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Master Bellman equation in the Wasserstein space: Uniqueness of viscosity solutions

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Abstract

We study the Bellman equation in the Wasserstein space arising in the study of mean field control problems, namely stochastic optimal control problems for McKean-Vlasov diffusion processes. Using the standard notion of viscosity solution à la Crandall-Lions extended to our Wasserstein setting, we prove a comparison result under general conditions, which coupled with the dynamic programming principle, implies that the value function is the unique viscosity solution of the Master Bellman equation. This is the first uniqueness result in such a second-order context. The classical arguments used in the standard cases of equations in finite-dimensional spaces or in infinite-dimensional separable Hilbert spaces do not extend to the present framework, due to the awkward nature of the underlying Wasserstein space. The adopted strategy is based on finite-dimensional approximations of the value function obtained in terms of the related cooperative n -player game, and on the construction of a smooth gauge-type function, built starting from a regularization of a sharpe estimate of the Wasserstein metric; such a gauge-type function is used to generate maxima/minima through a suitable extension of the Borwein-Preiss generalization of Ekeland's variational principle on the Wasserstein space.

Keywords: viscosity solutions, Bellman equation, Wasserstein space, comparison theorem, Ekeland's variational principle.

Mathematics Subject Classification (2020): 49L25, 35Q89, 35B51.

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1 Introduction

The main goal of this paper is to develop a viscosity theory for second-order partial differential equations on the Wasserstein space related to the so-called mean field (or McKean-Vlasov) control problems, namely stochastic optimal control problems for McKean-Vlasov diffusion processes. Such partial differential equations are also known as Master Bellman equations or Bellman equations in the Wasserstein space, see for instance [6, 14, 42]. The topic of mean field optimal control is a very recent area of research, on which there are however already many papers and the two monographs [5, 14], to which we refer for a thorough introduction. Mean field control problems are strictly related to mean field games, developed by Lasry and Lions in [31, 32, 33] (see also Lions' lectures at Collège de France [36]) and by Huang, Caines, Malhamé [28]. Both mean field control problems and mean field games can be interpreted as searches for equilibria of stochastic differential games with a continuum of players, symmetrically interacting each other through the empirical distribution of the entire population. These two problems differ because of the notion of equilibrium adopted. Mean field games arise when the concept of Nash's non-cooperative equilibrium is employed, while mean field control problems are related to Pareto optimality where players can be identified with a single "representative agent", see for instance [14, Section 6.2, pages 514-515]. In the latter case the stochastic differential game can be thought as an optimization problem of a central planner, who is looking for a common strategy in order to optimize some collective objective functional.

The state space of mean field control problems is the set of probability measures, and usually the Wasserstein space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures having finite second moment is adopted. Various notions of differentiability for maps defined on spaces of probability measures are available, and some of them are particularly relevant in the theory of optimal transportation, see [3, 41] for a detailed presentation of these geometric approaches. The Master Bellman equation (see equation (3.3) below) adopts instead the notion of differentiability introduced by Lions [36] (see also [12, 13, 14], and Section 3), whose nature is more functional analytic than geometric. Such a definition seems to be the natural choice in the study of second-order Bellman equations in the Wasserstein space and related stochastic optimal control problems. In fact, it gave rise to a stochastic calculus on the space of probability measures, and in particular to an Itô formula (chain rule) for maps defined on the Wasserstein space (we recall it in our Theorem 3.3), which allows to relate the value function of the control problem to the Bellman equation (we recall it in our Theorem 3.8). Regarding the relation between partial differential equations adopting the derivatives introduced by Lions (as in the present paper) and equations using notions of differentiability as those adopted in optimal transport theory, we mention results in this direction in the first-order case in [27] and in a second-order semi-linear case in [23] (see also Remark 3.6).

The theory of partial differential equations in the Wasserstein space is an emerging research topic, whose rigorous investigation is still at an early stage. There are already well-posedness results in the first-order case, see [2, 24, 1, 25, 26, 27], even for equations adopting different notions of derivative with respect to the measure. They however do not admit an extension to the second-order case, which is notoriously a different and more challenging

problem. Concerning second-order equations, papers [39, 40, 4, 16, 15] focus on the existence of viscosity solutions, proving that the value function solves in the viscosity sense the Master Bellman equation. All those articles adopt the notion of viscosity solution à la Crandall-Lions, properly adapted to the Wasserstein space, as we do in the present paper (see Definition 3.5). Notice that, even if these papers dealt with the uniqueness property, they established it only for the so-called lifted Bellman equation, which is formulated on the Hilbert space of corresponding random variables so that standard results apply. We also recall that the relation between such a lifted equation and the original Bellman equation in the Wasserstein space is not rigorously clarified, and in particular whether the lifted value function is a viscosity solution to the lifted equation. Actually, it is not yet clear under which conditions test functions in the lifting Hilbert space are related to test functions in the Wasserstein space, see discussion in Remark 3.6.

Uniqueness for second-order equations in the Wasserstein space is only addressed in the two papers [42] and [11]. In [42], a new notion of viscosity-type solution is adopted, which differs from the Crandall-Lions definition since the maximum/minimum condition is formulated on compact subsets of the Wasserstein space. This modification makes easier to prove uniqueness, which is completely established in some specific cases. On the other hand, [11] studies viscosity solutions à la Crandall-Lions for a class of integro-differential Bellman equations of particular type. More precisely, the coefficients of the McKean-Vlasov stochastic differential equations, as well as the coefficients of the reward functional, do not depend on the state process itself, but only on its probability distribution. This allows to consider only deterministic functions of time as control processes in the mean field control problem, so that the Master Bellman equation has a particular form. Moreover, in [11] the Master Bellman equation is formulated on the subset of the Wasserstein space of probability measures having finite exponential moments, equipped with the topology of weak convergence, which makes such a space σ -compact and allows establishing uniqueness in this context.

In the present paper we prove, under general conditions, existence and uniqueness of viscosity solutions for Master Bellman equations arising in the study of mean field optimal control problems. This is the first uniqueness result for such class of equations in the present general context. Classical arguments based on Ishii's lemma used in the standard cases of equations in finite-dimensional spaces or in infinite-dimensional separable Hilbert spaces seem hard to extend to the present framework, due to the awkward nature of the underlying Wasserstein space. The adopted strategy is instead based on refinements of early ideas from the theory of viscosity solutions [35] and relies on the existence of a candidate solution to the equation, which in our case is the value function v of the mean field control problem. In particular, we prove (see Theorem 5.1) that any viscosity subsolution u_1 (resp. supersolution u_2) is smaller (resp. greater) than the candidate solution v . In [35], the arguments for proving $u_1 \leq v$ (or, similarly, $v \leq u_2$) are as follows: one performs a smoothing v_n of v through its control representation, take a maximum of $u_1 - v_n$ (relying on the local compactness of the finite-dimensional space), and exploit the viscosity subsolution property of u_1 with v_n as test function. In [37] such a methodology is extended to the infinite-dimensional case, relying on Ekeland's variational principle in order to generate maxima/minima.

In the context of equations in the Wasserstein space, the above arguments require the following adjustments. Firstly, the smoothing of v is based on a propagation of chaos result [20], namely on a finite-dimensional approximation of the value function through value functions of non-degenerate cooperative n -player games. Secondly, in order to generate maxima/minima the idea is to perturb $u_1 - v_n$ (or $u_2 - v_n$) relying on a suitable extension of the Borwein-Preiss generalization of Ekeland's principle, see [9, Theorem 2.5.2]. According to the latter, $u_1 - v_n$ can be perturbed using a so-called gauge-type function (see Definition 4.1). For the proof of the comparison theorem, such a perturbation has to be smooth. In an infinite-dimensional Hilbert space setting, an example of smooth gauge-type function is the square of the norm. In the present context, the main issue is to construct a *smooth* gauge-type function. This is achieved in Section 4, starting from a sharp estimate of the square of the Wasserstein metric (see (4.4)) and performing a smoothing of such a quantity (see Lemma 4.4). Due to the complexity of the techniques employed, our results are formulated under boundedness assumptions on the coefficients. The extension to more general cases covering applications like the ones mentioned in [19, Introduction] seems possible and will be the object of future research.

The rest of the paper is organized as follows. In Section 2 we formulate the mean field optimal control problem and state the assumptions that are used throughout the paper; in such a section we also prove some properties of the value function v and state the dynamic programming principle. In Section 3 we recall the notion of differentiability introduced by Lions, we state the Itô formula, we introduce the Master Bellman equation, and we give the definition of viscosity solution. Section 4 is devoted to the construction of the smooth gauge-type function, from which we derive the smooth variational principle on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, namely Theorem 4.5. In Section 5 we prove the comparison theorem (Theorem 5.1), from which we deduce the uniqueness result (Corollary 5.2). Finally, in Appendix A we perform the smooth finite-dimensional approximation of the value function; in particular, in subsection A.1 we approximate the mean field control problem with non-degenerate control problems; then, in subsection A.2 we introduce the related cooperative n -player game and state the propagation of chaos result.

2 Mean field optimal control problem

Wasserstein spaces of probability measures. Given a Polish space (S, d_S) , we denote by $\mathcal{P}(S)$ the set of all probability measures on $(S, \mathcal{B}(S))$. We also define, for every $q \geq 1$,

$$\mathcal{P}_q(S) := \left\{ \mu \in \mathcal{P}(S) : \text{for some (and hence for all) } x_0 \in S, \int_S d_S(x_0, x)^q \mu(dx) < +\infty \right\}.$$

The set $\mathcal{P}_q(\mathbb{S})$ is endowed with the q -Wasserstein distance defined as

$$\mathcal{W}_q(\mu, \mu') := \inf \left\{ \int_{\mathbb{S} \times \mathbb{S}} d_{\mathbb{S}}(x, y)^q \pi(dx, dy) : \pi \in \mathcal{P}(\mathbb{S} \times \mathbb{S}) \right. \\ \left. \text{such that } \pi(\cdot \times \mathbb{S}) = \mu \text{ and } \pi(\mathbb{S} \times \cdot) = \mu' \right\}^{\frac{1}{q}}, \quad q \geq 1,$$

for every $\mu, \mu' \in \mathcal{P}_q(\mathbb{S})$. The space $(\mathcal{P}_q(\mathbb{S}), \mathcal{W}_q)$ is a Polish space, see for instance [41, Theorem 6.18].

Probabilistic setting and control processes. We fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a m -dimensional Brownian motion $B = (B_t)_{t \geq 0}$ is defined. We denote by $\mathbb{F}^B = (\mathcal{F}_t^B)_{t \geq 0}$ the \mathbb{P} -completion of the filtration generated by B , which is also right-continuous, so that it satisfies the usual conditions. We assume that there exists a sub- σ -algebra \mathcal{G} of \mathcal{F} satisfying the following properties.

- i) \mathcal{G} and \mathcal{F}_∞^B are independent.
- ii) \mathcal{G} is “rich enough”, namely $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi \text{ such that } \xi: \Omega \rightarrow \mathbb{R}^d, \text{ with } \xi \text{ being } \mathcal{G}\text{-measurable and } \mathbb{E}|\xi|^2 < \infty\}$. Recall from [15, Lemma 2.1] that such a requirement is equivalent to the existence of a \mathcal{G} -measurable random variable $U: \Omega \rightarrow \mathbb{R}$ having uniform distribution on $[0, 1]$.

We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the filtration defined as

$$\mathcal{F}_t = \mathcal{G} \vee \mathcal{F}_t^B, \quad t \geq 0.$$

We observe that \mathbb{F} satisfies the usual conditions of \mathbb{P} -completeness and right-continuity. Finally, we fix a finite time horizon $T > 0$ and a Polish space A . We then denote by \mathcal{A} the set of control processes, namely the family of all \mathbb{F} -progressively measurable processes $\alpha: [0, T] \times \Omega \rightarrow A$.

Assumptions and state equation. We consider the functions

$$b, \sigma, f: [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A \longrightarrow \mathbb{R}^d, \mathbb{R}^{d \times m}, \mathbb{R}, \quad g: \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R},$$

on which we impose the following assumptions.

Assumption (A).

- (i) *The functions b, σ, f, g are measurable.*
- (ii) *There exists a constant $K \geq 0$ such that*

$$|b(t, x, \mu, a) - b(t, x', \mu', a)| + |\sigma(t, x, \mu, a) - \sigma(t, x', \mu', a)| \leq K(|x - x'| + \mathcal{W}_2(\mu, \mu')), \\ |b(t, x, \mu, a)| + |\sigma(t, x, \mu, a)| + |f(t, x, \mu, a)| + |g(x, \mu)| \leq K,$$

for all $(t, a) \in [0, T] \times A$, $(x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, $|x - x'|$ denoting the Euclidean norm of $x - x'$ in \mathbb{R}^d , $|\sigma(t, x, \mu, a)| := (\text{tr}(\sigma\sigma^\top)(t, x, \mu, a))^{1/2} = (\sum_{i,j} |\sigma_{i,j}(t, x, \mu, a)|^2)^{1/2}$ denoting the Frobenius norm of the matrix $\sigma(t, x, \mu, a)$.

- (iii) The function f is locally uniformly continuous in (x, μ) uniformly with respect to (t, a) . Similarly, g is locally uniformly continuous. More precisely, it holds that: for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $(t, a) \in [0, T] \times A$, $(x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$,

$$\begin{aligned} |x - x'| + \mathcal{W}_2(\mu, \mu') \leq \delta \\ \implies |f(t, x, \mu, a) - f(t, x', \mu', a)| \leq \varepsilon \quad \text{and} \quad |g(x, \mu) - g(x', \mu')| \leq \varepsilon. \end{aligned}$$

Assumption (A'). There exists a constant $K \geq 0$ such that

$$|f(t, x, \mu, a) - f(t, x', \mu', a)| + |g(x, \mu) - g(x', \mu')| \leq K(|x - x'| + \mathcal{W}_2(\mu, \mu')),$$

for all $(t, a) \in [0, T] \times A$, $(x, \mu), (x', \mu') \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$.

Assumption (B). There exist constants $K \geq 0$ and $\gamma \in [0, 1]$ such that

$$\begin{aligned} |b(t, x, \mu, a) - b(s, x, \mu, a)| + |\sigma(t, x, \mu, a) - \sigma(s, x, \mu, a)| \\ + |f(t, x, \mu, a) - f(s, x, \mu, a)| \leq K|t - s|^\gamma, \end{aligned}$$

for all $t, s \in [0, T]$, $(x, \mu, a) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times A$.

Assumption (C). The Polish space A is compact.

Remark 2.1. Assumption (B) is required in the proof of Theorem A.5 in order to exploit regularity results for uniformly parabolic Bellman equations. On the other hand, Assumptions (A) and (C) are required in the propagation of chaos result, that is Theorem A.4. All these assumptions are therefore required in Theorem 5.1 and Corollary 5.2. Finally, notice that the results of the present section are stated under Assumption (A), however they hold under weaker assumptions, see [15].

For every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\alpha \in \mathcal{A}$, the state process evolves according to the following controlled McKean-Vlasov stochastic differential equation:

$$X_s = \xi + \int_t^s b(r, X_r, \mathbb{P}_{X_r}, \alpha_r) dr + \int_t^s \sigma(r, X_r, \mathbb{P}_{X_r}, \alpha_r) dB_r, \quad s \in [t, T]. \quad (2.1)$$

Proposition 2.2. Suppose that Assumption (A) holds. For every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\alpha \in \mathcal{A}$, there exists a unique (up to \mathbb{P} -indistinguishability) continuous \mathbb{F} -progressively measurable process $X^{t, \xi, \alpha} = (X_s^{t, \xi, \alpha})$ solution to equation (2.1) satisfying

$$\mathbb{E} \left[\sup_{s \in [t, T]} |X_s^{t, \xi, \alpha}|^2 \right]^{1/2} \leq C_1 \left(1 + \mathbb{E} [|\xi|^2]^{1/2} \right),$$

for some constant C_1 , independent of t, ξ, α .
Moreover, for every $\xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ it holds that

$$\mathbb{E} \left[\sup_{s \in [t, T]} |X_s^{t, \xi, \alpha} - X_s^{t, \xi', \alpha}|^2 \right]^{1/2} \leq C_2 \mathbb{E} [|\xi - \xi'|^2]^{1/2}, \quad (2.2)$$

for some constant C_2 , independent of t, ξ, ξ', α .

Proof. See [15, Proposition 2.8]. □

Reward functional and lifted value function. We consider the *reward functional* J , given by

$$J(t, \xi, \alpha) = \mathbb{E} \left[\int_t^T f(s, X_s^{t, \xi, \alpha}, \mathbb{P}_{X_s^{t, \xi, \alpha}}, \alpha_s) ds + g(X_T^{t, \xi, \alpha}, \mathbb{P}_{X_T^{t, \xi, \alpha}}) \right],$$

and the function V , to which we refer as the *lifted value function*, defined as

$$V(t, \xi) = \sup_{\alpha \in \mathcal{A}} J(t, \xi, \alpha), \quad \forall (t, \xi) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d).$$

Proposition 2.3. *Suppose that Assumption (A) holds. The function V satisfies the following properties.*

- 1) V is bounded.
- 2) V is jointly continuous, namely: for every $\{(t_n, \xi_n)\}_n, (t, \xi)$, with $t_n, t \in [0, T]$ and $\xi_n \in L^2(\Omega, \mathcal{F}_{t_n}, \mathbb{P}; \mathbb{R}^d), \xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, such that $|t_n - t| + \mathbb{E}|\xi_n - \xi|^2 \rightarrow 0$, it holds that $V(t_n, \xi_n) \rightarrow V(t, \xi)$.
- 3) The function V is uniformly continuous in ξ uniformly in $t \in [0, T]$: for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $t \in [0, T], \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$,

$$\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}} \leq \delta \implies |V(t, \xi) - V(t, \xi')| \leq \varepsilon. \quad (2.3)$$

- 4) If in addition (A') holds, there exists a constant $L \geq 0$ (depending only on T, K, C_2 in (2.2)) such that

$$|V(t, \xi) - V(t, \xi')| \leq L \mathbb{E}[|\xi - \xi'|^2]^{1/2}, \quad (2.4)$$

for all $t \in [0, T], \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$.

Proof. Item 1) is a direct consequence of the boundedness of f and g , while item 2) follows from [15, Proposition 3.3]. Concerning item 3), we begin noticing that

$$|V(t, \xi) - V(t, \xi')| \leq \sup_{\alpha \in \mathcal{A}} |J(t, \xi, \alpha) - J(t, \xi', \alpha)|. \quad (2.5)$$

We now observe that

$$\begin{aligned} |J(t, \xi, \alpha) - J(t, \xi', \alpha)| &\leq \int_t^T \mathbb{E} \left[|f(s, X_s^{t, \xi, \alpha}, \mathbb{P}_{X_s^{t, \xi, \alpha}}, \alpha_s) - f(s, X_s^{t, \xi', \alpha}, \mathbb{P}_{X_s^{t, \xi', \alpha}}, \alpha_s)| \right] ds \\ &\quad + \mathbb{E} \left[|g(X_T^{t, \xi, \alpha}, \mathbb{P}_{X_T^{t, \xi, \alpha}}) - g(X_T^{t, \xi', \alpha}, \mathbb{P}_{X_T^{t, \xi', \alpha}})| \right]. \end{aligned} \quad (2.6)$$

From **(A)**-(ii), (2.6), (2.2) and Markov inequality, there exists a constant $C > 0$ such that

$$\begin{aligned} |J(t, \xi, \alpha) - J(t, \xi', \alpha)| &\leq \\ &\int_t^T \mathbb{E} \left[|f(s, X_s^{t, \xi, \alpha}, \mathbb{P}_{X_s^{t, \xi, \alpha}}, \alpha_s) - f(s, X_s^{t, \xi', \alpha}, \mathbb{P}_{X_s^{t, \xi', \alpha}}, \alpha_s)| \mathbb{1}_{\sup_{s \in [t, T]} |X_s^{t, \xi, \alpha} - X_s^{t, \xi', \alpha}| \leq \delta} \right] ds \\ &\quad + \mathbb{E} \left[|g(X_T^{t, \xi, \alpha}, \mathbb{P}_{X_T^{t, \xi, \alpha}}) - g(X_T^{t, \xi', \alpha}, \mathbb{P}_{X_T^{t, \xi', \alpha}})| \mathbb{1}_{\sup_{s \in [t, T]} |X_s^{t, \xi, \alpha} - X_s^{t, \xi', \alpha}| \leq \delta} \right] + C \frac{\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}}}{\delta}, \end{aligned}$$

for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\alpha \in \mathcal{A}$ and $\delta > 0$. Fix now $\varepsilon > 0$. From **(A')**, there exists δ_1 such that

$$|J(t, \xi, \alpha) - J(t, \xi', \alpha)| \leq \varepsilon(T+1) + C \frac{\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}}}{\delta_1},$$

for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, $\alpha \in \mathcal{A}$, with $\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}} \leq \delta_1$. Using (2.5), we get by taking $\delta := \varepsilon \delta_1$

$$|V(t, \xi) - V(t, \xi')| \leq \varepsilon(T+1+C),$$

for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, such that $\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}} \leq \delta$.

Finally, concerning item 4), using the Lipschitz continuity of f and g , together with inequality

$$\mathcal{W}_2(X_s^{t, \xi, \alpha}, X_s^{t, \xi', \alpha}) \leq \mathbb{E}[|X_s^{t, \xi, \alpha} - X_s^{t, \xi', \alpha}|^2]^{1/2},$$

and (2.2), we deduce from (2.6) the Lipschitz continuity of J in ξ , uniformly with respect to t and α . Hence estimate (2.4) follows from (2.5). \square

Law invariance property and dynamic programming principle. We recall from [15] that V satisfies the fundamental *law invariance property*.

Theorem 2.4. *Suppose that Assumptions **(A)** (i)-(ii) and **(A')** hold. Then, the map V satisfies the law invariance property: for every $t \in [0, T]$ and $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with $\mathbb{P}_\xi = \mathbb{P}_{\xi'}$, it holds that*

$$V(t, \xi) = V(t, \xi').$$

Proof. See [15, Theorem 3.5]. \square

As a consequence of Theorem 2.4, if Assumptions **(A)** (i)-(ii) and **(A')** hold, we can define the *value function* $v: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ as

$$v(t, \mu) = V(t, \xi), \quad \forall (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \quad (2.7)$$

for any $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. By Proposition 2.5 we immediately deduce the following result.

Proposition 2.5. *Suppose that Assumption (A) holds. The function v satisfies the following properties.*

- 1) v is bounded.
- 2) v is jointly continuous on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.
- 3) The function v is uniformly continuous in ξ uniformly in $t \in [0, T]$: for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for every $t \in [0, T]$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that

$$\mathcal{W}_2(\mu, \mu') \leq \delta \implies |v(t, \mu) - v(t, \mu')| \leq \varepsilon.$$

- 4) If in addition (A') holds, for all $t \in [0, T]$, $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$|v(t, \mu) - v(t, \mu')| \leq L \mathcal{W}_2(\mu, \mu'),$$

with L as in (2.4).

Proof. The claim follows directly from Proposition 2.5, we only report the proof of items 3) and 4). Fix $\varepsilon > 0$ and $\delta_1 > 0$ such that

$$\mathbb{E}[|\xi - \xi'|^2]^{\frac{1}{2}} \leq \delta_1 \implies |V(t, \xi) - V(t, \xi')| \leq \varepsilon.$$

for all $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. Since \mathcal{F}_t contains the σ -algebra \mathcal{G} , we have

$$\mathcal{W}_2(\mu, \mu') = \inf \left\{ \mathbb{E}[|\xi - \xi'|^2]^{1/2} : \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \text{ with } \mathbb{P}_\xi = \mu \text{ and } \mathbb{P}_{\xi'} = \mu' \right\}.$$

We then get the result from (2.3) and (2.7) by taking $\delta = \frac{\varepsilon}{2}$.

Finally, suppose that in addition (A') holds. By (2.4) and (2.7), we have

$$|v(t, \mu) - v(t, \mu')| = |V(t, \xi) - V(t, \xi')| \leq L \mathbb{E}[|\xi - \xi'|^2]^{1/2},$$

for any $\xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$, with $\mathbb{P}_\xi = \mu$ and $\mathbb{P}_{\xi'} = \mu'$. Hence

$$\begin{aligned} |v(t, \mu) - v(t, \mu')| &\leq L \inf \left\{ \mathbb{E}[|\xi - \xi'|^2]^{1/2} : \xi, \xi' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d), \text{ with } \mathbb{P}_\xi = \mu \text{ and } \mathbb{P}_{\xi'} = \mu' \right\} \\ &= L \mathcal{W}_2(\mu, \mu'). \end{aligned}$$

□

Finally, we state the dynamic programming principle for v .

Theorem 2.6. *Suppose that Assumption (A) holds. Then, v satisfies the dynamic programming principle: for all $t, s \in [0, T]$, with $t \leq s$, $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, it holds that*

$$v(t, \mu) = \sup_{\alpha \in \mathcal{A}} \left\{ \mathbb{E} \left[\int_t^s f(r, X_r^{t, \xi, \alpha}, \mathbb{P}_{X_r^{t, \xi, \alpha}}, \alpha_r) dr \right] + v(s, \mathbb{P}_{X_s^{t, \xi, \alpha}}) \right\},$$

for any $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$ with $\mathbb{P}_\xi = \mu$.

Proof. See [15, Corollary 3.8].

□

3 Master Bellman equation

L -derivatives and Itô's formula along a flow of probability measures. We refer to [14, Section 5.2] for the definitions of the L -derivatives of first and second-order of a map $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ with respect to μ , which are given by $\partial_\mu u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $\partial_x \partial_\mu u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$. We recall that such definitions are based on the notion of *lifting* of a map $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, which is a map $U: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \rightarrow \mathbb{R}$ satisfying

$$U(t, \xi) = u(t, \mathbb{P}_\xi), \quad (3.1)$$

for every $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ (here, to alleviate notation, we have defined the lifting on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which the mean field control problem was defined; however, any other probability space supporting a random variable with uniform distribution on $[0, 1]$ can be used). We observe that derivatives in the present context can be defined in different ways: the so-called ‘‘flat’’ derivative or the intrinsic notion of differential in the Wasserstein space. We refer for instance to [14, Chapter 5] for a survey and some equivalence results.

Definition 3.1. $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ is the set of continuous functions $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ such that:

- 1) the lifting U of u admits a continuous Fréchet derivative $D_\xi U: [0, T] \times L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d) \rightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$, in which case there exists, for any $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, a measurable function $\partial_\mu u(t, \mu): \mathbb{R}^d \rightarrow \mathbb{R}^d$, such that $D_\xi U(t, \xi) = \partial_\mu u(t, \mu)(\xi)$, for any $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ with law μ .
- 2) The map $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_\mu u(t, \mu)(x) \in \mathbb{R}^d$ is jointly continuous;
- 3) $\partial_t u$ and $\partial_x \partial_\mu u$ exist and the maps $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_t u(t, \mu) \in \mathbb{R}$, $(t, x, \mu) \in [0, T] \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \mapsto \partial_x \partial_\mu u(t, \mu)(x) \in \mathbb{R}^{d \times d}$ are continuous.

Definition 3.2. $C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ is the subset of $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ of functions $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ satisfying, for some constant $C \geq 0$,

$$|\partial_\mu u(t, \mu)(x)| + |\partial_x \partial_\mu u(t, \mu)(x)| \leq C(1 + |x|^2),$$

for all $(t, \mu, x) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$.

Theorem 3.3 (Itô's formula). Let $u \in C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$, $t \in [0, T]$, $\xi \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d)$. Let also $\beta: [0, T] \times \Omega \rightarrow \mathbb{R}^d$ and $\vartheta: [0, T] \times \Omega \rightarrow \mathbb{R}^{d \times m}$ be bounded and \mathbb{F} -progressively measurable processes. Consider the d -dimensional Itô process

$$X_s = \xi + \int_t^s \beta_r dr + \int_t^s \vartheta_r dB_r, \quad \forall s \in [t, T].$$

Then, it holds that

$$\begin{aligned} u(s, \mathbb{P}_{X_s}) &= u(t, \mathbb{P}_\xi) + \int_t^s \partial_t u(r, \mathbb{P}_{X_r}) dr + \int_t^s \mathbb{E} \left[\langle \beta_r, \partial_\mu u(r, \mathbb{P}_{X_r})(X_r) \rangle \right] dr \\ &\quad + \frac{1}{2} \int_t^s \mathbb{E} \left[\text{tr} \left(\vartheta_r \vartheta_r^\top \partial_x \partial_\mu u(r, \mathbb{P}_{X_r})(X_r) \right) \right], \end{aligned}$$

for all $s \in [t, T]$.

Proof. The claim follows from [14, Proposition 5.102] (see also [15, Theorem 4.15 and Remark 4.16]). \square

Viscosity solutions. Now, consider the second-order partial differential equation on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$:

$$\begin{cases} \partial_t u(t, \mu) = F(t, \mu, u(t, \mu), \partial_\mu u(t, \mu)(\cdot), \partial_x \partial_\mu u(t, \mu)(\cdot)), & (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ u(T, \mu) = \int_{\mathbb{R}^d} g(x, \mu) \mu(dx), & \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases} \quad (3.2)$$

with $F: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R} \times L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d) \times L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^{d \times d}) \rightarrow \mathbb{R}$, where $\mathcal{B}(\mathbb{R}^d)$ are the Borel subsets of \mathbb{R}^d , and $L^2(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu; \mathbb{R}^d)$ is the set of $\mathcal{B}(\mathbb{R}^d)$ -measurable functions that are square-integrable with respect to μ .

Definition 3.4. A function $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a classical solution to equation (3.3) if $u \in C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ and satisfies (3.3).

Definition 3.5. A function $u: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ is a viscosity subsolution (resp. supersolution) to equation (3.2) if:

- $u(T, \mu) \leq$ (resp. \geq) $\int_{\mathbb{R}^d} g(x, \mu) \mu(dx)$, for every $\mu \in \mathcal{P}_2(\mathbb{R}^d)$;
- for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and any $\varphi \in C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ such that $u - \varphi$ has a maximum at (t, μ) (with value 0), then (3.2) is satisfied with the inequality \geq (resp. \leq) instead of the equality and with φ in place of u .

Finally, u is a viscosity solution of (3.2) if it is both a viscosity subsolution and a viscosity supersolution.

Remark 3.6. The above definition of viscosity solution is exactly in the spirit of the definition of Crandall and Lions for second-order equations (see for instance [17]) in finite dimension. In [17] it is proved that this definition is equivalent to the one using second-order semidifferentials (jets), while here such equivalence is not obvious.

We say that our definition is an “intrinsic” definition to distinguish it from the definition, adopted first in [39], which exploits the lifted equation (in the sense that a function is a viscosity solution if its lifting along (3.1) satisfies equation (3.2) with F substituted by its lifting \tilde{F}) and which, for this reason, we call “lifted” definition.

The relationship between these two definitions is not obvious. Indeed, as shown in Example 2.1 in [10], the lifted function of a smooth function on the Wasserstein space may not be smooth on the lifted Hilbert space, and so a viscosity solution in the intrinsic sense may not be a viscosity solution in the lifted sense. In the first-order case a kind of equivalence result between two related definitions is provided in [27, Theorem 4.4]. In a second-order semi-linear case some results in this direction are provided in [23, Section 5]. We are not aware of any results on the fully non-linear second-order case.

As we recalled in the introduction an intrinsic notion of viscosity solution is employed also in the papers [42] and [11]. The definition introduced in [42, Definition 4.4], differs from our definition since test functions must satisfy the maximum/minimum condition on suitable compact subsets of the Wasserstein space, denoted by \mathcal{P}_L . Using this modification the authors prove first a comparison result among regular sub/supersolutions (“partial comparison”) and then a general comparison result with the assumption that the supremum of classical subsolutions and the infimum of classical supersolutions coincide. On the other hand, [11] studies viscosity solutions à la Crandall-Lions for a class of integro-differential Bellman equations of particular type. More precisely, the coefficients of the McKean-Vlasov stochastic differential equations, as well as the coefficients of the reward functional, do not depend on the state process itself, but only on its probability distribution. This allows to consider only deterministic functions of time as control processes in the mean field control problem, so that the Master Bellman equation has a particular form. Moreover, in [11] the Master Bellman equation is formulated on the subset of the Wasserstein space of probability measures having finite exponential moments, equipped with the topology of weak convergence, which makes such a space σ -compact and allows establishing uniqueness in this context.

Now, we consider the Master Bellman equation, namely equation (3.2) with

$$F(t, \mu, r, p(\cdot), M(\cdot)) = - \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ f(t, x, \mu, a) + \langle b(t, x, \mu, a), p(x) \rangle + \frac{1}{2} \text{tr}[(\sigma \sigma^\top)(t, x, \mu, a) M(x)] \right\} \mu(dx).$$

Therefore, equation (3.2) becomes

$$\begin{cases} \partial_t u(t, \mu) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ f(t, x, \mu, a) + \langle b(t, x, \mu, a), \partial_\mu u(t, \mu)(x) \rangle + \frac{1}{2} \text{tr}[(\sigma \sigma^\top)(t, x, \mu, a) \partial_x \partial_\mu u(t, \mu)(x)] \right\} \mu(dx) = 0, & (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ u(T, \mu) = \int_{\mathbb{R}^d} g(x, \mu) \mu(dx), & \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{cases} \quad (3.3)$$

Remark 3.7. As described in [15, Section 5.2], to which we refer for more details, equation (3.3) can be written in various alternative forms. In particular, (3.3) corresponds to [15, equation (5.17)], the only difference being the presence of $\sup_{a \in A}$ which in [15, equation

(5.17)] is replaced by $\text{ess sup}_{a \in A}$. However, as described in [15, Remark 5.8], under assumption (A), $\text{ess sup}_{a \in A}$ can be replaced by $\text{sup}_{a \in A}$.

Finally, we mention that an alternative form of equation (3.3) is the following (corresponding to equation [15, equation (5.16)]):

$$\begin{cases} \partial_t u(t, \mu) + \sup_{a \in \mathcal{M}} \left\{ \int_{\mathbb{R}^d} f(t, x, \mu, a(x)) \mu(dx) + \int_{\mathbb{R}^d} \langle b(t, x, \mu, a(x)), \partial_\mu u(t, \mu)(x) \rangle \mu(dx) \right. \\ \left. + \frac{1}{2} \int_{\mathbb{R}^d} \text{tr}[(\sigma \sigma^\top)(t, x, \mu, a(x)) \partial_x \partial_\mu u(t, \mu)(x)] \mu(dx) \right\} = 0, & (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ u(T, \mu) = \int_{\mathbb{R}^d} g(x, \mu) \mu(dx), & \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

where \mathcal{M} is the set of Borel-measurable maps $a: \mathbb{R}^d \rightarrow A$.

Theorem 3.8. *Let Assumptions (A) and (B) hold. Then, the value function v , given by (2.7), is a viscosity solution of equation (3.3).*

Proof. See [15, Theorem 5.5]. Notice that in [15] a different definition of viscosity solution is adopted, with $\varphi \in C_b^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ rather than $\varphi \in C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$, but the arguments remain the same. \square

Remark 3.9. *From [15, Theorem 5.5] we have that Theorem 3.8 still holds if we replace (B) with the following weaker assumption: the functions b, σ, f are uniformly continuous in the time variable t , uniformly with respect to (x, μ, a) .*

4 Smooth variational principle

As described in the introduction, the comparison theorem (Theorem 5.1) relies on a smooth variational principle on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, to which the present section is devoted. Such a result is obtained from an extension of the Borwein-Preiss variational principle, for which we refer to [8] and, in particular, for its general form, to [9, Theorem 2.5.2]. An essential tool of [9, Theorem 2.5.2] is the concept of gauge-type function, whose definition is given below.

Definition 4.1. *Let d_2 be a metric on $\mathcal{P}_2(\mathbb{R}^d)$ such that $(\mathcal{P}_2(\mathbb{R}^d), d_2)$ is complete. Consider the set $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + d_2(\mu, \nu)$. A map $\rho: ([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2 \rightarrow [0, +\infty)$ is said to be a ***gauge-type function*** if the following holds.*

- a) $\rho((t, \mu), (t, \mu)) = 0$, for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.
- b) ρ is continuous on $([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2$.
- c) For all $\varepsilon > 0$, there exists $\eta > 0$ such that, for all $(t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, the inequality $\rho((t, \mu), (s, \nu)) \leq \eta$ implies $|t - s| + d_2(\mu, \nu) < \varepsilon$.

In the sequel we construct a gauge-type function on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, taking a particular metric d_2 on $\mathcal{P}_2(\mathbb{R}^d)$, namely the so-called Gaussian-smoothed 2-Wasserstein distance, see [38]. To this regard, we denote by $\mathcal{N}_\sigma := \mathcal{N}(0, \sigma I_d)$, for every $\sigma > 0$, the d -dimensional multivariate normal distribution with zero mean and covariance matrix σI_d , with I_d being the identity matrix of order d . Then, for every $\sigma > 0$, the Gaussian-smoothed 2-Wasserstein distance is defined as

$$\mathcal{W}_2^{(\sigma)}(\mu, \nu) := \mathcal{W}_2(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma), \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d),$$

where $*$ denotes the convolution of probability measures.

Lemma 4.2. *For every $\sigma > 0$, $\mathcal{W}_2^{(\sigma)}$ is a metric on $\mathcal{P}_2(\mathbb{R}^d)$, inducing the same topology as \mathcal{W}_2 . Moreover, $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2^{(\sigma)})$ is a complete metric space.*

Proof. The first part follows from [38, Proposition 1]. It remains to prove that the metric space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2^{(\sigma)})$ is complete. Let $\{\mu_n\}_n \subset \mathcal{P}_2(\mathbb{R}^d)$ be a Cauchy sequence with respect to $\mathcal{W}_2^{(\sigma)}$. Then, $\{\mu_n * \mathcal{N}_\sigma\}_n$ is a Cauchy sequence with respect to \mathcal{W}_2 . Since $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ is complete, there exists some $\bar{\nu} \in \mathcal{P}_2(\mathbb{R}^d)$ such that $\mathcal{W}_2(\mu_n * \mathcal{N}_\sigma, \bar{\nu}) \rightarrow 0$ as $n \rightarrow \infty$. It follows (see for instance [3, Proposition 7.1.5]) that $\{\mu_n * \mathcal{N}_\sigma\}_n$ has uniformly integrable second moments, namely

$$\limsup_{k \rightarrow \infty} \sup_n \int_{|y| \geq k} |y|^2 (\mu_n * \mathcal{N}_\sigma)(dy) = 0. \quad (4.1)$$

Now notice that, given $a, b, c \in \mathbb{R}, h \in \mathbb{N}$, if $a \leq b + c$ then $a 1_{\{a \geq h\}} \leq 2b 1_{\{b \geq h/2\}} + 2c 1_{\{c \geq h/2\}}$. Hence, by the elementary inequality $|x|^2 \leq 2|x+z|^2 + 2|z|^2$, valid for every $x, z \in \mathbb{R}^d$, we get

$$|x|^2 1_{\{|x| \geq \sqrt{h}\}} \leq 4|x+z|^2 1_{\{|x+z| \geq \sqrt{h/2}\}} + 4|z|^2 1_{\{|z| \geq \sqrt{h/2}\}}, \quad \forall x, z \in \mathbb{R}^d, h \in \mathbb{N}. \quad (4.2)$$

Integrating the above inequality on $\mathbb{R}^d \times \mathbb{R}^d$ with respect to the product measure $\mu_n(dx) \mathcal{N}_\sigma(dz)$, we obtain (setting $k := \sqrt{h}$, to simplify notation)

$$\begin{aligned} \int_{|x| \geq k} |x|^2 \mu_n(dx) &\leq 4 \iint_{|x+z| \geq k/\sqrt{2}} |x+z|^2 \mu_n(dx) \mathcal{N}_\sigma(dz) + 4 \int_{|z| \geq k/\sqrt{2}} |z|^2 \mathcal{N}_\sigma(dz) \\ &= 4 \int_{|y| \geq k/\sqrt{2}} |y|^2 (\mu_n * \mathcal{N}_\sigma)(dy) + 4 \int_{|z| \geq k/\sqrt{2}} |z|^2 \mathcal{N}_\sigma(dz). \end{aligned}$$

Then, by (4.1), we deduce that $\{\mu_n\}_n$ has uniformly integrable second moments. This implies that $\{\mu_n\}_n$ is tight, so that we can apply [3, Proposition 7.1.5], from which we deduce the existence of a subsequence $\{\mu_{n_k}\}_k$ converging to some $\nu \in \mathcal{P}_2(\mathbb{R}^d)$ with respect to \mathcal{W}_2 . Notice that (we denote by φ_π the characteristic function of the probability measure $\pi \in \mathcal{P}(\mathbb{R}^d)$)

$$\varphi_{\mu_{n_k} * \mathcal{N}_\sigma}(u) = \varphi_{\mu_{n_k}}(u) e^{-\frac{1}{2}\sigma^2|u|^2} \xrightarrow{k \rightarrow \infty} \varphi_\nu(u) e^{-\frac{1}{2}\sigma^2|u|^2} = \varphi_{\nu * \mathcal{N}_\sigma}(u).$$

Then, by Lévy's continuity theorem it follows that $\mathcal{W}_2(\mu_{n_k} * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) \rightarrow 0$ as $k \rightarrow \infty$. This implies that $\nu * \mathcal{N}_\sigma = \bar{\nu}$. By a standard argument, the entire sequence $\{\mu_n * \mathcal{N}_\sigma\}_n$ converges to $\nu * \mathcal{N}_\sigma$ with respect to \mathcal{W}_2 . This shows that $\mathcal{W}_2^{(\sigma)}(\mu_n, \nu) \rightarrow 0$ as $n \rightarrow \infty$ and concludes the proof. \square

Our aim is to find a gauge-type function $\rho = \rho((t, \mu), (t_0, \mu_0))$ smooth with respect to (t, μ) , for every fixed (t_0, μ_0) , on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$. The construction of our smooth gauge-type function (whose definition is given in Lemma 4.4 below) relies on a sharp upper bound of \mathcal{W}_2 obtained in [18, 22] (see also [14, Section 5.1.2]), which is valid in any dimension d and is reported in Lemma 4.3. Notice however that, in the particular case $d = 1$, ad hoc gauge-type functions may be constructed in easier ways, as for instance relying on the following inequality (see [7, Proposition 7.14]):

$$\mathcal{W}_2(\mu, \nu)^2 \leq 4 \int_{-\infty}^{+\infty} |x| |F_\mu(x) - F_\nu(x)| dx, \quad \forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}), \quad (4.3)$$

where F_μ and F_ν are the cumulative distribution functions of μ and ν , respectively. When $d \in \mathbb{N}$, the upper bound of Lemma 4.3 can be viewed as a d -dimensional analogue of (4.3).

Lemma 4.3. *For every integer $\ell \geq 0$, let \mathcal{P}_ℓ denote the partition of $(-1, 1]^d$ into $2^{d\ell}$ translations of $(-2^{-\ell}, 2^{-\ell}]^d$. Moreover, let $B_0 := (-1, 1]^d$ and, for every integer $n \geq 1$, $B_n := (-2^n, 2^n]^d \setminus (-2^{n-1}, 2^{n-1}]^d$. Then, for every $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the following inequality holds:*

$$(W_2(\mu, \nu))^2 \leq c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)|, \quad (4.4)$$

where $2^n B := \{2^n x \in \mathbb{R}^d : x \in B\}$ and $c_d > 0$ is a constant depending only on d .

Proof. Inequality (4.4) follows from [22, Lemma 5 and Lemma 6] (or, equivalently, [14, Lemma 5.11 and Lemma 5.12]). \square

Next lemma provides the claimed smooth gauge-type function and it is the main result of the present section. Notice that such a gauge-type function is obtained performing a smoothing of the right-hand side of (4.4), proceeding as follows.

- a) Firstly, the absolute value of the difference $\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)$ appearing in (4.4) is replaced by $\sqrt{|\mu((2^n B) \cap B_n) - \nu((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell}$, with $\delta_{n,\ell} = 2^{-(4n+2d\ell)}$. In other words, we replace $|\cdot|$ by the smooth function $\sqrt{\cdot + \delta_{n,\ell}^2} - \delta_{n,\ell}$. The particular choice of $\delta_{n,\ell}$ will be used to obtain the convergence of a certain series (see (4.16)).
- b) Secondly, as already mentioned, our function will be of gauge-type on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, with $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$. As a consequence, we consider (4.4) for $\mathcal{W}_2^{(1)}(\mu, \nu) = \mathcal{W}_2(\mu * \mathcal{N}_1, \nu * \mathcal{N}_1)$. This implies that $\mu((2^n B) \cap B_n)$ and $\nu((2^n B) \cap B_n)$ are replaced respectively by $(\mu * \mathcal{N}_1)((2^n B) \cap B_n)$ and $(\nu * \mathcal{N}_1)((2^n B) \cap B_n)$.

Lemma 4.4. *We adopt the same notations as in Lemma 4.3. Let $\rho_2: ([0, T] \times \mathcal{P}_2(\mathbb{R}^d))^2 \rightarrow [0, +\infty)$ be defined as*

$$\begin{aligned} \rho_2((t, \mu), (s, \nu)) &= |t - s|^2 + \\ &+ c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left(\sqrt{ |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right). \end{aligned} \quad (4.5)$$

with $\delta_{n,\ell} := 2^{-(4n+2d\ell)}$. Then, the following holds.

- 1) ρ_2 is a gauge-type function on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, with $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$;
- 2) for every fixed $(t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, the map $(t, \mu) \mapsto \rho_2((t, \mu), (t_0, \mu_0))$ is in $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$;
- 3) there exists a constant C_d (depending only on the dimension d) such that

$$|\rho_2((t, \mu), (t_0, \mu_0))| \leq T^2 + C_d \left(1 + \int_{\mathbb{R}^d} |x|^2 \mu(dx) + \int_{\mathbb{R}^d} |x|^2 \mu_0(dx) \right), \quad (4.6)$$

$$|\partial_t \rho_2((t, \mu), (t_0, \mu_0))| \leq 2T, \quad (4.7)$$

$$|\partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)| \leq C_d(1 + |x|^2), \quad (4.8)$$

$$|\partial_x \partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)| \leq C_d(1 + |x|^2), \quad (4.9)$$

for all $(t, \mu), (t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, $x \in \mathbb{R}^d$.

Proof. We split the proof into four steps.

Step I. Uniform convergence of the series in ρ_2 . We prove a preliminary result concerning the series in ρ_2 . Let \mathcal{M} be a subset of $\mathcal{P}_2(\mathbb{R}^d)$ such that $\{\mu * \mathcal{N}_1\}_{\mu \in \mathcal{M}}$ has uniformly integrable second moments. Our aim is to prove that the series appearing in the definition of ρ_2 converges uniformly with respect to $\mu, \nu \in \mathcal{M}$. More precisely, we prove that for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\mu, \nu \in \mathcal{M}} \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)| \leq \varepsilon. \quad (4.10)$$

Then, the claim follows from (this is a consequence of the elementary inequality $\sqrt{a^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \leq |a|$, valid for every $a \in \mathbb{R}$)

$$\begin{aligned} &\sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left(\sqrt{ |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right) \\ &\leq \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)|. \end{aligned}$$

Let us prove (4.10). First of all, notice that

$$\begin{aligned} & \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)| \\ & \leq \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_1)((2^n B) \cap B_n) + \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\nu * \mathcal{N}_1)((2^n B) \cap B_n). \end{aligned}$$

Observe also that $\sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_1)((2^n B) \cap B_n) = (\mu * \mathcal{N}_1)(B_n)$, therefore $\sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} (\mu * \mathcal{N}_1)((2^n B) \cap B_n) = (4/3)(\mu * \mathcal{N}_1)(B_n)$, since $\sum_{\ell \geq 0} 2^{-2\ell} = 4/3$. So, in particular, (4.10) follows if we prove that for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\sup_{\mu \in \mathcal{M}} \sum_{n \geq N} 2^{2n} (\mu * \mathcal{N}_1)(B_n) \leq \varepsilon.$$

Recalling that $B_n = (-2^n, 2^n]^d \setminus (-2^{n-1}, 2^{n-1}]^d$, we obtain $2^{2n} \leq 4|x|^2/d, \forall x \in B_n$. Hence, for every $N \in \mathbb{N}$,

$$\sum_{n \geq N} 2^{2n} (\mu * \mathcal{N}_1)(B_n) \leq \frac{4}{d} \int_{\mathbb{R}^d \setminus (-2^{N-1}, 2^{N-1}]^d} |x|^2 (\mu * \mathcal{N}_1)(dx).$$

Since the family $\{\mu * \mathcal{N}_1\}_{\mu \in \mathcal{M}}$ has uniformly integrable second moments, the claim follows.

Step II. ρ_2 is a gauge-type function on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ with respect to the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$. It is clear that ρ_2 satisfies item a) of Definition 4.1. Concerning items b) and c), we split the rest of the proof of Step II into two substeps.

ρ_2 satisfies item b) of Definition 4.1. Our aim is to prove that, given $\{(t_k, \mu_k)\}_k, \{(s_k, \nu_k)\}_k \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $(t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, if $|t_k - t| + \mathcal{W}_2^{(1)}(\mu_k, \mu) + |s_k - s| + \mathcal{W}_2^{(1)}(\nu_k, \nu) \rightarrow 0$ then $\rho_2((t_k, \mu_k), (s_k, \nu_k)) \rightarrow \rho_2((t, \mu), (s, \nu))$. In particular, we have to prove that, if $\mathcal{W}_2^{(1)}(\mu_k, \mu) + \mathcal{W}_2^{(1)}(\nu_k, \nu) \rightarrow 0$, then

$$\begin{aligned} & \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left(\sqrt{|((\mu_k * \mathcal{N}_1) - (\nu_k * \mathcal{N}_1))((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right) \\ & \xrightarrow{k \rightarrow \infty} \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left(\sqrt{|((\mu * \mathcal{N}_1) - (\nu * \mathcal{N}_1))((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right). \end{aligned}$$

Since $\mathcal{W}_2^{(1)}(\mu_k, \mu) = \mathcal{W}_2(\mu_k * \mathcal{N}_1, \mu * \mathcal{N}_1)$ and $\mathcal{W}_2^{(1)}(\nu_k, \nu) = \mathcal{W}_2(\nu_k * \mathcal{N}_1, \nu * \mathcal{N}_1)$, we have that $\mathcal{W}_2(\mu_k * \mathcal{N}_1, \mu * \mathcal{N}_1) + \mathcal{W}_2(\nu_k * \mathcal{N}_1, \nu * \mathcal{N}_1) \rightarrow 0$. Now, recall from [3, Proposition 7.1.5] that this implies that $\{\mu_n * \mathcal{N}_1\}_n$ (resp. $\{\nu_n * \mathcal{N}_1\}_n$) weakly converges to $\mu * \mathcal{N}_1$ (resp. $\nu * \mathcal{N}_1$) and has uniformly integrable second moments. Since both $\mu * \mathcal{N}_1$ and $\nu * \mathcal{N}_1$ are absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^d , by the weak convergence (and, in particular, by the portmanteau theorem) we deduce that

$$\lim_{k \rightarrow \infty} (\mu_k * \mathcal{N}_1)((2^n B) \cap B_n) = (\mu * \mathcal{N}_1)((2^n B) \cap B_n).$$

Similarly $(\nu_k * \mathcal{N}_1)((2^n B) \cap B_n) \rightarrow (\nu * \mathcal{N}_1)((2^n B) \cap B_n)$. In addition, since $\{\mu_n * \mathcal{N}_1\}_n$ and $\{\nu_n * \mathcal{N}_1\}_n$ have uniformly integrable second moments, from Step I we can interchange the limit with the series, so that the claim follows.

ρ_2 satisfies item c) of Definition 4.1. Our aim is to prove the following: for every $\varepsilon > 0$, there exists $\eta = \eta_\varepsilon > 0$ such that, for all $(t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, the inequality $\rho((t, \mu), (s, \nu)) \leq \eta$ implies

$$|t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |(\mu * \mathcal{N}_1)((2^n B) \cap B_n) - (\nu * \mathcal{N}_1)((2^n B) \cap B_n)| \leq \frac{\varepsilon^2}{2}. \quad (4.11)$$

As a matter of fact, recalling that $\mathcal{W}_2^{(1)}(\mu, \nu) = \mathcal{W}_2(\mu * \mathcal{N}_1, \nu * \mathcal{N}_1)$, from inequality (4.4) we conclude that (using the elementary inequality $x + y \leq \sqrt{2x^2 + 2y^2}$, valid for every $x, y \in \mathbb{R}$)

$$|t - s| + \mathcal{W}_2^{(1)}(\mu, \nu) \leq \sqrt{2|t - s|^2 + 2\mathcal{W}_2^{(1)}(\mu, \nu)^2} \leq \varepsilon.$$

Let us prove that (4.11) holds with $\eta = \eta_\varepsilon$ given by

$$\eta_\varepsilon := \left(\sqrt{8c_d + \varepsilon^2/2} - \sqrt{8c_d} \right)^2. \quad (4.12)$$

To this end, denote by $d_2((t, \mu), (s, \nu))$ the left-hand side of (4.11), namely

$$d_2((t, \mu), (s, \nu)) := |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |((\mu * \mathcal{N}_1) - (\nu * \mathcal{N}_1))((2^n B) \cap B_n)|.$$

Moreover, for every $n \geq 0, \ell \geq 0, B \in \mathcal{P}_\ell, \mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, denote

$$\begin{aligned} a_{n,\ell}(\mu, \nu, B) &:= \left(\sqrt{|((\mu * \mathcal{N}_1) - (\nu * \mathcal{N}_1))((2^n B) \cap B_n)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \right), \\ b_{n,\ell}(\mu, \nu, B) &:= |((\mu * \mathcal{N}_1) - (\nu * \mathcal{N}_1))((2^n B) \cap B_n)|. \end{aligned}$$

Notice that

$$\rho_2((t, \mu), (s, \nu)) = |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} a_{n,\ell}(\mu, \nu, B), \quad (4.13)$$

$$d_2((t, \mu), (s, \nu)) = |t - s|^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} b_{n,\ell}(\mu, \nu, B). \quad (4.14)$$

Now, consider $(t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that $\rho_2((t, \mu), (s, \nu)) \leq \eta_\varepsilon$, with η_ε given by (4.12). Then

$$c_d 2^{2(n-\ell)} a_{n,\ell}(\mu, \nu, B) \leq \rho_2((t, \mu), (s, \nu)) \leq \eta_\varepsilon. \quad (4.15)$$

Since $a_{n,\ell}(\mu, \nu, B) = \sqrt{|b_{n,\ell}(\mu, \nu, B)|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell}$, we obtain

$$b_{n,\ell}(\mu, \nu, B) = \sqrt{|a_{n,\ell}(\mu, \nu, B)|^2 + 2\delta_{n,\ell} a_{n,\ell}(\mu, \nu, B)} \leq a_{n,\ell}(\mu, \nu, B) + \sqrt{2\delta_{n,\ell} a_{n,\ell}(\mu, \nu, B)},$$

where we have used the elementary inequality $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$, valid for every $x, y \geq 0$. Therefore, by (4.15) we get

$$b_{n,\ell}(\mu, \nu, B) \leq a_{n,\ell}(\mu, \nu, B) + \sqrt{\frac{2}{c_d} \eta_\varepsilon} \sqrt{\delta_{n,\ell} 2^{-2(n-\ell)}} = a_{n,\ell}(\mu, \nu, B) + \sqrt{\frac{2}{c_d} \eta_\varepsilon} 2^{-(3n+(d-1)\ell)}$$

where the last equality follows from the fact that $\delta_{n,\ell} = 2^{-(4n+2d\ell)}$. Hence, from (4.13) and (4.14) we obtain

$$d_2((t, \mu), (s, \nu)) \leq \rho_2((t, \mu), (s, \nu)) + c_d \sqrt{\frac{2}{c_d} \eta_\varepsilon} \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} 2^{-(3n+(d-1)\ell)}$$

Recalling that $\rho_2((t, \mu), (s, \nu)) \leq \eta_\varepsilon$ and also that \mathcal{P}_ℓ contains $2^{d\ell}$ sets (see the statement of Lemma 4.3), we get

$$\begin{aligned} d_2((t, \mu), (s, \nu)) &\leq \eta_\varepsilon + \sqrt{2 c_d \eta_\varepsilon} \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{2n} 2^{-2\ell} 2^{d\ell} 2^{-(3n+(d-1)\ell)} \\ &= \eta_\varepsilon + \sqrt{2 c_d \eta_\varepsilon} \sum_{n \geq 0} \sum_{\ell \geq 0} 2^{-n} 2^{-\ell} = \eta_\varepsilon + 4\sqrt{2 c_d \eta_\varepsilon} = \frac{\varepsilon^2}{2}, \end{aligned} \quad (4.16)$$

where the last equality follows from the definition of η_ε .

Step III. The map $(t, \mu) \mapsto \rho_2((t, \mu), (t_0, \mu_0))$ is in $C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$. We denote by $\zeta: \mathbb{R}^d \rightarrow \mathbb{R}$ the density function of the standard multivariate normal distribution \mathcal{N}_1 , given by

$$\zeta(z) = \frac{1}{(2\pi)^{d/2}} e^{-\frac{1}{2}|z|^2}, \quad \forall z \in \mathbb{R}^d.$$

Then, the map ρ_2 can be written as

$$\begin{aligned} \rho_2((t, \mu), (t_0, \mu_0)) &= |t - t_0|^2 + \\ &+ c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \left(\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy) \right|^2 + \delta_{n,\ell}^2 - \delta_{n,\ell}} \right), \end{aligned}$$

where

$$\phi_n^B(x) := \int_{(2^n B) \cap B_n} \zeta(z - x) dz, \quad \forall x \in \mathbb{R}^d.$$

We split the rest of the proof of Step III into two substeps.

First-order derivatives. By direct calculation, we have $\partial_t \rho_2((t, \mu), (t_0, \mu_0)) = 2(t - t_0)$. Moreover, we claim that $\partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)$ is given by

$$\begin{aligned} &c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)}{\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|^2 + \delta_{n,\ell}^2}} \partial_x \phi_n^B(x) \\ &= c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{(\mu * \mathcal{N}_1 - \mu_0 * \mathcal{N}_1)((2^n B) \cap B_n)}{\sqrt{\left| (\mu * \mathcal{N}_1 - \mu_0 * \mathcal{N}_1)((2^n B) \cap B_n) \right|^2 + \delta_{n,\ell}^2}} \partial_x \phi_n^B(x), \end{aligned} \quad (4.17)$$

where $\partial_x \phi_n^B$ denotes the gradient of ϕ_n^B . In order to prove (4.17), we denote, for every $n, \ell \geq 0$, $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, $B \in \mathcal{P}_\ell$,

$$u_{n,\ell}^{B,\mu_0}(\mu) = \sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) \mu(dy) - \int_{\mathbb{R}^d} \phi_n^B(y) \mu_0(dy) \right|^2 + \delta_{n,\ell}^2} - \delta_{n,\ell}, \quad \forall \mu \in \mathcal{P}_2(\mathbb{R}^d).$$

Let us determine $\partial_\mu u_{n,\ell}^{B,\mu_0}$. To this end, let us consider the lifting $U_{n,\ell}^{B,\mu_0}: L^2(\Omega; \mathbb{R}^d) \rightarrow \mathbb{R}$ of $u_{n,\ell}^{B,\mu_0}$, given by $U_{n,\ell}^{B,\mu_0}(\xi) = u_{n,\ell}^{B,\mu_0}(\mu)$, for every $\xi \in L^2(\Omega; \mathbb{R}^d)$ having distribution μ . Recall from the definition of $\partial_\mu u_{n,\ell}^{B,\mu_0}$ that, for every $\{\eta_k\}_k \subset L^2(\Omega; \mathbb{R}^d)$ such that $|\eta_k|_{L^2(\Omega; \mathbb{R}^d)} \rightarrow 0$, it holds that

$$\lim_{k \rightarrow \infty} \frac{|U_{n,\ell}^{B,\mu_0}(\xi + \eta_k) - U_{n,\ell}^{B,\mu_0}(\xi) - \mathbb{E}[\langle \partial_\mu u_{n,\ell}^{B,\mu_0}(\mu)(\xi), \eta_k \rangle]|}{|\eta_k|_{L^2(\Omega; \mathbb{R}^d)}} = 0, \quad (4.18)$$

where $\xi \in L^2(\Omega; \mathbb{R}^d)$ has distribution μ . Then, we have

$$\partial_\mu u_{n,\ell}^{B,\mu_0}(\mu)(x) = \frac{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)}{\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|^2 + \delta_{n,\ell}^2}} \partial_x \phi_n^B(x), \quad (4.19)$$

for every $(\mu, x) \in \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. Now, by (4.18) we see that (4.17) follows if we prove that the series

$$c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{|U_{n,\ell}^{B,\mu_0}(\xi + \eta_k) - U_{n,\ell}^{B,\mu_0}(\xi) - \mathbb{E}[\langle \partial_\mu u_{n,\ell}^{B,\mu_0}(\mu)(\xi), \eta_k \rangle]|}{|\eta_k|_{L^2(\Omega; \mathbb{R}^d)}} \quad (4.20)$$

converges uniformly with respect to k . To this end, denote

$$h(\lambda) := U_{n,\ell}^{B,\mu_0}(\xi + \lambda \eta_k), \quad 0 \leq \lambda \leq 1.$$

Since $h(1) = h(0) + \int_0^1 h'(\lambda) d\lambda$, we get

$$U_{n,\ell}^{B,\mu_0}(\xi + \eta) = U_{n,\ell}^{B,\mu_0}(\xi) + \int_0^1 \mathbb{E}[\langle \partial_\mu u_{n,\ell}^{B,\mu_0}(\mu_{k,\lambda})(\xi + \lambda \eta_k), \eta_k \rangle] d\lambda,$$

where $\mu_{k,\lambda}$ is the distribution of $\xi + \lambda \eta_k$. Then, (4.20) is bounded from above by

$$\begin{aligned} c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \int_0^1 |\mathbb{E}[\langle \partial_\mu u_{n,\ell}^{B,\mu_0}(\mu_{k,\lambda})(\xi + \lambda \eta_k), \eta_k / |\eta_k|_{L^2(\Omega; \mathbb{R}^d)} \rangle]| d\lambda \\ + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |\mathbb{E}[\langle \partial_\mu u_{n,\ell}^{B,\mu_0}(\mu)(\xi), \eta_k / |\eta_k|_{L^2(\Omega; \mathbb{R}^d)} \rangle]|. \end{aligned} \quad (4.21)$$

Notice that $\{\eta_k\}_{k \in \mathbb{N}}$ has uniformly integrable second moments (see for instance [29, Theorem 4.12]), so that $\{\xi + \lambda \eta_k\}_{k \in \mathbb{N}, \lambda \in [0,1]}$ also has uniformly integrable second moments. Therefore, the two series in (4.21) converge uniformly if we prove that (ν denotes the distribution of η)

$$c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \mathbb{E}[|\partial_\mu u_{n,\ell}^{B,\mu_0}(\nu)(\eta)|]$$

converges uniformly with respect to ν , whenever ν belongs to a subset \mathcal{M} of $\mathcal{P}_2(\mathbb{R}^d)$ with uniformly integrable second moments, namely

$$\lim_{M \rightarrow \infty} \sup_{\nu \in \mathcal{M}} \int_{|x| \geq M} |x|^2 \nu(dx) = 0. \quad (4.22)$$

Then, the claim follows if we prove that for every $\varepsilon > 0$, there exists $N = N(\varepsilon, \mathcal{M}) \in \mathbb{N}$ such that, for every $\nu \in \mathcal{M}$, it holds that

$$c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \mathbb{E}[|\partial_\mu u_{n,\ell}^{B,\mu_0}(\nu)(\eta)|] \leq \varepsilon, \quad (4.23)$$

with $\eta \in L^2(\Omega; \mathbb{R}^d)$ having distribution ν . Firstly, notice from (4.19) that $|\partial_\mu u_{n,\ell}^{B,\mu_0}(\mu)(x)| \leq |\partial_x \phi_n^B(x)|$, $\forall x \in \mathbb{R}^d$. Moreover $\partial_x \phi_n^B(x) = \int_{(2^n B) \cap B_n} (z - x) \zeta(z - x) dz$. Therefore, the series (4.23) is bounded from above by

$$\begin{aligned} & c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \mathbb{E} \left[\int_{(2^n B) \cap B_n} |z - \eta| \zeta(z - \eta) dz \right] \\ &= c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \mathbb{E} \left[\int_{B_n} |z - \eta| \zeta(z - \eta) dz \right] = \frac{4}{3} c_d \sum_{n \geq N} 2^{2n} \mathbb{E} \left[\int_{B_n} |z - \eta| \zeta(z - \eta) dz \right]. \end{aligned}$$

Recalling that $B_n = (-2^n, 2^n]^d \setminus (-2^{n-1}, 2^{n-1}]^d$, we obtain $2^{2n} \leq 4|z|^2/d$, $\forall z \in B_n$, so that

$$\begin{aligned} & \frac{4}{3} c_d \sum_{n \geq N} 2^{2n} \mathbb{E} \left[\int_{B_n} |z - \eta| \zeta(z - \eta) dz \right] \leq \frac{16}{3d} c_d \mathbb{E} \left[\int_{\mathbb{R}^d \setminus (-2^{N-1}, 2^{N-1}]^d} |z|^2 |z - \eta| \zeta(z - \eta) dz \right] \\ & \leq \frac{16}{3d} c_d \mathbb{E} \left[\int_{|z| \geq 2^{N-1}} |z|^2 |z - \eta| \zeta(z - \eta) dz \right] \\ & = \frac{16}{3d} c_d \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} 1_{\{|z| \geq 2^{N-1}\}} |z|^2 |z - x| \zeta(z - x) dz \right) \nu(dx) \\ & = \frac{16}{3d} c_d \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1_{\{|y+x| \geq 2^{N-1}\}} |y+x|^2 |y| \zeta(y) dy \nu(dx). \end{aligned}$$

Applying the elementary inequality (4.2) (with $x, x+z, z, \sqrt{h}$ replaced respectively by $y+x, y, x, 2^{N-1}$), we obtain

$$\begin{aligned} & \frac{16}{3d} c_d \iint_{\mathbb{R}^d \times \mathbb{R}^d} 1_{\{|y+x| \geq 2^{N-1}\}} |y+x|^2 |y| \zeta(y) dy \nu(dx) \\ & \leq \frac{64}{3d} c_d \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(1_{\{|y| \geq 2^{N-3/2}\}} |y|^2 + 1_{\{|x| \geq 2^{N-3/2}\}} |x|^2 \right) |y| \zeta(y) dy \nu(dx) \\ & = \frac{64}{3d} c_d \int_{|y| \geq 2^{N-3/2}} |y|^3 \zeta(y) dy + \frac{64}{3d} c_d \left(\int_{\mathbb{R}^d} |y| \zeta(y) dy \right) \left(\int_{|x| \geq 2^{N-3/2}} |x|^2 \nu(dx) \right). \end{aligned}$$

Then, (4.23) follows from (4.22).

Second-order derivatives. We claim that $\partial_x \partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)$ is equal to

$$\begin{aligned} & c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{\int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy)}{\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|^2 + \delta_{n,\ell}^2}} \partial_{xx}^2 \phi_n^B(x) \\ &= c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{(\mu * \mathcal{N}_1 - \mu_0 * \mathcal{N}_1)((2^n B) \cap B_n)}{\sqrt{\left| (\mu * \mathcal{N}_1 - \mu_0 * \mathcal{N}_1)((2^n B) \cap B_n) \right|^2 + \delta_{n,\ell}^2}} \partial_{xx}^2 \phi_n^B(x), \end{aligned} \quad (4.24)$$

where $\partial_{xx}^2 \phi_n^B$ denotes the Hessian matrix of ϕ_n^B . Proceeding as in the previous substep, we see that this follows if we prove that the series $(|\partial_{xx}^2 \phi_n^B(x)|)$ stands for the Frobenius norm of the $d \times d$ matrix $\partial_{xx}^2 \phi_n^B(x)$

$$c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|}{\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|^2 + \delta_{n,\ell}^2}} |\partial_{xx}^2 \phi_n^B(x)|$$

converges uniformly with respect to x , whenever x belongs to a bounded subset of \mathbb{R}^d . More precisely, we prove that for all $\varepsilon > 0$ and $M \in \mathbb{N}$, there exists $N = N(\varepsilon, M) \in \mathbb{N}$ such that, for every $x \in \mathbb{R}^d$, with $|x| \leq M$, it holds that

$$c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \frac{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|}{\sqrt{\left| \int_{\mathbb{R}^d} \phi_n^B(y) (\mu - \mu_0)(dy) \right|^2 + \delta_{n,\ell}^2}} |\partial_{xx}^2 \phi_n^B(x)| \leq \varepsilon. \quad (4.25)$$

We begin noting that the latter series is bounded from above by

$$c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} |\partial_{xx}^2 \phi_n^B(x)|. \quad (4.26)$$

We also observe that

$$\partial_{xx}^2 \phi_n^B(x) = I_d \int_{(2^n B) \cap B_n} \zeta(z - x) dz - \int_{(2^n B) \cap B_n} (z - x) \otimes (z - x) \zeta(z - x) dz, \quad (4.27)$$

where I_d denotes the identity matrix of order d , while $(z - x) \otimes (z - x)$ is the $d \times d$ matrix with (i, j) -component equal to $(z_i - x_i)(z_j - x_j)$. Then, (4.26) is bounded from above by (notice that the Frobenius norms $|I_d|$ and $|(z - x) \otimes (z - x)|$ are given respectively by \sqrt{d} and $|z - x|^2$, where $|z - x|$ denotes the Euclidean norm of $z - x$)

$$\begin{aligned} & c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \int_{(2^n B) \cap B_n} (\sqrt{d} + |z - x|^2) \zeta(z - x) dz \\ &= c_d \sum_{n \geq N} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \int_{B_n} (\sqrt{d} + |z - x|^2) \zeta(z - x) dz \\ &= \frac{4}{3} c_d \sum_{n \geq N} 2^{2n} \int_{B_n} (\sqrt{d} + |z - x|^2) \zeta(z - x) dz. \end{aligned}$$

Recalling that $2^{2n} \leq 4|z|^2/d, \forall z \in B_n$, we find

$$\begin{aligned} & \frac{4}{3}c_d \sum_{n \geq N} 2^{2n} \int_{B_n} (\sqrt{d} + |z - x|^2) \zeta(z - x) dz \\ & \leq \frac{16}{3d}c_d \int_{\mathbb{R}^d \setminus (-2^{N-1}, 2^{N-1}]^d} |z|^2 (\sqrt{d} + |z - x|^2) \zeta(z - x) dz. \end{aligned} \quad (4.28)$$

Since $|x| \leq M$, from the right-hand side of (4.28) we see that (4.25) follows.

Step IV. Bounds. From the definition of ρ_2 and the elementary inequality $\sqrt{x^2 + \delta_{n,\ell}^2} - \delta_{n,\ell} \leq |x|$, valid for every $x \in \mathbb{R}$, we get

$$\begin{aligned} \rho_2((t, \mu), (t_0, \mu_0)) & \leq T^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} ((\mu * \mathcal{N}_1) + (\mu_0 * \mathcal{N}_1))((2^n B) \cap B_n) \\ & = T^2 + c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} ((\mu * \mathcal{N}_1) + (\mu_0 * \mathcal{N}_1))(B_n) \\ & = T^2 + \frac{4}{3}c_d \sum_{n \geq 0} 2^{2n} ((\mu * \mathcal{N}_1) + (\mu_0 * \mathcal{N}_1))(B_n). \end{aligned}$$

Using that $2^{2n} \leq 4|x|^2/d, \forall x \in B_n$, we obtain

$$\rho_2((t, \mu), (t_0, \mu_0)) \leq T^2 + \frac{16}{3d}c_d \int_{\mathbb{R}^d} |x|^2 (\mu * \mathcal{N}_1)(dx) + \frac{16}{3d}c_d \int_{\mathbb{R}^d} |x|^2 (\mu_0 * \mathcal{N}_1)(dx). \quad (4.29)$$

Notice that (recall that ζ denotes the probability density function of the standard multivariate normal distribution)

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^2 (\mu * \mathcal{N}_1)(dx) & = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |z + y|^2 \zeta(y) dy \right) \mu(dz) \\ & \leq 2 \int_{\mathbb{R}^d} |z|^2 \mu(dz) + 2 \int_{\mathbb{R}^d} |y|^2 \zeta(y) dy. \end{aligned} \quad (4.30)$$

From (4.29) and (4.30) we see that (4.6) follows.

Now, concerning the time derivative, recalling that $\partial_t \rho_2((t, \mu), (t_0, \mu_0)) = 2(t - t_0)$, we obtain $|\partial_t \rho_2((t, \mu), (t_0, \mu_0))| \leq 2T$. On the other hand, recalling that $\partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)$ is given by (4.17), we obtain (notice that $\partial_x \phi_n^B(x) = \int_{(2^n B) \cap B_n} (z - x) \zeta(z - x)$)

$$\begin{aligned} |\partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)| & \leq c_d \sum_{n \geq 0} 2^{2n} \sum_{\ell \geq 0} 2^{-2\ell} \sum_{B \in \mathcal{P}_\ell} \int_{(2^n B) \cap B_n} |z - x| \zeta(z - x) dz \\ & = \frac{4}{3}c_d \sum_{n \geq 0} 2^{2n} \int_{B_n} |z - x| \zeta(z - x) dz. \end{aligned}$$

Since $2^{2n} \leq 4|z|^2/d, \forall z \in B_n$, we get

$$\begin{aligned} |\partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)| &\leq \frac{16}{3d} c_d \int_{\mathbb{R}^d} |z|^2 |z-x| \zeta(z-x) dz = \frac{16}{3d} c_d \int_{\mathbb{R}^d} |y+x|^2 |y| \zeta(y) dy \\ &\leq \frac{32}{3d} c_d \int_{\mathbb{R}^d} |y|^3 \zeta(y) dy + \frac{32}{3d} c_d |x|^2 \int_{\mathbb{R}^d} |y| \zeta(y) dy, \end{aligned}$$

which gives (4.8).

From similar calculations, by (4.24) and (4.27), we deduce that $\partial_x \partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)$ is bounded by

$$\begin{aligned} |\partial_x \partial_\mu \rho_2((t, \mu), (t_0, \mu_0))(x)| &\leq \frac{16}{3d} c_d \int_{\mathbb{R}^d} |z|^2 (\sqrt{d} + |z-x|^2) \zeta(z-x) dz \\ &= \frac{16}{3d} c_d \int_{\mathbb{R}^d} |y+x|^2 (\sqrt{d} + |y|^2) \zeta(y) dy \\ &\leq \frac{32}{3d} c_d \int_{\mathbb{R}^d} |y|^2 (\sqrt{d} + |y|^2) \zeta(y) dy + \frac{32}{3d} c_d |x|^2 \int_{\mathbb{R}^d} (\sqrt{d} + |y|^2) \zeta(y) dy. \end{aligned}$$

We conclude that (4.9) holds. \square

We are in a position to state the smooth variational principle on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.

Theorem 4.5. *Fix $\delta > 0$ and let $G: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ be upper semicontinuous and bounded from above. Given $\varepsilon > 0$, let $(t_0, \mu_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ be such that*

$$\sup G - \varepsilon \leq G(t_0, \mu_0).$$

Then, there exist $(\tilde{t}, \tilde{\mu}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and a sequence $\{(t_k, \mu_k)\}_{k \geq 1} \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that:

- (i) $\rho_2((\tilde{t}, \tilde{\mu}), (t_k, \mu_k)) \leq \frac{\varepsilon}{2^k \delta}$, for every $k \geq 0$;
- (ii) $G(t_0, \mu_0) \leq G(\tilde{t}, \tilde{\mu}) - \delta \varphi(\tilde{t}, \tilde{\mu})$, with $\varphi: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow [0, +\infty)$ given by

$$\varphi(t, \mu) = \sum_{k \geq 0} \frac{1}{2^k} \rho_2((t, \mu), (t_k, \mu_k)), \quad \forall (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d);$$

- (iii) $G(t, \mu) - \delta \varphi(t, \mu) < G(\tilde{t}, \tilde{\mu}) - \delta \varphi(\tilde{t}, \tilde{\mu})$, for every $(t, \mu) \in ([0, T] \times \mathcal{P}_2(\mathbb{R}^d)) \setminus \{(\tilde{t}, \tilde{\mu})\}$.

Furthermore, the function φ satisfies the following properties.

- 1) $\varphi \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ and

$$|\varphi(t, \mu)| \leq 2T^2 + 2C_d \left(1 + \sup_{k \geq 0} \int_{\mathbb{R}^d} |x|^2 \mu_k(dx) \right) + 2C_d \int_{\mathbb{R}^d} |x|^2 \mu(dx),$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, where C_d is the same constant appearing in (4.8) and $\sup_k \int_{\mathbb{R}^d} |x|^2 \mu_k(dx) < \infty$;

- 2) its time derivative, $\partial_t \varphi(t, \mu)$, is bounded by $4T$;
- 3) its measure derivative, $\partial_\mu \varphi(t, \mu)(x)$, is bounded by $2C_d(1 + |x|^2)$;
- 4) its second-order measure derivative, $\partial_x \partial_\mu \varphi(t, \mu)(x)$, is bounded by $2C_d(1 + |x|^2)$.

Proof. Items (i)-(ii)-(iii) follow directly from the Borwein-Preiss variational principle [9, Theorem 2.5.2] applied on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ with gauge-type function ρ_2 (we only remark that, concerning the sequence $\{\delta_k\}_{k \geq 0}$ appearing in the statement of [9, Theorem 2.5.2], here we take $\delta_0 = \delta$ and $\delta_k = \delta/2^k$, $k \geq 1$). On the other hand, items 2), 3), 4) follow respectively from (4.7), (4.8), (4.9). Finally, concerning item 1), from (4.6) we have

$$|\varphi(t, \mu)| \leq 2T^2 + 2C_d + 2C_d \int_{\mathbb{R}^d} |x|^2 \mu(dx) + C_d \sum_{k \geq 0} \frac{1}{2^k} \int_{\mathbb{R}^d} |x|^2 \mu_k(dx),$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. From item (i) we have $\rho_2((\bar{t}, \bar{\mu}), (t_k, \mu_k)) \rightarrow 0$ as $k \rightarrow +\infty$. Since ρ_2 is a gauge-type function, we deduce that $\mathcal{W}_2^{(1)}(\bar{\mu}, \mu_k) \rightarrow 0$. By Lemma 4.2 it follows that $\mathcal{W}_2(\bar{\mu}, \mu_k) \rightarrow 0$. This implies that the sequence $\{\mu_k\}_k$ is bounded in $\mathcal{P}_2(\mathbb{R}^d)$ and concludes the proof of item 1). \square

5 Comparison theorem and uniqueness

Theorem 5.1 (Comparison). *Let Assumptions (A), (B), (C) hold. Consider bounded and continuous functions $u_1: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ and $u_2: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$, with u_1 (resp. u_2) being a viscosity subsolution (resp. supersolution) to equation (3.3). Then, it holds that $u_1 \leq u_2$ on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.*

Proof. Let v_0 be the map defined by (A.1) with $\varepsilon = 0$ (see also Remark A.1). Our aim is to prove that $u_1 \leq v_0$ and $v_0 \leq u_2$ on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, from which the claim follows. We only report the proof of $u_1 \leq v_0$, as the proof of $v_0 \leq u_2$ can be done proceeding along the same lines. We split the proof of $u_1 \leq v_0$ into five steps.

STEP I. By contradiction, we suppose that there exists $(t_0, \tilde{\mu}_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that

$$(u_1 - v_0)(t_0, \tilde{\mu}_0) > 0.$$

Since both u_1 and v_0 are continuous, we can find $q > 2$ and $\mu_0 \in \mathcal{P}_q(\mathbb{R}^d)$ such that

$$(u_1 - v_0)(t_0, \mu_0) > 0. \tag{5.1}$$

As a matter of fact, let $\xi \in L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ be such that $\mathbb{P}_\xi = \tilde{\mu}_0$. For every $k \in \mathbb{N}$, let $\mu_0^k \in \mathcal{P}_2(\mathbb{R}^d)$ be the distribution of $\xi_k := \xi \mathbf{1}_{\{|\xi| \leq k\}}$. We see that $\mu_0^k \in \mathcal{P}_q(\mathbb{R}^d)$, for any $q \geq 1$. Moreover, it holds that

$$\mathcal{W}_2(\mu_0^k, \tilde{\mu}_0)^2 \leq \mathbb{E}[|\xi_k - \xi|^2] = \int_{|x| > k} |x|^2 \tilde{\mu}_0(dx) \xrightarrow{k \rightarrow +\infty} 0,$$

from which we deduce that (5.1) holds with $\mu_0 := \mu_0^k$ for some k large enough.

Finally, we notice that $t_0 < T$, in fact $u_1(T, \mu_0) \leq g(\mu_0) = v_0(T, \mu_0)$.

STEP II. For every $\varepsilon > 0$ and $n \in \mathbb{N}$, let $v_{\varepsilon, n, m}$ be the map given by (A.4). Recall from Theorem A.5 that $v_{\varepsilon, n, m} \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$.

Now, for every $\delta > 0$, we define $\check{u}_1(t, \mu) := e^{\sqrt{\delta}(t+1)}u_1(t, \mu)$, for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, and similarly $\check{v}_{\varepsilon, n, m}$, $\check{f}_{n, m}$, \check{f} from $v_{\varepsilon, n, m}$, $f_{n, m}$, f , respectively. We also define $\check{g}(x, \mu) := e^{\sqrt{\delta}(T+1)}g(x, \mu)$ and $\check{g}_{n, m}(x, \mu) := e^{\sqrt{\delta}(T+1)}g_{n, m}(x, \mu)$, for every $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. We observe that \check{u}_1 is a viscosity subsolution of the following equation:

$$\begin{cases} \partial_t \check{u}_1(t, \mu) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \check{f}(t, x, \mu, a) + \frac{1}{2} \text{tr}[(\sigma \sigma^\top)(t, x, \mu, a) \partial_x \partial_\mu \check{u}_1(t, \mu)(x)] \right. \\ \left. + \langle b(t, x, \mu, a), \partial_\mu \check{u}_1(t, \mu)(x) \rangle \right\} \mu(dx) = \sqrt{\delta} \check{u}_1(t, \mu), & (t, \mu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \\ \check{u}_1(T, \mu) = \int_{\mathbb{R}^d} \check{g}(x, \mu) \mu(dx), & \mu \in \mathcal{P}_2(\mathbb{R}^d). \end{cases} \quad (5.2)$$

Moreover, by Theorem A.5 we deduce that $\check{v}_{\varepsilon, n, m}$ solves the following equation:

$$\begin{cases} \partial_t \check{v}_{\varepsilon, n, m}(t, \mu) - \sqrt{\delta} \check{v}_{\varepsilon, n, m}(t, \mu) + \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \langle b(t, x_i, \widehat{\mu}^{n, \bar{x}}, a_i), \partial_{x_i} \check{v}_{\varepsilon, n, m}(t, \bar{x}) \rangle \right. \\ \left. + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(t, x_i, \widehat{\mu}^{n, \bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \check{v}_{\varepsilon, n, m}(t, \bar{x}) \right] \right. \\ \left. + \frac{1}{n} \check{f}_{n, m}(t, x_1, \dots, x_n, a_i) \right\} \mu(dx_1) \otimes \dots \otimes \mu(dx_n) = 0, & (t, \mu) \in [0, T) \times \mathcal{P}_2(\mathbb{R}^d), \\ \check{v}_{\varepsilon, n, m}(T, \mu) = \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^{dn}} \check{g}_{n, m}(x_1, \dots, x_n) \mu(dx_1) \otimes \dots \otimes \mu(dx_n), & \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{cases} \quad (5.3)$$

where $\widehat{\mu}^{n, \bar{x}}$ is given by (A.9) and $\check{v}_{\varepsilon, n, m}(t, \bar{x}) := e^{\sqrt{\delta}(t+1)}\bar{v}_{\varepsilon, n, m}(t, \bar{x})$, for every $(t, \bar{x}) \in [0, T] \times \mathbb{R}^{dn}$, with $\bar{v}_{\varepsilon, n, m}$ being the same function appearing in Theorem A.5.

Finally, notice that $v_{\varepsilon, n, m}$ is bounded by a constant independent of ε, n, m . As a consequence, there exists a constant $\lambda \geq 0$, independent of ε, n, m , satisfying

$$\sup(\check{u}_1 - \check{v}_{\varepsilon, n, m}) \leq (\check{u}_1 - \check{v}_{\varepsilon, n, m})(t_0, \mu_0) + \lambda. \quad (5.4)$$

STEP III. Since $\check{u}_1 - \check{v}_{\varepsilon, n, m}$ is bounded and continuous, by (5.4) and Theorem 4.5 with $G = \check{u}_1 - \check{v}_{\varepsilon, n, m}$, we obtain that for any $\delta > 0$ there exists $\{(t_k, \mu_k)\}_{k \geq 1} \subset [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ converging to some $(\tilde{t}, \tilde{\mu}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, such that the following properties hold.

- i) $\rho_2((t_k, \mu_k), (\tilde{t}, \tilde{\mu})) \leq \frac{\lambda}{2^k \delta}$, for any $k \geq 0$, where ρ_2 is given by (4.5).
- ii) $(\check{u}_1 - \check{v}_{\varepsilon, n, m})(t_0, \mu_0) \leq (\check{u}_1 - (\check{v}_{\varepsilon, n, m} + \delta \varphi))(\tilde{t}, \tilde{\mu})$, with φ defined as

$$\varphi(t, \mu) := \sum_{k=0}^{+\infty} \frac{1}{2^k} \rho_2((t, \mu), (t_k, \mu_k)) \quad \forall (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d).$$

iii) It holds that

$$(\check{u}_1 - (\check{v}_{\varepsilon,n,m} + \delta\varphi))(\tilde{t}, \tilde{\mu}) = \sup_{(t,x) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)} (\check{u}_1 - (\check{v}_{\varepsilon,n,m} + \delta\varphi))(t, \mu). \quad (5.5)$$

We also have $\varphi \in C_2^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$ and φ satisfies items 1)-2)-3)-4) of Theorem 4.5. Notice that, taking $k = 0$ in item i), we find

$$\sqrt{\delta} |\tilde{t} - t_0| \leq \sqrt{\lambda}, \quad \forall \delta > 0. \quad (5.6)$$

Moreover, recall from item 1) of Lemma 4.4 that ρ_2 is a gauge-type function on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, with $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$. As a consequence, by item c) of Definition 4.1, there exists $\eta > 0$ (not depending on $\varepsilon, n, m, \delta$) such that, for all $(t, \mu), (s, \nu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, the inequality $\rho_2((t, \mu), (s, \nu)) \leq \eta$ implies $|t - s| + \mathcal{W}_2^{(1)}(\mu, \nu) < 1$. Hence, taking $k = 0$ in item i) above, we get

$$\mathcal{W}_2^{(1)}(\tilde{\mu}, \mu_0) \leq 1, \quad \forall \delta \geq \frac{\lambda}{\eta}.$$

Finally, by [38, Lemma 1] we obtain

$$\mathcal{W}_2(\tilde{\mu}, \mu_0) \leq \mathcal{W}_2^{(1)}(\tilde{\mu}, \mu_0) + 2\sqrt{d+2} \leq 1 + 2\sqrt{d+2}, \quad \forall \delta \geq \frac{\lambda}{\eta}. \quad (5.7)$$

STEP IV. It holds that $\tilde{t} < T$, for δ large enough. Indeed, when $\delta \rightarrow +\infty$, we get from item i) of STEP III with $k = 0$ and the fact that ρ_2 is a gauge-type function on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ (with $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ endowed with the metric $((t, \mu), (s, \nu)) \mapsto |t - s| + \mathcal{W}_2^{(1)}(\mu, \nu)$) that $|\tilde{t} - t_0| + \mathcal{W}_2^{(1)}(\tilde{\mu}, \mu_0) \rightarrow 0$. Recalling that $t_0 < T$, we deduce that $\tilde{t} < T$, for δ large enough.

STEP V. From (5.5) and the fact that \check{u}_1 is a viscosity subsolution of (5.2), we find

$$\begin{aligned} & -\partial_t(\check{v}_{\varepsilon,n,m} + \delta\varphi)(\tilde{t}, \tilde{\mu}) - \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu(\check{v}_{\varepsilon,n,m} + \delta\varphi)(\tilde{t}, \tilde{\mu})(x) \rangle \right. \\ & \left. + \frac{1}{2} \text{tr}[(\sigma\sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu(\check{v}_{\varepsilon,n,m} + \delta\varphi)(\tilde{t}, \tilde{\mu})(x)] + \check{f}(\tilde{t}, x, \tilde{\mu}, a) \right\} \tilde{\mu}(dx) + \sqrt{\delta} \check{u}_1(\tilde{t}, \tilde{\mu}) \leq 0. \end{aligned}$$

Then

$$\begin{aligned} \sqrt{\delta} \check{u}_1(\tilde{t}, \tilde{\mu}) & \leq \delta \partial_t \varphi(\tilde{t}, \tilde{\mu}) + \delta \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x) \rangle \right. \\ & \left. + \frac{1}{2} \text{tr}[(\sigma\sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x)] \right\} \tilde{\mu}(dx) \\ & + \partial_t \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu}) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x) \rangle \right. \\ & \left. + \frac{1}{2} \text{tr}[(\sigma\sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x)] + \check{f}(\tilde{t}, x, \tilde{\mu}, a) \right\} \tilde{\mu}(dx). \end{aligned}$$

Recalling the $\check{v}_{\varepsilon,n,m}$ satisfies equation (5.3), we obtain

$$\begin{aligned}
& \sqrt{\delta} (\check{u}_1 - \check{v}_{\varepsilon,n,m})(\tilde{t}, \tilde{\mu}) \leq \delta \partial_t \varphi(\tilde{t}, \tilde{\mu}) \\
& + \delta \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x) \rangle + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x)] \right\} \tilde{\mu}(dx) \\
& + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x) \rangle + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x)] \right. \\
& + \check{f}(\tilde{t}, x, \tilde{\mu}, a) \left. \right\} \tilde{\mu}(dx) - \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} \check{f}_{n,m}(\tilde{t}, \bar{x}, a_i) + \langle b(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x}) \rangle \right. \\
& + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x}) \right] \left. \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n),
\end{aligned} \tag{5.8}$$

with $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^{dn}$. Subtracting $\delta^{3/2} \varphi(\tilde{t}, \tilde{\mu})$ from the left-hand side of (5.8), recalling item ii) of STEP III and that $\varphi \geq 0$, we find

$$\begin{aligned}
& \sqrt{\delta} e^{\sqrt{\delta}(t_0+1)} (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) = \sqrt{\delta} (\check{u}_1 - \check{v}_{\varepsilon,n,m})(t_0, \mu_0) \\
& \leq \delta \partial_t \varphi(\tilde{t}, \tilde{\mu}) + \delta \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x) \rangle \right. \\
& + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x)] \left. \right\} \tilde{\mu}(dx) \\
& + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x) \rangle + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x)] \right. \\
& + \check{f}(\tilde{t}, x, \tilde{\mu}, a) \left. \right\} \tilde{\mu}(dx) - \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} \check{f}_{n,m}(\tilde{t}, \bar{x}, a_i) + \langle b(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x}) \rangle \right. \\
& + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x}) \right] \left. \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n).
\end{aligned}$$

Now, recalling that b and σ are bounded, by items 2)-3)-4) of Theorem 4.5 we deduce that the quantity

$$\partial_t \varphi(\tilde{t}, \tilde{\mu}) + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x) \rangle + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \varphi(\tilde{t}, \tilde{\mu})(x)] \right\} \tilde{\mu}(dx)$$

is bounded by $M(1 + \int_{\mathbb{R}^d} |x|^2 \tilde{\mu}(dx))$, with $M \geq 0$ being a constant independent of $\varepsilon, n, m, \delta$.

Hence

$$\begin{aligned}
& \sqrt{\delta} e^{\sqrt{\delta}(t_0+1)} (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \delta M \left(1 + \int_{\mathbb{R}^d} |x|^2 \tilde{\mu}(dx) \right) \\
& + \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ \langle b(\tilde{t}, x, \tilde{\mu}, a), \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x) \rangle + \frac{1}{2} \text{tr} [(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a) \partial_x \partial_\mu \check{v}_{\varepsilon,n,m}(\tilde{t}, \tilde{\mu})(x)] \right. \\
& + \check{f}(\tilde{t}, x, \tilde{\mu}, a) \left. \right\} \tilde{\mu}(dx) - \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} \check{f}_{n,m}(\tilde{t}, \bar{x}, a_i) + \langle b(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x}) \rangle \right. \\
& + \left. \frac{1}{2} \text{tr} [((\sigma \sigma^\top)(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \check{v}_{\varepsilon,n,m}(\tilde{t}, \bar{x})] \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n).
\end{aligned}$$

By estimates (A.6) and (A.7), we have

$$\begin{aligned}
& \sqrt{\delta} e^{\sqrt{\delta}(t_0+1)} (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \delta M \left(1 + \int_{\mathbb{R}^d} |x|^2 \tilde{\mu}(dx) \right) \\
& + e^{\sqrt{\delta}(\tilde{t}+1)} C_{\varepsilon,n,m} \int_{\mathbb{R}^d} \sup_{a \in A} \left\{ |b(\tilde{t}, x, \tilde{\mu}, a)| + \frac{1}{2} |(\sigma \sigma^\top)(\tilde{t}, x, \tilde{\mu}, a)| + |f(\tilde{t}, x, \tilde{\mu}, a)| \right\} \mu(dx) \\
& + e^{\sqrt{\delta}(\tilde{t}+1)} \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} |f_{n,m}(\tilde{t}, \bar{x}, a_i)| + |b(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i)| \right. \\
& + \left. \frac{1}{2} |(\sigma \sigma^\top)(\tilde{t}, x_i, \hat{\mu}^{n,\bar{x}}, a_i)| \right\} \tilde{\mu}(dx_1) \otimes \cdots \otimes \tilde{\mu}(dx_n),
\end{aligned}$$

with $C_{\varepsilon,n,m}$ as in (A.6)-(A.7). Since $b, \sigma, f, f_{n,m}$ are uniformly bounded, we obtain (possibly enlarging $C_{\varepsilon,n,m}$)

$$\sqrt{\delta} e^{\sqrt{\delta}(t_0+1)} (u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \delta M \left(1 + \int_{\mathbb{R}^d} |x|^2 \tilde{\mu}(dx) \right) + e^{\sqrt{\delta}(\tilde{t}+1)} C_{\varepsilon,n,m}.$$

Then

$$(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq \sqrt{\delta} e^{-\sqrt{\delta}(t_0+1)} M \left(1 + \int_{\mathbb{R}^d} |x|^2 \tilde{\mu}(dx) \right) + \frac{1}{\sqrt{\delta}} e^{\sqrt{\delta}(\tilde{t}-t_0)} C_{\varepsilon,n,m}.$$

Letting $\delta \rightarrow +\infty$, and using (5.6) and (5.7), we find

$$(u_1 - v_{\varepsilon,n,m})(t_0, \mu_0) \leq 0.$$

Recalling Lemma A.2 and Theorem A.4 letting $m \rightarrow +\infty$, $n \rightarrow +\infty$ and then $\varepsilon \rightarrow 0^+$, we conclude that

$$(u_1 - v_0)(t_0, \mu_0) \leq 0,$$

which contradicts (5.1). \square

Corollary 5.2 (Uniqueness). *Let Assumptions (A), (B), (C) hold. Then, the value function v , given by (2.7), is the unique bounded and continuous viscosity solution of equation (3.3).*

Proof. From Proposition 2.5 and Theorem 3.8 we know that v is bounded, continuous, and it is a viscosity solution of equation (3.3). Now, let u be another bounded and continuous viscosity solution of equation (3.3). Then, by Theorem 5.1 we deduce that $u \leq v$ and $v \leq u$ (in fact, both v and u are viscosity sub/supersolution of equation (3.3)), from which we conclude that $v \equiv u$. \square

A Smooth finite-dimensional approximations of the value function

A.1 Mean field control problem on a different probabilistic setting and approximation by non-degenerate control problems

In the present appendix we formulate the mean field control problem on a different probabilistic setting, supporting an independent d -dimensional Brownian motion \hat{W} .

Let $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ be a complete probability space on which a m -dimensional Brownian motion $\hat{B} = (\hat{B}_t)_{t \geq 0}$ and a d -dimensional Brownian motion $\hat{W} = (\hat{W}_t)_{t \geq 0}$ are defined, with \hat{B} and \hat{W} being independent. We denote by $\hat{\mathbb{F}}^{B,W} = (\hat{\mathcal{F}}_t^{B,W})_{t \geq 0}$ the $\hat{\mathbb{P}}$ -completion of the filtration generated by \hat{B} and \hat{W} . We also assume that there exists a sub- σ -algebra $\hat{\mathcal{G}}$ of $\hat{\mathcal{F}}$ satisfying the following properties.

- i) $\hat{\mathcal{G}}$ and $\hat{\mathcal{F}}_\infty^{B,W}$ are independent.
- ii) $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi \text{ such that } \xi: \hat{\Omega} \rightarrow \mathbb{R}^d, \text{ with } \xi \text{ being } \hat{\mathcal{G}}\text{-measurable and } \hat{\mathbb{E}}|\xi|^2 < \infty\}$.

We denote by $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t \geq 0}$ the filtration defined as $\hat{\mathcal{F}}_t = \hat{\mathcal{G}} \vee \hat{\mathcal{F}}_t^{B,W}$, for all $t \geq 0$. Finally, we denote by $\hat{\mathcal{A}}$ the set of control processes, namely the family of all $\hat{\mathbb{F}}$ -progressively measurable processes $\hat{\alpha}: [0, T] \times \hat{\Omega} \rightarrow A$.

Now, for every $\varepsilon \geq 0$, $t \in [0, T]$, $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$, $\hat{\alpha} \in \hat{\mathcal{A}}$, let $\hat{X}^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} = (\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}})_{s \in [t, T]}$ be the unique solution to the following controlled McKean-Vlasov stochastic differential equation:

$$\hat{X}_s = \hat{\xi} + \int_t^s b(r, \hat{X}_r, \mathbb{P}_{\hat{X}_r}, \hat{\alpha}_r) dr + \int_t^s \sigma(r, \hat{X}_r, \mathbb{P}_{\hat{X}_r}, \hat{\alpha}_r) d\hat{B}_r + \varepsilon(\hat{W}_s - \hat{W}_t), \quad \forall s \in [t, T].$$

Moreover, consider the lifted value function

$$V_\varepsilon(t, \hat{\xi}) = \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \hat{\mathbb{E}} \left[\int_t^s f(r, \hat{X}_r^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_r^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}}, \hat{\alpha}_r) dr + g(\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}}) \right],$$

for every $t \in [0, T]$, $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$. Under Assumption (A), from Theorem 2.4 applied in the present probabilistic setting, with σ and B replaced respectively by $(\sigma, \varepsilon I_d)$ and (\hat{B}, \hat{W}) , we know that V_ε satisfies the law invariance property. Therefore we can define the value function $v_\varepsilon: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ as follows:

$$v_\varepsilon(t, \mu) = V_\varepsilon(t, \hat{\xi}), \tag{A.1}$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and any $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$ such that $\mathbb{P}_{\hat{\xi}} = \mu$. Moreover, applying Proposition 2.5 in the present probabilistic setting, it follows that v_ε is bounded, jointly continuous on $[0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, and uniformly continuous: for any $\eta > 0$ there exists $\delta > 0$ such that

$$\mathcal{W}_2(\mu, \mu') \leq \delta \implies |v_\varepsilon(t, \mu) - v_\varepsilon(t', \mu')| \leq \eta,$$

for any $t \in [0, T]$ and $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$.

Remark A.1. Notice that, under Assumption (A), it is not immediately clear if $v_0 \equiv v$. However, under Assumptions (A) and (B), applying Theorem 3.8 in the present probabilistic setting we deduce that v_0 is a viscosity solution of the Master Bellman equation (3.3). As a consequence, under Assumptions (A)-(B)-(C), by Corollary 5.2 we conclude that $v_0 \equiv v$.

Lemma A.2. Suppose that Assumptions (A) and (A') hold. Then, there exists a function $\rho: [0, +\infty) \rightarrow [0, +\infty)$ such that $\rho(0) = 0$,

$$\rho(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 0,$$

and, for every $\varepsilon \geq 0$,

$$|v_\varepsilon(t, \mu) - v_0(t, \mu)| \leq \rho(\varepsilon),$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.

If in addition Assumption (A') holds, then there exists a constant $C_{K,T} \geq 0$, depending only on K and T , such that, for every $\varepsilon \geq 0$,

$$|v_\varepsilon(t, \mu) - v_0(t, \mu)| \leq C_{K,T} \varepsilon,$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$.

Proof. By usual calculations (as in [30, Theorem 2.5.9]), we obtain

$$\hat{\mathbb{E}} \left[\sup_{t \leq s \leq T} |\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}|^2 \right] \leq C_K T e^{C_K T} \varepsilon^2, \quad (\text{A.2})$$

for every $\varepsilon \geq 0$, $t \in [0, T]$, $\hat{\xi} \in L^2(\hat{\Omega}, \hat{\mathcal{F}}_t, \hat{\mathbb{P}}; \mathbb{R}^d)$, $\hat{\alpha} \in \hat{\mathcal{A}}$, for some constant $C_K \geq 0$, depending only on K . From (A.2) and Markov's inequality, there exists a constant $C > 0$ such that

$$\begin{aligned} |v_\varepsilon(t, \mu) - v_0(t, \mu)| &\leq \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \hat{\mathbb{E}} \left[\left(\int_t^T |f(s, \hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s}) - f(s, \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s})| ds \right. \right. \\ &\quad \left. \left. + |g(\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s}) - g(\hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s})| \right) \mathbb{1}_{\sup_{t \leq s \leq T} |\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}| \leq \delta} \right] \\ &\quad + C \frac{\varepsilon}{\delta}, \end{aligned}$$

for any $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ and $\delta > 0$. From **(A')**, we get a constant C' such that for any $\eta > 0$ there exists $\delta = \delta(\eta) > 0$ such that

$$|v_\varepsilon(t, \mu) - v_0(t, \mu)| \leq C' \left(\eta + \frac{\varepsilon}{\delta} \right),$$

for any $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. The claim follows taking $\varepsilon \leq \eta\delta(\eta)$. Now, suppose that Assumption **(A')** holds. Then, we have

$$\begin{aligned} |v_\varepsilon(t, \mu) - v_0(t, \mu)| &\leq \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \hat{\mathbb{E}} \left[\int_t^T |f(s, \hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s} - f(s, \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}, \hat{\alpha}_s})| ds \right. \\ &\quad \left. + |g(\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}} - g(\hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}, \mathbb{P}_{\hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}})| \right] \\ &\leq K \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \left\{ \int_t^T \left\{ \hat{\mathbb{E}}[|\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}|] + \mathcal{W}_2(\mathbb{P}_{\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}}, \mathbb{P}_{\hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}}) \right\} ds \right. \\ &\quad \left. + \hat{\mathbb{E}}[|\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}|] + \mathcal{W}_2(\mathbb{P}_{\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}}}, \mathbb{P}_{\hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}}) \right\} \\ &\leq K \sup_{\hat{\alpha} \in \hat{\mathcal{A}}} \left\{ \int_t^T \left\{ \hat{\mathbb{E}}[|\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}|^2]^{1/2} + \mathbb{E}[|\hat{X}_s^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_s^{0, t, \hat{\xi}, \hat{\alpha}}|^2]^{1/2} \right\} ds \right. \\ &\quad \left. + \hat{\mathbb{E}}[|\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}|^2]^{1/2} + \mathbb{E}[|\hat{X}_T^{\varepsilon, t, \hat{\xi}, \hat{\alpha}} - \hat{X}_T^{0, t, \hat{\xi}, \hat{\alpha}}|^2]^{1/2} \right\} \\ &\leq 2K(T+1) \sqrt{C_K T e^{C_K T} \varepsilon}, \end{aligned}$$

where the last inequality follows from estimate **(A.2)**. \square

A.2 Cooperative n -player stochastic differential game and propagation of chaos result

Let $n \in \mathbb{N}$ and let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a complete probability space, supporting independent Brownian motions $\bar{B}^1, \dots, \bar{B}^n, \bar{W}^1, \dots, \bar{W}^n$, with \bar{B}^i (resp. \bar{W}^i) being m -dimensional (resp. d -dimensional). Let also $\bar{\mathbb{F}}^{B, W} = (\bar{\mathcal{F}}_t^{B, W})_{t \geq 0}$ denote the $\bar{\mathbb{P}}$ -completion of the filtration generated by \bar{B} and \bar{W} , with $\bar{B} = (\bar{B}^1, \dots, \bar{B}^n)$ and $\bar{W} = (\bar{W}^1, \dots, \bar{W}^n)$. Moreover, let $\bar{\mathcal{G}}$ be a sub- σ -algebra of $\bar{\mathcal{F}}$ satisfying the following properties.

i) $\bar{\mathcal{G}}$ and $\bar{\mathcal{F}}_\infty^{B, W}$ are independent.

ii) $\mathcal{P}_2(\mathbb{R}^d) = \{\mathbb{P}_\xi \text{ such that } \bar{\xi}: \bar{\Omega} \rightarrow \mathbb{R}^d, \text{ with } \bar{\xi} \text{ being } \bar{\mathcal{G}}\text{-measurable and } \bar{\mathbb{E}}|\bar{\xi}|^2 < \infty\}$.

Furthermore, let $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t \geq 0}$ be given by $\bar{\mathcal{F}}_t := \bar{\mathcal{G}} \vee \bar{\mathcal{F}}_t^B$, for every $t \geq 0$. Finally, let $\bar{\mathcal{A}}^n$ be the family of all $\bar{\mathbb{F}}$ -progressively measurable processes $\bar{\alpha} = (\bar{\alpha}^1, \dots, \bar{\alpha}^n): [0, T] \times \bar{\Omega} \rightarrow A^n$. Now, for every $\varepsilon > 0$, $t \in [0, T]$, $\bar{\alpha} \in \bar{\mathcal{A}}^n$, $\bar{\xi}^1, \dots, \bar{\xi}^n \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^d)$, with $\bar{\xi} := (\bar{\xi}^1, \dots, \bar{\xi}^n)$, let

$\bar{X}^{\varepsilon,t,\bar{\xi},\bar{\alpha}} = (\bar{X}^{1,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \dots, \bar{X}^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}})$ be the unique solution to the following system of controlled stochastic differential equations:

$$\bar{X}_s^i = \bar{\xi}^i + \int_t^s b(r, \bar{X}_r^i, \hat{\mu}_r^n, \bar{\alpha}_r^i) dr + \int_t^s \sigma(r, \bar{X}_r^i, \hat{\mu}_r^n, \bar{\alpha}_r^i) dB_r^i + \varepsilon (\bar{W}_s^i - \bar{W}_t^i), \quad \forall s \in [t, T],$$

for $i = 1, \dots, n$, with

$$\hat{\mu}_r^n = \frac{1}{n} \sum_{j=1}^n \delta_{\bar{X}_r^j}, \quad \forall r \in [t, T].$$

We denote $\hat{\mu}_r^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}} = \frac{1}{n} \sum_{j=1}^n \delta_{\bar{X}_r^{j,\varepsilon,t,\bar{\xi},\bar{\alpha}}}$. Then, the value function $\tilde{v}_{\varepsilon,n}: [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn}) \rightarrow \mathbb{R}$ of the cooperative n -player game is given by

$$\tilde{v}_{\varepsilon,n}(t, \bar{\mu}) = \sup_{\bar{\alpha} \in \bar{\mathcal{A}}^n} \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f(s, \bar{X}_s^{i,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \hat{\mu}_s^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \bar{\alpha}_s^i) ds + g(\bar{X}_T^{i,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \hat{\mu}_T^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}) \right], \quad (\text{A.3})$$

for every $t \in [0, T]$, $\bar{\mu} \in \mathcal{P}_2(\mathbb{R}^{dn})$, with $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$ such that $\mathbb{P}_{\bar{\xi}} = \bar{\mu}$.

We also introduce the approximated value function $\tilde{v}_{\varepsilon,n,m}: [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn}) \rightarrow \mathbb{R}$ of the cooperative n -player game, which is given by

$$\begin{aligned} \tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) = \sup_{\bar{\alpha} \in \bar{\mathcal{A}}^n} \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}) ds \right. \\ \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}) \right], \end{aligned}$$

where $f_{n,m}: [0, T] \times \mathbb{R}^{dn} \times A \rightarrow \mathbb{R}$ and $g_{n,m}: \mathbb{R}^{dn} \rightarrow \mathbb{R}$ are smooth approximations of f and g defined as follows:

$$\begin{aligned} f_{n,m}(t, x_1, \dots, x_n, a) &= m^{nd} \int_{\mathbb{R}^{dn}} f \left(t, y_i - x_i, \frac{1}{n} \sum_{j=1}^n \delta_{y_j - x_j}, a \right) \prod_{j=1}^n \Phi(my_j) dy_j, \\ g_{n,m}(x_1, \dots, x_n) &= m^{nd} \int_{\mathbb{R}^{dn}} g \left(y_i - x_i, \frac{1}{n} \sum_{j=1}^n \delta_{y_j - x_j} \right) \prod_{j=1}^n \Phi(my_j) dy_j, \end{aligned}$$

for all $n, m \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{R}^d$, $(t, a) \in [0, T] \times A$, with $\Phi: \mathbb{R}^d \rightarrow [0, +\infty)$ being a C^∞ function with compact support satisfying $\int_{\mathbb{R}^d} \Phi(y) dy = 1$.

Remark A.3. Notice that it is not a priori clear the fact that the right-hand side of (A.3) depends on $\bar{\xi}$ only through its law $\bar{\mu}$. However, as the cooperative n -player game is an example of mean field control problem (indeed, it is a standard stochastic optimal control problem) we can apply the results of Section 2 to it. In particular, from Theorem 2.4 we deduce the law invariance property, which explains why we can consider the value function $\tilde{v}_{\varepsilon,n}$ (and, similarly, $\tilde{v}_{\varepsilon,n,m}$), which depends only on $\bar{\mu}$ rather than on $\bar{\xi}$.

We also consider the functions $v_{\varepsilon,n,m}, v_{\varepsilon,n}: [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \rightarrow \mathbb{R}$ defined as

$$v_{\varepsilon,n,m}(t, \mu) = \tilde{v}_{\varepsilon,n,m}(t, \mu \otimes \cdots \otimes \mu) \quad \text{and} \quad v_{\varepsilon,n}(t, \mu) = \tilde{v}_{\varepsilon,n}(t, \mu \otimes \cdots \otimes \mu), \quad (\text{A.4})$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. We can then state the following propagation of chaos result for $v_{\varepsilon,n,m}$ (a more general propagation of chaos result holds for $\tilde{v}_{\varepsilon,n,m}$, see [20, Theorem 3.3]).

Theorem A.4. *Suppose that Assumptions (A) and (C) hold. Let $\varepsilon > 0$ and $(t, \mu) \in \mathcal{P}_2(\mathbb{R}^d)$. If there exists $q > 2$ such that $\mu \in \mathcal{P}_q(\mathbb{R}^d)$, then*

$$\lim_{n \rightarrow +\infty} \lim_{m \rightarrow +\infty} v_{\varepsilon,n,m}(t, \mu) = v_{\varepsilon}(t, \mu).$$

Proof. Since f and g satisfy Assumptions (A), from the definitions of $f_{n,m}$ and $g_{n,m}$ we easily get the convergence

$$\lim_{m \rightarrow +\infty} v_{\varepsilon,n,m}(t, \mu) = v_{\varepsilon,n}(t, \mu).$$

Then, the convergence

$$\lim_{n \rightarrow +\infty} v_{\varepsilon,n}(t, \mu) = v_{\varepsilon}(t, \mu)$$

is a consequence of Theorem 3.3 in [20]. □

A.3 Smooth finite-dimensional approximations

We consider the same probabilistic setting as in Section A.2.

Theorem A.5. *Suppose that Assumptions (A) and (B) hold. Then, for every $\varepsilon > 0$, $n, m \in \mathbb{N}$, there exists $\bar{v}_{\varepsilon,n,m}: [0, T] \times \mathbb{R}^{dn} \rightarrow \mathbb{R}$ such that*

$$v_{\varepsilon,n,m}(t, \mu) = \int_{\mathbb{R}^{dn}} \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \mu(dx_1) \cdots \mu(dx_n), \quad (\text{A.5})$$

for every $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$, with $v_{\varepsilon,n,m}$ given by (A.4), and the following holds.

- 1) $\bar{v}_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathbb{R}^{dn})$ and $v_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$.
- 2) For all $(t, \mu, \bar{x}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^{dn}$, it holds that

$$|\partial_{\bar{x}} \bar{v}_{\varepsilon,n,m}(t, \bar{x})| + |\partial_{\bar{x}\bar{x}}^2 \bar{v}_{\varepsilon,n,m}(t, \bar{x})| \leq C_{\varepsilon,n,m}, \quad (\text{A.6})$$

$$|\partial_{\mu} v_{\varepsilon,n,m}(t, \mu)(x)| + |\partial_x \partial_{\mu} v_{\varepsilon,n,m}(t, \mu)(x)| \leq C_{\varepsilon,n,m}, \quad (\text{A.7})$$

for some constant $C_{\varepsilon,n,m} \geq 0$, possibly depending on ε, n, m .

3) $v_{\varepsilon,n,m}$ solves the following equation:

$$\left\{ \begin{array}{l} \partial_t v_{\varepsilon,n,m}(t, \mu) + \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \langle b(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \rangle \right. \\ \left. + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right] \right. \\ \left. + \frac{1}{n} f_{n,m}(t, x_1, \dots, x_n, a_i) \right\} \mu(dx_1) \otimes \dots \otimes \mu(dx_n) = 0, \quad (t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d), \\ v_{\varepsilon,n,m}(T, \mu) = \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^{dn}} g_{n,m}(x_1, \dots, x_n) \mu(dx_1) \otimes \dots \otimes \mu(dx_n), \quad \mu \in \mathcal{P}_2(\mathbb{R}^d), \end{array} \right. \quad (\text{A.8})$$

with

$$\widehat{\mu}^{n,\bar{x}} := \frac{1}{n} \sum_{j=1}^n \delta_{x_j} \quad (\text{A.9})$$

for every $n, m \in \mathbb{N}$, $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^{dn}$ and $x_1, \dots, x_n \in \mathbb{R}^d$.

Proof. We split the proof into four steps.

Step I. Definition of $\bar{v}_{\varepsilon,n,m}$ and its properties. Fix $\varepsilon > 0$ and $n, m \in \mathbb{N}$. For every $t \in [0, T]$, $\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^{dn}$, let $\bar{v}_{\varepsilon,n,m}: [0, T] \times \mathbb{R}^{dn} \rightarrow \mathbb{R}$ be given by

$$\bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) = \tilde{v}_{\varepsilon,n,m}(t, \delta_{x_1} \otimes \dots \otimes \delta_{x_n}), \quad (\text{A.10})$$

with $\tilde{v}_{\varepsilon,n,m}$ defined by (A.3). In other words, $\bar{v}_{\varepsilon,n,m}$ corresponds to the value function of the cooperative n -player game (see Section A.2) with deterministic initial state \bar{x} in place of the random vector $\tilde{\xi}$. Hence

$$\begin{aligned} & \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \\ &= \sup_{\bar{\alpha} \in \bar{A}^n} \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x},\bar{\alpha}}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x},\bar{\alpha}}, \bar{\alpha}_s^i) ds + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x},\bar{\alpha}}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x},\bar{\alpha}}) \right]. \end{aligned}$$

This optimal control problem involve coefficients satisfying Assumption (A). Therefore $\bar{v}_{\varepsilon,n,m}$ is bounded, jointly continuous, and Lipschitz with respect to \bar{x} . Moreover, $\bar{v}_{\varepsilon,n,m}$ is a viscosity solution of the following Bellman equation:

$$\left\{ \begin{array}{l} \partial_t \bar{v}_{\varepsilon,n,m}(t, \bar{x}) + \sup_{(a_1, \dots, a_n) \in A^n} \left\{ \frac{1}{n} \sum_{i=1}^n f_{n,m}(t, \bar{x}, a_i) \right. \\ \left. + \sum_{i=1}^n \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right] \right. \\ \left. + \sum_{i=1}^n \langle b(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \rangle \right\} = 0, \quad \forall (t, \bar{x}) \in [0, T] \times \mathbb{R}^{dn}, \\ \bar{v}_{\varepsilon,n,m}(t, \bar{x}) = \frac{1}{n} \sum_{i=1}^n g_{n,m}(\bar{x}), \quad \forall \bar{x} \in \mathbb{R}^{dn}, \end{array} \right. \quad (\text{A.11})$$

with $\widehat{\mu}^{n,\bar{x}}$ given by (A.9). We notice that equation (A.11) is uniformly parabolic, with coefficients satisfying Assumptions (A) and (B). It follows that $\bar{v}_{\varepsilon,n,m} \in C^{1,2}([0, T] \times \mathbb{R}^{dn})$ (see [34, Theorem 14.15] and the comments just below Theorem 14.15 regarding the case with linear operators “ L_ν ”). In addition, from [34, Theorem 14.18] we deduce that estimate (A.6) holds. More precisely, notice that [34, Theorem 14.18] gives an estimate for the supremum norm of the derivatives on a bounded domain, since it is formulated for a parabolic PDE on a bounded domain, while here we need a pointwise estimate and we work on the entire space. Then, in order to deduce (A.6) from [34, Theorem 14.18], we can use a localization argument (proceeding for instance as in [35, see just after Corollary II.2]) to go from our PDE on the entire space to a PDE on a ball of radius 1 centered at a generic point (t, \bar{x}) . Since our coefficients are globally bounded and Hölder continuous, such an estimate does not depend on (t, \bar{x}) , therefore we conclude that (A.6) holds.

Finally, we observe that equation (A.11) can be equivalently written as

$$\begin{aligned} \partial_t \bar{v}_{\varepsilon,n,m}(t, \bar{x}) + \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} f_{n,m}(t, \bar{x}, a_i) + \langle b(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i), \partial_{x_i} \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \rangle \right. \\ \left. + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(t, x_i, \widehat{\mu}^{n,\bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \bar{v}_{\varepsilon,n,m}(t, \bar{x}) \right] \right\} = 0, \end{aligned} \quad (\text{A.12})$$

for all $(t, \bar{x}) \in [0, T] \times \mathbb{R}^{dn}$.

Step II. Proof of equality (A.5). We prove the more general equality

$$\tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) = \int_{\mathbb{R}^{dn}} \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \bar{\mu}(dx_1, \dots, dx_n), \quad (\text{A.13})$$

for every $(t, \bar{\mu}) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^{dn})$, from which (A.5) follows. Notice that equality (A.13) can be equivalently written as

$$\tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) = \mathbb{E}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})],$$

for every $t \in [0, T]$, $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$, with $\mathbb{P}_{\bar{\xi}} = \bar{\mu}$. We split the rest of the proof of Step II into two substeps.

Step II-a. General case: $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$. Observe that we can apply Proposition 2.5 to the cooperative n -player game, from which we deduce that $\tilde{v}_{\varepsilon,n,m}$ is bounded, jointly continuous, and Lipschitz with respect to $\bar{\mu}$. Moreover, recall from Step I above that $\bar{v}_{\varepsilon,n,m}$ is also bounded, jointly continuous, and Lipschitz with respect to \bar{x} . As a consequence, the general case with $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$ can be deduced, relying on an approximation argument, from the case where $\bar{\xi}$ takes only a finite number of values, namely from the next Step II-b.

Step II-b. $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$ taking only a finite number of values. Firstly, we fix some notation. For every $t \in [0, T]$, let $\bar{\mathbb{F}}^{B,W,t} = (\bar{\mathcal{F}}_s^{B,W,t})_{s \geq 0}$ be the $\bar{\mathbb{P}}$ -completion of the filtration generated by $(\bar{B}_{s \vee t} - \bar{B}_t)_{s \geq 0}$ and $(\bar{W}_{s \vee t} - \bar{W}_t)_{s \geq 0}$, where we recall that $\bar{B} = (\bar{B}^1, \dots, \bar{B}^n)$ and $\bar{W} = (\bar{W}^1, \dots, \bar{W}^n)$. Let also $\text{Prog}(\bar{\mathbb{F}}^{B,W,t})$ denote the σ -algebra of $[0, T] \times \bar{\Omega}$ of all $\bar{\mathbb{F}}^{B,W,t}$ -progressive sets.

Proof of the inequality $\tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) \leq \mathbb{E}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})]$. Suppose that $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$ takes only a finite number of values. In such a case, by [15, Lemma B.3] there exists a $\bar{\mathcal{F}}_t$ -measurable random variable $\bar{U}: \bar{\Omega} \rightarrow \mathbb{R}$, having uniform distribution on $[0, 1]$ and being independent of $\bar{\xi}$. Then, by [15, Lemma B.2], for every $\bar{\alpha} \in \bar{\mathcal{A}}^n$ there exists a measurable function

$$a: ([0, T] \times \bar{\Omega} \times \mathbb{R}^{dn} \times [0, 1], \text{Prog}(\bar{\mathbb{F}}^{B,W,t}) \otimes \mathcal{B}(\mathbb{R}^{dn}) \otimes \mathcal{B}([0, 1])) \longrightarrow (A^n, \mathcal{B}(A^n))$$

such that $\bar{\beta} := (a_s(\bar{\xi}, \bar{U}))_{s \in [0, T]} \in \bar{\mathcal{A}}^n$ and

$$\begin{aligned} & \left(\bar{\xi}, (a_s(\bar{\xi}, \bar{U}))_{s \in [t, T]}, (\bar{B}_s - \bar{B}_t)_{s \in [t, T]}, (\bar{W}_s - \bar{W}_t)_{s \in [t, T]} \right) \\ & \stackrel{\mathcal{L}}{=} \left(\bar{\xi}, (\bar{\alpha}_s)_{s \in [t, T]}, (\bar{B}_s - \bar{B}_t)_{s \in [t, T]}, (\bar{W}_s - \bar{W}_t)_{s \in [t, T]} \right), \end{aligned}$$

where $\stackrel{\mathcal{L}}{=}$ stands for equality in law. As a consequence, proceeding along the same lines as in [21, Proposition 1.137], we deduce that

$$\left(\bar{X}_s^{\varepsilon, t, \bar{\xi}, \bar{\alpha}}, \bar{\alpha}_s \right)_{s \in [t, T]} \stackrel{\mathcal{L}}{=} \left(\bar{X}_s^{\varepsilon, t, \bar{\xi}, \bar{\beta}}, \bar{\beta}_s \right)_{s \in [t, T]}.$$

Moreover, since $\bar{\xi}$ takes only a finite number of values, it holds that

$$\bar{\xi} = \sum_{k=1}^K \bar{x}_k 1_{\bar{E}_k}, \quad (\text{A.14})$$

for some $K \in \mathbb{N}$, $\bar{x}_k \in \mathbb{R}^{dn}$, $\bar{E}_k \in \sigma(\bar{\xi})$, with $\{\bar{E}_k\}_{k=1, \dots, K}$ being a partition of $\bar{\Omega}$. Let also

$$\bar{\beta}_{k,s} := a_s(\bar{x}_k, \bar{U}), \quad \forall s \in [0, T], k = 1, \dots, K.$$

It is easy to see that $\bar{X}^{\varepsilon, t, \bar{\xi}, \bar{\beta}}$ and $\bar{X}^{\varepsilon, t, \bar{x}_1, \bar{\beta}_1} 1_{\bar{E}_1} + \dots + \bar{X}^{\varepsilon, t, \bar{x}_K, \bar{\beta}_K} 1_{\bar{E}_K}$ satisfy the same system of controlled stochastic differential equations, therefore, by pathwise uniqueness, they are $\bar{\mathbb{P}}$ -indistinguishable. Hence

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1, \varepsilon, t, \bar{\xi}, \bar{\alpha}}, \dots, \bar{X}_s^{n, \varepsilon, t, \bar{\xi}, \bar{\alpha}}, \bar{\alpha}_s^i) ds + g_{n,m}(\bar{X}_T^{1, \varepsilon, t, \bar{\xi}, \bar{\alpha}}, \dots, \bar{X}_T^{n, \varepsilon, t, \bar{\xi}, \bar{\alpha}}) \right] \\ & = \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1, \varepsilon, t, \bar{\xi}, \bar{\beta}}, \dots, \bar{X}_s^{n, \varepsilon, t, \bar{\xi}, \bar{\beta}}, \bar{\beta}_s^i) ds + g_{n,m}(\bar{X}_T^{1, \varepsilon, t, \bar{\xi}, \bar{\beta}}, \dots, \bar{X}_T^{n, \varepsilon, t, \bar{\xi}, \bar{\beta}}) \right] \\ & = \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\sum_{k=1}^K \left(\int_t^T f_{n,m}(s, \bar{X}_s^{1, \varepsilon, t, \bar{x}_k, \bar{\beta}_k}, \dots, \bar{X}_s^{n, \varepsilon, t, \bar{x}_k, \bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\ & \quad \left. \left. + g_{n,m}(\bar{X}_T^{1, \varepsilon, t, \bar{x}_k, \bar{\beta}_k}, \dots, \bar{X}_T^{n, \varepsilon, t, \bar{x}_k, \bar{\beta}_k}) \right) 1_{\bar{E}_k} \right]. \end{aligned}$$

Since both $\{\bar{X}^{t,\varepsilon,\bar{x}_k,\bar{\beta}_k}\}_k$ and $\{\bar{\beta}_k\}_k$ are independent of $\{\bar{E}_k\}_k$, we have

$$\begin{aligned}
& \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\sum_{k=1}^K \left(\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\
& \quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}) 1_{\bar{E}_k} \right) \right] \\
&= \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\sum_{k=1}^K \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\
& \quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}) \right) 1_{\bar{E}_k} \right] \\
&= \sum_{k=1}^K \bar{\mathbb{E}} \left[\frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\
& \quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}) \right) 1_{\bar{E}_k} \right] \leq \sum_{k=1}^K \bar{\mathbb{E}} \left[\bar{v}_{\varepsilon,n,m}(t, \bar{x}_k) 1_{\bar{E}_k} \right] = \bar{\mathbb{E}}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})].
\end{aligned}$$

As $\bar{\alpha}$ was arbitrary, we obtain (denoting by $\bar{\mu}$ the law of $\bar{\xi}$)

$$\begin{aligned}
\tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) &= \sup_{\bar{\alpha} \in \bar{\mathcal{A}}^n} \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \bar{\alpha}_s^i) ds \right. \\
& \quad \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{\xi},\bar{\alpha}}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{\xi},\bar{\alpha}}) \right] \leq \bar{\mathbb{E}}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})].
\end{aligned}$$

Proof of the inequality $\bar{\mathbb{E}}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})] \leq \tilde{v}_{\varepsilon,n,m}(t, \bar{\mu})$. Let $\bar{\mathcal{A}}_t^n$ be the subset of $\bar{\mathcal{A}}^n$ of all $\bar{\mathbb{F}}^{B,W,t}$ -progressively measurable processes $\bar{\alpha} = (\bar{\alpha}^1, \dots, \bar{\alpha}^n): [0, T] \times \bar{\Omega} \rightarrow A^n$. Then, it is well-known that the value function $\bar{v}_{\varepsilon,n,m}$ in (A.10) is also given by

$$\begin{aligned}
& \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \tag{A.15} \\
&= \sup_{\bar{\alpha} \in \bar{\mathcal{A}}_t^n} \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x},\bar{\alpha}}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x},\bar{\alpha}}, \bar{\alpha}_s^i) ds + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x},\bar{\alpha}}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x},\bar{\alpha}}) \right],
\end{aligned}$$

where the supremum is taken on $\bar{\mathcal{A}}_t^n$ rather than on $\bar{\mathcal{A}}^n$. Now, let $\bar{\xi} \in L^2(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{\mathbb{P}}; \mathbb{R}^{dn})$ be given by (A.14). By (A.15), for every $\delta > 0$ and $k = 1, \dots, K$, there exists $\bar{\beta}_k \in \bar{\mathcal{A}}_t^n$ (possibly depending on δ) such that

$$\begin{aligned}
\bar{v}_{\varepsilon,n,m}(t, \bar{x}_k) &\leq \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}} \left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \\
& \quad \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}) \right] + \delta.
\end{aligned}$$

Then, define

$$\bar{\beta} := \sum_{k=1}^K \bar{\beta}_k 1_{\bar{E}_k}.$$

Notice that $\bar{\beta} \in \bar{\mathcal{A}}^n$. Moreover, it is easy to see that $\bar{X}^{\varepsilon,t,\bar{\xi},\bar{\beta}}$ and $\bar{X}^{\varepsilon,t,\bar{x}_1,\bar{\beta}_1} 1_{\bar{E}_1} + \dots + \bar{X}^{\varepsilon,t,\bar{x}_K,\bar{\beta}_K} 1_{\bar{E}_K}$ satisfy the same system of controlled stochastic differential equations, therefore, by pathwise uniqueness, they are $\bar{\mathbb{P}}$ -indistinguishable. Hence (using the independence of both $\{\bar{X}^{t,\varepsilon,\bar{x}_k,\bar{\beta}_k}\}_k$ and $\{\bar{\beta}_k\}_k$ from $\{\bar{E}_k\}_k$)

$$\begin{aligned} \bar{\mathbb{E}}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})] &= \sum_{k=1}^K \bar{\mathbb{E}}\left[\bar{v}_{\varepsilon,n,m}(t, \bar{x}_k) 1_{\bar{E}_k}\right] \\ &\leq \sum_{k=1}^K \bar{\mathbb{E}}\left[\frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}}\left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\ &\quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k})\right] 1_{\bar{E}_k}\right] + \delta \\ &= \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}}\left[\sum_{k=1}^K \bar{\mathbb{E}}\left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\ &\quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k})\right] 1_{\bar{E}_k}\right] + \delta \\ &= \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}}\left[\sum_{k=1}^K \left(\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \bar{\beta}_{k,s}^i) ds \right. \right. \\ &\quad \left. \left. + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{x}_k,\bar{\beta}_k}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{x}_k,\bar{\beta}_k})\right) 1_{\bar{E}_k}\right] + \delta \\ &= \frac{1}{n} \sum_{i=1}^n \bar{\mathbb{E}}\left[\int_t^T f_{n,m}(s, \bar{X}_s^{1,\varepsilon,t,\bar{\xi},\bar{\beta}}, \dots, \bar{X}_s^{n,\varepsilon,t,\bar{\xi},\bar{\beta}}, \bar{\beta}_s^i) ds + g_{n,m}(\bar{X}_T^{1,\varepsilon,t,\bar{\xi},\bar{\beta}}, \dots, \bar{X}_T^{n,\varepsilon,t,\bar{\xi},\bar{\beta}})\right] + \delta \\ &\leq \tilde{v}_{\varepsilon,n,m}(t, \bar{\mu}) + \delta, \end{aligned}$$

with $\bar{\mu}$ being the law of $\bar{\xi}$. From the arbitrariness of δ , we conclude that the inequality $\bar{\mathbb{E}}[\bar{v}_{\varepsilon,n,m}(t, \bar{\xi})] \leq \tilde{v}_{\varepsilon,n,m}(t, \bar{\mu})$ holds.

Step III. Proof of item 1) and estimate (A.7). We begin noting that, by equality (A.5), we have

$$\partial_t v_{\varepsilon,n,m}(t, \mu) = \int_{\mathbb{R}^{dn}} \partial_t \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \mu(dx_1) \cdots \mu(dx_n), \quad (\text{A.16})$$

which proves that $\partial_t v_{\varepsilon,n,m}$ exists and is continuous. Now, for every $\varepsilon > 0$ and $n \geq 2$, let $\hat{v}_{\varepsilon,n,m} : [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n \rightarrow \mathbb{R}$ be given by

$$\hat{v}_{\varepsilon,n,m}(t, \mu_1, \dots, \mu_n) := \tilde{v}_{\varepsilon,n,m}(t, \mu_1 \otimes \cdots \otimes \mu_n) = \int_{\mathbb{R}^{dn}} \bar{v}_{\varepsilon,n,m}(t, x_1, \dots, x_n) \mu_1(dx_1) \cdots \mu_n(dx_n),$$

for every $(t, \mu_1, \dots, \mu_n) \in [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n$. Then, by direct calculation, we obtain

$$\begin{aligned} & \partial_{\mu_i} \hat{v}_{\varepsilon, n, m}(t, \mu_1, \dots, \mu_n)(x) \\ &= \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i} \bar{v}_{\varepsilon, n, m}(t, x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \mu_1(dx_1) \cdots \mu_{i-1}(dx_{i-1}) \mu_{i+1}(dx_{i+1}) \cdots \mu_n(dx_n), \end{aligned}$$

for every $(t, \mu_1, \dots, \mu_n, x) \in [0, T] \times (\mathcal{P}_2(\mathbb{R}^d))^n \times \mathbb{R}^d$, $i = 1, \dots, n$. Since $v_{\varepsilon, n, m}(t, \mu) = \hat{v}_{\varepsilon, n, m}(t, \mu, \dots, \mu)$, we obtain

$$\begin{aligned} & \partial_{\mu} v_{\varepsilon, n, m}(t, \mu)(x) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i} \bar{v}_{\varepsilon, n, m}(t, x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \mu(dx_1) \cdots \mu(dx_{i-1}) \mu(dx_{i+1}) \cdots \mu(dx_n), \end{aligned}$$

for every $(t, \mu, x) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. Hence

$$\begin{aligned} & \partial_x \partial_{\mu} v_{\varepsilon, n, m}(t, \mu)(x) \\ &= \sum_{i=1}^n \int_{\mathbb{R}^{d(n-1)}} \partial_{x_i}^2 \bar{v}_{\varepsilon, n, m}(t, x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) \mu(dx_1) \cdots \mu(dx_{i-1}) \mu(dx_{i+1}) \cdots \mu(dx_n), \end{aligned}$$

for every $(t, \mu, x) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d$. In conclusion, we see that $v_{\varepsilon, n, m} \in C^{1,2}([0, T] \times \mathcal{P}_2(\mathbb{R}^d))$. In addition, from (A.6) we deduce that (A.7) holds.

Step IV. Proof of item 3). Recall that $\bar{v}_{\varepsilon, n, m}$ solves equation (A.11). Fix $(t, \mu) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$. When $t = T$, integrating the terminal condition of (A.11) with respect to $\mu \otimes \cdots \otimes \mu$ on \mathbb{R}^{dn} , we get

$$v_{\varepsilon, n, m}(T, \mu) = \frac{1}{n} \sum_{i=1}^n \int_{\mathbb{R}^{dn}} g_{n, m}(x_1, \dots, x_n) \mu(dx_1) \otimes \cdots \otimes \mu(dx_n),$$

which corresponds to the terminal condition of equation (A.8). On the other hand, when $t < T$, integrating equation (A.12) with respect to $\mu \otimes \cdots \otimes \mu$ on \mathbb{R}^{dn} , and using (A.16), we find

$$\begin{aligned} & \partial_t v_{\varepsilon, n, m}(t, \mu) + \int_{\mathbb{R}^{dn}} \sum_{i=1}^n \sup_{a_i \in A} \left\{ \frac{1}{n} f_{n, m}(t, \bar{x}, a_i) + \langle b(t, x_i, \hat{\mu}^{n, \bar{x}}, a_i), \partial_{x_i} \bar{v}_{\varepsilon, n, m}(t, \bar{x}) \rangle \right. \\ & \left. + \frac{1}{2} \text{tr} \left[((\sigma \sigma^\top)(t, x_i, \hat{\mu}^{n, \bar{x}}, a_i) + \varepsilon^2) \partial_{x_i x_i}^2 \bar{v}_{\varepsilon, n, m}(t, \bar{x}) \right] \right\} \mu(dx_1) \otimes \cdots \otimes \mu(dx_n) = 0, \end{aligned}$$

which corresponds to equation (A.8). □

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