, y, z are parameters, the notation $C(x, y, z)$ or $\kappa(x, y, z)$ stand for positive constants only depending on the parameters x, y, z. We write $A \lesssim B$ when there is an absolute constant C such that $A \leq C \times B$. In the case where C depends on parameters x, y, z we will write $A \lesssim_{x,y,z} B$. We denote by R the collection containing all rectangles of $\mathbb{R}^2$; for $r \in R$ we define its *orientation* as the angle $\omega_r \in [0, \pi)$ that its longest side makes with the Ox-axis and its *eccentricity* as the ratio $\kappa_r \in (0, 1]$ of its shortest side by its longest side.

1 Introduction

If G is an arbitrary family in R, we define the *derivation basis* G^* as

$$G^* = \left\{ \vec{t} + hr : \vec{t} \in \mathbb{R}^2, h > 0, r \in G \right\}.$$

The derivation basis G^* is the smallest collection invariant by dilation and translation that contains any element of G. We define the *geometric maximal operator M_G generated by G* as

$$M_G f(x) := \sup_{x \in r \in G^*} \frac{1}{|r|} \int_r |f|$$

for any $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ and $x \in \mathbb{R}^2$. The supremum is taken on elements r in G^* that contain the point x. Hence we have by definition

$$M_G = M_{G^*}$$

and so any maximal operators encountered in this note are by definition invariant by translation and dilation. For $p \in (1, \infty]$ we define the operator norm $\|M_G\|_p$ of M_G by

$$\|M_G\|_p = \sup_{\|f\|_p = 1} \|M_G f\|_p.$$

If $\|M_G\|_p < \infty$ we say that M_G is bounded on $L^p(\mathbb{R}^2)$. We will say that the operator M_G is a **good operator** when it is bounded on L^p for any $p > 1$. On the other side, we say that the operator M_G is a **bad operator** when it is unbounded on L^p for any $1 < p < \infty$. The L^p-boundeness of a maximal operator M_G is related to the *geometry* that the family G exhibits ; we try to recall what is known so far.

Directional basis

A lot of research has been done in the case where G is equal to

$$R^\Omega := \{r \in R : \omega_r \in \Omega\}$$

where Ω is an arbitrary set of direction in $[0, \pi)$. In other words, R^Ω contains all rectangles whose longest side make an angle $\omega \in \Omega$ with the Ox-axis. We say that R^Ω is a *directional basis* and to alleviate the notation we write

$$M_{R^\Omega} := M^\Omega.$$

In the literature, the operator M^Ω is said to be a *directional maximal operator*. The study of those operators goes back at least to Cordoba and Fefferman's article [8] *On differentiation of integrals* in which they use geometric techniques to show that if $\Omega = \left\{\frac{\pi}{2^k}\right\}_{k\geq 1}$ then M^Ω has weak-type $(2, 2)$.

In [5] *Kakeya sets and directional maximal operators in the plane*, M. Bateman completely characterized the L^p-boundeness of directional operators. Precisely he proved the following theorem.

Theorem 1. *Fix an arbitrary set of directions $\Omega \subset [0, \pi)$. The directional maximal operator M^Ω is either bounded on L^p for any $1 < p < \infty$ or either unbounded on L^p for any $1 < p < \infty$. In other words, an operator M^Ω is either good or bad.*

To prove this, M. Bateman associates to a set of directions a sub-tree of the dyadic tree and introduces a quantity called *split*. We invite the reader to look at this article and also to look out for [6] where M. Bateman and N. Katz deploy the same techniques but in a context more particular. In this note, we fully exploit the tree vocabulary in order to study geometric maximal operators that are not necessarily directional.

Theorem 1 precisely states that a set of directions Ω yields a directional operator M^Ω that is either good or bad. Merging the vocabulary, we will say that a set of directions Ω is a *good set of directions* when M^Ω is good and that it is a *bad set of directions* when M^Ω is bad. The notion of good/bad is well understood for a set of directions Ω. To say it bluntly, Ω is good if and only if it can be included in a finite union of lacunary sets of finite order. If this is not possible, Ω is a bad set of directions.

Question

A simple question arises : can one find a family G included in R such that the operator M_G is unbounded on $L^{1+\epsilon}$ for some small $\epsilon > 0$ and bounded on L^{1+A} for some large $A > 1$? We believe that such a family does not exist and we thank P. Hagelstein and A. Stokolos for communicating us the following conjecture.

Conjecture. *Any maximal operator M_G generated by a family G included in R is either bounded on L^p for all finite $p > 1$ or either unbounded on L^p for all finite $p > 1$. In other words, given an family G included in R, can one prove that the maximal operator M_G is either good or bad ?*

In this direction, P. Hagelstein and A. Stokolos proved in [10] *Tauberian conditions for geometric maximal operators* that if the operator M_G satisfies a *tauberian inequality* then it is bounded on L^p for large $p < \infty$. Precisely, they show that if there exist $0 < \eta_0 < 1$ and $C_0 > 0$ such that

$$|\{M_G \mathbb{1}_E > \eta_0\}| \leq C_0 \times |E|$$

for any bounded measurable set E, then M_G is at least bounded on L^{p_0} for a large p_0. The problem here is that p_0 depends *a priori* on the basis G.

Another interesting problem can be phrased as follows : given G included in R find the optimal Orlicz space L^Φ on which the operator M_G is bounded. This question won't be adressed here but the interested reader might consult Stokolos' article [13] *On the differentiation of integrals of functions from $L\phi(L)$* or more recently D'Aniello, Moonens and Rosenblatt's article [3] *Differentiating Orlicz spaces with rare bases of rectangles* or D'Aniello and Moonens' article [4] *Differentiating Orlicz spaces with rectangles having fixed shapes in a set of directions.*

Results

To any family G contained in R we associate a quantity $\lambda_{[G]} \in \mathbb{N} \cup \infty$ that we call *analytic split* of G. Loosely speaking, the analytic split $\lambda_{[G]}$ indicates whether or not G contains a lot of rectangles in terms of *orientation* and *eccentricity*. Our main contribution is the following theorem.

Theorem 2. *For any family G and any $1 < p < \infty$ we have*

$$A_p \times \log(\lambda_{[G]}) \leq \|M_G\|_p^p$$

where A_p is a constant only depending on p.

An important feature of this inequality is that we do not make any assumption on the family G. Observe that the analytic split of a family G indicates if the family G is large *i.e.* if M_G is an operator with large L^p-norms. We give an application of theorem 2 to *rarefied directional basis*. Fix an arbitrary set of directions Ω in $[0, \pi)$; we will say that a basis G composed of rectangles is a *rarefied directional basis* of R^Ω if we have

$$\{\omega_r : r \in G\} = \Omega$$

and if moreover for any $\omega \in \Omega$ we have

$$\inf_{r \in G, \omega_r = \omega} \kappa_r = 0.$$

This means that for any $\omega \in \Omega$ and $\epsilon > 0$, the family G contains a rectangle r such that $\omega_r = \omega$ and $\kappa_r < \epsilon$. Loosely speaking the basis G is a *rarefied* version of the directional basis R^Ω while keeping all of its directions available with arbitrary thin rectangles. The following theorem is corollary of theorem 2 ; it completely characterizes the L^p-boundeness of rarefied directional maximal operator.

Theorem 3. *Suppose that Ω is a bad set of directions that is to say suppose that M^Ω is a bad operator. With the above notations, the maximal operator M_G associated to any rarefied basis G of R^Ω is a bad operator i.e. it is unbounded on L^p for any $1 < p < \infty$.*

A priori since we only have $G \subset R^\Omega$ we have $M_G \leq M^\Omega$ and so for any $1 < p < \infty$ we have $\|M_G\|_p \leq \|M^\Omega\|_p$. However Theorem 3 states if we have $\|M^\Omega\|_p = \infty$ then actually $\|M_G\|_p = \infty$.

Acknowledgments

I would like to thank Laurent Moonens and Emmanuel Russ for their kind advices.

Plan

This note is structured as follow.

- In section 2 we introduce the notion of Kakeya-type sets. Loosely speaking, a Kakeya-type set $A \subset \mathbb{R}^2$ for M_G is a geometric set that yields interesting lower bounds on the quantity $\|M_G\|_p^p$ for any $1 < p$. Then we define a collection T composed of parallelograms. The collection T has a natural structure of binary tre and for technical reasons, we prefer to work with family G included in T rather than included in R.
- In section 3 we develop a vocabulary adapted to the dyadic structure of the collection T. For example, we will consider special subsets of T like *path* or (sub-)*tree*. This language is convenient for us and we will use it to define precisely the *analytic split*

$$\lambda_{[G]} \in \mathbb{N} \cup \{\infty\}$$

of an arbitrary family G included in T in section 4. Our main goal is to prove the following estimate for any $1 < p < \infty$

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p.$$

Observe that G is an arbitrary family.

- In section 5 we recall how M. Bateman proves the following estimate for any $1 < p < \infty$

$$\log(\lambda_{[G]}) \lesssim_p \|M_{[G]}\|_p^p.$$

In this case, $[G]$ is not an arbitrary family since it has a tree structure. In section 6, we make a brief pause to comment Bateman's work in [5] ; this will be useful for applications.

- In section 7, we prove different geometric estimates on rectangles which are necessary in order to prove theorem 2. Those estimates are of independent interest.

- Finally in section 8 we prove that for any family G and any $1 < p < \infty$ we have

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p.$$

Namely we prove theorem 2. To conclude, as an application of theorem 2, we characterize the L^p-boundeness of rarefied directional maximal operators. Precisely we prove theorem 3.

2 Kakeya-type sets and parallelograms

In this section we introduce the notion of Kakeya-type sets and the collection

$$T = \{u_n(k) : n \geq 0, 0 \leq k \leq 2^n - 1\}$$

where $u_n(k)$ is the parallelogram whose vertices are the points $(0,0), (0, \frac{1}{2^n}), (1, \frac{k-1}{2^n})$ and $(1, \frac{k}{2^n})$.

Kakeya-type sets

We detail how to obtain a lower bound for $\|M_G\|_p$ using a characteristic function $\mathbb{1}_A$. Specifically, we explain how we can construct a set A with elements of G^* that gives non trivial lower bound.

We say that a maximal operator M_G admits a Kakeya-type set $A \subset \mathbb{R}^2$ of level (η, ϵ) with $\epsilon, \eta > 0$ when we have

$$|A| \leq \epsilon \times |\{M_G \mathbb{1}_A > \eta\}|.$$

In this case, for any $p > 1$ we have

$$\|M_G\|_p \geq \eta \epsilon^{-\frac{1}{p}}.$$

Indeed, we have $\int (M_G \mathbb{1}_A)^p \geq \eta^p \epsilon^{-1} |A|$; since $|A| = \|\mathbb{1}_A\|_p^p$. This leads to the following proposition.

Proposition 1. *Suppose there exists $\eta_0 > 0$ such that for any $\epsilon > 0$ the maximal operator M_G admits a Kakeya-type set A of level (η_0, ϵ). In this case, the maximal operator M_G is unbounded on L^p for any $1 \leq p < \infty$.*

We formally give a method to construct interesting Kakeya-type sets for M_G with elements of G^*. Suppose there is a collection $\{p_i\}_{i \in I} \subset G^*$ such that for each $i \in I$ there is a subset $s_i \subset p_i$ satisfying

$$|s_i| \geq \eta |p_i|$$

and

$$\left| \bigcup_{i \in I} s_i \right| < \epsilon \left| \bigcup_{i \in I} p_i \right|.$$

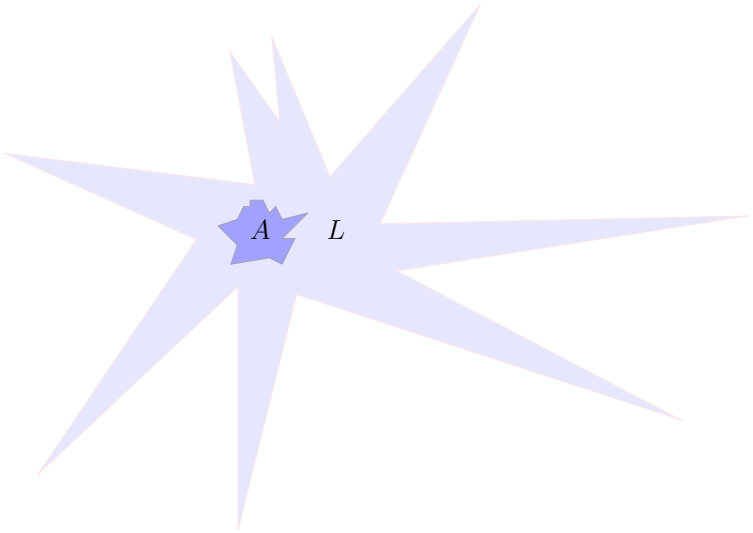

Figure 1: Suppose that for any $\epsilon > 0$ one can find a set A such that the set $L := \{M_G \mathbb{1}_A > \frac{1}{2}\}$ is much bigger than A *i.e.* $|A| \ll |L|$. In this case, M_G is a bad operator.

In this case, the set $A := \bigcup_{i\in I} s_i$ is a Kakeya-type set of level (η, ϵ). Indeed, we have the following inclusion

$$\bigcup_{i\in I} p_i \subset \{M_G \mathbb{1}_A > \eta\}$$

because $p_i \in G^*$ for any $i \in I$ and so

$$|A| \leq \epsilon \left|\{M_G \mathbb{1}_A > \eta\}\right|.$$

Discretization of a basis

Instead of working with rectangles we will consider that our family G is in the collection T composed of *pulled-out parallelograms* which is defined as follow. For $n \geq 0$ and $0 \leq k \leq 2^n - 1$ consider the parallelogram $u_n(k)$ whose vertices are the points $(0,0), (0, \frac{1}{2^n}), (1, \frac{k-1}{2^n})$ and $(1, \frac{k}{2^n})$. We say that $u_n(k)$ is a *pulled-out parallelogram of scale* n and we define the collection T as

$$T = \{u_n(k) : n \geq 0, 0 \leq k \leq 2^n - 1\}.$$

The following proposition precises that we do not lose information if we consider that our family are contained in T and not in R. We won't prove it since this kind of reduction is well known in the literature, see [5] or [1] for example.

Proposition 2. *Fix an arbitrary family G in R. Without loss of generality, we can suppose that we have*

$$\{\omega_r : r \in G\} \subset [0, \frac{\pi}{4}).$$

There exists at least one family G_a contained in T satisfying the following inequality

$$\frac{1}{c_d} \times M_{G_a} \leq M_G \leq c_d \times M_{G_a}$$

*where $c_d = c_2$ is a constant who only depends on the dimension $d = 2$. We will say that the family G_a is an **approximation** of the family G.*

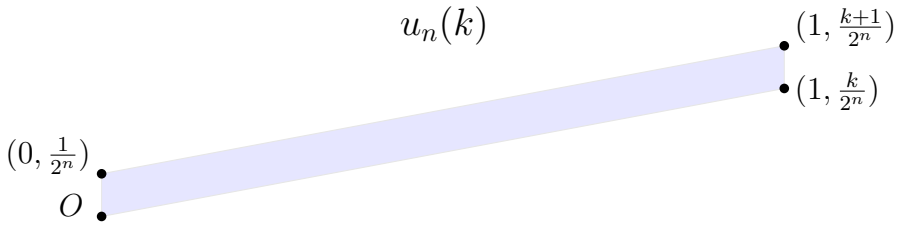

Figure 2: A parallelogram of scale n.

We give an example : consider the family $G = R^{\{0\}}$. In this case, we denote the operator M_G by M_S : in the the literature, M_S is called the *strong* maximal operator. We would like an explicit pointwise approximation of M_S by an operator M_{G_0} where G_0 is a family in T, as announced in proposition 2. Observe that the family G_0 defined as

$$G_0 := \{u_n(0) \in T : n \geq 0\}$$

satisfies proposition 2 in this case. More generally, we can construct an approximation G_a of G as follow. For any $r \in G$ consider the parallelogram $u_r \in T^*$ that is the biggest, in term of measure, parallelogram included in r. We claim that the family G_a^* defined as

$$G_a^* = \{u_r \in T^* : r \in G^*\}$$

is an approximation of G. In other words, this proposition means that we can discretize any basis

$$G \subset R$$

into a basis

$$G_a \subset T$$

at the loss of a dimensional constant. In regard of the L^p-boundeness, M_G or M_{G_a} are equivalent and thus this substitution is irrelevant. Hence, we can work with T instead of R without loss of generality.

3 Structure of T

The collection of parallelograms T has a natural structure of binary tree and we develop a vocabulary adapted to this structure. Our goal is to find a way to detect family G in T such that $\|M_G\|_p^p$ is large.

Tree, root, path and leaves

For any $u \in T$ of scale $n \geq 1$, there exist a unique $u_f \in T$ of scale $n-1$ such that $u \subset u_f$. We say that u_f is the *parent* of u. In the same fashion, observe that there are only two elements $u_h, u_l \in T$ of scale $n+1$ such that $u_h, u_l \subset u$. We say that u_h and u_l are the *children* of u. Observe that $u \in T$ is the child of $v \in T$ if and only if $u \subset v$ and $2|u| = |v|$: we will often use those two conditions. We take advantage of this structure to define particular structured subsets of T : paths, trees and leaves.

We say that a sequence (finite or infinite) $\{u_i\}_{i \in \mathbb{N}} \subset T$ is a *path* if it satisfies $u_{i+1} \subset u_i$ and $2|u_{i+1}| = |u_i|$ for any i *i.e.* if u_i is the parent of u_{i+1} for any i. Different situations can occur. A finite path P has a first element u and a last element v (defined in a obvious fashion) and we will write $P(u, v) := P$. On the other hand, an infinte path P has no endpoint.

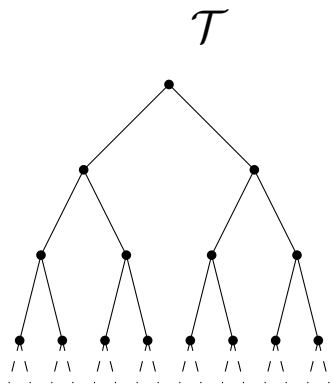

Figure 3: The family T composed of pulled-out parallelograms. We develop a language adapted to its structure of binary tree.

For any family G contained in T, there is a unique parallelogram $r \in T$ such that any $u \in G$ is included in r and $|r|$ is minimal. We say that this element $r_G := r$ is the *root* of G and we define the set $[G]$ as

$$[G] := \{u \in T : \exists v \in G, v \subset u \subset r_G\}.$$

A subset of T of the form $[G]$ is called a *tree* generated by G. We define the set L_G as

$$L_G = \{u \in G : \forall v \in G, v \subset u \Rightarrow v = u\}.$$

An element of L_G is called a *leaf* of G. Observe that we have the following identity

$$[G] = [L_G]$$

and

$$L_G = L_{[G]}.$$

The first identity says that the leaves of a tree $[G]$ can be seen as the minimal set that generates $[G]$. The second identity states that $[G]$ is not bigger than G in the sense that it does not have more leaves. If P is an infinite path, we have by definition $L_P = \emptyset$.

Linear composition

Let G be an arbitrary family in T and let r be the root of $[G]$. We fix an arbitrary element $\tilde{r}$ in T and we consider the family $\tilde{G}$ defined as follow : the family $\tilde{G}$ has the same disposition than G in T but $[\tilde{G}]$ is rooted at $\tilde{r}$. In order to formulate it precisely consider the unique bijective linear map with positive determinant $L : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$L(r) = \tilde{r}$$

and define the family $\tilde{G}$ as

$$\tilde{G} := \{L(u) : u \in G\} \subset T.$$

Now, it is routine to show that we have for any $f \in L^1_{\mathrm{loc}}$

$$M_{\tilde{G}} f = \frac{1}{|\det(L)|} \times M_G(f \circ L)$$

and so we have for any $1 < p < \infty$

$$\|M_{\tilde{G}}\|_p = \|M_G\|_p.$$

Hence, what truly matters when considering a family G contained in T is not its *absolute position* in the tree T but its *structural disposition* in the binary tree.

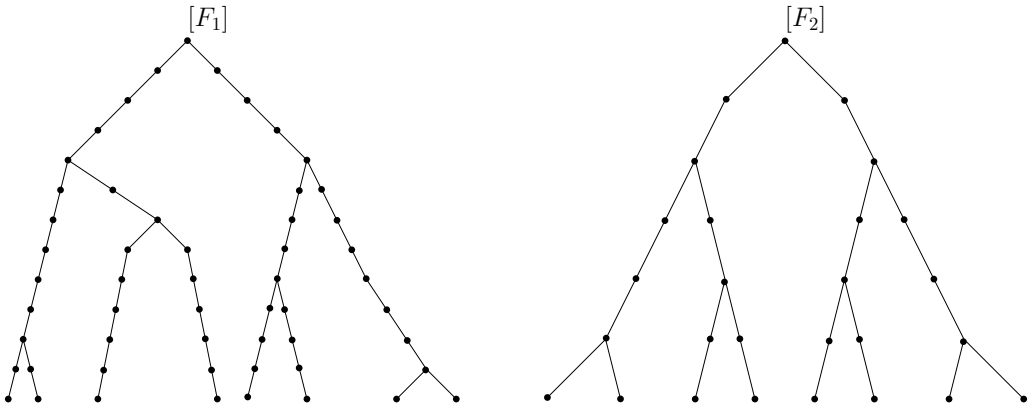

Figure 4: Two fig trees of scale $n = 2$ but with different heights.

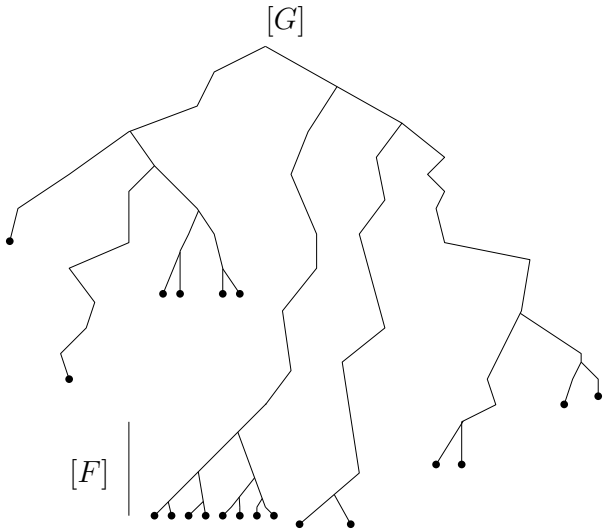

Figure 5: We can compute on this example the analytic split of G ; we have $\lambda_{[G]} = 3$.

4 Analytic split of a family G

We associate to any family G included in T a natural number $\lambda_{[G]} \in \mathbb{N} \cup \{\infty\}$ that we call *analytic split*. At the end of this note, we will prove that we have for any $1 < p < \infty$

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p.$$

Hence the analytic split does indicate if the family G yields a maximal operator M_G with large L^p-norm.

For any tree $[G]$, we define its *boundary* $\partial[G]$ as the set of path in $[G]$ that are maximal for the inclusion *i.e.* $P \in \partial[G]$ if and only if P is a path included in $[G]$ such that if $P' \subset [G]$ is a path that contains P then $P = P'$. Observe that if $[G]$ is finite then there is a natural bijection between $L_{[G]}$ and $\partial[G]$ as any leaf $u \in L_{[G]}$ defines a path $P(r_G, u) \in \partial[G]$ and any path $P \in \partial[G]$ has an endpoint u that belongs to $L_{[G]}$. For any tree $[G]$ and path $P \in \partial[G]$ we define the *splitting number* of P relatively to $[G]$ as

$$s_{P,[G]} := \#\left\{u \in [G] \setminus P : \exists v \in P, u \subset v, 2|u| = |v|\right\}.$$

One can easily compute the splitting number of a path with a representation of the tree. Loosely speaking, while walking down along the path P, the quantity $s_{P,[G]}$ is the number of times that one can leave P and remains in $[G]$. Observe that the splitting number of a path P is defined relatively to a tree $[G]$ *i.e.* we might have $s_{P,[G]} \neq s_{P,[E]}$ for different trees.

We say that a tree $[F]$ is a **fig tree of scale n and height h** when

- $[F]$ is finite and $\#\partial[F] = 2^n$
- for any $P \in \partial[F]$ we have $s_{P,[F]} = n$ and $\#P = h$.

Observe that by construction we always have

$$h \geq n.$$

A basic example of fig tree of scale n is the tree $[B_n]$ defined as $[B_n] = \left\{u \in T : |u| \geq \frac{1}{2^n}\right\}$. In this case, the height of $[B_n]$ is n. However this is the only fig tree satisfying this. One can see any fig tree $[F]$ of scale n as a uniformly stretched version of $[B_n]$.

Finally we say that a tree $[G]$ has an **analytic split**

$$\lambda_{[G]} = n$$

when $[G]$ contains a fig tree $[F]$ of scale n and do not contains any fig tree of scale $n+1$. In the case where $[G]$ contains fig trees of arbitrary high scale, we set

$$\lambda_{[G]} = \infty.$$

More generally for any family G contained in T (*i.e.* when G is not necessarily a tree), we define its analytic split as

$$\lambda_G := \lambda_{[G]}.$$

Observe then that because $[G] = [L_G]$ we have $\lambda_G = \lambda_{L_G}$. In the following section, we explain why the notion of analytic split is relevant.

5 Bateman's construction

We detail why a fig tree $[F]$ of scale n yields a maximal operator $M_{[F]}$ satisfying for any $1 < p < \infty$

$$\log(n) \lesssim_p \|M_{[F]}\|_p^p.$$

Combined with the definition of the analytic split, the following proposition comes easily. Pay attention that in Bateman's work, this estimate is only available for a tree $[G]$.

Proposition 3. *Fix a tree $[G]$ in T. For any $1 < p < \infty$ we have*

$$\|M_{[G]}\|_p^p \geq B_p \times \log(\lambda_{[G]})$$

where $B_p > 0$ is a constant only depending on p.

Proof. For any collection $X, Y \subset T$, if $X \subset Y$ then by definition we have $M_X \leq M_Y$. The conclusion comes from the fact that $[G]$ contains a fig tree $[F]$ of scale $\lambda_{[G]}$ by definition. □

We indicate how one can prove the following theorem. From now on, the constant $C > 1$ is a fixed constant, say $C = 2600$.

Theorem 4 (Bateman's construction)**.** *Suppose that $[F]$ is a fig tree of scale n and height h. In this case the maximal operator $M_{[F]}$ admits a Kakeya-type set, that we denote A_2, of level*

$$\left(\frac{1}{4}, C^2 \log(n)^{-1}\right) \simeq \left(\frac{1}{4}, \log(n)^{-1}\right).$$

Using results of section 2, this implies that for any $1 < p < \infty$ we have

$$\|M_{[F]}\|_p^p \geq B_p \times \log(n)$$

with $B_p = 4^{-p} C^{-2}$.

To prove theorem 4, M. Bateman explicitly constructs a Kakeya-type set of the desired level ; we recall how he achieves this construction. Fix a arbitrary fig tree $[F]$ of scale n and height h rooted at $u_0(0)$; we are looking for a Kakeya-type set A_2 of level

$$\left(\frac{1}{4}, C^2 \log(n)^{-1}\right).$$

We shall construct this Kakeya-type set A_2 as a realisation of a random set that we denote, in the same fashion, $A_2(\omega)$. This is done in three steps.

Step 1 : construction of $A_2(\omega)$

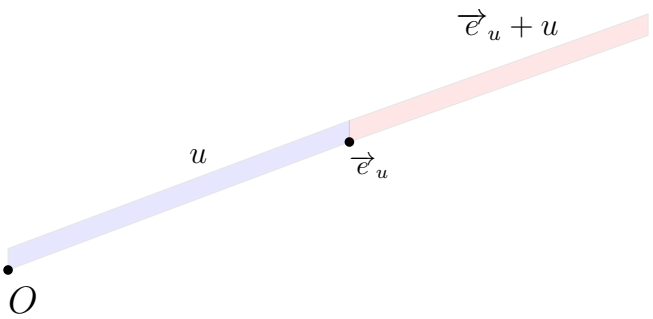

Figure 6: A representation of $d(u)$.

For $u \in T$, we denote by $\vec{e}_u \in \mathbb{R}^2$ its lower right vertice and by $d(u)$ the parallelogram defined as

$$d(u) := u \cup \{\vec{e}_u + u\}.$$

We fix a 2^h mutually independent random variables

$$r_k : (\Omega, \mathcal{F}, \mathbb{P}) \to L_{[F]}$$

who are uniformly distributed in the set $L_{[F]}$ *i.e.* for any $k \leq 2^h$ and any $u \in L_{[F]}$ we have $\mathbb{P}(r_k = u) = \frac{1}{2^n}$. We define then the random set A as

$$A = \bigcup_{k \leq 2^h} \left(\vec{t}_k + d(r_k)\right)$$

where $\vec{t}_k = (0, \frac{k-1}{2^h})$ is a deterministic vector. Define the first and second halves of A as $A_1 := A \cap ([0,1] \times \mathbb{R})$ and $A_2 := A \cap ([1,2] \times \mathbb{R})$. By definition we have

$$A_1 = \bigcup_{k \leq 2^h} (\vec{t}_k + r_k)$$

and

$$A_2 = \bigcup_{k \leq 2^h} (\vec{t}_k + \vec{e}_{r_k} + r_k)\,.$$

Step 2 : Bateman's estimate

We state Bateman's main result in [5]. This estimate states that, with positive probability, $|A_1|$ is much bigger than $|A_2|$.

Theorem 5 (Bateman's estimate)**.** *We have*

$$\mathbb{P}\left(|A_1(\omega)| \geq \frac{\log(n)}{Cn}, |A_2| \leq \frac{C}{n}\right) > 0.$$

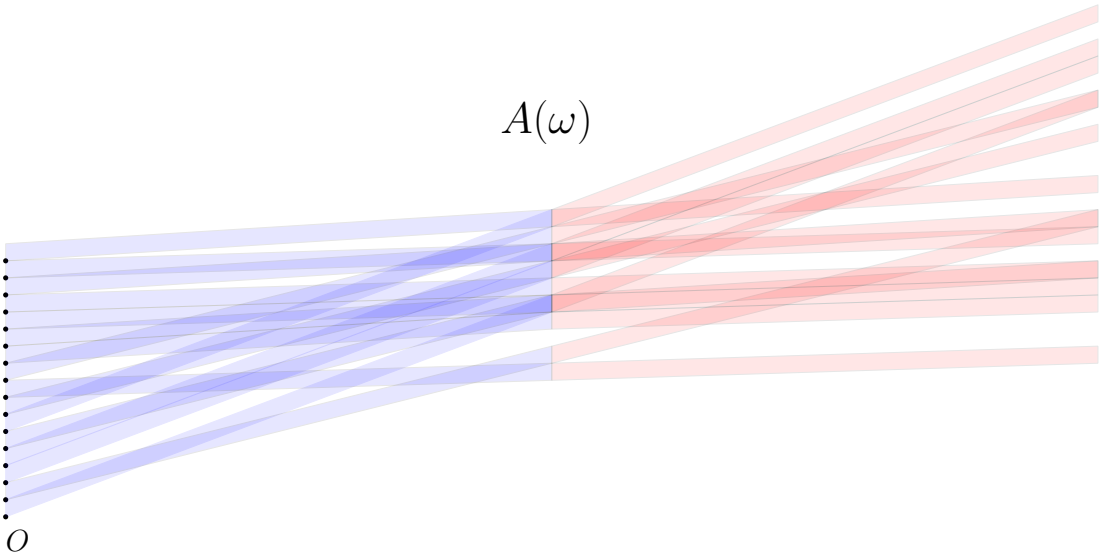

Figure 7: A realisation of $A(\omega)$. Even if this is not flagrant here, the point is that the blue area tends to becomes $\log(n)$ times bigger than the red area.

The proof of this theorem is quite difficult. It involves fine geometric estimates, percolation theory and the use of the so-called notion of *stickiness* of thin tubes of the euclidean plane. We refer to [5] for its proof and for more information but we would suggest to take a look at [6] first. Indeed, in [6], M. Bateman and N. Katz built a scheme of proof that is similar to the one in [5] but in a simpler setting.

Step 3 : the set A_2 is a Kakeya-type set of level $(\frac{1}{4}, C^2 \log(n)^{-1})$

With positive probability the set A_2 is a Kakeya-type set of level

$$(\frac{1}{4}, C^2 \log(n)^{-1})$$

for $M_{[F]}$. Indeed, pick any realisation

$$\omega \in \left\{ |A_1(\omega)| \geq \frac{\log(n)}{Cn}, |A_2| \leq \frac{C}{n} \right\}$$

and we show that

$$A_2 := A_2(\omega)$$

is a Kakeya-type set as announced. To do this, observe that by construction, for any $x \in \vec{t_k}+r_k(\omega) := \vec{t_k} + r_k$, we have

$$\frac{1}{|\vec{t_k} + 2r_k|} \int_{\vec{t_k}+2r_k} \mathbb{1}_{A_2}(y)dy > \frac{|\left\{\vec{t_k} + 2r_k\right\} \cap \left\{\vec{t_k} + \vec{e}_{r_k} + r_k\right\}|}{4|r_k|} = \frac{1}{4}$$

and so

$$A_1 \subset \left\{ M_{[F]}\mathbb{1}_{A_2} > \frac{1}{4} \right\}.$$

Since we also have

$$|A_1| \geq \frac{\log(n)}{C^2}|A_2|$$

this shows that A_2 is a Kakeya-type set of level $(\frac{1}{4}, C^2 \log(n)^{-1})$.

6 Bateman's main result

The work of Bateman in [5] (and also Bateman and Katz in [6]) is of particular importance for us for different reasons.

- The vocabulary and notion that we have developed for T is largely inspired by [5].
- As we have explained, Bateman proved that if $[F]$ is a fig tree of scale n then for any $1 < p < \infty$ we have
$$\log(n) \lesssim_p \|M_{[F]}\|_p^p.$$
- However in [5], the author only considers tree $[T]$ with the following features : the tree $[T]$ is infinite and has no leaf *i.e.* $L_{[T]} = \emptyset$.

In contrast, in this note, we consider arbitrary family G included in T (*i.e.* G is not necessarily a tree or a path or any structured subset) and we will prove that we still have for any $1 < p < \infty$ we have

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p.$$

Before we prove this estimate, we would like to detail the work of Bateman in [5]. Consider a directional basis R^Ω and an approximation of this family G. We aim to know more about the structure of G ; we simply construct an explicit approximation G. For this purpose, we define the tree $[T_\Omega]$ as follow

$$[T_\Omega] := \{u_n(k) \in T : \tan(\Omega) \cap [\frac{k-1}{2^n}, \frac{k}{2^n}) \neq \emptyset\}.$$

Observe that by construction the tree $[T_\Omega]$ is an infinite tree that has no leaf *i.e.*

$$L_{[T_\Omega]} = \emptyset.$$

We claim that we have the following pointwise inequality

$$\frac{1}{c_d} \times M_{[T_\Omega]} \leq M^\Omega \leq c_d \times M_{[T_\Omega]}$$

where $c_d = c_2$ is a constant who only depends on the dimension $d = 2$. In other words, we have the following proposition.

Proposition 4. *For any set of directions $\Omega \subset [0, \frac{\pi}{4})$, the family $[T_\Omega]$ included in T is an approximation of the directional family R^Ω. Moreover, the family $[T_\Omega]$ is an infinite tree with no leaf.*

The notion of split according to Bateman

In [5], M. Bateman introduces a notion of *split* adapted to trees that are infinite and who have no leaf. We are going to recall how he defines this notion. Let $[T]$ an infinite tree with no leaf *i.e.*

$$L_{[T]} = \emptyset.$$

We recall that for a path $P \in \partial[T]$ we have defined the *splitting number* of P relatively to $[T]$ as

$$s_{P,[T]} := \#\left\{u \in [T] \setminus P : \exists v \in P, u \subset v, 2|u| = |v|\right\}.$$

Define then the following quantity

$$S([T]) := \min_{P \in \partial[T]} s_{P,[T]}.$$

As in [5], we can finally define the *split* $s_{[T]}$ of the tree $[T]$ as

$$s_{[T]} := \sup_{[T'] \subset [T]} S([T'])$$

where the supremum is taken on all infinite tree $[T']$ included in $[T]$ that has no leaf. The following proposition states that on infinite tree with no leaf, the notion of *split* and *analytic split* coincide.

Proposition 5. *Let $[T]$ an infinite tree with no leaf. We have*

$$s_{[T]} = \lambda_{[T]}.$$

We leave the proof of this proposition to the reader. In this note, we have defined the analytic split because it is more adapted to our setting since we consider arbitrary family G. Indeed, the analytic split involves the *orientation* and the *eccentricity* that a family G exhibits whereas the split $s_{[T]}$ is more suited for infinite tree with no leaf, but in this case, the *eccentricity* plays no role.

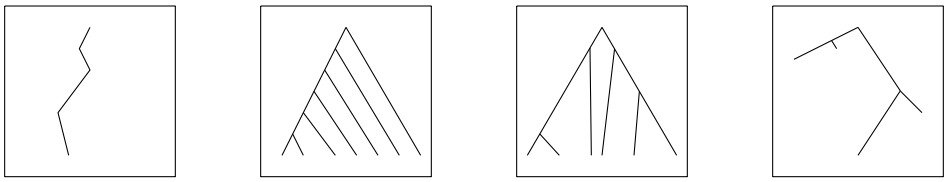

Figure 8: The first three trees are infinite and have no leaf : one can compute their split, it is respectively $s_{[T]} = \lambda_{[T]} = 0, 1, 3$. The last tree is not infinite and do have leaves. However we have $\lambda_{[T]} = 1$.

A theorem of Bateman

In [5] Bateman completely characterizes the boundeness on L^p for any $1 < p < \infty$ of the operator $M_{[T_\Omega]}$ where $\Omega \subset [0, \frac{\pi}{4})$ is an arbitrary set of directions. He proved the following theorem.

Theorem 6. *Suppose that $[T]$ is an infinite tree with no leaf. Then either we have $s_{[T]} = \infty$ and in this case the operator $M_{[T]}$ is bad or we have $s_{[T]} < \infty$ and in this case the operator $M_{[T]}$ is good.*

One can see that what allows Bateman to characterizes the operator $M_{[T]}$ is the analytic split of $[T]$ *i.e.* the structure of $[T]$. We invite the reader to look at [5] for a proof of the theorem. We also suggest to look at [1] and [6] for more details.

7 Geometric estimates

In order to prove theorem 2 *i.e.* to prove that for any $1 < p < \infty$ we have

$$\log(\lambda_{[G]}) \lesssim_p \|M_G\|_p^p$$

where G is an arbitrary family in T, we need different geometric estimates. Indeed, our strategy will consist in exploiting Bateman's construction of the (random) set A_2 but with thinner parallelograms. We begin by proving geometric estimates on $\mathbb{R}$ which will help us to prove geometric estimates on $\mathbb{R}^2$. Finally we prove a geometric estimate on $\mathbb{R}^2$ involving maximal operators that is crucial. Those different estimates are of independant interest.

Geometric estimates on $\mathbb{R}$

If I is a bounded interval on $\mathbb{R}$ and $\tau > 0$ we denote by τI the interval that has the same center as I and τ times its length *i.e.*

$$|\tau I| = \tau\,|I|\,.$$

The following lemma can be found in [2].

Lemma 1 (Austin's covering lemma)**.** *Let $\{I_\alpha\}_{\alpha\in A}$ a finite family of bounded intervals on $\mathbb{R}$. There is a disjoint subfamily*

$$\{I_{\alpha_k}\}_{k\leq N}$$

such that

$$\bigcup_{\alpha\in A} I_\alpha \subset \bigcup_{k\leq N} 3I_{\alpha_k}$$

We apply Austin's covering lemma to prove two geometric estimates on intervals of the real line. The first one concerns union of dilated intervals.

Lemma 2. *Fix $\tau > 0$ and let $\{I_\alpha\}_{\alpha\in A}$ a finite family of bounded intervals on $\mathbb{R}$. We have*

$$B_\tau \times \left|\bigcup_{\alpha\in A} \tau I_\alpha\right| \leq \left|\bigcup_{\alpha\in A} I_\alpha\right| \leq C_\tau \times \left|\bigcup_{\alpha\in A} \tau I_\alpha\right|$$

where $C_\tau = \sup\{\tau, \frac{1}{\tau}\}$ and $B_\tau = \inf\{\tau, \frac{1}{\tau}\}$. In other words we have

$$\left|\bigcup_{\alpha\in A} I_\alpha\right| \simeq_\tau \left|\bigcup_{\alpha\in A} \tau I_\alpha\right|.$$

Proof. Suppose that $\tau > 1$. We just need to prove that

$$B_\tau \times \left|\bigcup_{\alpha\in A} \tau I_\alpha\right| \leq \left|\bigcup_{\alpha\in A} I_\alpha\right|$$

since it is obvious that $I \subset \tau I$ in this case. We apply Austin' covering lemma to $\{I_\alpha\}_{\alpha\in A}$ which gives a disjoint subfamily

$$\{I_{\alpha_k}\}_{k\leq N}$$

such that

$$\bigcup_{\alpha\in A} I_\alpha \subset \bigcup_{k\leq N} 3I_{\alpha_k}.$$

Obviously we have

$$\left|\bigcup_{\alpha\in A} I_\alpha\right| \geq \sum_{k\leq N} |I_{\alpha_k}|$$

and so

$$\left|\bigcup_{\alpha\in A} I_\alpha\right| \geq \sum_{k\leq N} \frac{1}{3\tau}\left|3\tau I_{\alpha_k}\right| \geq \frac{1}{3\tau}\left|\bigcup_{k\leq N} 3\tau I_{\alpha_k}\right|.$$

Hence it suffices to prove that

$$\bigcup_{\alpha\in A} \tau I_\alpha \subset \bigcup_{k\leq N} 3\tau I_{\alpha_k}$$

to conclude. Let $x \in \tau I_\alpha$ we write

$$x = c_\alpha + \tau y$$

where c_α is the center of I_α and such that $c_\alpha + y \in I_\alpha$. There exists a $k \leq N$ such that

$$c_\alpha + y \in 3I_{\alpha_k}$$

and so by definition $x \in 3\tau I_{\alpha_k}$ which concludes. The case $\tau < 1$ is similar. □

Now that we have dealt with union of dilated intervals we consider union of translated intervals.

Lemma 3. *Let $\mu > 0$ be a positive constant. For any finite family of intervals $\{I_\alpha\}_{\alpha\in A}$ on $\mathbb{R}$ and any finite family of scalars $\{t_\alpha\}_{\alpha\in A} \subset \mathbb{R}$ such that, for all $\alpha \in A$*

$$|t_\alpha| < \mu \times |I_\alpha|$$

we have

$$\left|\bigcup_{\alpha\in A} I_\alpha\right| \simeq_\mu \left|\bigcup_{\alpha\in A} (t_\alpha + I_\alpha)\right|.$$

Proof. To begin with, observe that we have for any $\alpha \in A$

$$t_\alpha + I_\alpha \subset (1+\mu)I_\alpha.$$

We apply Austin's covering lemma to the family $\{I_\alpha\}_{\alpha\in A}$ which gives a disjoint subfamily $\{I_{\alpha_k}\}_{k\leq N}$ such that

$$\bigcup_{\alpha\in A} I_\alpha \subset \bigcup_{k\leq N} 3I_{\alpha_k}.$$

In particular we have

$$\left|\bigsqcup_{k\leq N} I_{\alpha_k}\right| \simeq \left|\bigcup_{\alpha\in A} I_\alpha\right|.$$

We consider now the family

$$\{(1+\mu)I_{\alpha_k}\}_{k\leq N}$$

which is *a priori* not disjoint. We apply again Austin's covering lemma which gives a disjoint subfamily that we will denote $\left\{(1+\mu)I_{\alpha_{k_l}}\right\}_{l\leq M}$ who satisfies

$$\bigcup_{k\leq N} (1+\mu)I_{\alpha_k} \subset \bigcup_{l\leq M} 3(1+\mu)I_{\alpha_{k_l}}.$$

In particular we have

$$\left|\bigsqcup_{l\leq M} (1+\mu)I_{\alpha_{k_l}}\right| \simeq \left|\bigcup_{k\leq N} (1+\mu)I_{\alpha_k}\right|.$$

To conclude, it suffices to observe that for any $\alpha \in A$ we have

$$t_\alpha + I_\alpha \subset (1+\mu) I_\alpha$$

because $|t_\alpha| \leq \mu \times |I_\alpha|$. Hence the family

$$\{t_{\alpha_{k_l}} + I_{\alpha_{k_l}}\}_{l \leq M}$$

is disjoint and so finally

$$\left| \bigsqcup_{l \leq M} \left(t_{\alpha_{k_l}} + I_{\alpha_{k_l}} \right) \right| = \sum_{l \leq M} \left| I_{\alpha_{k_l}} \right| \geq \frac{1}{3(1+\mu)} \left| \bigcup_{l \leq M} 3(1+\mu) I_{\alpha_{k_l}} \right| \simeq_\mu \left| \bigcup_{\alpha \in A} I_\alpha \right|$$

where we have used lemma 2 in the last step. □

Geometric estimates on $\mathbb{R}^2$

We denote by $\mathcal{P}$ the set containing all parallelogram $p \subset \mathbb{R}^2$ whose vertices are of the form $(t,a),(t,b),(s,c)$ and (s,d) where $t-s>0$ and $b-a=d-c>0$. We say that $l_p := t-s$ is the *length* of p and that $w_p := b-a$ is the *width* of p. We do not have necessarily $l_p \geq w_p$. For $p \in \mathcal{P}$ and and a positive ratio $0<\tau<1$ we denote by $\mathcal{P}(p,\tau)$ the collection defined as

$$\mathcal{P}(p,\tau) := \{ s \in \mathcal{P} : s \subset p, l_s = l_p, |s| \geq \tau |p| \}.$$

In other words, we have $s \in \mathcal{P}(p,\tau)$ if and only if s is included in p and has the same length as p and its area is greater than $\tau|p|$. We won't use directly the following proposition but its proof is instructive.

Proposition 6 (geometric estimate I)**.** *Fix $\tau > 1$ and any finite family of parallelograms $\{p_i\}_{i \in I} \subset \mathcal{P}$. For each $i \in I$, select an element $s_i \in \mathcal{P}(p_i,\tau)$. The following holds*

$$\left| \bigcup_{i \in I} s_i \right| \geq \frac{\tau}{3} \left| \bigcup_{i \in I} p_i \right|.$$

Proof. We let $U = \bigcup_{i \in I} p_i$ and $V = \bigcup_{i \in I} s_i$. Fix $x \in \mathbb{R}$ and for $i \in I$, denote by p_i^x and s_i^x the segments $p_i \cap \{x \times \mathbb{R}\}$ and $s_i \cap \{x \times \mathbb{R}\}$. Observe that we have by hypothesis

$$|s_i^x| \geq \tau |p_i^x|.$$

By definition, we have the following equality

$$\left| \bigcup_{i \in I} p_i^x \right| = \int \mathbb{1}_U(x,y) dy$$

and as well as

$$\left| \bigcup_{i \in I} s_i^x \right| = \int \mathbb{1}_V(x,y) dy.$$

We apply Austin's covering lemma to the family $\{p_i^x\}_{i \in I}$ which gives a subfamily $J \subset I$ such that the segments $\left\{p_j^x\right\}_{j \in J}$ are disjoint intervals satisfying

$$\bigcup_{i \in I} p_i^x \subset \bigcup_{j \in J} 3 p_j^x.$$

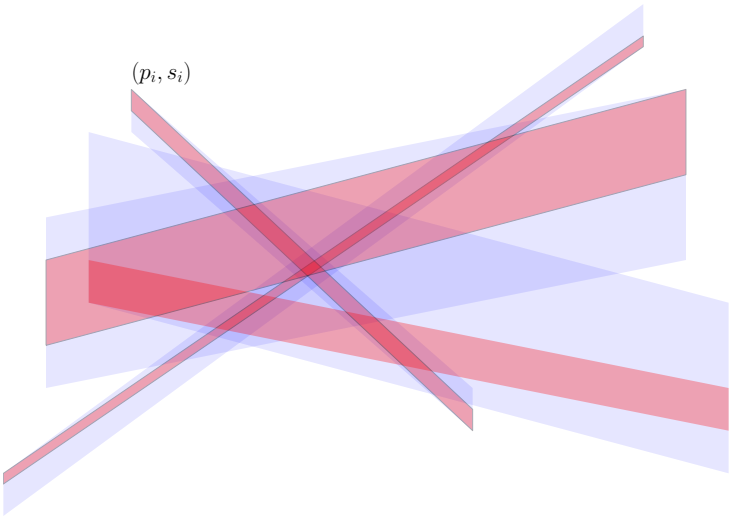

Figure 9: Each red parallelogram s_i satisfy $s_i \subset p_i$ and $|s_i| \geq \frac{1}{4}|p_i$ where p_i is a blue parallelogram. Hence, the red shaded area takes at least $\simeq \frac{1}{4}$ of the blue one.

This yields

$$\left|\bigcup_{i\in I} s_i^x\right| \geq \sum_{j\in J} |s_j^x| \geq \frac{\tau}{3}\left|\bigcup_{i\in I} p_i^x\right|.$$

An integration over $x \in \mathbb{R}$ concludes the proof. □

We aim to give a more general version of proposition 6 using lemma 2 and 3. We can prove the following proposition. For $p \in \mathcal{P}$, we define the parallelogram $h(p) \in \mathcal{P}$ as

$$h(p) := \bigcup_{k=-2}^{2} (0, kw_p) + p.$$

In other words, $h(p)$ is a parallelogram who has same length and orientation than p but is 5 times wider *i.e.* $w_{h(p)} = 5w_p$.

Proposition 7 (geometric estimate II)**.** *Fix $0 < \tau < 1$ and any finite family of parallelograms $\{p_i\}_{i\in I} \subset \mathcal{P}$. For each $i \in I$, select an element $s_i \in \mathcal{P}(h(p_i), \tau)$. The following estimate holds*

$$\left|\bigcup_{i\in I} s_i\right| \geq \frac{\tau}{54}\left|\bigcup_{i\in I} p_i\right|.$$

Proof. As in the proof of lemma 6, denote $U = \bigcup_{i\in I} p_i$ and $V = \bigcup_{i\in I} s_i$. Fix $x \in \mathbb{R}$ and for $i \in I$, denote by p_i^x and s_i^x the segments $p_i \cap \{x \times \mathbb{R}\}$ and $s_i \cap \{x \times \mathbb{R}\}$. For any $i \in I$, observe that there is a scalar t_i satisfying $|t_i| \leq \mu \times |p_i|$ with

$$\mu = 5$$

such that

$$t_i + \tau p_i^x \subset s_i^x.$$

Applying lemma 3, we then have (since $9 \times (1 + \mu) = 54$)

$$\left|\bigcup_{i\in I} s_i^x\right| \geq \left|\bigcup_{i\in I} (t_i + \tau p_i^x)\right| \geq \frac{1}{54}\left|\bigcup_{i\in I} \tau p_i^x\right|.$$

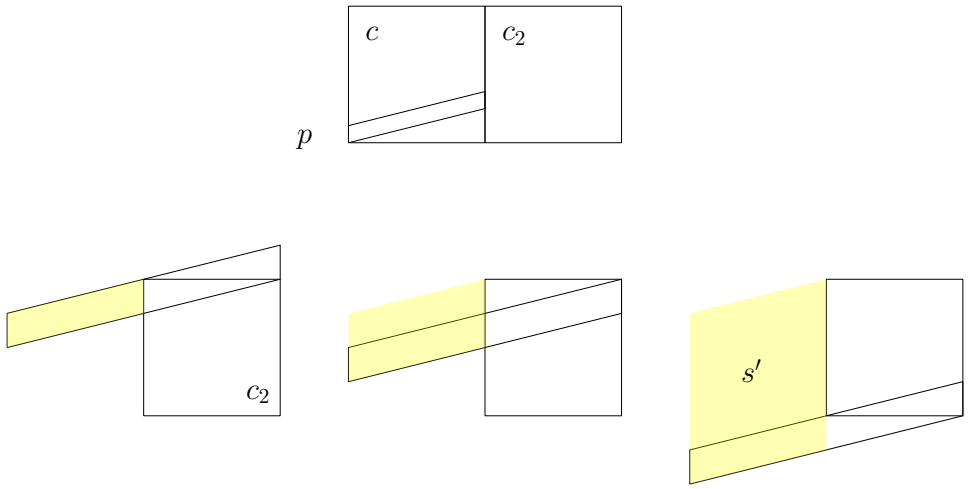

Figure 10: An illustration of the situation in proposition 9. We have $s' \subset \{M_p \mathbb{1}_{c_2} > \frac{1}{4}\}$ and $s' \subset h(c)$. Moreover $|s'|$ is quite large *i.e.* we have $|s'| \simeq |c|$.

We conclude using lemma 2

$$\frac{1}{54}\left|\bigcup_{i\in I} \tau p_i^x\right| \geq \frac{\tau}{54}\left|\bigcup_{i\in I} p_i^x\right|$$

by integrating on x as before. □

Geometric estimate involving a maximal operator

We state a last geometric estimate involving maximal operator that will turn out to be crucial. We begin by a specific case. Consider $c = [0,1]^2$ and $c_2 = \overrightarrow{(1,0)} + c$ and any element $p \in \mathcal{P}$ included in c such that

$$l_p = l_c$$

and

$$|p| \leq \frac{1}{2}|c|.$$

Proposition 8. *There is a parallelogram $s \in \mathcal{P}(h(c), \frac{1}{4})$ depending on p such that the following inclusion holds*

$$s \subset \left\{ M_p \mathbb{1}_{c_2} > \frac{1}{16} \right\}.$$

Proof. Without loss of generality, we can suppose that the lower left corner of p is O. The upper left corner of p is the point $(0, w_p)$ and we denote by $(d, 1)$ and $(d + w_p, 1)$ its lower right and upper right corners. Since $p \subset c$ we have

$$d + w_p \leq 1.$$

The upper right corner of $\frac{1}{2}p$ is the point $(\frac{1}{2}(d + w_p), \frac{1}{2})$ and so for any $0 \leq y \leq 1 - \frac{1}{2}(d + w_p)$ we have

$$(0, y) + \frac{1}{2}p \subset c.$$

This yields our inclusion as follow. Let $\vec{t} \in \mathbb{R}^2$ be a vector such that the center of the parallelogram $\tilde{p} = \vec{t} + 2p$ is the point $(1, 0)$. By construction we directly have

$$|\tilde{p} \cap c_2| \geq \frac{1}{16}$$

but moreover for any $0 \leq y \leq \frac{1}{2}$ we have

$$|\{(0,y)+\tilde{p}\} \cap c_2| \geq \frac{1}{16}$$

since the upper right quarter of $\tilde{p}$ is relatively to c_2 in the same position than p relatively to c. Finally, denoting by p^* the parallelogram $\tilde{p} \cap [0,1] \times \mathbb{R}$, the parallelogram s defined as

$$s := \bigcup_{0 \leq y \leq \frac{1}{2}} ((0,y)+p^*)$$

satisfies the condition claimed. This concludes the proof. □

We state now the previous proposition in its general form. We fix an arbitrary element $c \in \mathcal{P}$ whose lower left vertex is the point O, and we denote by $\vec{e_c} \in \mathbb{R}^2$ its lower right vertex and by $l_c > 0$ its width. We consider the parallelogram $c_2 := \vec{e_c} + c$ and any element $p \in \mathcal{P}$ included in c such that

$$l_p = l_c$$

and

$$|p| \leq \frac{1}{2}|c|.$$

Proposition 9. *There is parallelogram $s \in \mathcal{P}(h(c), \frac{1}{4})$ depending on p such that the following inclusion holds*

$$s \subset \left\{ M_p \mathbb{1}_{c_2} > \frac{1}{4} \right\}.$$

Proof. There is a unique linear function $f : \mathbb{R}^2 \to \mathbb{R}^2$ with positive determinant such that

$$f(c) = [0,1]^2.$$

Using this function and the previous lemma, the conclusion comes. □

8 Proof of theorem 2

We are now ready to prove that for an arbitrary family G contained in T and any $1 < p < \infty$ one has

$$B_p \times \log(\lambda_{[G]}) \leq \|M_G\|_p^p.$$

To do so, we prove that M_G admits a Kakeya-type set of level

$$\left(\frac{1}{16}, 84C^2 \log(n)^{-1}\right) \simeq \left(\frac{1}{2}, \log(n)^{-1}\right).$$

Our strategy is simple : the family G generates a tree $[G]$ and we denote as in section 5

$$A_2 := A_2(\omega)$$

the Kakeya-type set for $M_{[G]}$. We prove that this set A_2 is also a Kakeya-type set of desired level for M_G.

Using Bateman's construction

The family G generates a tree $[G]$; we fix a fig tree $[F] \subset [G]$ of scale $\lambda_{[G]}$ and we denote by $h \in \mathbb{N}$ its height. Consider as before the random set A associated to $[F]$

$$A := \bigcup_{k \leq 2^h} \vec{t}_k + d(r_k).$$

We fix a realisation $\omega \in \Omega$ such that

$$|A_1(\omega)| \geq \frac{\log(n)}{C^2} |A_2(\omega)|.$$

We want to take advantage of $A_2 := A_2(\omega)$ but this time using elements of G and not elements of $[F]$.

Applying proposition 9

For any $u \in L_{[F]}$ denote by $\vec{e}_u$ its lower right vertice and fix g_u an element of G such that

$$g_u \subset u.$$

Once again, we insist on the fact that we have

$$g_u \in G$$

but not necessarily $u \in G$. What is known for sure is that

$$u \in [G].$$

To each couple (u, g_u) we can apply the proposition 9 because $g_u \subset u$ and so

$$|g_u| \leq \frac{1}{2}|u|.$$

This gives a parallelogram $s_u \in \mathcal{P}(h(u), \frac{1}{4})$ such that

$$s_u \subset \left\{ M_{g_u} \mathbb{1}_{\vec{e}_u + u} > \frac{1}{16} \right\}.$$

Because $g_u \in G$ we obviously have

$$M_{\{g_u\}} \leq M_G$$

and so

$$s_u \subset \left\{ M_G \mathbb{1}_{\vec{e}_u + u} > \frac{1}{16} \right\}.$$

Considering the union over $k \leq 2^h$ we have

$$B_1 := \bigcup_{k \leq 2^h} \vec{t}_k + s_{r_k} \subset \left\{ M_G \mathbb{1}_{A_2} > \frac{1}{16} \right\}$$

and so finally

$$|B_1| \leq \left| \left\{ M_G \mathbb{1}_{A_2} > \frac{1}{16} \right\} \right|.$$

Estimating $|B_1|$

It remains to compute $|B_1|$. To do so, observe that we can use proposition 7 with the families $\left\{\vec{t_k} + r_k\right\}_{k\leq 2^h}$ and $\left\{\vec{t_k} + s_{r_k}\right\}_{k\leq 2^h}$. This yields

$$|B_1| \geq \frac{1}{21\times 4}|A_1|$$

and so we finally have

$$|A_2| \leq \frac{84C^2}{\log(n)}\left|\left\{M_G\mathbb{1}_{A_2} > \frac{1}{16}\right\}\right|.$$

In other words, the set A_2 is a Kakeya-type set of level $(\frac{1}{16}, 84C^2\log(n)^-1)$ for the maximal operator M_G. As before, this implies a lower bound on $\|M_G\|_p$ and this concludes the proof.

9 Proof of theorem 3

We finally detail how one can derive theorem 3 from theorem 2. The proof of the claims are routine and we leave them to the reader. We recall that a basis G is called a *rarefaction* of R^Ω when we have $\{\omega_R : R \in G\} = \Omega$ and moreover for any $\omega \in \Omega$

$$\inf_{R\in G, \omega_R=\omega} \kappa_R = 0.$$

This means that for any $\omega \in \Omega$ and $\epsilon > 0$, the family G contains a rectangle R such that

$$\omega_R = \omega$$

and

$$\kappa_R < \epsilon.$$

We fix a rarefied family G of R^Ω and let $G_a \subset T$ be an approximation of the family G. We claim that we can choose G_a such that

$$[G_a] = [T_\Omega]$$

i.e. the tree generated by G_a is the tree $[T_\Omega]$. Indeed, we have $G \subset R^\Omega$ and so we can choose G_a such that we have

$$G_a \subset [T_\Omega].$$

Now the hypothesis on G implies that we have in this case

$$[G_a] = [T_\Omega].$$

We can finally prove theorem 3. We suppose that Ω is a bad set of directions ; in other words we suppose that we have

$$\lambda_{[T_\Omega]} = \infty.$$

For any $1 < p < \infty$, we have the following inequalities

$$\|M_G\|_p^p \simeq \|M_{G_a}\|_p^p \gtrsim_p \log(\lambda_{[G_a]}) = \log(\lambda_{[T_\Omega]}) = \infty$$

since we have $[G_a] = [T_\Omega]$. Hence we have for any $1 < p < \infty$

$$\|M_G\|_p^p = \infty$$

which concludes.

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