

**ON THE INTEGER TRANSFINITE DIAMETER  
OF INTERVALS OF THE FORM  $\left[\frac{r}{s}, u\right]$  OR  $[0, (\sqrt{a} - \sqrt{b})^2]$   
AND OF FAREY INTERVALS**

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**Abstract**

Firstly, we consider intervals of the form  $\left[\frac{r}{s}, u\right]$  where  $r, s$  are positive integers with  $\gcd(r, s)=1$  and  $u$  is a real number, or of the form  $[0, (\sqrt{a} - \sqrt{b})^2]$  where  $a, b$  are positive integers. Thanks to a lemma of Chudnovsky, we give first a lower bound of the integer transfinite diameter of such intervals. Then, using the method of explicit auxiliary functions and our recursive algorithm, we explain how to get an upper bound for this quantity. We finish with some numerical examples. Secondly, we prove inequalities on the integer transfinite diameter of Farey intervals, i.e., intervals of the type  $\left[\frac{a}{q}, \frac{b}{s}\right]$  where  $|as - bq| = 1$ .

## 1 Introduction

Let  $K$  be a compact subset of  $\mathbb{C}$ . The *transfinite diameter* of  $K$  is defined by

$$t(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{C}[X] \\ P \text{ monic} \\ \deg(P) = n}} |P|_{\infty, K}^{\frac{1}{n}},$$

where  $|P|_{\infty, K} = \sup_{z \in K} |P(z)|$  for  $P \in \mathbb{C}[X]$ , while the *integer transfinite diameter* of  $K$  is defined by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \geq 1 \\ n \rightarrow \infty}} \inf_{\substack{P \in \mathbb{Z}[X] \\ \deg(P) = n}} |P|_{\infty, K}^{\frac{1}{n}}$$

If  $I$  is a real interval of length  $|I|$ , we have the well-known properties  $t(I) = \frac{|I|}{4}$  and, if  $|I| \geq 4$ ,  $t_{\mathbb{Z}}(I) = t(I) = \frac{|I|}{4}$  (for more details, see [F1]). Therefore, we restrict our attention to intervals  $I$  with  $|I| < 4$ . In this case, we know that  $t(I) \leq t_{\mathbb{Z}}(I) \leq \sqrt{t(I)} = \frac{1}{2}\sqrt{|I|}$ . The first inequality came from the definition and the second one is due to D. Hilbert [H]. Many authors have been interested in the study of  $t_{\mathbb{Z}}(I)$ . The case  $I = [0, 1]$  was particularly studied because of a theoretical result of Gelfond and Schnirelman [L] showing that, if  $t_{\mathbb{Z}}([0, 1]) = 1/e = 0.3678$  were to hold, then a simple proof of the prime number theorem would follow. However, we know that  $0.4213 \leq t_{\mathbb{Z}}([0, 1]) \leq 0.422685$ . The lower bound is due to I. Pritsker [Pr] while the author proved the upper bound [F2]. The exact value of  $t_{\mathbb{Z}}(I)$  remains unknown for  $I$  with  $|I| < 4$ .

In [F1], we gave lower and upper bounds for several Farey intervals i.e.,  $I = \left[\frac{r}{s}, \frac{p}{q}\right]$  with  $|ps - qr| = 1$ . Using more polynomials than us (polynomials found heuristically), J. Aguirre and J.C. Peral [AP] have improved the upper bound of each our intervals. Thanks to our recursive algorithm described in Section 3, we can improve their bounds slightly, but not by enough to be worth including in this paper. For example, we can obtain  $t_{\mathbb{Z}}\left(\left[0, \frac{1}{64}\right]\right) \leq 0.01520467$ , whereas J. Aguirre and J.C. Peral had  $t_{\mathbb{Z}}\left(\left[0, \frac{1}{64}\right]\right) \leq 0.01520616$ . Thus, we turn our attention to the integer transfinite diameter of intervals of the type  $\left[\frac{r}{s}, u\right]$ , where  $r, s$  are positive integers with

$\gcd(r, s)=1$  and  $u$  is a real number or of the type  $[0, (\sqrt{a} - \sqrt{b})^2]$  where  $a, b$  are positive integers. Using methods developed in [F1] and our recursive algorithm, we obtain lower and upper bounds for them. To illustrate the method, we give some numerical examples.

Then we consider an interval  $I = \left[ \frac{a}{q}, \frac{b}{s} \right]$ , where  $|as - bq| = 1$ , and an interval  $J = \left[ \frac{c}{qs}, \frac{d}{(q+s)^2} \right]$  with  $|c(q+s)^2 - dqs| = 1$ . We prove the following inequalities:

**Theorem 1.**

$$t_{\mathbf{Z}}(J) \leq t_{\mathbf{Z}}(I)^2 \leq \max\left(\frac{q}{s}, \frac{s}{q}\right)t_{\mathbf{Z}}(J).$$

## 2 The lower bounds

In [F1], we proved the following result:

**Theorem 2.**

$$t_{\mathbf{Z}}\left(\left[\frac{r}{s}, \frac{p}{q}\right]\right) \geq \frac{1}{q+s} \prod_{i=0}^{\infty} (1 + \lambda_i)^{-\frac{1}{2^{i+1}}},$$

where

$$\lambda_0 = \frac{qs}{(q+s)^2} \quad \text{and} \quad \lambda_{i+1} = \frac{\lambda_i}{(1 + \lambda_i)^2}.$$

The essential tool of the proof is a lemma of Chudnovsky [C] that is applied to a particular family of polynomials introduced by C. J. Smyth [S1].

Now, consider  $I = \left[ \frac{r}{s}, u \right]$  where  $r, s$  are positive integers with  $\gcd(r, s)=1$  and  $u$  is a real number. Euclid's algorithm allows us to find the largest fraction  $\frac{p}{q}$  such that  $\left[ \frac{r}{s}, \frac{p}{q} \right] \subset I$ , so that  $t_{\mathbf{Z}}\left(\left[\frac{r}{s}, \frac{p}{q}\right]\right) \leq t_{\mathbf{Z}}(I)$ . Then, we apply Theorem 2 to the interval  $\left[ \frac{r}{s}, \frac{p}{q} \right]$  and we deduce a lower bound for  $I$ .

Let  $I$  be the interval  $[0, (\sqrt{a} - \sqrt{b})^2]$  where  $a, b$  are positive integers. We search for the smallest integer  $q > 0$  such that  $\left[0, \frac{1}{q}\right] \subset I$ . Again, we apply Theorem 2 to the interval  $\left[0, \frac{1}{q}\right]$  to obtain a lower bound for  $I$ .

## 3 The upper bounds

### 3.1 The case $I = \left[ \frac{r}{s}, u \right]$

Let  $I$  be an interval of the form  $\left[ \frac{r}{s}, u \right]$ , where  $r, s$  are integers and  $u$  is a real number. Here, we will use the principle of explicit auxiliary functions and our recursive algorithm to get an upper bound for  $I$ .

First, we choose two integers  $p, q$  such that  $x(t) = \frac{qt - p}{r - ts}$  maps  $\left[ \frac{r}{s}, u \right]$  into  $[b, \infty)$  with the conditions  $x(u) = b$  and  $0 < b < 1$ , and  $x\left(\frac{r}{s}\right) \rightarrow \infty$ . We know that our recursive algorithm is very efficient on  $[0, \infty)$ , which is why we impose the latter condition. The first condition implies that  $qr - ps = -1$ .

Now, suppose that we can find a sequence of polynomials  $(Q_j)_{1 \leq j \leq J}$  in  $\mathbb{Z}[X]$ , a sequence of positive real numbers  $(c_j)_{1 \leq j \leq J}$  and a real  $m$  such that

$$(3.1) \quad \forall x \in [b, \infty), f(x) = \log |q + sx| - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \geq m,$$

which can also be written

$$\forall x \in [b, \infty), \prod_{j=1}^J |Q_j(x)|^{b_j} \leq e^{-Nm} |q + sx|^N,$$

where  $(b_j)_{1 \leq j \leq J}$  is a sequence of integers and  $N$  is an integer such that  $\sum_{1 \leq j \leq J} b_j \deg(Q_j) \leq N$ .

We deduce from the previous inequality

$$\forall x \in [b, \infty), |R(x)| \leq e^{-mN} |q + sx|^N,$$

where we put  $R(x) = \prod_{j=1}^J |Q_j(x)|^{b_j}$  and note that  $\deg(R) \leq N$ .

Here we make a change of variable to get back to the interval  $I$ . We define  $y = \frac{p + rx}{q + sx}$ . Since  $x > b$  we have  $y \in I$ . We define  $P(y) = R\left(\frac{qy - p}{r - sy}\right) (r - sy)^N$ . Thus  $\deg(P) = \deg(R) + (N - \deg(R)) = N$  and the previous inequality becomes

$$\forall y \in I, |P(y)| \leq e^{-mN},$$

because  $qr - ps = -1$ .

Hence,  $\max_{y \in I} |P(y)|^{\frac{1}{\deg(P)}} \leq e^{-m}$ .

The polynomial  $P$  is explicit and we obtain  $t_{\mathbb{Z}}(I) \leq e^{-m}$ .

Now, we have to find the polynomials  $Q_j$ , the real numbers  $c_j$  and the minimum  $m$  of the function  $f$  which is called an *auxiliary function*. This notion was introduced into number theory by C. J. Smyth in [S2].

We need the following definition: let  $K$  be a compact subset of  $\mathbb{C}$ , if  $\varphi$  is a positive function defined on  $K$ , the  $\varphi$ -generalized integer transfinite diameter of  $K$  is defined by

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{n \rightarrow \infty} \inf_{\substack{n \geq 1 \\ \deg(P) = n}} \sup_{P \in \mathbb{Z}[X]} \sup_{z \in K} \left( |P(z)|^{\frac{1}{n}} \varphi(z) \right).$$

This version of weighted integer transfinite diameter was introduced by F. Amoroso [A] and is an important tool in the study of rational approximations of logarithms of rational numbers.

In the auxiliary function (3.1), we replace the coefficients  $c_j$  by rational numbers  $a_j/q$ , where  $q$  is a positive integer such that  $q \cdot c_j$  is an integer for all  $1 \leq j \leq J$ . Then we can write:

$$(3.2) \quad \forall x > b, f(x) = \log |q + sx| - \frac{t}{r} \log |Q(x)| \geq m.$$

where  $Q = \prod_{j=1}^J Q_j^{a_j} \in \mathbb{Z}[X]$  is of degree  $r = \sum_{j=1}^J a_j \deg Q_j$  and  $t = \sum_{j=1}^J c_j \deg Q_j$  (this formulation was introduced by J. P. Serre). Thus we seek a polynomial  $Q \in \mathbb{Z}[X]$  such that

$$\sup_{x > b} |Q(x)|^{t/r} |q + sx|^{-1} \leq e^{-m}.$$

If we suppose that  $t$  is fixed, it is equivalent to find an effective upper bound for the weighted integer transfinite diameter over the interval  $[b, \infty)$  with the weight  $\varphi(x) = |q + sx|^{-1}$ :

$$t_{\mathbf{Z},\varphi}([b, \infty)) = \liminf_{\substack{r \geq 1 \\ r \rightarrow \infty}} \inf_{\substack{P \in \mathbf{Z}[X] \\ \deg(P) = r}} \sup_{x > b} \left( |P(x)|^{\frac{t}{r}} \varphi(x) \right).$$

**Remark:** Even though we have replaced the compact set  $K$  by the infinite interval  $[b, \infty)$ , the weight  $\varphi$  ensures that the quantity  $t_{\mathbf{Z},\varphi}([b, \infty))$  is finite.

The main challenge is to find a set of “good” polynomials  $Q_j$ , i.e., which gives the best possible value for  $m$ . Until 2003, the polynomials were found heuristically. This was the case in [F1] when we have bounded the integer transfinite diameter of Farey intervals. In 2003, Q. Wu [Wu] developed an algorithm to systematically search for “good” polynomials. His method was the following. We consider an auxiliary function as defined by (3.1). We fix a set  $E_0$  of control points, uniformly distributed on the real interval  $L = [b, A]$  where  $A$  is “sufficiently large”. Thanks to the LLL algorithm, we find a polynomial  $Q$  small on  $E_0$  with respect to the quadratic norm. We test this polynomial in the auxiliary function and we keep only the factors of  $Q$  which have a nonzero exponent. The convergence of this new function gives local minima that we add to the set of points  $E_0$  to get a new set of control points  $E_1$ . We use again the LLL algorithm with the set  $E_1$  and the process is repeated.

In 2006, we made two improvements to this previous algorithm in the use of the LLL algorithm. The first one is, at each step, to take into account not only the new control point but also the new polynomials of the best auxiliary function. The second one is the introduction of a corrective coefficient  $t$ . The idea is to get good polynomials  $Q_j$  by induction. Thus, we call this algorithm the *recursive algorithm*. The first step consists in the optimization of the auxiliary function  $f_1 = \log(q + sx) - t \log x$ . We have  $t = c_1$  where  $c_1$  is the value that gives the best function  $f_1$ . We suppose that we have some polynomials  $Q_1, Q_2, \dots, Q_J$  and a function  $f$  as good as possible for this set of polynomials in the form (??)E2. We seek a polynomial  $R \in \mathbf{Z}[x]$  of degree  $k$  ( $k = 10$  for instance) such that

$$\sup_{x \in L} |Q(x)R(x)|^{\frac{t}{r+k}} |q + sx|^{-1} \leq e^{-m},$$

where  $Q = \prod_{j=1}^J Q_j$ . We want the quantity

$$\sup_{x \in L} |Q(x)R(x)| |q + sx|^{\frac{(r+k)}{t}}$$

to be as small as possible. We apply the LLL algorithm to the linear forms

$$Q(x_i)R(x_i) |q + sx_i|^{\frac{(r+k)}{t}}.$$

The  $x_i$  are control points, which are uniformly distributed points on the interval  $L$  to which we have added points where  $f$  has local minima. Thus we find a polynomial  $R$  whose irreducible factors  $R_j$  are good candidates to enlarge the set  $\{Q_1, \dots, Q_J\}$ . We only keep the factors  $R_j$  that have a nonzero coefficient in the newly optimized auxiliary function  $f$ . After optimization, some previous polynomials  $Q_j$  may have a zero exponent and so are removed. We stop when two consecutive steps do not give any new polynomial.

Besides, we have to optimize the  $c_j$ , that is to solve a problem of the following form: find

$$\max_C \min_{x \in X} f(x, C)$$

where  $f(x, C)$  is a linear form with respect to  $C = (c_0, c_1, \dots, c_k)$  ( $c_0$  is the coefficient of  $x$  and is equal to 1) and  $X$  is a compact domain of  $\mathbb{C}$ , the maximum is taken over  $c_j \geq 0$  for  $j = 0, \dots, k$ . A classical solution consists in taking very many control points  $(x_i)_{1 \leq i \leq N}$  and in solving the standard problem of linear programming:

$$\max_C \min_{1 \leq j \leq N} f(x_j, C).$$

The result depends then on the choice of the control points.

The idea of the semi infinite linear programming (introduced into Number Theory by C. J. Smyth [S2]) consists in repeating the previous process adding at each step new control points and verifying that this process converges to  $m$ , the value of the linear form for an optimum choice of  $C$ . The algorithm is the following:

- (1) We choose an initial value for  $C$  i.e.,  $C^0$  and we calculate

$$m'_0 = \min_{x \in X} f(x, C^0).$$

- (2) We choose a finite set  $X_0$  of control points belonging to  $X$ , and we have

$$m'_0 \leq m \leq m_0 = \min_{x \in X_0} f(x, C^0).$$

- (3) We add to  $X_0$  the points where  $f(x, C^0)$  has local minima to get a new set  $X_1$  of control points.  
(4) We solve the usual linear programming problem:

$$\max_C \min_{x \in X_1} f(x, C).$$

We get a new value for  $C$  denoted by  $C^1$  and a result of the linear programming equal to  $m'_1 = \min_{x \in X} f(x, C^1)$ . Then we have

$$m'_0 \leq m'_1 \leq m \leq m_1 = \min_{x \in X_1} f(x, C^1) \leq m_0,$$

- (5) We repeat the steps from (2) to (4) and thus we get two sequences  $(m_i)$  et  $(m'_i)$  which satisfy

$$m'_0 \leq m'_1 \leq \dots \leq m'_i \leq m \leq m_i \leq \dots \leq m_1 \leq m_0,$$

We stop when there is a good enough convergence, for example when  $m_i - m'_i \leq 10^{-6}$ . Supposing that  $p$  iterations are sufficient, we then take  $m = m'_p$ .

### 3.2 The case $I = [0, (\sqrt{a} - \sqrt{b})^2]$

Let  $I$  be an interval  $[0, (\sqrt{a} - \sqrt{b})^2]$  where  $a, b$  are positive integers. We define  $\alpha$  to be the integer part of  $\left(\frac{\sqrt{a} + \sqrt{b}}{a - b}\right)^2$ . Then, the change of variable  $x(t) = \frac{\alpha t - 1}{-t}$  maps  $I$  to  $[b, \infty)$  where  $b$  is

the fractional part of  $\left(\frac{\sqrt{a} + \sqrt{b}}{a - b}\right)^2$ . Here, we consider the auxiliary function defined on  $[b, \infty)$

by:

$$f(x) = \log |\alpha + x| - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)|.$$

Then we proceed in a similar way to the method described in the paragraph above to get an upper bound for  $t_{\mathbb{Z}}(I)$ .

## 4 Numerical examples

Applying the method describe above, we prove that:

**Results 1.**

$$0.097580 \leq t_{\mathbf{Z}} \left( \left[ \frac{1}{2}, \frac{1}{\sqrt{3}} \right] \right) \leq 0.104355$$

$$0.039854 \leq t_{\mathbf{Z}} \left( \left[ \frac{7}{5}, \sqrt{2} \right] \right) \leq 0.047285$$

$$0.0496984 \leq t_{\mathbf{Z}} \left( \left[ \frac{27}{10}, e \right] \right) \leq 0.054964$$

$$0.102371 \leq t_{\mathbf{Z}}([3, \pi]) \leq 0.115497$$

$$0.1295 \leq t_{\mathbf{Z}}([0, (\sqrt{2} - 1)^2]) \leq 0.133449$$

$$0.284642 \leq t_{\mathbf{Z}}([0, (\sqrt{3} - 1)^2]) \leq 0.295697$$

**Proof:**

We detail the proof for the first interval. We search  $(p_0, q_0) \in \mathbf{Z}^2$  such that  $q_0 - 2p_0 = -1$  ( $\star$ ) with  $q_0 > 0$  as small as possible:  $p_0 = q_0 = 1$  is convenient. Then, the general solution of the equation ( $\star$ ) is of the type:  $(p_k = 1 + k, q_k = 1 + 2k)$ ,  $k \in \mathbf{Z}$ . For  $k = 3$ , we have:

$\left[ \frac{1}{2}, \frac{4}{7} \right] \subset \left[ \frac{1}{2}, \frac{1}{\sqrt{3}} \right]$  so that we apply Theorem 2 to this interval to get the lower bound. We are

in the case where  $\frac{1}{\sqrt{3}} < \frac{p_0}{q_0}$  thus  $b = \frac{q_2/\sqrt{3} - p_2}{1 - 2/\sqrt{3}} = \sqrt{3} - 1$ . We optimize the auxiliary function

$$f(x) = \log |5 + 2x| - \sum_{1 \leq j \leq J} c_j \log |Q_j(x)| \text{ on } [\sqrt{3} - 1, \infty).$$

For the third interval, we have  $(p_0, q_0) = (19, 7)$  then here,  $\frac{p_0}{q_0} < e$ . In this case, we put

$$b = \frac{q_0 e - p_0}{27 - 10e}.$$

The polynomials involved in the previous inequalities can be read off from Table 1 to Table 6 below.

## 5 Proof of Theorem 1

We put  $I = \left[ \frac{a}{q}, \frac{b}{s} \right]$  where  $|as - bq| = 1$  and  $J = \left[ \frac{c}{qs}, \frac{d}{(q+s)^2} \right]$  with  $|c(q+s)^2 - dqs| = 1$ .

### 5.1 The left hand-side inequality

We suppose that  $t_{\mathbf{Z}}(I) = e^{-m_I}$ . Thus, for all  $\epsilon > 0$ , sufficiently small for  $m = m_I - \epsilon > 0$ , there exists a polynomial  $H_1 \in \mathbf{Z}[t]$ ,  $\deg(H_1) = D$ , satisfying

$$(5.1) \quad \max_{t \in I} |H_1(t)|^{1/D} \leq e^{-m}.$$

We can write  $H_1(t) = (ts - b)^{D-k}K(t)$  where  $\deg(K) = k \leq D$  and  $ts - b$  does not divide  $K$ . Now, put  $x = \frac{a - tq}{ts - b}$  then  $x \geq 0$ .  $H_1$  becomes

$$H_1\left(\frac{a + bx}{q + sx}\right) = \left(\frac{\pm 1}{q + sx}\right)^D \underbrace{(q + sx)^k K\left(\frac{a + bx}{q + sx}\right)}_{H_2(x)}$$

where  $\deg(H_2) = \deg(K) \leq D$ . From (5.1), we deduce that  $\forall x \geq 0, |H_2(x)|^{1/D} \leq e^{-m}(q + sx)$ . If we put  $f(x) = \log(q + sx) - \frac{1}{D} \log |H_2(x)|$  for all  $x \geq 0$ , we have  $\inf_{x \geq 0} f(x) \geq m$ .

We define the function  $g$  by :  $\forall x > 0, g(x) = \frac{1}{2}[f(x) + f(\frac{1}{x})]$ . There exists a polynomial  $H_3 \in \mathbb{Z}[x]$ ,  $\deg(H_3) \leq \deg(H_2)$  such that

$$g(x) = \frac{1}{2} \left[ \log \left( (q + s)^2 + qs(x + \frac{1}{x} - 2) \right) - \frac{1}{D} \log |H_3(x + \frac{1}{x} - 2)| \right].$$

Now, put  $y = x + \frac{1}{x} - 2$  and  $h(y) = \log((q + s)^2 + qsy) - \frac{1}{D} \log |H_3(y)|$ . Then

$$(5.2) \quad \inf_{y \geq 0} h(y) \geq 2m.$$

Next, put  $z = \frac{cy + d}{qsy + (q + s)^2}$ , so that  $z$  belongs to the interval  $J$ . From (5.2), we deduce that

$$\forall z \in J, |qsz - c| |H_3\left(\frac{d - z(q + s)^2}{qsz - c}\right)|^{1/D} \leq e^{-2m},$$

i.e.,

$$\forall z \in J, \underbrace{|(qsz - c)^{D - \deg(H_3)} H_3\left(\frac{d - z(q + s)^2}{qsz - c}\right) (qsz - c)^{\deg(H_3)}|^{1/D}}_{H_4(z)} \leq e^{-2m},$$

where  $\deg(H_4) = \deg(H_3)$ .

Hence, there exists a polynomial  $H_5(z) = (qsz - c)^{D - \deg(H_4)} H_4(z) \in \mathbb{Z}[z]$ ,  $\deg(H_5) = D$  such that  $\forall z \in J, |H_5(z)|^{1/D} \leq e^{-2m}$ . It means that

$$t_{\mathbb{Z}}(J) \leq |H_5(z)|^{1/D} \leq (e^{-(mI - \epsilon)})^2.$$

Finally, for  $\epsilon$  that tends towards 0, we obtain

$$t_{\mathbb{Z}}(J) \leq t_{\mathbb{Z}}(I)^2.$$

## 5.2 The right-hand side inequality

We suppose that  $t_{\mathbb{Z}}(J) = e^{-mJ}$ . Thus, for all  $\epsilon > 0$ , sufficiently small for  $m = m_J - \epsilon > 0$ , there exists a polynomial  $H_1 \in \mathbb{Z}[t]$ ,  $\deg(H_1) = D$ , satisfying

$$(5.3) \quad \max_{t \in J} |H_1(t)|^{1/D} \leq e^{-m}.$$

We can write  $H_1(t) = (qst - c)^{D-k}K(t)$  where  $\deg(K) = k \leq D$  and  $qst - c$  does not divide  $K$ . Now, put  $x = \frac{d - t(q + s)^2}{qst - c}$  then  $x \geq 0$ . then  $H_1$  becomes

$$H_1\left(\frac{cx + d}{qsx + (q + s)^2}\right) = \left(\frac{\pm 1}{qsx + (q + s)^2}\right)^D \underbrace{(qsx + (q + s)^2)^k K\left(\frac{cx + d}{qsx + (q + s)^2}\right)}_{H_2(x)},$$

where  $\deg(H_2) = \deg(K) \leq D$ . From (5.3), we deduce that

$$(5.4) \quad \forall x \geq 0, |H_2(x)|^{1/D} \leq e^{-m}(qsx + (q+s)^2).$$

For all  $x \geq 0$ , it is possible to find  $y \geq 0$  such that  $x = y + \frac{1}{y} - 2$ . Thus, (5.4) becomes:

$$(5.5) \quad \left| y^D H_2 \left( y + \frac{1}{y} - 2 \right) \right|^{1/D} \leq e^{-m}(q + sy)(qy + s).$$

At last, put  $z = \frac{a + by}{q + sy}$  so that  $z$  belongs to the interval  $I$ . From (5.5), we deduce

$$\left| (a - qz)^D (sz - b)^D H_2 \left( \frac{a - qz}{sz - b} + \frac{sz - b}{a - qz} - 2 \right) \right|^{1/D} \leq e^{-m} |(s^2 - q^2)z + aq - bs|,$$

i.e., there exists a polynomial  $H_3 \in \mathbb{Z}[z]$ ,  $\deg(H_3) = 2 \deg(H_2)$  such that

$$\underbrace{|(a - qz)^{D - \deg(H_2)} (sz - b)^{D - \deg(H_2)} H_3(z)|^{1/D}}_{H_4(z)} \leq e^{-m} |(s^2 - q^2)z + aq - bs|,$$

with  $\deg(H_4) = 2D$ . Thus, we have  $|H_4(z)|^{1/2D} \leq (e^{-m} |(s^2 - q^2)z + aq - bs|)^{1/2}$ . As the function  $z \rightarrow (s^2 - q^2)z + aq - bs$  is monotonic on the interval  $I$ , it follows that  $\forall z \in I$ ,  $|H_4(z)|^{1/2D} \leq e^{-m/2} \sqrt{\max\left(\frac{q}{s}, \frac{s}{q}\right)}$ . This means that

$$t_{\mathbb{Z}}(I) \leq e^{-(m_J - \epsilon)/2} \sqrt{\max\left(\frac{q}{s}, \frac{s}{q}\right)}.$$

Now, letting  $\epsilon \rightarrow 0$ , we get

$$t_{\mathbb{Z}}(I) \leq t_{\mathbb{Z}}(I)^{1/2} \sqrt{\max\left(\frac{q}{s}, \frac{s}{q}\right)}.$$



**Table 1:**  $t_{\mathbf{z}}\left(\left[\frac{1}{2}, \frac{1}{\sqrt{3}}\right]\right)$

j	$c_j$	$Q_j$
1	0.194694226959	$x - 1$
2	0.035494760974	$x - 2$
3	0.002816285615	$x - 3$
4	0.004358713735	$2x^3 - 12x^2 + 20x - 9$
5	0.018800399043	$x^3 - 7x^2 + 14x - 7$
6	0.008005832569	$x^4 - 10x^3 + 33x^2 - 41x + 16$
7	0.001938639658	$x^4 - 10x^3 + 32x^2 - 39x + 15$
8	0.004174431074	$x^4 - 10x^3 + 33x^2 - 42x + 17$
9	0.001311916586	$x^5 - 11x^4 + 47x^3 - 95x^2 + 88x - 29$
10	0.000845050740	$x^5 - 14x^4 + 72x^3 - 168x^2 + 174x - 62$
11	0.000018382517	$x^5 - 13x^4 + 60x^3 - 123x^2 + 113x - 37$
12	0.000659169385	$x^6 - 15x^5 + 87x^4 - 245x^3 + 347x^2 - 232x + 58$
13	0.000522369339	$x^6 - 15x^5 + 84x^4 - 224x^3 + 297x^2 - 185x + 43$
14	0.000661169596	$x^6 - 16x^5 + 96x^4 - 274x^3 + 389x^2 - 260x + 65$
15	0.000060014755	$x^7 - 18x^6 + 129x^5 - 474x^4 + 958x^3 - 1059x^2 + 592x - 130$
16	0.000046641048	$2x^7 - 33x^6 + 217x^5 - 736x^4 + 1388x^3 - 1453x^2 + 781x - 167$
17	0.000968791114	$x^7 - 18x^6 + 129x^5 - 475x^4 + 966x^3 - 1080x^2 + 613x - 137$
18	0.000283740742	$x^7 - 19x^6 + 145x^5 - 571x^4 + 1240x^3 - 1469x^2 + 873x - 202$
19	0.001171298002	$x^8 - 21x^7 + 180x^6 - 820x^5 + 2170x^4 - 3416x^3 + 3124x^2 - 1518x + 301$
20	0.000599474735	$x^8 - 20x^7 + 165x^6 - 731x^5 + 1896x^4 - 2938x^3 + 2648x^2 - 1267x + 247$
21	0.000513081115	$x^{12} - 31x^{11} + 422x^{10} - 3333x^9 + 17000x^8 - 58960x^7 + 142509x^6 - 241765x^5 + 285610x^4 - 229089x^3 + 118442x^2 - 35464x + 4657$
22	0.000266864041	$x^{12} - 31x^{11} + 422x^{10} - 3333x^9 + 17001x^8 - 58975x^7 + 142601x^6 - 242068x^5 + 286200x^4 - 229789x^3 + 118939x^2 - 35658x + 4689$

**Table 2:**  $t_{\mathbf{z}}\left(\left[\frac{7}{5}, \sqrt{2}\right]\right)$

j	$c_j$	$Q_j$
1	0.162145981678	$x - 1$
2	0.016639017547	$2x - 1$
3	0.047659908308	$x^2 - 3x + 1$
4	0.008735631418	$x^2 - 4x + 2$
5	0.006909486855	$x^2 - 5x + 2$
6	0.008890960380	$x^3 - 6x^2 + 9x - 3$
7	0.007016724017	$x^4 - 9x^3 + 20x^2 - 14x + 3$
8	0.000210255230	$x^4 - 7x^3 + 13x^2 - 7x + 1$
9	0.001167637570	$x^5 - 10x^4 + 33x^3 - 46x^2 + 26x - 5$
10	0.005671721498	$x^6 - 13x^5 + 60x^4 - 121x^3 + 111x^2 - 46x + 7$
11	0.000866481352	$x^6 - 13x^5 + 61x^4 - 132x^3 + 138x^2 - 65x + 11$
12	0.000552782161	$x^7 - 15x^6 + 86x^5 - 239x^4 + 344x^3 - 256x^2 + 93x - 13$
13	0.001261761511	$x^7 - 14x^6 + 77x^5 - 209x^4 + 297x^3 - 219x^2 + 79x - 11$
14	0.001985253256	$x^7 - 15x^6 + 85x^5 - 229x^4 + 315x^3 - 222x^2 + 76x - 10$
15	0.000621304236	$2x^7 - 27x^6 + 138x^5 - 343x^4 + 446x^3 - 303x^2 + 101x - 13$
16	0.001330105312	$x^9 - 20x^8 + 159x^7 - 657x^6 + 1551x^5 - 2168x^4 + 1799x^3 - 863x^2 + 220x - 23$
17	0.000656024029	$x^9 - 19x^8 + 147x^7 - 597x^6 + 1388x^5 - 1908x^4 + 1555x^3 - 733x^2 + 184x - 19$
18	0.000658635577	$x^{10} - 23x^9 + 217x^8 - 1104x^7 + 3347x^6 - 6301x^5 + 7439x^4 - 5436x^3 + 2366x^2 - 559x + 55$
19	0.000367168471	$x^{10} - 22x^9 + 198x^8 - 959x^7 + 2763x^6 - 4941x^5 + 5550x^4 - 3873x^3 + 1617x^2 - 368x + 35$
20	0.000486211368	$x^{10} - 23x^9 + 217x^8 - 1106x^7 + 3364x^6 - 6357x^5 + 7531x^4 - 5518x^3 + 2406x^2 - 569x + 56$
21	0.000294675856	$x^{10} - 22x^9 + 199x^8 - 968x^7 + 2795x^6 - 4999x^5 + 5608x^4 - 3905x^3 + 1626x^2 - 369x + 35$
22	0.000155171828	$x^{12} - 27x^{11} + 309x^{10} - 1978x^9 + 7867x^8 - 20424x^7 + 35410x^6 - 41270x^5 + 32141x^4 - 16392x^3 + 5234x^2 - 947x + 74$
23	0.000751307129	$x^{12} - 26x^{11} + 288x^{10} - 1791x^9 + 6945x^8 - 17655x^7 + 30136x^6 - 34807x^5 + 27052x^4 - 13857x^3 + 4467x^2 - 819x + 65$
24	0.000018796797	$x^{12} - 28x^{11} + 332x^{10} - 2201x^9 + 9074x^8 - 24482x^7 + 44308x^6 - 54232x^5 + 44648x^4 - 24206x^3 + 8246x^2 - 1594x + 133$
25	0.000014657665	$x^{13} - 31x^{12} + 416x^{11} - 3191x^{10} + 15570x^9 - 50902x^8 + 114432x^7 - 178754x^6 + 193820x^5 - 144150x^4 + 71665x^3 - 22651x^2 + 4101x - 323$

**Table 3:**  $t_{\mathbf{z}}\left(\left[\frac{27}{10}, e\right]\right)$

j	$c_j$	$Q_j$
1	0.550041283463	$x - 1$
2	0.042040774975	$x - 2$
3	0.006830062504	$x - 3$
4	0.000301725058	$x^3 - 9x^2 + 21x - 12$
5	0.011449194208	$x^3 - 8x^2 + 19x - 11$
6	0.001470121850	$x^4 - 12x^3 + 45x^2 - 62x + 27$
7	0.000195767550	$x^4 - 13x^3 + 48x^2 - 65x + 28$
8	0.001517499543	$x^4 - 13x^3 + 52x^2 - 79x + 37$
9	0.002387068906	$x^4 - 11x^3 + 40x^2 - 56x + 25$
10	0.000714217606	$x^4 - 11x^3 + 39x^2 - 53x + 23$
11	0.001613407579	$2x^4 - 18x^3 + 56x^2 - 70x + 29$
12	0.009942904485	$2x^4 - 19x^3 + 60x^2 - 75x + 31$
13	0.000242092296	$x^5 - 13x^4 + 62x^3 - 136x^2 + 136x - 49$
14	0.002838151941	$x^5 - 14x^4 + 73x^3 - 175x^2 + 190x - 73$
15	0.003090337490	$x^5 - 14x^4 + 69x^3 - 153x^2 + 153x - 55$
16	0.000877501362	$x^5 - 14x^4 + 70x^3 - 157x^2 + 158x - 57$
17	0.000783458036	$x^7 - 21x^6 + 174x^5 - 732x^4 + 1678x^3 - 2091x^2 + 1315x - 325$
18	0.000718382052	$x^8 - 23x^7 + 216x^6 - 1080x^5 + 3143x^4 - 5451x^3 + 5503x^2 - 2959x + 651$
19	0.000776048203	$x^9 - 25x^8 + 264x^7 - 1542x^6 + 5481x^5 - 12283x^4 + 17346x^3 - 14882x^2 + 7040x - 1401$
20	0.000669081482	$x^{13} - 39x^{12} + 673x^{11} - 6800x^{10} + 44870x^9 - 204169x^8 + 659163x^7 - 1529181x^6 + 2549851x^5 - 3019341x^4 + 2468843x^3 - 1320786x^2 + 414697x - 57780$
21	0.000505379075	$x^{13} - 39x^{12} + 673x^{11} - 6801x^{10} + 44892x^9 - 204377x^8 + 660280x^7 - 1532961x^6 + 2558282x^5 - 3031903x^4 + 2481202x^3 - 1328481x^2 + 417440x - 58206$

**Table 4:**  $t_{\mathbf{z}}([3, \pi])$

j	$c_j$	$Q_j$
1	0.069920470860	$x$
2	0.054008763288	$x - 1$
3	0.004496196393	$x - 2$
4	0.019096822557	$x^2 - 3x + 1$
5	0.001274142158	$x^2 - 4x + 1$
6	0.000255157554	$x^2 - 4x + 2$
7	0.000316881818	$x^3 - 6x^2 + 9x - 3$
8	0.005641656329	$x^3 - 5x^2 + 6x - 1$
9	0.000422361372	$x^3 - 6x^2 + 8x - 1$
10	0.002719098798	$x^4 - 7x^3 + 14x^2 - 8x + 1$
11	0.003581114060	$x^4 - 7x^3 + 13x^2 - 7x + 1$
12	0.000394795544	$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$
13	0.000135078176	$x^5 - 9x^4 + 27x^3 - 31x^2 + 12x - 1$
14	0.000337538478	$x^5 - 8x^4 + 22x^3 - 25x^2 + 10x - 1$
15	0.000497738418	$x^5 - 9x^4 + 27x^3 - 32x^2 + 13x - 1$
16	0.000244309435	$x^6 - 11x^5 + 42x^4 - 68x^3 + 47x^2 - 13x + 1$
17	0.000453661539	$x^6 - 11x^5 + 42x^4 - 67x^3 + 45x^2 - 12x + 1$
18	0.000391877242	$x^7 - 13x^6 + 63x^5 - 143x^4 + 157x^3 - 78x^2 + 16x - 1$
19	0.000449585658	$x^7 - 13x^6 + 63x^5 - 144x^4 + 160x^3 - 80x^2 + 16x - 1$
20	0.000405766077	$x^7 - 13x^6 + 62x^5 - 137x^4 + 147x^3 - 73x^2 + 15x - 1$
21	0.000993500494	$x^7 - 12x^6 + 55x^5 - 120x^4 + 129x^3 - 65x^2 + 14x - 1$
22	0.000277846069	$x^7 - 12x^6 + 55x^5 - 121x^4 + 133x^3 - 69x^2 + 15x - 1$
23	0.000032921798	$x^7 - 13x^6 + 61x^5 - 131x^4 + 136x^3 - 66x^2 + 14x - 1$
24	0.000059915334	$x^7 - 13x^6 + 63x^5 - 143x^4 + 158x^3 - 80x^2 + 16x - 1$
25	0.000437650741	$x^8 - 15x^7 + 89x^6 - 267x^5 + 431x^4 - 371x^3 + 160x^2 - 31x + 2$
26	0.000125350780	$x^9 - 16x^8 + 102x^7 - 337x^6 + 629x^5 - 676x^4 + 409x^3 - 132x^2 + 20x - 1$
27	0.000204978013	$x^9 - 16x^8 + 103x^7 - 345x^6 + 651x^5 - 703x^4 + 424x^3 - 135x^2 + 20x - 1$
28	0.000197079509	$x^{10} - 18x^9 + 134x^8 - 538x^7 + 1273x^6 - 1822x^5 + 1560x^4 - 766x^3 + 200x^2 - 24x + 1$
29	0.000155065443	$x^{11} - 20x^{10} + 166x^9 - 748x^8 + 2013x^7 - 3361x^6 + 3513x^5 - 2276x^4 + 891x^3 - 200x^2 + 23x - 1$
30	0.000278363869	$x^{12} - 21x^{11} + 189x^{10} - 958x^9 + 3021x^8 - 6178x^7 + 8291x^6 - 7242x^5 + 4017x^4 - 1359x^3 + 263x^2 - 26x + 1$
31	0.000070028708	$x^{14} - 25x^{13} + 273x^{12} - 1718x^{11} + 6923x^{10} - 18777x^9 + 35084x^8 - 45474x^7 + 40682x^6 - 24738x^5 + 9961x^4 - 2548x^3 + 387x^2 - 31x + 1$

**Table 5:**  $t_{\mathbf{Z}}([0, (\sqrt{2} - 1)^2])$

j	$c_j$	$Q_j$
1	0.115071100342	$x - 1$
2	0.042666343105	$x - 2$
3	0.006877529069	$x - 3$
4	0.008229041311	$x^2 - 5x + 5$
5	0.005431600967	$x^3 - 8x^2 + 18x - 10$
6	0.000180696893	$x^4 - 11x^3 + 40x^2 - 55x + 24$
7	0.000156842585	$x^4 - 11x^3 + 39x^2 - 53x + 23$
8	0.000204367141	$2x^4 - 18x^3 + 56x^2 - 70x + 29$
9	0.001513115862	$x^4 - 10x^3 + 34x^2 - 45x + 19$
10	0.000644176350	$x^5 - 12x^4 + 55x^3 - 119x^2 + 119x - 43$
11	0.001573791306	$x^5 - 13x^4 + 63x^3 - 141x^2 + 144x - 53$
12	0.000719661252	$x^5 - 13x^4 + 62x^3 - 135x^2 + 133x - 47$
13	0.000865530666	$x^5 - 13x^4 + 62x^3 - 136x^2 + 136x - 49$
14	0.001236127494	$x^5 - 13x^4 + 63x^3 - 140x^2 + 141x - 51$
15	0.000391029270	$x^5 - 14x^4 + 73x^3 - 175x^2 + 190x - 73$
16	0.000918506387	$x^6 - 16x^5 + 99x^4 - 302x^3 + 479x^2 - 375x + 113$
17	0.000697517226	$x^6 - 16x^5 + 98x^4 - 296x^3 + 466x^2 - 363x + 109$
18	0.000463327673	$x^7 - 18x^6 + 132x^5 - 510x^4 + 1119x^3 - 1392x^2 + 908x - 239$
19	0.000367368001	$x^7 - 18x^6 + 133x^5 - 520x^4 + 1156x^3 - 1455x^2 + 957x - 253$
20	0.000732687113	$x^7 - 18x^6 + 132x^5 - 511x^4 + 1125x^3 - 1404x^2 + 917x - 241$
21	0.000025639888	$2x^{11} - 55x^{10} + 667x^9 - 4704x^8 + 21414x^7 - 66001x^6 + 140395x^5 - 205912x^4 + 203889x^3 - 129715x^2 + 47698x - 7679$
22	0.000716635362	$x^{11} - 29x^{10} + 370x^9 - 2738x^8 + 13041x^7 - 41928x^6 + 92748x^5 - 141021x^4 + 144313x^3 - 94600x^2 + 35735x - 5893$
23	0.000397521950	$x^{11} - 29x^{10} + 369x^9 - 2717x^8 + 12853x^7 - 40986x^6 + 89835x^5 - 135252x^4 + 136992x^3 - 88857x^2 + 33207x - 5417$
24	0.000524876998	$x^{11} - 29x^{10} + 370x^9 - 2738x^8 + 13040x^7 - 41913x^6 + 92655x^5 - 140713x^4 + 143727x^3 - 93962x^2 + 35368x - 5807$
25	0.000612892287	$x^{11} - 29x^{10} + 369x^9 - 2717x^8 + 12852x^7 - 40973x^6 + 89766x^5 - 135058x^4 + 136679x^3 - 88566x^2 + 33062x - 5387$
26	0.000148377159	$x^{12} - 33x^{11} + 484x^{10} - 4167x^9 + 23428x^8 - 90515x^7 + 246146x^6 - 474230x^5 + 641859x^4 - 594750x^3 + 357931x^2 - 125575x + 19423$
27	0.000103216390	$x^{12} - 33x^{11} + 485x^{10} - 4192x^9 + 23699x^8 - 92192x^7 + 252704x^6 - 491157x^5 + 671055x^4 - 627975x^3 + 381809x^2 - 135365x + 21163$
28	0.000133150862	$x^{12} - 32x^{11} + 455x^{10} - 3799x^9 + 20732x^8 - 77852x^7 + 206129x^6 - 387456x^5 + 512803x^4 - 465781x^3 + 275478x^2 - 95228x + 14551$
29	0.000246669901	$x^{13} - 35x^{12} + 552x^{11} - 5191x^{10} + 32453x^9 - 142327x^8 + 450113x^7 - 1038385x^6 + 1745715x^5 - 2110202x^4 + 1780812x^3 - 992518x^2 + 327128x - 48119$
30	0.000039809568	$x^{14} - 38x^{13} + 655x^{12} - 6783x^{11} + 47114x^{10} - 232039x^9 + 835104x^8 - 2229770x^7 + 4435843x^6 - 6539179x^5 + 7027795x^4 - 5336585x^3 + 2705220x^2 - 819024x + 111689$

**Table 6:**  $t_{\mathbf{Z}}([0, (\sqrt{3} - 1)^2])$

j	$c_j$	$Q_j$
1	0.270018919329	$x - 1$
2	0.077813140539	$x - 2$
3	0.011175462644	$x - 3$
4	0.011249920871	$x^2 - 5x + 5$
5	0.001033589541	$x^2 - 6x + 7$
6	0.000260925664	$x^3 - 9x^2 + 24x - 19$
7	0.002443270637	$x^4 - 9x^3 + 27x^2 - 31x + 11$
8	0.000528294702	$4x^5 - 44x^4 + 180x^3 - 343x^2 + 305x - 101$
9	0.000696152224	$x^5 - 14x^4 + 72x^3 - 167x^2 + 173x - 64$
10	0.009486871917	$x^5 - 13x^4 + 63x^3 - 141x^2 + 144x - 53$
11	0.000692925074	$x^5 - 14x^4 + 70x^3 - 158x^2 + 161x - 59$
12	0.001337857384	$x^5 - 14x^4 + 71x^3 - 163x^2 + 168x - 62$
13	0.000171441668	$x^5 - 15x^4 + 82x^3 - 203x^2 + 225x - 88$
14	0.000196328975	$x^6 - 17x^5 + 111x^4 - 355x^3 + 586x^2 - 475x + 148$
15	0.000542120909	$x^6 - 17x^5 + 111x^4 - 355x^3 + 586x^2 - 474x + 147$
16	0.005326448972	$x^6 - 16x^5 + 100x^4 - 311x^3 + 506x^2 - 408x + 127$
17	0.000722607689	$x^7 - 19x^6 + 146x^5 - 587x^4 + 1332x^3 - 1705x^2 + 1140x - 307$
18	0.001404371641	$x^7 - 19x^6 + 146x^5 - 587x^4 + 1333x^3 - 1709x^2 + 1145x - 309$
19	0.000116819748	$x^7 - 19x^6 + 146x^5 - 588x^4 + 1340x^3 - 1727x^2 + 1165x - 317$
20	0.000502577474	$x^7 - 19x^6 + 146x^5 - 588x^4 + 1339x^3 - 1722x^2 + 1157x - 313$
21	0.000262250259	$x^7 - 20x^6 + 161x^5 - 677x^4 + 1606x^3 - 2148x^2 + 1498x - 419$
22	0.000701618498	$x^7 - 18x^6 + 133x^5 - 522x^4 + 1171x^3 - 1495x^2 + 1002x - 271$

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