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# ON THE ZHANG-ZAGIER MEASURE

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## Abstract

We first improve the known lower bound for the absolute Zhang-Zagier measure in the general case. Then we restrict our study to totally positive algebraic integers. In this case, we are able to find six points for the related spectrum. At last, we give inequalities involving the Zhang-Zagier measure, the Mahler measure and the length of such integers.

## 1 Introduction

Let  $\alpha$  be an algebraic number of degree  $d$  and minimal polynomial  $P(z) = z^n + \dots + a_n = \prod_{j=1}^n (z - \alpha_j)$ . The *Mahler measure* of  $\alpha$  is

$$M(\alpha) = \prod_{j=1}^n \max(1, |\alpha_j|).$$

Now, let  $h(\alpha)$  be the absolute logarithmic height on  $\overline{\mathbb{Q}}$ . S. Zhang [Zh] established the existence of a constant  $C$  such that

$$h(\alpha) + h(1 - \alpha) \geq C.$$

Shortly thereafter, D. Zagier [Za] made effective this result and showed that if  $\alpha$  is an algebraic number different from the roots of  $z(z-1)(z^2-z+1)$  then

$$h(\alpha) + h(1 - \alpha) \geq \frac{1}{2} \log \frac{1 + \sqrt{5}}{2}.$$

In the following, we define the *Zhang-Zagier measure* by  $Z(\alpha) = M(\alpha)M(1 - \alpha)$  and the *absolute Zhang-Zagier measure* by  $\zeta(\alpha) = Z(\alpha)^{1/\deg(\alpha)}$ . In 2001, C. Doche [D] improved the above lower bound and found  $\zeta(\alpha) \geq 1.281777$  for all algebraic integers different from the roots of  $(z^2 - z)(z^2 - z + 1)\phi_{10}(z)\phi_{10}(1 - z)$ . He first established a list of polynomials with small Zhang-Zagier measure and then tested them in his auxiliary function. His search allowed him to find a polynomial of degree 32 and measure 1.287527. In this paper, we use our recursive algorithm to find good polynomials involved in our auxiliary functions and we get a slightly improvement of Doche's result:

**Theorem 1.** *If  $\alpha$  is an algebraic integer different from the roots of  $(z^2 - z)(z^2 - z + 1)\phi_{10}(z)\phi_{10}(1 - z)$  then*

$$\zeta(\alpha) \geq 1.282416.$$

The polynomials involved in this result can be read off from Table 1.

The improvement is not sufficient to get the next point of the spectrum that we conjecture to be the algebraic integer with measure 1.287527 found firstly by C. Doche.

From now, we work with totally positive algebraic integers  $\alpha$ . This means that all the conjugates of  $\alpha$  are positive real numbers. We get the following result:

**Theorem 2.** *If  $\alpha$  is a totally positive algebraic integer different from the roots of  $x(x-1)(x-2)(x^2-3x+1)(x^3-5x^2+6x-1)(x^4-7x^3+13x^2-7x+1)(x^5-9x^4+26x^3-29x^2+11x-1)$  then*

$$\zeta(\alpha) \geq 2.352316$$

**Corollary 1.** *The six first points of the spectrum of the quantities  $\zeta(\alpha)$  where  $\alpha$  is a totally positive algebraic integer are:*

$$\begin{aligned} 1 &= \zeta(x) = \zeta(x-1) \\ 2 &= \zeta(x-2) \\ 2.058171\dots &= \zeta(x^2-3x+1) \\ 2.246979\dots &= \zeta(x^3-5x^2+6x-1) \\ 2.286958\dots &= \zeta(x^4-7x^3+13x^2-7x+1) \\ 2.349434\dots &= \zeta(x^5-9x^4+26x^3-29x^2+11x-1) \end{aligned}$$

We conjecture that the following point has a Zhang-Zagier measure equal to 2.364136 and minimal polynomial  $x^4 - 7x^3 + 14x^2 - 8x + 1$ .

The polynomials involved in this result can be read off from Table 2.

At last, for a totally positive algebraic integer  $\alpha$ , we get lower and upper bounds for its Zhang-Zagier measure, bounds that involve the length or the Mahler measure of  $\alpha$ . Remember that the *length* of  $\alpha$  is the sum of the absolute value of the coefficients of its minimal polynomial. In the case of a totally positive algebraic integer  $\alpha$ , we also have

$L(\alpha) = \prod_{i=1}^d (1 + \alpha_i)$  where  $\alpha_1 = \alpha, \dots, \alpha_d$  are the conjugates of  $\alpha$ . More precisely, we

prove the following results:

**Theorem 3.** *If  $\alpha$  is a totally positive algebraic integer of degree  $d$  different from the roots of  $x(x-1)(x-2)(x^2-3x+1)(x^2-4x+1)(x^3-5x^2+6x-1)(x^3-6x^2+9x-3)(x^3-6x^2+5x-1)$  then*

$$0.943791^d L(\alpha)^{1.040116} \leq Z(\alpha) \leq 0.0019202^d L(\alpha)^{5.733298} \quad (1.1)$$

**Theorem 4.** *If  $\alpha$  is a totally positive algebraic integer of degree  $d$  different from the roots of  $x(x-1)(x-2)(x^2-3x+1)(x^2-4x+1)(x^3-5x^2+6x-1)(x^3-6x^2+5x-1)$  then*

$$1.076655^d M(\alpha)^{1.403799} \leq Z(\alpha) \leq 0.252693^d M(\alpha)^{4.311210} \quad (1.2)$$

The paper is organized as follows. In Section 2, we recall the principle of the explicit auxiliary functions and their link with the integer transfinite diameter. Then we explain

our recursive algorithm that allows us to get the lower bound of Theorem 1. The scheme of the proof for Theorem 2 is exactly the same as those for Theorem 1, so we do not recall it. We only give in Table 2 the polynomials involved and their coefficients. In Section 3, we mention the different auxiliary functions used to obtain Theorem 3 and Theorem 4. We also give the polynomials and their coefficients.

## 2 Proof of Theorem 1

### 2.1 The explicit auxiliary function

Let  $\alpha$  be an algebraic number of degree  $d$  with conjugates  $\alpha_1 = \alpha, \dots, \alpha_d$  and minimal polynomial  $P$ . The auxiliary function involved in Theorem 1 is of the following type:

$$\text{for } z \text{ in } \mathbb{C}, f(z) = \log \max(1, |z|) + \log \max(1, |1 - z|) - \sum_{1 \leq j \leq J} c_j \log |Q_j(z(1 - z))| \quad (2.1)$$

where the  $c_j$  are positive real numbers and the polynomials  $Q_j$  are non zero polynomials in  $\mathbb{Z}[z]$ . The main point is to choose the numbers  $c_j$  and the polynomials  $Q_j$  in order to maximize the minimum  $m$  of  $f(z)$ . Since the function  $f$  is harmonic outside the union of arbitrarily small disks around the roots of the polynomials  $Q_j$ , the minimum is taken on  $|z| = 1$  or  $|1 - z| = 1$ . But  $f$  is invariant under the transformation  $z \rightarrow 1 - z$ , so it is sufficient to search the minimum of  $f$  on  $\mathcal{C} = \{|z| = 1\}$ .

We have

$$\sum_{i=1}^d f(\alpha_i) \geq md$$

i.e.,

$$\log M(\alpha)M(1 - \alpha) \geq md + \sum_{1 \leq j \leq J} c_j \log \left| \prod_{i=1}^d Q_j(\alpha_i(1 - \alpha_i)) \right|.$$

We assume that  $P$  does not divide any  $Q_j$ , hence  $\prod_{i=1}^d Q_j(\alpha_i(1 - \alpha_i))$  is a nonzero integer because it is the resultant of  $P(z)$  and  $Q_j(z(1 - z))$  and the resultant of  $P(z)$  and  $Q_j(z(1 - z))$  is the determinant of a matrix composed of integers.

Therefore, if  $\alpha$  is not a root  $Q_j(z(1 - z))$ , we get

$$\zeta(\alpha) \geq m.$$

Now, the main problem is to find a good list of polynomials  $Q_j$  which gives a value of  $m$  as large as possible. For this purpose, we link the auxiliary function to the integer transfinite diameter in order to find our polynomials by the recursive algorithm.

## 2.2 Auxiliary functions and the integer transfinite diameter

In this section, we will need the following definition:

Let  $K$  be a compact subset of  $\mathbb{C}$ . We define *the integer transfinite diameter of  $K$*  by

$$t_{\mathbb{Z}}(K) = \liminf_{n \rightarrow \infty} \inf_{\substack{n \geq 1 \\ P \in \mathbb{Z}[z] \\ \deg(P) = n}} |P|_{\infty, K}^{\frac{1}{n}},$$

where  $|P|_{\infty, K} = \sup_{z \in K} |P(z)|$ .

If  $\varphi$  is a positive function defined on  $K$ , the  $\varphi$ -*integer transfinite diameter of  $K$*  is defined as

$$t_{\mathbb{Z}, \varphi}(K) = \liminf_{n \rightarrow \infty} \inf_{\substack{n \geq 1 \\ P \in \mathbb{Z}[z] \\ \deg(P) = n}} \sup_{z \in K} \left( |P(z)|^{\frac{1}{n}} \varphi(z) \right).$$

This weighted version of the integer transfinite diameter was introduced by F. Amoroso [A]. It is an important tool in the study of rational approximations of logarithms of rational numbers.

Inside the auxiliary function (2.1), we replace the numbers  $c_j$  by rational numbers. Then we can write:

$$\text{for } z \in \mathbb{C}, \quad f(z) = \log \max(1, |z|) + \log \max(1, |1 - z|) - \frac{t}{r} \log |Q(z)| \quad (2.2)$$

where  $Q \in \mathbb{Z}[z]$  is of degree  $r$  and  $t$  is a positive real number. We want to get a function whose minimum  $m$  in  $\mathcal{C} = \{|z| = 1\}$  is as large as possible. Thus we search for a polynomial  $Q \in \mathbb{Z}[z]$  such that

$$\sup_{z \in \mathcal{C}} |Q(z)|^{t/r} (\max(1, |z|) \max(1, |1 - z|))^{-1} \leq e^{-m}.$$

If we suppose that  $t$  is fixed, it is clear that we need an effective upper bound for the quantity

$$t_{\mathbb{Z}, \varphi}(\mathcal{C}) = \liminf_{r \rightarrow +\infty} \inf_{\substack{r \geq 1 \\ P \in \mathbb{Z}[z] \\ \deg(P) = r}} \sup_{z \in \mathcal{C}} \left( |P(z)|^{\frac{t}{r}} \varphi(z) \right).$$

where we use the weight  $\varphi(z) = (\max(1, |z|) \max(1, |1 - z|))^{-1}$ .

## 2.3 Construction of the auxiliary function

The improvement compared with Wu's algorithm is that our polynomials are obtained by induction. Suppose that we have  $Q_1, Q_2, \dots, Q_J$ . Then we use semi-infinite linear programming (introduced into number theory by C. Smyth [Sm]) to optimize  $f$  for this set of polynomials (i.e., to get the greatest possible  $m$ ). We obtain the numbers  $c_1, c_2, \dots, c_J$

and  $f$  in the form (2.2) with  $t = \sum_{j=1}^J c_j \deg(Q_j)$ .

In the following, we put  $w = z(1 - z)$ . Hence, the auxiliary function becomes

$$f(w) = \log \max(1, |w|) - \sum_{1 \leq j \leq J} c_j \log |Q_j(w)|.$$

On the set  $\mathcal{C}$  where  $z = e^{it}$  with  $0 \leq t \leq \pi$ , we have

$$f(w) = \log \max(1, |v|) - \sum_{1 \leq j \leq J} c_j \log |Q_j(v)|$$

where we put  $v = e^{it} - e^{2it}$ . Therefore, for several values of the integer  $k$ , we search a polynomial  $R(z) = \sum_{l=0}^k a_l z^l \in \mathbb{Z}[z]$  such that

$$\sup_{0 \leq t \leq \pi} |Q(v)R(v)|^{\frac{t}{r+k}} \exp(\log \max(1, |v|)) \leq e^{-m}$$

i.e., such that

$$\sup_{0 \leq t \leq \pi} |Q(v)R(v)| \exp\left(-\frac{r+k}{2t} \log \max(1, |v|)\right)$$

is as small as possible.

Since  $R(v)$  is not a real linear form in the unknown coefficients  $a_l$ , we replace it by its real part and its imaginary part. Then, we apply the LLL algorithm to the linear forms

$$|Q(v_n)\operatorname{Re}(R(v_n))| \exp\left(-\frac{r+k}{2t} \log \max(1, |v_n|^2)\right),$$

$$|Q(v_n)\operatorname{Im}(R(v_n))| \exp\left(-\frac{r+k}{2t} \log \max(1, |v_n|^2)\right).$$

The numbers  $v_n = e^{it_n} - e^{2it_n}$  are control points where the numbers  $t_n$  are suitable points in the intervals  $(0, \pi)$ , including the points where  $f$  has its least local minima.

We get a polynomial  $R$  whose factors  $R_j$  are good candidates to enlarge the set of polynomials  $(Q_1, Q_2, \dots, Q_J)$ . We only keep the polynomials  $R_j$  which have a nonzero coefficient  $c_j$  in the new optimized auxiliary function  $f$ . After optimization, some previous polynomials  $Q_j$  may have a zero coefficient  $c_j$  and so are removed.

In order to get the constant in Theorem 1, we take  $k$  from 5 to 32 successively.

### 3 Proofs of Theorem 3 and Theorem 4

The scheme of the proofs follow those of Theorem 1. So we do not detail it again. We only give the different auxiliary functions used here.

For the lower bound of the inequality (1.1) (resp. (1.2)), we work with: for  $x > 0$ ,

$$f(x) = \log \max(1, x) + \log \max(1, |1-x|) - c_0 \log(x+1) \text{ (resp. } -c_0 \log(\max(1, x))) - \sum_{j=1}^J c_j \log |Q_j(x)|$$

whereas for the upper bound, we work with: for  $x > 0$ ,

$$f(x) = -\log \max(1, x) - \log \max(1, |1-x|) + c_0 \log(x+1) \text{ (resp. } +c_0 \log(\max(1, x))) - \sum_{j=1}^J c_j \log |Q_j(x)|.$$

The polynomials involved in the different auxiliary functions are:

$$\begin{aligned} Q_1 &= x \\ Q_2 &= x - 1 \\ Q_3 &= x - 2 \\ Q_4 &= 2x - 1 \\ Q_5 &= x^2 - 3x + 1 \\ Q_6 &= x^2 - 4x + 1 \\ Q_7 &= 2x^2 - 4x + 1 \\ Q_8 &= 3x^2 - 7x + 3 \\ Q_9 &= x^3 - 5x^2 + 6x - 1 \\ Q_{10} &= x^3 - 6x^2 + 5x - 1 \\ Q_{11} &= x^3 - 6x^2 + 9x - 3 \end{aligned}$$

The following table gives for each inequality the polynomials  $Q_j$  and the coefficients  $c_j$ .

Theorem	$Q_i$	$c_i$
Theorem 3 Left hand side inequality	$Q_1 \ Q_2 \ Q_3$ $Q_5 \ Q_7 \ Q_8$ $Q_9 \ Q_{11}$	0.05017587 0.46925178 0.21095580 0.04636319 0.00506568 0.00143313 0.01577735 0.01293106
Theorem 3 Right hand side inequality	$Q_1 \ Q_2 \ Q_4$ $Q_5 \ Q_6 \ Q_9$	1.61855077 0.06337982 0.05745574 0.07993953 0.04961051 0.08758098
Theorem 4 Left hand side inequality	$Q_1 \ Q_2 \ Q_3$ $Q_5 \ Q_9$	0.07403818 0.11915061 0.21388470 0.01672947 0.02864617
Theorem 4 Right hand side inequality	$Q_1 \ Q_2 \ Q_4$ $Q_5 \ Q_6 \ Q_{11}$	0.61332402 0.71445144 0.08155391 0.03990899 0.01436664 0.09322176

**Table 1:** Polynomials  $Q_j$  and their coefficients  $c_j$  involved in Theorem 1.

$j$	$c_j$	$Q_j$
1	0.233410801570	$w$
2	0.181822930849	$w - 1$
3	0.001206393184	$w^3 + w^2 - 2w + 1$
4	0.000374649587	$w^3 - 4w^2 + 3w - 1$
5	0.005894024785	$w^4 - 2w^3 + 4w^2 - 3w + 1$
6	0.000183896015	$2w^7 - 5w^6 + 6w^5 + 2w^4 - 11w^3 + 11w^2 - 5w + 1$
7	0.000078343816	$w^8 - 2w^7 + 4w^6 - 7w^5 + 13w^4 - 16w^3 + 12w^2 - 5w + 1$
8	0.002193099635	$w^8 - 3w^7 + 8w^6 - 16w^5 + 26w^4 - 27w^3 + 17w^2 - 6w + 1$
9	0.000147754755	$w^8 - w^7 - 3w^5 + 15w^4 - 22w^3 + 16w^2 - 6w + 1$
10	0.000000733657	$2w^{10} - 4w^9 + 6w^8 - 13w^7 + 40w^6 - 78w^5 + 91w^4 - 66w^3 + 30w^2 - 8w + 1$
11	0.000165661820	$w^{11} - w^{10} - w^9 + 5w^8 - 13w^7 + 40w^6 - 78w^5 + 91w^4 - 66w^3 + 30w^2 - 8w + 1$
12	0.000048632149	$w^{12} - 3w^{11} + 7w^{10} - 14w^9 + 30w^8 - 58w^7 + 96w^6 - 123w^5 + 114w^4 - 73w^3 + 31w^2 - 8w + 1$
13	0.000256750154	$w^{12} - 3w^{11} + 8w^{10} - 18w^9 + 36w^8 - 62w^7 + 97w^6 - 123w^5 + 114w^4 - 73w^3 + 31w^2 - 8w + 1$
14	0.000495658299	$w^{13} + w^{12} - 7w^{11} + 22w^{10} - 56w^9 + 134w^8 - 265w^7 + 384w^6 - 393w^5 + 283w^4 - 142w^3 + 48w^2 - 10w + 1$
15	0.000383867623	$w^{16} - 4w^{15} + 10w^{14} - 17w^{13} + 26w^{12} - 47w^{11} + 119w^{10} - 298w^9 + 592w^8 - 878w^7 + 963w^6 - 780w^5 + 464w^4 - 199w^3 + 59w^2 - 11w + 1$
16	0.000263423929	$w^{16} - 4w^{15} + 10w^{14} - 19w^{13} + 39w^{12} - 85w^{11} + 179w^{10} - 337w^9 + 554w^8 - 761w^7 + 830w^6 - 691w^5 + 427w^4 - 190w^3 + 58w^2 - 11w + 1$
17	0.000044659874	$w^{16} - 4w^{15} + 11w^{14} - 23w^{13} + 44w^{12} - 82w^{11} + 165w^{10} - 338w^9 + 614w^8 - 885w^7 + 964w^6 - 780w^5 + 464w^4 - 199w^3 + 59w^2 - 11w + 1$
18	0.000149649503	$w^{17} - 5w^{16} + 15w^{15} - 32w^{14} + 55w^{13} - 77w^{12} + 89w^{11} - 70w^{10} - 23w^9 + 237w^8 - 522w^7 + 702w^6 - 645w^5 + 417w^4 - 189w^3 + 58w^2 - 11w + 1$
19	0.000020914109	$w^{17} - 4w^{16} + 10w^{15} - 16w^{14} + 21w^{13} - 23w^{12} + 11w^{11} + 60w^{10} - 253w^9 + 569w^8 - 871w^7 + 962w^6 - 780w^5 + 464w^4 - 199w^3 + 59w^2 - 11w + 1$
20	0.000168295233	$w^{20} - 6w^{19} + 23w^{18} - 67w^{17} + 169w^{16} - 387w^{15} + 832w^{14} - 1668w^{13} + 3051w^{12} - 4971w^{11} + 7095w^{10} - 8738w^9 + 9111w^8 - 7867w^7 + 5503w^6 - 3052w^5 + 1310w^4 - 421w^3 + 96w^2 - 14w + 1$
21	0.000170895189	$2w^{22} - 8w^{21} + 17w^{20} - 23w^{19} + 29w^{18} - 55w^{17} + 134w^{16} - 240w^{15} + 235w^{14} - 162w^{13} + 911w^{12} - 4057w^{11} + 9940w^{10} - 16043w^9 + 18608w^8 - 16177w^7 + 10746w^6 - 5478w^5 + 2124w^4 - 611w^3 + 124w^2 - 16w + 1$
22	0.000331485602	$w^{22} - 3w^{21} + 5w^{20} - 5w^{19} + 9w^{18} - 23w^{17} + 56w^{16} - 182w^{15} + 742w^{14} - 2563w^{13} + 6761w^{12} - 13674w^{11} + 21508w^{10} - 26588w^9 + 26002w^8 - 20172w^7 + 12392w^6 - 5982w^5 + 2233w^4 - 626w^3 + 125w^2 - 16w + 1$
23	0.000011094132	$2w^{22} - 10w^{21} + 32w^{20} - 81w^{19} + 192w^{18} - 434w^{17} + 904w^{16} - 1694w^{15} + 2952w^{14} - 5123w^{13} + 9063w^{12} - 15256w^{11} + 22324w^{10} - 26895w^9 + 26082w^8 - 20185w^7 + 12393w^6 - 5982w^5 + 2233w^4 - 626w^3 + 125w^2 - 16w + 1$
24	0.000032649587	$w^{22} - 5w^{21} + 15w^{20} - 28w^{19} + 35w^{18} - 42w^{17} + 108w^{16} - 294w^{15} + 616w^{14} - 1364w^{13} + 3632w^{12} - 8719w^{11} + 16032w^{10} - 22143w^9 + 23302w^8 - 18946w^7 + 11984w^6 - 5887w^5 + 2219w^4 - 625w^3 + 125w^2 - 16w + 1$

**Table 2:** Polynomials  $Q_j$  and their coefficients  $c_j$  involved in Theorem 2.

$j$	$c_j$	$Q_j$
1	0.318213692889	$x$
2	0.560989550740	$x - 1$
3	0.146712879739	$x - 2$
4	0.085581998806	$x^2 - 3x + 1$
5	0.009989741209	$2x^2 - 4x + 1$
6	0.027466055396	$x^3 - 5x^2 + 6x - 1$
7	0.001627355914	$x^3 - 6x^2 + 9x - 3$
8	0.019745453086	$x^4 - 7x^3 + 13x^2 - 7x + 1$
9	0.002696297374	$x^4 - 7x^3 + 14x^2 - 8x + 1$
10	0.006409808197	$x^5 - 9x^4 + 26x^3 - 29x^2 + 11x - 1$
11	0.000734349441	$x^5 - 10x^4 + 32x^3 - 41x^2 + 20x - 3$
12	0.002606265352	$x^5 - 10x^4 + 29x^3 - 32x^2 + 12x - 1$
13	0.001597428179	$x^6 - 8x^5 + 21x^4 - 19x^3 + 5x - 1$
14	0.002071094499	$x^6 - 12x^5 + 44x^4 - 67x^3 + 44x^2 - 12x + 1$
15	0.004552695030	$x^7 - 12x^6 + 54x^5 - 117x^4 + 127x^3 - 65x^2 + 14x - 1$
16	0.000684338722	$x^7 - 12x^6 + 56x^5 - 125x^4 + 136x^3 - 67x^2 + 14x - 1$
17	0.000498131006	$x^7 - 13x^6 + 63x^5 - 143x^4 + 157x^3 - 78x^2 + 16x - 1$
18	0.004439944492	$x^8 - 15x^7 + 89x^6 - 269x^5 + 445x^4 - 402x^3 + 187x^2 - 40x + 3$
19	0.000101829613	$x^8 - 15x^7 + 84x^6 - 228x^5 + 323x^4 - 240x^3 + 91x^2 - 16x + 1$
20	0.000392957443	$x^9 - 17x^8 + 111x^7 - 368x^6 + 680x^5 - 717x^4 + 424x^3 - 134x^2 + 20x - 1$
21	0.000038759169	$x^9 - 16x^8 + 101x^7 - 327x^6 + 592x^5 - 612x^4 + 356x^3 - 113x^2 + 18x - 1$
22	0.000917704297	$x^9 - 17x^8 + 112x^7 - 374x^6 + 692x^5 - 725x^4 + 422x^3 - 130x^2 + 19x - 1$
23	0.000643449147	$x^9 - 16x^8 + 101x^7 - 330x^6 + 610x^5 - 650x^4 + 390x^3 - 125x^2 + 19x - 1$
24	0.002297151596	$x^9 - 16x^8 + 101x^7 - 328x^6 + 598x^5 - 625x^4 + 368x^3 - 117x^2 + 18x - 1$
25	0.000490756544	$x^9 - 17x^8 + 112x^7 - 376x^6 + 704x^5 - 750x^4 + 444x^3 - 138x^2 + 20x - 1$
26	0.000245201839	$x^{10} - 19x^9 + 149x^8 - 632x^7 + 1592x^6 - 2458x^5 + 2317x^4 - 1290x^3 + 397x^2 - 59x + 3$
27	0.000304626485	$x^{11} - 20x^{10} + 166x^9 - 751x^8 + 2041x^7 - 3459x^6 + 3676x^5 - 2410x^4 + 942x^3 - 207x^2 + 23x - 1$
28	0.000489242607	$x^{12} - 22x^{11} + 204x^{10} - 1050x^9 + 3322x^8 - 6752x^7 + 8944x^6 - 7677x^5 + 4177x^4 - 1388x^3 + 265x^2 - 26x + 1$
29	0.000207437845	$x^{12} - 23x^{11} + 221x^{10} - 1170x^9 + 3783x^8 - 7815x^7 + 10471x^6 - 9046x^5 + 4921x^4 - 1617x^3 + 300x^2 - 28x + 1$
30	0.000345260406	$x^{12} - 22x^{11} + 203x^{10} - 1037x^9 + 3253x^8 - 6557x^7 + 8624x^6 - 7366x^5 + 4003x^4 - 1337x^3 + 259x^2 - 26x + 1$
31	0.001407389680	$x^{12} - 22x^{11} + 203x^{10} - 1038x^9 + 3263x^8 - 6598x^7 + 8713x^6 - 7476x^5 + 4080x^4 - 1365x^3 + 263x^2 - 26x + 1$
32	0.001628850329	$x^{12} - 22x^{11} + 204x^{10} - 1048x^9 + 3304x^8 - 6687x^7 + 8823x^6 - 7553x^5 + 4108x^4 - 1369x^3 + 263x^2 - 26x + 1$
33	0.000264084319	$x^{12} - 21x^{11} + 187x^{10} - 934x^9 + 2902x^8 - 5860x^7 + 7790x^6 - 6764x^5 + 3744x^4 - 1270x^3 + 248x^2 - 25x + 1$
34	0.000964834732	$x^{13} - 24x^{12} + 245x^{11} - 1403x^{10} + 5008x^9 - 11706x^8 + 18314x^7 - 19270x^6 + 13531x^5 - 6218x^4 + 1804x^3 - 310x^2 + 28x - 1$
35	0.000025335981	$x^{16} - 30x^{15} + 395x^{14} - 3024x^{13} + 15030x^{12} - 51263x^{11} + 123608x^{10} - 213858x^9 + 266720x^8 - 239024x^7 + 152390x^6 - 67977x^5 + 20692x^4 - 4139x^3 + 512x^2 - 35x + 1$
36	0.000079042303	$x^{16} - 31x^{15} + 420x^{14} - 3293x^{13} + 16677x^{12} - 57645x^{11} + 140053x^{10} - 242641x^9 + 301038x^8 - 266579x^7 + 166930x^6 - 72829x^5 + 21653x^4 - 4239x^3 + 516x^2 - 35x + 1$

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