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Expected utility maximization with stochastically ordered returns

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Abstract

Expected utility is an influential theory to study rational choice among risky assets. For each investment, an economic agent expects to receive a random payoff and therefore maximizes its expected utility. To the best of our knowledge, there exists no general procedure to take the derivative of the expected utility as a function of the investment without heavy assumptions on the underlying processes. This article considers expected utility maximization when payoffs are modeled by a family of random variables increasing with investment for the convolution order such as Poisson, Gamma or Exponential distributions. For several common utility functions, with the help of fractional calculus, we manage to obtain closed-form formulas for the expected utility derivative. The paper also provides two economic applications: production of competitive firms and investment in prevention.

Keywords: Convolution order, expected utility, Fractional calculus, prevention.

1 Introduction

Many economic studies consider the following optimization problem: Given an economic environment, an agent tries to maximize his expected utility, by choosing an optimal value for a control parameter. In order to determine his optimal choice, he needs to appropriately take into the consequences of his actions.

This problem is common in economics and has been extensively studied in different contexts, such as firms under uncertainty (Sandmo, 1971, Klemperer and Meyer, 1989), prevention (Ehrlich and Becker, 1972, Lee, 1998, Courbage, 2001), field choice studies (Altonji (1993)) or workers training (Loewenstein and Spletzer (1998)). However, most of these models involve that the agent exactly knows the consequences of his investment.

However, due to many externalities, it is usually impossible to predict exactly the effects of a decision. One can only estimate its consequences with an error margin. For example, a company can invest on training in order to increase production. However, even if this investment will effectively increase workers' efficiency, the final production will also vary due to external events, such as device failures or occupational diseases.

Most of these models can be unified into a general framework. We consider an agent admitting a utility function u for which decisions are based on a control parameter λ . This parameter can potentially model prevention investment, training expenses, etc. This parameter chosen by the

agent has a deterministic effect $C(\lambda)$ which can be considered as the cost of λ . Most of the time, $C(\lambda)$ will take the form $\tilde{C}(\lambda) - W$, with W the initial wealth such that $C(\lambda) < 0$ and \tilde{C} a real function of λ . Due to exogenous phenomena, the agent also considers a random variable family $(X_\lambda)_{\lambda>0}$ which is controlled by the parameter λ . The agent is then concerned by evaluating

$$u(X_\lambda - C(\lambda)). \quad (1)$$

The way the agent evaluates the quantity (1) has been the topic of many theories. One of the simplest models is the expected utility of Morgenstern and Von Neumann (1953)¹. In this model, the agent is searching for

$$\lambda^* = \operatorname{argmax}_\lambda \mathbb{E}[u(X_\lambda - C(\lambda))]. \quad (2)$$

However, Problem (2) is very complex to solve in general. If you assume that X_λ admits a density function $f_{X_\lambda}(\lambda, \cdot)$ and that both C and f can be differentiated with respect to λ , this problem amounts to find λ^* solving

$$\int_0^\infty u(x - C(\lambda^*)) \frac{\partial f_{X_\lambda}(\lambda^*, x)}{\partial \lambda} dx = \int_0^\infty u'(x - C(\lambda^*)) C'(\lambda^*) f_{X_\lambda}(\lambda^*, x) dx. \quad (3)$$

Finding an explicit solution for (3) is impossible without more assumptions. Moreover, Equation (3) is difficult to interpret since the term $\frac{\partial f_{X_\lambda}(\lambda, \cdot)}{\partial \lambda}$ is difficult to construe.

In order to deal with this problem, most authors specify exactly how X_λ depends on λ (see Sections 4.1 and 4.2 for examples) even though resulting models can lack of generality and flexibility. In a recent work, Bensalem et al. (2020) proposed a general model in which variations of X_λ are controlled, only assuming that $\lambda_1 < \lambda_2$ implies X_{λ_1} is dominated in the first stochastic order by X_{λ_2} . However, the authors worked with the dual theory of Yaari (1987), thus only considering the utility function $u(x) = x$.

In this paper, we study more complicated utility functions that involve sophisticated mathematical tools. As shown in Section 3, we use fractional calculus to derive the expected utility derivative in case of power utility functions. Among the various possible definitions of fractional calculus (see Yang (2019) for a survey), we consider the Weyl (or Liouville-Weyl) approach that has been popularized by Cressie and Borkent (1986) to study the moment of any order of a random variable.

In this paper, the family of random variables X_λ is supposed to be ordered for the convolution order. The main result of this paper consists in using this assumption to obtain two new expressions for the derivative of $\mathbb{E}[u(X_\lambda - C(\lambda))]$ with respect to λ , in case where u is an entire function or u is a power utility function. Two economic examples are provided in order to show how these formulas can be applied to concrete economic problems, and discuss the related economic intuition. Moreover, we provide several new mathematical results which establish a close link between the convolution order and cumulants.

¹For more details, we refer to Quiggin (2012)

The article is organized as follows. In Section 2, we introduce the convolution order and its properties. In Section 3.1, we provide some preliminary results, including how the cumulants are naturally linked with the convolution order. In Section 3.2, we derive two formulas for the derivative of the expected utility. In Section 4.1, formulas are applied to the problem of industrial firm trying to maximise their profit (competitive firm under uncertainty context as stated by Sandmo (1971)). In Section 4.2, results are applied to the prevention problem, where an agent has to choose between investing in an insurance contract or in prevention (Ehrlich and Becker, 1972).

2 The convolution order

In this section we present the main results of the literature on the convolution order.

Let X and Y be two random variables. Their cumulating distribution functions are denoted by F_X and F_Y , and their Laplace transform are denoted by L_X and L_Y (for all $s > 0$, we have $L_X(s) = \mathbb{E}(e^{-sX})$). Since we deal with random variables, the expression “ Y is smaller than X ” can have different meanings. Hence, many partial orders have been proposed to allow comparison between two random variables.

The standard stochastic order is called *the first stochastic order*. The random variable Y is said smaller than X in the first stochastic order (denoted by $Y \leq_{st} X$) if and only if for all $x \in \mathbb{R}$, $F_X(x) \leq F_Y(x)$.

Perhaps the best way to perceive this order is to consider the economical point of view : $Y \leq_{st} X$ if and only if for all increasing function u , $\mathbb{E}(u(Y)) \leq \mathbb{E}(u(X))$. Thus, in the expected utility theory, any rational agent will prefer the random variable X to Y , whatever his risk aversion (Denuit et al., 2006).

In a deterministic context, another intuitive way to perceive a quantity y as smaller than x consists of considering the difference between these two quantities : if $x - y$ is positive, then y is smaller than x . In other words, there exists a non-negative z such as $x = y + z$.

In a stochastic context, a new order is needed in order to capture this type of differences : the stochastic convolution order.

Definition 1 (Convolution order). A random variable Y is said smaller than X in the convolution order (denoted by $Y \leq_{conv} X$) if there exists a non-negative random variable Z , independent of Y , such that

$$X =_{\mathcal{L}} Y + Z, \tag{4}$$

with $=_{\mathcal{L}}$ meaning that for all $x \in \mathbb{R}$, $F_X(x) = F_{Y+Z}(x)$.

This order has first been introduced by Shaked and Suarez-Llorens (2003). It means that X is so much preferable to Y that you have to add a positive payoff to Y if you want an agent to choose Y over X . However, even if this order seems intuitive, it has not extensively been studied in the literature. Examples of articles involving the convolution order are Zhang (2018) and Castaño-Martínez et al. (2013)

Several common distributions can be ordered with the stochastic convolution order:

Lemma 2. *The following properties hold:*

- *If $k \in \mathbb{R}$, then $X \leq_{conv} X + k$.*
- *If X is exponentially distributed with parameter λ_1 and Y is exponentially distributed with parameter $\lambda_2 \geq \lambda_1$, then $Y \leq_{conv} X$.*
- *If X follows a gamma distribution with parameters α_1, β and Y follows a gamma distribution with parameters $\alpha_2 \leq \alpha_1, \beta$, then $Y \leq_{conv} X$.*
- *If X follows a Poisson distribution of parameter λ_1 and Y follows a Poisson distribution of parameter $\lambda_2 \geq \lambda_1$, then $Y \geq_{conv} X$.*

Proof.

Since these results have already been proved (e.g. Bowers et al. (1984)), we will only show how to find the random variable Z for the exponential case, which will be of interest later. Let $x \in \mathbb{R}^+$. We have

$$\mathbb{P}(X \leq x) = 1 - e^{-\lambda_1 x} = \mathbb{P}(Y + Z \leq x), \quad (5)$$

which leads to

$$1 - e^{-\lambda_1 x} = \int_0^x \mathbb{P}(Z < x - t) \lambda_2 e^{-\lambda_2 t} dt. \quad (6)$$

From a variable change follows

$$1 - e^{-\lambda_1 x} = \int_0^x \mathbb{P}(Z < z) \lambda_2 e^{-\lambda_2(x-z)} dz. \quad (7)$$

Deriving (7) with respect to x gives

$$1 - \frac{\lambda_2 - \lambda_1}{\lambda_2} e^{-\lambda_1 x} = \mathbb{P}(Z < x). \quad (8)$$

Thus,

$$Z = \begin{cases} 0 & \text{with probability } 1 - \frac{\lambda_2 - \lambda_1}{\lambda_2}, \\ K & \text{with probability } \frac{\lambda_2 - \lambda_1}{\lambda_2}, \end{cases}$$

with K a random variable exponentially distributed with parameter λ_1 . ■

The convolution order also presents several useful properties, of which two are in the next proposition. The proofs can be found in Shaked and Suarez-Llorens (2003)

Proposition 3. *The following properties hold:*

1. *If X and Y are nonnegative random variables, $Y \leq_{conv} X$ if and only if $\frac{L_X(s)}{L_Y(s)}$ is a completely monotone function in s .*
2. *$Y \leq_{conv} X \Rightarrow Y \leq_{st} X$.*

3 Optimizing the expected utility

As mentioned in Section 1, it is necessary to state how the variation of the control parameter λ affects the resulting random variable $X_{\lambda+h}$ in order to solve Problem 2. In this perspective, the convolution order appears as an appropriate tool. Indeed, for a variation of the control parameter, it provides the existence of an intermediate positive random variable which captures the effect on the random variable of interest.

Let us denote by $(X_\lambda)_{\lambda \in \mathbb{R}_+}$ a family of random variables. We assume that this family is increasing² for the convolution order, i.e. for all $\lambda, h \in \mathbb{R}_+$, we have $X_\lambda \leq_{conv} X_{\lambda+h}$. The parameter λ represents the control parameter. For example, increasing the investment λ will increase the expected return on investment.

From these assumptions, it follows that for all $\lambda, h \in \mathbb{R}_+$, there exists $Z_{\lambda, \lambda+h}$ such that $X_{\lambda+h} =_{\mathcal{L}} X_\lambda + Z_{\lambda, \lambda+h}$. As said before, $Z_{\lambda, \lambda+h}$ captures the differences between $X_{\lambda+h}$ and X_λ . Thus, we will first study the random variable $Z_{\lambda, \lambda+h}$, and in particular, the behaviour of $Z_{\lambda, \lambda+h}$ when h is getting small ($h \rightarrow 0$) in order to obtain results on the derivative of the expected utility.

3.1 Some preliminary results

Thanks to the convolution order, for all $\lambda, h \in \mathbb{R}_+$, the random variable $X_{\lambda+h}$ can be decomposed as the sum of two independent variables, X_λ and $Z_{\lambda, \lambda+h}$. Thus, it is natural to consider risk measures verifying $\rho(X + Y) = \rho(X) + \rho(Y)$ for two independent X and Y . A well-known family of risk measures verifying this property are the cumulants. Section 3.2 shows that cumulants are the quantities capturing well the variation of "randomness" of random variables $(X_\lambda)_{\lambda > 0}$.

Definition 4 (Cumulant-generating function). The cumulant-generating function of a random variable X is the function defined by

$$g_X(t) = \log(\mathbb{E}(e^{Xt})) = \sum_{n=1}^{\infty} \kappa_n(X) \frac{t^n}{n!}.$$

We call the n -th cumulant of X the risk measure $\kappa_n(X) = g_X^{(n)}(0)$, with $g_X^{(n)}$ the n -th derivative of the function g_X .

Cumulants possess several interesting properties. We present three of them without demonstration (the first one follows from the definition, the second one can be found in Smith (1995), the third one is Theorem 12.1 in Gut (2013)). The reader could refer to Gut (2013) for more details.

Proposition 5. *Let $n \in \mathbb{N}^*$. The cumulants satisfy the following properties:*

1. *If X and Y are independent random variables, $\kappa_n(X + Y) = \kappa_n(X) + \kappa_n(Y)$.*

2. *$\kappa_n(X) = m_n(X) - \sum_{i=1}^{n-1} \binom{n-1}{i-1} \kappa_i(X) m_{n-i}(X)$, with $m_n(X) = \mathbb{E}(X^n)$ the n -th moment of X .*

²The case of a decreasing family is also of interest to model an agent trying to diminish his risk. This case is very similar to the increasing case, and is thus only developed in Appendix B and in Section 4.2.

3. $m_n(X)$ and $\kappa_n(X)$ exist if and only if $\sum_{m=1}^{\infty} m^{n-1}P(X \geq m) < \infty$.

From the second and third proposition, we notice that a necessary and sufficient condition for the n -th cumulant to exist is that the n -th moment exists. We will thus make the following assumption for the remainder of the paper.

Assumption 6. There exists $\Lambda > 0$ such that

$$\Lambda = \min(\sup \left\{ \lambda \in \mathbb{R}_+^* \text{ such that for all } n \in \mathbb{N}, t > 0, \mathbb{E}(X_\lambda^n e^{tX_\lambda}) < \infty \right\}, \lambda_{max}) \quad (9)$$

The quantity $\lambda_{max} \in \mathbb{R}_+^*$ represents the maximum investment, since nobody can invest an infinite amount of money. For example, λ_{max} could be the smaller λ verifying $C(\lambda) = 0$. Assumption 6 is a slightly stronger assumption than assuming that for all $n \in \mathbb{N}$, $|\mathbb{E}(X_\lambda^n)| < \infty$, but this will happen to be useful for proving Proposition 12.

Cumulants are of interest for our problem since the derivative of the n -th cumulant of X_λ is linked with the limit of the n -th cumulant of $Z_{\lambda, \lambda+h}$, as stated in the next proposition.

Proposition 7. *If (X_λ) is a family of random variables increasing for the convolution order, for all $\lambda < \Lambda$ and for all $n \in \mathbb{N}^*$ we have*

$$\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = \lim_{h \rightarrow 0} \frac{\kappa_n(Z_{\lambda, \lambda+h})}{h}. \quad (10)$$

Proof.

Let $\lambda < \Lambda$. From the first cumulant property follows, for all $h > 0$,

$$\kappa_n(X_{\lambda+h}) = \kappa_n(X_\lambda + Z_{\lambda, \lambda+h}) = \kappa_n(X_\lambda) + \kappa_n(Z_{\lambda, \lambda+h}). \quad (11)$$

Thus,

$$\lim_{h \rightarrow 0} \frac{\kappa_n(X_{\lambda+h}) - \kappa_n(X_\lambda)}{h} = \lim_{h \rightarrow 0} \frac{\kappa_n(Z_{\lambda, \lambda+h})}{h}, \quad (12)$$

Which ends the proof by definition of the derivative. ■

More surprisingly, the next result shows that the derivative of the n -th cumulant of X_λ is linked with the limit of the n -th moment of $Z_{\lambda, \lambda+h}$.

Proposition 8. *If (X_λ) is an increasing family for the convolution order, for all $\lambda < \Lambda$ and for all $n \in \mathbb{N}^*$, we have*

$$\lim_{h \rightarrow 0} \mathbb{E}(Z_{\lambda, \lambda+h}^n) = 0. \quad (13)$$

Moreover,

$$\lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^n)}{h} = \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda}. \quad (14)$$

Proof.

Let $\lambda < \Lambda$. Using Proposition 7, we just have to show that

$$\lim_{h \rightarrow 0} \frac{\kappa_n(Z_{\lambda, \lambda+h})}{h} = \lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^n)}{h}. \quad (15)$$

We proceed by induction. By definition of (X_λ) , for all $h > 0$ we have

$$\mathbb{E}(X_{\lambda+h}) - \mathbb{E}(X_\lambda) = \mathbb{E}(Z_{\lambda, \lambda+h}), \quad (16)$$

which proves both results for $n = 1$. Let $N > 1 \in \mathbb{N}^*$. We assume that the result is true for all $n < N \in \mathbb{N}^*$.

Using the point 2) of Proposition 5, we have

$$\kappa_N(Z_{\lambda, \lambda+h}) = m_N(Z_{\lambda, \lambda+h}) - \sum_{i=1}^{N-1} \binom{N-1}{i-1} \kappa_i(Z_{\lambda, \lambda+h}) m_{N-i}(Z_{\lambda, \lambda+h}). \quad (17)$$

From the induction hypothesis it follows that

$$\lim_{h \rightarrow 0} \kappa_N(Z_{\lambda, \lambda+h}) = \lim_{h \rightarrow 0} m_N(Z_{\lambda, \lambda+h}). \quad (18)$$

Yet,

$$\lim_{h \rightarrow 0} \kappa_N(Z_{\lambda, \lambda+h}) = \lim_{h \rightarrow 0} \kappa_N(X_{\lambda+h}) - \kappa_N(X_\lambda) = 0. \quad (19)$$

This shows that $\lim_{h \rightarrow 0} m_N(Z_{\lambda, \lambda+h}) = 0$. Moreover, from equation (17) follows, for all $h > 0$,

$$\frac{\kappa_N(Z_{\lambda, \lambda+h})}{h} = \frac{m_N(Z_{\lambda, \lambda+h})}{h} - \sum_{i=1}^{N-1} \binom{N-1}{i-1} \frac{\kappa_i(Z_{\lambda, \lambda+h})}{h} m_{N-i}(Z_{\lambda, \lambda+h}). \quad (20)$$

Proposition 7 proves that for all $i < N$, $\frac{\kappa_i(Z_{\lambda, \lambda+h})}{h} \xrightarrow{h \rightarrow 0} \frac{\partial \kappa_i(X_\lambda)}{\partial \lambda} \in \mathbb{R}$.

Moreover, by induction hypothesis, $m_{N-i}(Z_{\lambda, \lambda+h}) \xrightarrow{h \rightarrow 0} 0$ for all $i < N$.

Finally,

$$\lim_{h \rightarrow 0} \frac{\kappa_N(Z_{\lambda, \lambda+h})}{h} = \lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^N)}{h}. \quad (21)$$

■

Proposition 7 and Proposition 8 show that variations of X_λ are well captured by the variable $Z_{\lambda, \lambda+h}$. Thus, studying $Z_{\lambda, \lambda+h}$ is a way to study X_λ . These Propositions are well illustrated by the exponential case.

Example 9. Let $(X_\lambda)_{\lambda>0}$ be the family of exponentially distributed random variables, let $n \in \mathbb{N}^*$. For all $\lambda \in [0, \Lambda[$, $h \in]0, \Lambda - \lambda[$ the random variable $Z_{\lambda, \lambda+h}$ is given by

$$Z_{\lambda, \lambda+h} = \begin{cases} 0 & \text{with probability } 1 - \frac{h}{\lambda+h}, \\ K & \text{with probability } \frac{h}{\lambda+h}, \end{cases} \quad (22)$$

With K a random variable exponentially distributed with parameter λ . It is easy to see that the probability of $Z_{\lambda, \lambda+h}$ being positive approaches 0 when h decreased.

From (22), it follows that

$$\mathbb{E}(Z_{\lambda, \lambda+h}^n) = \frac{h}{\lambda+h} \mathbb{E}(K^n). \quad (23)$$

Thus, $\lim_{h \rightarrow 0} \mathbb{E}(Z_{\lambda, \lambda+h}^n) = 0$. Moreover, since $\mathbb{E}(K^n) = \frac{n!}{\lambda^n}$, we finally have

$$\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = -\frac{n!}{\lambda^{n+1}}. \quad (24)$$

since X_λ is decreasing for the convolution order (see Proposition 16 in Appendix B).

We end this section with several corollaries of Proposition 8. The first one shows that, when a family of random variables (X_λ) is increasing in the sense of the convolution order, then its "randomness" can only increase with the parameter λ .

Corollary 10. *Let $(X_\lambda)_\lambda$ be a family of random variables increasing in the sense of the convolution order. Then :*

1. $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \geq 0$ for all $n \in \mathbb{N}^*$.
2. If for all $\lambda \in [0, \Lambda[$, X_λ takes his value on \mathbb{N} then for all $n \in \mathbb{N}^*$, $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \leq \frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda}$
3. If $\frac{\partial \kappa_1(X_\lambda)}{\partial \lambda} < \frac{\partial \kappa_2(X_\lambda)}{\partial \lambda}$, then for all $n \in \mathbb{N}^*$, $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} < \frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda}$. Moreover, $\lim_{n \rightarrow \infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = \infty$.

Proof. The point 1) is a direct consequence of Proposition 8 since $Z_{\lambda, \lambda+h}$ is positive by definition.

Regarding point 2), if for all $\lambda \in [0, \Lambda[$, X_λ takes its value on \mathbb{N} , then necessarily for all $\lambda \in [0, \Lambda[$, $h \in]0, \Lambda - \lambda[$, $Z_{\lambda, \lambda+h}$ takes its value on \mathbb{N} . Thus, for all $n \in \mathbb{N}^*$, for all $\lambda \in [0, \Lambda[$, $h \in]0, \Lambda - \lambda[$,

$$\mathbb{E}(Z_{\lambda, \lambda+h}^n) = \sum_{k=1}^{\infty} k^n \mathbb{P}(Z_{\lambda, \lambda+h} = k) \leq \sum_{k=1}^{\infty} k^{n+1} \mathbb{P}(Z_{\lambda, \lambda+h} = k) = \mathbb{E}(Z_{\lambda, \lambda+h}^{n+1}) \quad (25)$$

Proposition 8 finishes the demonstration.

As for point 3, $\frac{\partial \kappa_1(X)}{\partial \lambda} < \frac{\partial \kappa_2(X)}{\partial \lambda}$ and Proposition 8 imply that there exists $\varepsilon > 0$ such that for all $h \in [0, \varepsilon]$, $\mathbb{E}(Z_{\lambda, \lambda+h}) \leq \mathbb{E}(Z_{\lambda, \lambda+h}^2)$.

Newton binomial formula shows that since for all $n \in \mathbb{N}^*$, $\lambda \in [0, \Lambda[$, $\mathbb{E}(X_\lambda^n) < \infty$, then for all $h \in [0, \Lambda - \lambda[$, $\mathbb{E}(Z_{\lambda, \lambda+h}^n) < \infty$.

Moreover, $\mathbb{E}(\ln(Z_{\lambda,\lambda+h})^2 Z_{\lambda,\lambda+h}^n) = \int_0^1 \ln(x)^2 x^n d\mathbb{P}(Z_{\lambda,\lambda+h}) + \int_1^\infty \ln(x)^2 x^n d\mathbb{P}(Z_{\lambda,\lambda+h})$ with

$$\lim_{x \rightarrow 0} \ln(x)^2 x^n = 0.$$

Since for all $x > 1$, $\ln(x)^2 < x^2$, we have

$$\int_1^\infty \ln(x)^2 x^n d\mathbb{P}(Z_{\lambda,\lambda+h}) < \int_1^\infty x^{n+2} d\mathbb{P}(Z_{\lambda,\lambda+h}) \leq \mathbb{E}(Z_\lambda^{n+2}) < \infty.$$

Finally, both terms converge and thus $\mathbb{E}(\ln(Z_{\lambda,\lambda+h})^2 Z_{\lambda,\lambda+h}^n) < \infty$.

This shows that the function

$$\begin{aligned} u &: \mathbb{R}_+ \rightarrow \mathbb{R} \\ \alpha &\mapsto \mathbb{E}(Z_{\lambda,\lambda+h}^\alpha) \end{aligned}$$

can be differentiated twice and its second derivative $u''(\alpha) = \mathbb{E}(\ln(Z_{\lambda,\lambda+h})^2 Z_{\lambda,\lambda+h}^\alpha)$ is nonnegative since $Z_{\lambda,\lambda+h}$ is nonnegative. Thus, the function u is convex. Because

$$\mathbb{E}(Z_{\lambda,\lambda+h}) < \mathbb{E}(Z_{\lambda,\lambda+h}^2), \tag{26}$$

Rolle's theorem shows that there exists $c \in [1, 2[$ such that $\forall \alpha > c, u'(\alpha) > 0$. This and Proposition 8 show that for all $n \in \mathbb{N}^*$, $\frac{\partial \kappa_n(X)}{\partial \lambda} < \frac{\partial \kappa_{n+1}(X)}{\partial \lambda}$ and since $\forall \alpha > c, u''(\alpha) > 0$, $\lim_{n \rightarrow \infty} \frac{\partial \kappa_n(X)}{\partial \lambda} = \infty$. ■

3.2 Deriving the expected utility

In this section, we consider a rational agent with a Von Neumann and Morgenstern utility function u . He has access to an investment λ . This investment results to a profit X_λ . The family of random variables $(X_\lambda)_{\lambda \geq 0}$ is supposed increasing for the convolution order. The agent also considers a deterministic function $C(\lambda)$, which models all the deterministic parameter, such as the cost of investment, or the initial wealth. Finally, the agent tries to solve the following optimization problem :

$$\max_{\lambda \geq 0} \mathbb{E}[u(X_\lambda - C(\lambda))] \tag{27}$$

In a first step, we suppose that u is entire³ (i.e the function $u(x)$ is equal to its Taylor series for all $x \in \mathbb{R}$). This covers for example the cases of exponential and polynomial utility functions. We also suppose that the function C is differentiable (i.e. the function C admits a derivative at any point). The following result gives a formula for the derivative of $\mathbb{E}[u(X_\lambda - C(\lambda))]$.

Proposition 11. *If u is entire, C is differentiable, and $(X_\lambda)_{\lambda \geq 0}$ is an increasing random variables family for the convolution order, then, for all $\lambda \in [0, \Lambda[$, $\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda}$ exists and*

³Commonly, the notion of entire function is used for complex functions, but this poses no problem since it is always possible to consider complex functions restricted to the real set. Notice that it is stronger than assuming functions are analytic.

$$\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda} = -C'(\lambda) \mathbb{E}(u'(X_\lambda - C(\lambda))) + \sum_{n=1}^{\infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(X_\lambda - C(\lambda)))}{n!}$$

Proof.

Let $\lambda \in [0, \Lambda[$. Since (X_λ) is increasing for the convolution order, there exists $h > 0$ such that

$$X_{\lambda+h} - C(\lambda + h) =_{\mathcal{L}} X_\lambda + Z_{\lambda, \lambda+h} - C(\lambda) - (C(\lambda + h) - C(\lambda)). \quad (28)$$

Composing equation (28) by u , and using its Taylor development gives

$$u(X_{\lambda+h} - C(\lambda + h)) =_{\mathcal{L}} u(X_\lambda - C(\lambda)) + \sum_{n=1}^{\infty} \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n \quad (29)$$

Taking the expectation in (29) and dividing by h gives

$$\frac{\mathbb{E}(u(X_{\lambda+h} - C(\lambda + h))) - \mathbb{E}(u(X_\lambda - C(\lambda)))}{h} = \mathbb{E} \left[\sum_{n=1}^{\infty} \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!h} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n \right]. \quad (30)$$

According to Bourbaki (2007), Corollary 2 p. 144, in order to swap the expectation and the sum in Equation (30), a sufficient condition is to show that $\sum_{n=1}^{\infty} \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!h} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n$ is convergent, and that there exists a function g such that for all $N \in \mathbb{N}$, we have

$$\left| \sum_{n=1}^N \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!h} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n \right| \leq g.$$

The first condition comes directly from the fact that u is an entire function. Regarding the second point, we know that

$$\left| \sum_{n=1}^{\infty} \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n \right| = |u(X_{\lambda+h} - C(\lambda + h)) - u(X_\lambda - C(\lambda))| \quad (31)$$

Let us first choose $\varepsilon > 0$. Since $\sum_{n=1}^N \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n$ converges, there exists $\tilde{N} \in \mathbb{N}$ such that for all $N > \tilde{N}$,

$$\sum_{n=1}^N \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n \leq |u(X_{\lambda+h} - C(\lambda + h)) - u(X_\lambda - C(\lambda)) + \varepsilon|. \quad (32)$$

Let us define the random variable

$$g = \max(g_1, g_2),$$

with

$$g_1 = |u(X_{\lambda+h} - C(\lambda + h)) - u(X_\lambda - C(\lambda)) + \varepsilon|$$

$$g_2 = \max_{N \in \{1, \dots, \tilde{N}\}} \sum_{n=1}^N \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda + h))^n$$

By definition of g , for all $N \in \mathbb{N}$, $\sum_{n=1}^N \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!} (Z_{\lambda, \lambda+h} + C(\lambda) - C(\lambda+h))^n \leq g$. Moreover, g is the maximum of a finite number of integrable random variables, so g is integrable. Therefore, we can swap the sum and the expectation in Equation (30).

Using Newton's binomial theorem and taking the limits in (30) leads to

$$\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda} = \lim_{h \rightarrow 0} \sum_{n=1}^{\infty} \frac{\mathbb{E}[u^{(n)}(X_\lambda - C(\lambda))]}{n!} A_{h,n}(\lambda), \quad (33)$$

with

$$A_{h,n}(\lambda) = \sum_{k=0}^n \binom{n}{k} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^k)}{h} (C(\lambda) - C(\lambda+h))^{n-k}. \quad (34)$$

We know from Proposition 8 that for all $k > 0$, $\lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^k)}{h} = \frac{\partial \kappa_k(X_\lambda)}{\partial \lambda}$.

Thus,

$$\lim_{h \rightarrow 0} \sum_{k=1}^n \binom{n}{k} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^k)}{h} (C(\lambda) - C(\lambda+h))^{n-k} = \lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^n)}{h} \quad (35)$$

Moreover, when $k = 0$, using L'Hôpital's rule, we have

$$\lim_{h \rightarrow 0} \frac{(C(\lambda) - C(\lambda+h))^n}{h} = \lim_{h \rightarrow 0} -n (C(\lambda) - C(\lambda+h))^{n-1} C'(\lambda+h), \quad (36)$$

which is equal to $-C'(\lambda)$ if $n = 1$, 0 otherwise. Combining the equations (35) and (36) give

$$\lim_{h \rightarrow 0} A_{h,n}(\lambda) = \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} - C'(\lambda) 1_{n=1}. \quad (37)$$

Incorporating equation (37) into equation (33) gives

$$\frac{\partial \mathbb{E}[u(X_\lambda - C(\lambda))]}{\partial \lambda} = -C'(\lambda) \mathbb{E}[u'(X_\lambda - C(\lambda))] + \sum_{n=1}^{\infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}[u^{(n)}(X_\lambda - C(\lambda))]}{n!}. \quad (38)$$

■

This result shows that the first derivative of the expectation depends on every cumulants of X_λ .

Even if some common utility function are concerned by the precedent result, some others are not, such as CRRA functions. Nevertheless, the following proposition shows that it is still possible to obtain a formula when the utility function is a power function.

Proposition 12. *If C is differentiable, $(X_\lambda)_{\lambda \geq 0}$ is an increasing random variables family for the convolution order, and $\sum_{n=k}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$ exists then, for all $\lambda \in [0, \Lambda[$, for all $\alpha = k - \delta$, $k \in \mathbb{N}^*$, $\delta \in]0, 1]$,*

$$\begin{aligned}
\frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{\partial \lambda} &= -C'(\lambda)\alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) \\
&+ \sum_{n=1}^k \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \binom{\alpha}{n} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \\
&+ \sum_{n=k+1}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}
\end{aligned} \tag{39}$$

Proof.

The development in power series of $(a+b)^\alpha$ is only available if $a < b$ or $b < a$. Since without additional assumptions, we cannot guarantee that there exists a certain $h > 0$ such that $Z_{\lambda, \lambda+h} < X_\lambda$ (see the Example 2 for a counter example), the methodology used for Proposition 11 cannot be used here.

For $\lambda \in [0, \Lambda[$, quantity $m_\alpha(X_\lambda) = \mathbb{E}((X_\lambda - C(\lambda))^\alpha)$ is the moment of order α of the random variable $(X_\lambda - C(\lambda))$. These non-integer moments have been studied by Cressie and Borkent (1986), and the proof of Proposition 12 heavily relies on their approach.

This approach is based on the moment generating function of X_λ , given by $\mathcal{M}_{X_\lambda}(t) = \mathbb{E}(e^{tX_\lambda})$. It is well-known that, for all $n \in \mathbb{N}^*$, $\frac{d^n \mathcal{M}_{X_\lambda}}{dt^n}(0) = m_n(X_\lambda)$. The idea of Cressie and Borkent is to introduce fractional calculus, in order to generalize this result to $\alpha \in \mathbb{R}^*$. To do so, they needed to use a derivative operator D such that, for all $c \in \mathbb{R}_+^*$, $\frac{D^\alpha e^{ct}}{Dt^\alpha} = c^\alpha e^{ct}$. To obtain such property, the authors worked with the framework of Weyl fractional calculus (also called Liouville-Weyl fractional calculus).

Let f be a function $\mathcal{C}^\infty(\mathbb{R})$ such that for all $p \in [1; \infty[$, all $t < \infty$, $\int_{-\infty}^t |f(x)|^p < \infty$. Following Kilbas et al. (1993)-p.94, we can then define the Weyl integral of order μ , $\mu > 0$, as

$$\frac{D^{-\mu}}{D t^{-\mu}} f(t) \equiv \Gamma(\mu)^{-1} \int_{-\infty}^t (t-z)^{\mu-1} f(z) dz. \tag{40}$$

Here, Γ denote the Gamma function, i.e.

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt. \tag{41}$$

We also define the Weyl derivative of order $\alpha = k - \delta$, with $k \in \mathbb{N}_*$ and $\delta \in]0, 1]$, as

$$\frac{D^\alpha f(t)}{D t^\alpha} \equiv \Gamma(\delta)^{-1} \int_{-\infty}^t (t-z)^{\delta-1} (d^k f(z)/dz^k) dz. \tag{42}$$

When $\alpha = k$, we retrieve the usual derivative. In the following, we are interested in computing the derivative

$$\frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{\partial \lambda},$$

for a real $\alpha > 0$.

The proof of Proposition 12 requires to compute $\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0)$ using Weyl fractional derivatives. This result is a particular case of a result given in Raina (1986). However, both the proof and the initial formula provided by Raina (1986) are quite complex. The following lemma gives a simpler proof for our specific context.

Lemma 13. *Let $n, k \in \mathbb{N}_*, \delta \in]0, 1]$ $\alpha = k - \delta$, $c \in \mathbb{R}_+$. Then, if $k < n$,*

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = \begin{cases} (-1)^{n-k} c^{\alpha-n} \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)} & \text{if } 0 < \delta < 1, \\ 0 & \text{if } \delta = 1. \end{cases} \quad (43)$$

If $k \geq n$,

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = c^{\alpha-n} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}. \quad (44)$$

Proof.

By definition,

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = \frac{1}{\Gamma(\delta)} \int_{-\infty}^0 (-z)^{\delta-1} \frac{D^k z^n e^{cz}}{Dz^k} dz. \quad (45)$$

Leibniz formula gives

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = \frac{1}{\Gamma(\delta)} \sum_{j=0}^{\min(n,k)} \binom{k}{j} \int_{-\infty}^0 (-z)^{\delta-1} \frac{n!}{n-j!} z^{n-j} c^{k-j} e^{cz} dz. \quad (46)$$

Using the variable change $u = cz$ in the integral gives

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = \frac{1}{\Gamma(\delta)} \sum_{j=0}^{\min(n,k)} \binom{k}{j} c^{k-j} \frac{n!}{n-j!} (-1)^{n-j} \int_{-\infty}^0 \left(\frac{-u}{c}\right)^{n-j+\delta-1} e^u \frac{du}{c}. \quad (47)$$

Definition of the Γ function gives

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = c^{k-n-\delta} \frac{1}{\Gamma(\delta)} \sum_{j=0}^{\min(n,k)} \binom{k}{j} \frac{n!}{n-j!} (-1)^{n-j} \Gamma(n-j+\delta). \quad (48)$$

Let suppose that $n \leq k$. Remarking that $\frac{\Gamma(n-j+\delta)}{\Gamma(\delta)(n-j)!} = \binom{n-j+\delta-1}{n-j} = (-1)^{n-j} \binom{-\delta}{n-j}$, we have

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = c^{k-n-\delta} n! \sum_{j=0}^n \binom{k}{j} \binom{-\delta}{n-j}. \quad (49)$$

Applying Chu-Vandermonde identity shows that

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = c^{k-n-\delta} n! \binom{k-\delta}{n}, \quad (50)$$

which finally leads to

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = c^{k-n-\delta} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}. \quad (51)$$

Similarly, if $k \leq n$, we can rewrite Equation (48) as

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = (-1)^{n-k} c^{k-n-\delta} \frac{\Gamma(n-k+\delta)\Gamma(k+1)}{\Gamma(\delta)} \sum_{j=0}^k \binom{n}{j} \binom{k-n-\delta}{k-j}. \quad (52)$$

Using Chu-Vandermonde identity⁴ gives

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = (-1)^{n-k} c^{k-n-\delta} \frac{\Gamma(n-k+\delta)\Gamma(k+1)}{\Gamma(\delta)} \binom{k-\delta}{k}. \quad (53)$$

From the last expression, we can see that for $\delta = 1$, the derivative is zero since

$$\binom{k-1}{k} = 0, \quad (54)$$

by definition of the binomial coefficient. In case $0 < \delta < 1$, we find that

$$\frac{D^\alpha t^n e^{ct}}{Dt^\alpha}(0) = (-1)^{n-k} c^{k-n-\delta} \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)}. \quad (55)$$

■

It is now possible to demonstrate Proposition 12 We have :

$$\frac{D\mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{D\lambda} = \frac{D}{D\lambda} \frac{D^\alpha \mathcal{M}_{(X_\lambda - C(\lambda))}}{D t^\alpha}(\lambda, 0) \quad (56)$$

It is possible to swap the derivatives operators since :

- There exists $l > 0$ such that for all $k \in \mathbb{N}^*$, $0 < t < l$, $0 \leq \lambda \leq \Lambda$ the integral

$$\int_{-\infty}^t (t-z)^{\delta-1} \mathbb{E}((X_\lambda - C(\lambda))^k e^{z(X_\lambda - C(\lambda))}) dz$$

is convergent. This is a natural consequence of the convergence of Weyl derivative, which is proven by Proposition 5 of Cressie and Borkent (1986) when Assumption 6 holds.

- For all $k \in \mathbb{N}^*$, $t > 0$, $\mathbb{E}((X_\lambda - C(\lambda))^k e^{t(X_\lambda - C(\lambda))})$ can be differentiated with respect to λ since it is an entire function (Proposition 11).

⁴We recall the Chu-Vandermonde identity : if $s, t \in \mathbb{R}$, $n \in \mathbb{N}$, then

$$\binom{s+t}{n} = \sum_{k=0}^n \binom{s}{k} \binom{t}{n-k}$$

- There exists $l > 0$ and a function $g > 0$ such that for all $k \in \mathbb{N}^*$, $z > 0$, $y \in]0, l[$, $\lambda \in]0, \Lambda[$,

$$\frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^k e^{z(X_\lambda - C(\lambda))})}{\partial \lambda} (t - z)^\delta < g(z).$$

Indeed, since $[0, \Lambda]$ is a compact and that $\frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^k e^{z(X_\lambda - C(\lambda))})}{\partial \lambda} (t - z)^\delta$ is finite for all $\lambda \in [0, \Lambda]$, it suffices to choose

$$g(z) = \max_{\lambda \in [0, \Lambda]} \frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^k e^{z(X_\lambda - C(\lambda))})}{\partial \lambda} (t - z)^\delta.$$

Thus,

$$\frac{D\mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{D\lambda} = \frac{D^\alpha}{D} \frac{D\mathbb{E}(e^{t(X_\lambda - C(\lambda))})}{Dt^\alpha} (\lambda, 0). \quad (57)$$

Since the function $x \mapsto e^{tx}$ is an entire function, and the family (X_λ) is ordered in the sense of the convolution order, it is possible to apply Proposition 11 to compute $\frac{D\mathbb{E}(e^{t(X_\lambda - C(\lambda))})}{D\lambda}$. We have :

$$\frac{D\mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{D\lambda} = \frac{D^\alpha}{Dt^\alpha} \left[-C'(\lambda) \mathbb{E}(te^{t(X_\lambda - C(\lambda))}) + \sum_{n=1}^{\infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(t^n e^{t(X_\lambda - C(\lambda))})}{n!} \right] (\lambda, 0) \quad (58)$$

Fubini theorem ensures that the expectancy and the Weyl derivative operator can commute. Moreover, assuming that $\sum_{n=k}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$ exists allows to distribute the Weyl derivative across the series thanks to Fubini theorem and Lemma 13. Then

$$\frac{D\mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{D\lambda} = \left[-C'(\lambda) \mathbb{E}\left(\frac{D^\alpha}{Dt^\alpha}(te^{t(X_\lambda - C(\lambda))})\right) + \sum_{n=1}^{\infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}\left(\frac{D^\alpha}{Dt^\alpha}(t^n e^{t(X_\lambda - C(\lambda))})\right)}{n!} \right] (\lambda, 0) \quad (59)$$

Since $k \geq 1$, using Lemma 13 in Equation (59) gives

$$\begin{aligned} \frac{D\mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{D\lambda} &= -C'(\lambda) \alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) \\ &+ \sum_{n=1}^k \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \binom{\alpha}{n} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \\ &+ \sum_{n=k+1}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!} \end{aligned} \quad (60)$$

■

Remark. An example of sufficient condition for

$$\sum_{n=k}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$$

to exist is when the sequence $(\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda})$ is non increasing, and $C(\lambda) < -1$.

An interesting case is when (X_λ) is the family of Poisson distributions of parameter λ . It is well known that the family of Poisson distributions is an increasing random variables family for the convolution order (in this case, $Z_{\lambda, \lambda+h}$ follows a Poisson distribution of parameter h), and that for all $n \in \mathbb{N}^*$, $\kappa_n(X_\lambda) = \lambda$ (e.g. Patil (1963)).

Proposition 11 and 12 can be thus be rewritten :

Corollary 14. *If u is entire, C is differentiable, and $(X_\lambda)_{\lambda \geq 0}$ is the family of Poisson distribution of parameter λ , then, for all $\lambda < \Lambda$,*

$$\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda} = -C'(\lambda) \mathbb{E}(u'(X_\lambda - C(\lambda))) + \mathbb{E}(u(X_\lambda - C(\lambda) + 1)) - \mathbb{E}(u(X_\lambda - C(\lambda))). \quad (61)$$

Moreover, if $C(\lambda) \leq -1$, for all $\alpha = k - \delta$, $k \geq 1$, $\delta \in]0, 1[$,

$$\begin{aligned} \frac{\partial \mathbb{E}(X_\lambda - C(\lambda))^\alpha}{\partial \lambda} &= -C'(\lambda) \alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) \\ &+ \sum_{n=1}^{k-1} \binom{\alpha}{n} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \\ &+ \mathbb{E}((X_\lambda - C(\lambda) + 1)^\alpha) \end{aligned} \quad (62)$$

Proof. Since (X_λ) is the Poisson distribution family, for all $n > 0$, $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = 1$.

Thus, Equation 61 follows directly from Proposition 11 and the fact that u is an entire function, because $u(X_\lambda - C(\lambda) + 1) = u(X_\lambda - C(\lambda)) + \sum_{n=1}^{\infty} \frac{u^{(n)}(X_\lambda - C(\lambda))}{n!}$.

Moreover, for all $x \in]-1, 1[$,

$$(1+x)^{-\delta} = \sum_{n=0}^{+\infty} (-\delta)(-\delta-1)\dots(-\delta-n+1) \frac{x^n}{n!} = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma(n+\delta)}{\Gamma(\delta)} \frac{x^n}{n!} \quad (63)$$

Integrating Equation (63) k times gives for all $x \in]-1, 1[$,

$$(1+x)^\alpha \frac{\Gamma(1-\delta)}{\Gamma(\alpha+1)} = \sum_{n=0}^{+\infty} (-1)^n \frac{\Gamma(n+\delta)}{\Gamma(\delta)} \frac{x^{n+k}}{(n+k)!} \quad (64)$$

Proposition 12 leads to

$$\begin{aligned} \frac{\partial \mathbb{E}(X_\lambda - C(\lambda))^\alpha}{\partial \lambda} &= -C'(\lambda) \alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) \\ &+ \sum_{n=1}^{k-1} \binom{\alpha}{n} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \\ &+ \sum_{n=k}^{\infty} (-1)^{n-k} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}, \end{aligned} \quad (65)$$

Moreover,

$$\sum_{n=k}^{\infty} (-1)^{n-k} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!} = \mathbb{E} \left((X_\lambda - C(\lambda))^\alpha \frac{\Gamma(\alpha+1)}{\Gamma(1-\delta)} \sum_{n=0}^{\infty} (-1)^n (X_\lambda - C(\lambda))^{-n-k} \frac{\Gamma(n+\delta)}{\Gamma(\delta)(n+k)!} \right). \quad (66)$$

The hypotheses $C(\lambda) < -1$ ensures that $\frac{1}{(X_\lambda - C(\lambda))} < 1$, thus, using Equation 64 gives

$$\sum_{n=k}^{\infty} (-1)^{n-k} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!} = \mathbb{E}((X_\lambda - C(\lambda) + 1)^\alpha), \quad (67)$$

which end the demonstration. ■

Notice that $C(\lambda) \leq -1$ has a realistic economic meaning if we consider an agent with initial wealth. Indeed, if one economic agent starts with an initial wealth or revenue of W , the function $C(\lambda)$ can be rewritten as $C(\lambda) = \tilde{C}(\lambda) - W$, with $\tilde{C}(\lambda)$ an increasing positive function. Thus, $C(\lambda) \leq -1$ is equivalent to $\tilde{C}(\lambda) \leq W - 1$. Thus, this assumption is nearly equivalent to assuming that the agent cannot borrow money to make his initial investment.

If the agent makes his choice according to a power utility function and is risk adverse (i.e. $\alpha < 1$), Equation (62) reduces to

$$\frac{\partial \mathbb{E}(X_\lambda - C(\lambda))^\alpha}{\partial \lambda} = -C'(\lambda)\alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) + \mathbb{E}((X_\lambda - C(\lambda) + 1)^\alpha) \quad (68)$$

This formula can be useful, especially since to the best of the authors knowledge there exists no closed formula for $\mathbb{E}(X_\lambda - C(\lambda))^\alpha$ in the case where X_λ follows a Poisson distribution.

4 Economic examples

In this section, we consider applications of the above formula for two classical economical problems. The first example in Section 4.1 studies the investment of a company that aims to determine the optimal quantity of goods to produce. We extend a result from Sandmo (1971) stating that in an uncertainty context, the company decides to produce less than in a certainty context. The second example in Section 4.2 studies the investment made by an agent in prevention actions when he has access to both prevention services and insurance contracts. We show that, contrary to the original model of Ehrlich and Becker (1972), but following the results of Briys et al. (1991), it is not possible to conclude whether prevention and insurance are substitutes.

4.1 First problem: Competitive firm under uncertainty

In this example, we consider a firm trying to maximize his expected profit, as presented in Sandmo (1971). This firm wants to produce a quantity λ of a good, with a unity selling price of p . The firm is subject to two type of costs : fix costs B (such as wages, rent,...), and operational costs $C(\lambda)$. C is supposed twice differentiable and increasing. The firm makes his choice according to a Von Neumann and Morgenstern utility function u .

In the classical deterministic model, the firm wants to maximize the quantity $u(p\lambda - C(\lambda) - B)$. Since

$$\frac{\partial u(p\lambda - C(\lambda) - B)}{\partial \lambda} = (p - C'(\lambda))u'(p\lambda - C(\lambda) - B), \quad (69)$$

the solution is the λ verifying $C'(\lambda) = p$ (if the second order condition is verified).

There are two ways to generalize this model:

- The first one is to suppose that the price P is a random variable as proposed by Sandmo (1971). However, in this model, the price P does not depend on λ . In our framework, it would be possible to generalize the model of Sandmo by considering an imperfect market model where the production of the firm impacts the price of the produced good in the market: the more the firm produces, the lower the price. This is not studied in this paper, since it would suppose that the random variable family is decreasing (in the sense of the convolution order), a case which will be considered in the next example.

- The second approach to generalize the classical model is to consider the amount of produced goods as uncertain. Indeed, many random factors could affect the production : strikes, absenteeism, machine failure, etc. Let the random variable family (X_λ) denote the total production. (X_λ) is supposed to be an increasing family in the sense of the convolution order. In this case (pX_λ) is also an increasing family of random variables for the convolution order. Notice that the random variable family $(X_\lambda) = \lambda$ also verify all our hypotheses : thus, the classical model is a particular case of this generalized model.

In this model, the firm tries to maximize

$$\mathbb{E}(u(pX_\lambda - C(\lambda) - B)). \quad (70)$$

If u is an entire function, Proposition 11 gives

$$\begin{aligned} \frac{\partial \mathbb{E}(u(pX_\lambda - C(\lambda) - B))}{\partial \lambda} &= -C'(\lambda) \mathbb{E}(u'(pX_\lambda - C(\lambda) - B)) \\ &+ \sum_{n=1}^{\infty} p^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(pX_\lambda - C(\lambda) - B))}{n!}, \end{aligned} \quad (71)$$

and thus

$$\begin{aligned} \frac{\partial \mathbb{E}(u(pX_\lambda - C(\lambda) - B))}{\partial \lambda} &= (p \frac{\partial \mathbb{E}(X_\lambda)}{\partial \lambda} - C'(\lambda)) \mathbb{E}(u'(pX_\lambda - C(\lambda) - B)) \\ &+ \sum_{n=2}^{\infty} p^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(pX_\lambda - C(\lambda) - B))}{n!}. \end{aligned} \quad (72)$$

Remark. Equation (72) shows that if X_λ can be written as $X_\lambda = X + f(\lambda)$, with X a random variable and f a deterministic real function, then the model boils down to the deterministic one.

In order to compare these results with the classical deterministic model, it is possible to impose $\mathbb{E}(X_\lambda) = \lambda$. This is a similar assumption than the one made by Sandmo, when he compared the case of a stochastic price of mean p to the classical model. This assumption is verified if, for example, $X_\lambda = \lambda$, if X_λ follows a Poisson distribution of parameter λ , if X_λ follows a Gaussian distribution of mean λ , or if X_λ follows a negative binomial distribution of parameters λ and $1/2$.

In this case, the derivative simplifies to

$$\begin{aligned} \frac{\partial \mathbb{E}(u(pX_\lambda - C(\lambda) - B))}{\partial \lambda} &= (p - C'(\lambda)) \mathbb{E}(u'(pX_\lambda - C(\lambda) - B)) \\ &+ \sum_{n=2}^{\infty} p^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(pX_\lambda - C(\lambda) - B))}{n!}. \end{aligned} \quad (73)$$

By comparing (69) and (73), we note that the general case (73) is similar to the one obtained in the classical case (69) but with the presence of the additional term:

$$\sum_{n=2}^{\infty} p^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(pX_\lambda - C(\lambda) - B))}{n!}.$$

This sum relies heavily on the cumulant derivatives $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda}$ which capture the randomness of the process X_λ .

Since u is entire, u is absolutely continuous, thus using the integral form of the remainder of the Taylor formula gives, for all $\lambda \in [0, \Lambda[$, for all $h \in [0, \Lambda - \lambda[$:

$$\frac{1}{h} \sum_{n=2}^{\infty} (pZ_{\lambda, \lambda+h})^n \frac{\mathbb{E}(u^{(n)}(pX_{\lambda} - C(\lambda) - B))}{n!} = \frac{1}{h} \int_{pX_{\lambda} - C(\lambda) - B}^{pX_{\lambda} - C(\lambda) - B + Z_{\lambda, \lambda+h}} \frac{u''(t)}{2!} (pX_{\lambda} - C(\lambda) - B + Z_{\lambda, \lambda+h} - t) dt. \quad (74)$$

If, as supposed by Sandmo, u is concave, then Equation 74 shows that, for all $\lambda, h > 0$:

$$\frac{1}{h} \sum_{n=2}^{\infty} (pZ_{\lambda, \lambda+h})^n \frac{\mathbb{E}(u^{(n)}(pX_{\lambda} - C(\lambda) - B))}{n!} \leq 0. \quad (75)$$

Taking the expectancy and the limits shows that

$$\sum_{n=2}^{\infty} p^n \frac{\partial \kappa_n(X_{\lambda})}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(pX_{\lambda} - C(\lambda) - B))}{n!} \leq 0. \quad (76)$$

Consequently, if $\frac{\partial \mathbb{E}(u(pX_{\lambda} - C(\lambda) - B))}{\partial \lambda}$ is decreasing (implying that $\frac{\partial^2 \mathbb{E}(u(pX_{\lambda} - C(\lambda) - B))}{\partial \lambda^2} \leq 0$), then the solution $\frac{\partial \mathbb{E}(u(pX_{\lambda} - C(\lambda) - B))}{\partial \lambda} = 0$ is smaller than the solution of $(p - C'(\lambda))\mathbb{E}(u'(pX_{\lambda} - C(\lambda) - B))$. Similarly to Sandmo (1971), which stated that “*under price uncertainty, the output is smaller than the certainty output*”, it is thus possible to state the following proposition :

Proposition 15. *If the random output is increasing for the convolution order, and if the utility function u is entire and concave, then, under production uncertainty, the output is smaller than the certainty output.*

4.2 Second problem : relationship between prevention investment and insurance coverage

In this example, we propose to work in a framework similar to Briys et al. (1991), which propose an extension of the model of Ehrlich and Becker (1972). Notice that Lee (1998) also proposed a model with a random prevention effect, but did not consider the interaction with insurance.

Briys et al. (1991) consider an economic agent making his decisions according to a utility function u and the expected utility theory. This agent possesses a wealth denoted by W . The authors considered that, with probability $p \in (0, 1)$, this agent will lose an amount $X \in \mathbb{R}_+$ (and with probability $1 - p$, he loses nothing). The agent has access to a proportional insurance, that is in exchange of a premium $(1 + \theta)\alpha p \mathbb{E}(X)$, he will be refunded of an amount αX by the insurer if a loss happens. Here, the parameter $0 < \alpha < 1$ is the proportion of the cover and the parameter $\theta > 0$ is the safety loading chosen by the insurer. They also consider that the agent can produce a prevention effort λ at the cost $C(\lambda)$. Prevention can produce two effects : it can either reduce the probability of occurrence of the claim (called “auto-protection”) or the amount of the loss (called “auto-insurance”).

In this example, we consider the auto-insurance framework. Briys et al. (1991) model the loss by $X = f(\lambda Z)$, with f a decreasing convex function and Z a random variable taking values in $[0, 1]$. Contrary to Briys et al., we do not explicit directly the dependence of the random variable: we

consider a random variable family (X_λ) decreasing for the convolution order.⁵ As in the original model, we consider that the insurer does not observe the prevention effort, and consequently that the insurer fixes his insurance premium by being cautious and considering the a priori risk, that is assuming there is no prevention effort with $\lambda = 0$.

Hence, let us denote by $W_X(\alpha, \lambda) = W - C(\lambda) - (1 + \theta)\alpha p \mathbb{E}(X_0) - (1 - \alpha)X_\lambda$ the wealth of the agent in the case a loss occurs, and by $W_{NX}(\alpha, \lambda) = W - C(\lambda) - (1 + \theta)\alpha p \mathbb{E}(X_0)$ the wealth otherwise.

The agent tries to solve the problem

$$\operatorname{argmax}_{\alpha, \lambda} p \mathbb{E}(u(W_X(\alpha, \lambda))) + (1 - p) \mathbb{E}(u(W_{NX}(\alpha, \lambda))). \quad (77)$$

First, we study the problem in the case where $X_\lambda = f(\lambda)$, with f a deterministic decreasing differentiable function, in order to compare the formula between the deterministic and the stochastic cases.

In this case, solutions (α^*, λ^*) of the problem (77) verify the two equations

$$- \frac{(1 - p)u'(W_{NX}(\alpha^*, \lambda^*))}{pu'(W_X(\alpha^*, \lambda^*))} = \frac{C'(\lambda^*) + (1 - \alpha^*)f'(\lambda^*)}{C'(\lambda^*)}, \quad (78)$$

and

$$- \frac{(1 - p)u'(W_{NX}(\alpha^*, \lambda^*))}{pu'(W_X(\alpha^*, \lambda^*))} = \frac{(1 + \theta)pf(0) - f(\lambda^*)}{(1 + \theta)pf(0)}. \quad (79)$$

In the stochastic case, when X_λ is a decreasing family for the convolution order, solutions (α^*, λ^*) of the problem (77) verify the two equations

$$- \frac{(1 - p)\mathbb{E}(u'(W_{NX}(\alpha^*, \lambda^*)))}{p\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))} = \frac{C'(\lambda^*) + \sum_{n=1}^{\infty} (1 - \alpha^*)^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(W_X(\alpha^*, \lambda^*)))}{\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))n!}}{C'(\lambda^*)}, \quad (80)$$

and

$$- \frac{(1 - p)\mathbb{E}(u'(W_{NX}(\alpha^*, \lambda^*)))}{p\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))} = \frac{(1 + \theta)pf(0) - \frac{\mathbb{E}(X_{\lambda^*} u'(W_X(\alpha^*, \lambda^*)))}{\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))}}{(1 + \theta)pf(0)}. \quad (81)$$

Only the terms directly depending of f change between Equations (78), (79) and (80), (81) :

$$\begin{aligned} (1 - \alpha^*)f' & \text{ is changed into } \sum_{n=1}^{\infty} (1 - \alpha^*)^n \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(W_X(\alpha^*, \lambda^*)))}{\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))n!} \\ f & \text{ is replaced by } \frac{\mathbb{E}(X_{\lambda^*} u'(W_X(\alpha^*, \lambda^*)))}{\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))}, \end{aligned}$$

which is linked to the covariance of X_{λ^*} and $u'(W_X(\alpha^*, \lambda^*))$.

⁵See Appendix B for more details on the decreasing case.

Similarly to Briys et al. (1991), and contrary to the model of Ehrlich and Becker (1972), it is not possible in this framework to prove that self insurance and insurance are substitutes.

Let's consider the much simpler model where $X_\lambda = X + f(\lambda)$, with X a positive random variable and f an analytic decreasing positive function. In that case, it is clear that Equation (80) amounts to Equation (78). However, this is not the case for Equation (81), which yields

$$-\frac{(1-p)\mathbb{E}(u'(W_{NX}(\alpha^*, \lambda^*)))}{p\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))} = \frac{(1+\theta)pf(0) - f(\lambda^*) - \frac{\mathbb{E}(Xu'(W_X(\alpha^*, \lambda^*)))}{\mathbb{E}(u'(W_X(\alpha^*, \lambda^*)))}}{(1+\theta)pf(0)}. \quad (82)$$

Contrary to the Example in the Section 4.1, the case $X_\lambda = X + f(\lambda)$ is thus not equivalent to the deterministic case anymore.

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A Proof of sufficient conditions for the existence of the expected utility derivative

Two simple sufficient conditions are given :

- The sequence $(\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda})$ is nonincreasing, and $C(\lambda) < -1$.
- The sequence $(\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda})$ is increasing, bounded, and $C(\lambda) < -1$.

Since $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} > 0$ (Corollary 10), the alternating series test shows that in the first case, the sequence $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$ is decreasing and tends to 0 at the limit.

In the second case, since $\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda}$ is increasing and bounded, it admits a positive limit K . Moreover, the sequence $(\mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!})$ is decreasing and its limit is null. Thus

$$\lim_{n \rightarrow \infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!} = 0. \quad (83)$$

Moreover, since $\frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))} > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \in]K - \frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))}, K]$. Since

$$\frac{\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda}}{(X_\lambda - C(\lambda))^{n+1-\alpha}} - \frac{\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda}}{(X_\lambda - C(\lambda))^{n-\alpha}} = \frac{\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda} - \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} (X_\lambda - C(\lambda))}{(X_\lambda - C(\lambda))^{n+1-\alpha}}, \quad (84)$$

the sequence $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$ is decreasing for all $n > N$ if

$$\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda} - \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} (X_\lambda - C(\lambda)) \leq 0 \quad (85)$$

for all $n > N$. Since, for all $n > N$,

$$\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda} - \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \leq \frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))}, \quad (86)$$

we have

$$\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda} - \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} (X_\lambda - C(\lambda)) \leq \frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))} - (X_\lambda - C(\lambda) - 1) \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda}. \quad (87)$$

Because $(X_\lambda - C(\lambda) - 1)$ is positive, and $\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} > K - \frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))}$, we have

$$\frac{\partial \kappa_{n+1}(X_\lambda)}{\partial \lambda} - \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} (X_\lambda - C(\lambda)) < \frac{K(X_\lambda - C(\lambda) - 1)}{(X_\lambda - C(\lambda))} (X_\lambda - C(\lambda)) - K(X_\lambda - C(\lambda) - 1) = 0. \quad (88)$$

The sequence is thus nonincreasing after $n > N$, and the sequence converges.

B Case of a decreasing family for the convolution order

Since the proofs are similar to the ones in the increasing case, only the main results are presented here.

Proposition 16. *If (X_λ) is a family of random variable decreasing for the convolution order, for all $\lambda < \Lambda$ and for all $n \in \mathbb{N}^*$, we have*

1.
$$\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = - \lim_{h \rightarrow 0} \frac{\kappa_n(Z_{\lambda, \lambda+h})}{h} \quad (89)$$

2.
$$\lim_{h \rightarrow 0} \mathbb{E}(Z_{\lambda, \lambda+h}^n) = 0 \quad (90)$$

3.
$$\frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} = - \lim_{h \rightarrow 0} \frac{\mathbb{E}(Z_{\lambda, \lambda+h}^n)}{h} \quad (91)$$

Proposition 17. *If u is entire, C is differentiable, and $(X_\lambda)_{\lambda \geq 0}$ is a decreasing random variables family for the convolution order, then, for all $\lambda < \Lambda$, $\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda}$ exists and*

$$\frac{\partial \mathbb{E}(u(X_\lambda - C(\lambda)))}{\partial \lambda} = -C'(\lambda) \mathbb{E}(u'(X_\lambda - C(\lambda))) + \sum_{n=1}^{\infty} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \frac{\mathbb{E}(u^{(n)}(X_\lambda - C(\lambda)))}{n!}.$$

Proposition 18. *If C is differentiable, $(X_\lambda)_{\lambda \geq 0}$ is an decreasing random variables family for the convolution order, and $\sum_{n=k}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!}$ exists then, for all λ , for all $\alpha = k - \delta > 0$, $k \in \mathbb{N}^*$, $\delta \in]0, 1]$,*

$$\begin{aligned} \frac{\partial \mathbb{E}((X_\lambda - C(\lambda))^\alpha)}{\partial \lambda} &= -C'(\lambda) \alpha \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-1}) \\ &+ \sum_{n=1}^k \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \binom{\alpha}{n} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \\ &+ \sum_{n=k+1}^{\infty} (-1)^{n-k} \frac{\partial \kappa_n(X_\lambda)}{\partial \lambda} \mathbb{E}((X_\lambda - C(\lambda))^{\alpha-n}) \frac{\Gamma(n-\alpha)\Gamma(\alpha+1)}{\Gamma(\delta)\Gamma(1-\delta)n!} \end{aligned} \quad (92)$$