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Is the Syracuse falling time bounded by 12?

Shalom Eliahou*, Jean Fromentin† and Régnald Simonetto‡

Abstract

Let $T: \mathbb{N} \rightarrow \mathbb{N}$ denote the $3x + 1$ function, where $T(n) = n/2$ if n is even, $T(n) = (3n + 1)/2$ if n is odd. As an accelerated version of T , we define a *jump* at $n \geq 1$ by $\text{jp}(n) = T^{(\ell)}(n)$, where ℓ is the number of digits of n in base 2. We present computational and heuristic evidence leading to surprising conjectures. The boldest one, inspired by the study of $2^\ell - 1$ for $\ell \leq 500\,000$, states that for any $n \geq 2^{500}$, at most four jumps starting from n are needed to fall below n , a strong form of the Collatz conjecture.

Keywords. Collatz conjecture, $3x + 1$ problem, stopping time, glide record, jump function.

1 Introduction

We denote by \mathbb{N} the set of positive integers. Let $T: \mathbb{N} \rightarrow \mathbb{N}$ be the notorious $3x + 1$ function, defined by $T(n) = n/2$ if n is even, $T(n) = (3n + 1)/2$ if n is odd. For $k \geq 0$, denote by $T^{(k)}$ the k th iterate of T . The *orbit* of n under T is the sequence

$$\mathcal{O}_T(n) = (n, T(n), T^{(2)}(n), \dots).$$

The famous Collatz conjecture states that for all $n \geq 1$, there exists $r \geq 1$ such that $T^{(r)}(n) = 1$. The least such r is denoted $\sigma_\infty(n)$ and called the *total stopping time* of n . An equivalent version of the Collatz conjecture states

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that for all $n \geq 2$, there exists $s \geq 1$ such that $T^{(s)}(n) < n$. The least such s is denoted by $\sigma(n)$ and called the *stopping time* of n . For instance, we have

$$\sigma(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \equiv 1 \pmod{4}, \end{cases} \quad (1)$$

as is well known and easy to check. A *stopping time record* is an integer $n \geq 2$ such that $\sigma(m) < \sigma(n)$ for all $2 \leq m \leq n - 1$.

For the original slower version $C: \mathbb{N} \rightarrow \mathbb{N}$, where $C(n) = n/2$ or $3n + 1$ according as n is even or odd, the analog of the stopping time is called the *glide* in [10]. The list of all currently known *glide records*, complete up to at least 2^{60} , is maintained in [11]. It is quite likely that glide records and stopping time records coincide; we have verified it by computer up to 2^{32} .

It is well known that $\sigma(n)$ is unbounded as n grows. For instance, since

$$T^{(\ell)}(2^\ell - 1) = 3^\ell - 1, \quad (2)$$

as follows from the formula $T(2^a 3^b - 1) = 2^{a-1} 3^{b+1} - 1$ for $a \geq 1$, we have $\sigma(2^\ell - 1) \geq \ell$ for all $\ell \geq 2$.

In this paper, we propose an accelerated version of the function T . The idea, somewhat as in [13], is to apply an iterate of T to n depending on the number of digits of n in base 2. Accordingly, we introduce the following function.

Definition 1.1 *The jump function $\text{jp}: \mathbb{N} \rightarrow \mathbb{N}$ is defined for $n \in \mathbb{N}$ by*

$$\text{jp}(n) = T^{(\ell)}(n),$$

where $\ell = \lfloor \log_2(n) + 1 \rfloor$ is the number of digits of n in base 2.

Example 1.2 We have $\text{jp}(1) = T^{(1)}(1) = 2$, and $\text{jp}(2) = T^{(2)}(2) = 2$ since 2 is of length $\ell = 2$ in base 2. For $n = 27$, written 11011 in base 2, hence of length $\ell = 5$, we have $\text{jp}(27) = T^{(5)}(27) = 71$. In turn, 71 is of length $\ell = 7$ in base 2 since $2^6 \leq 71 < 2^7$, whence $\text{jp}(71) = T^{(7)}(71) = 137$. The orbit of 27 under jumps is displayed below in (4).

Example 1.3 A single jump at $n = 2^\ell - 1$ with $\ell \geq 1$ yields

$$\text{jp}(2^\ell - 1) = 3^\ell - 1. \quad (3)$$

This follows from the equalities $\ell = \lfloor \log_2(2^\ell - 1) + 1 \rfloor$ and (2).

Example 1.4 We have $\text{jp}(2n) = \text{jp}(n)$ for all $n \geq 1$. Indeed, $2n$ is of length one more than n in base 2.

In analogy with the stopping time relative to T , we now introduce the falling time relative to jumps. As $\text{jp}(1) = \text{jp}(2) = 2$, we only consider $n \geq 3$.

Definition 1.5 Let $n \geq 3$. The falling time of n , denoted $\text{ft}(n)$, is the least $k \geq 1$ such that $\text{jp}^{(k)}(n) < n$, or ∞ if there is no such k .

Note that, for a presumed cyclic orbit under T with minimum $m \geq 3$, we would have $\text{jp}(m) = \infty$.

There is no tight comparison between stopping time and falling time. It may happen that $\sigma(a) < \sigma(b)$ whereas $\text{ft}(a) > \text{ft}(b)$. For instance, for $a = 41$ and $b = 43$, we have

$$\begin{aligned}\sigma(41) &= 2 < \sigma(43) = 5, \\ \text{ft}(41) &= 8 > \text{ft}(43) = 2.\end{aligned}$$

It may also happen that $\text{ft}(n) > \sigma(n)$, as shown by the case $n = 41$.

Of course, the Collatz conjecture is equivalent to $\text{ft}(n) < \infty$ for all $n \geq 3$. In Section 2, we provide computational evidence leading us to a stronger conjecture, namely that $\text{ft}(n)$ is in fact bounded for all $n \geq 3$. Specifically, all integers n we have tested so far satisfy $\text{ft}(n) \leq 16$. See Conjecture 2.4. In Section 3, in analogy with the falling time, we introduce the *Syracuse falling time* $\text{sft}(n)$, and corresponding conjectures, by only considering the odd terms in the orbits $\mathcal{O}_T(n)$. In Section 4, we report surprising computational results on $\text{ft}(2^\ell - 1)$ and $\text{sft}(2^\ell - 1)$ for $\ell \leq 500\,000$, and we formulate corresponding conjectures. In the last Section 5, inspired by the case $n = 2^\ell - 1$, we formulate still stronger conjectures on $\text{ft}(n)$ and $\text{sft}(n)$ for very large integers n . We conclude the paper with some supporting heuristics.

For a wealth of information, developments and commented references related to the $3x + 1$ problem, see the webpage and book of J. C. Lagarias [8, 9]. To date, the Collatz conjecture has been verified by computer up to 2^{68} by D. Barina [1]. Using this bound, it follows from [4] that any non-trivial cycle of T must have length at least 114 208 327 604.

2 Falling time records

In this section, we only consider those positive integers n satisfying $\sigma(n) \geq 3$, i.e. such that $n \equiv 3 \pmod{4}$ by (1). Let us denote by $4\mathbb{N} + 3$ the set of those integers. Here is our first computational evidence that the falling time remains small.

Proposition 2.1 *We have $\text{ft}(n) \leq 14$ for all $n \in [1, 2^{44} - 1]$ such that $n \equiv 3 \pmod{4}$.*

Proof. In a few days of computing time with CALCULCO [2]. ■

As shown in Table 1, the smallest $n \in 4\mathbb{N} + 3$ such that $\text{ft}(n) \geq 14$, namely $n = 12\,235\,060\,455$, actually satisfies $\text{ft}(n) = 14$ and $n > 2^{33}$.

Definition 2.2 *A falling time record is an integer $n \in 4\mathbb{N} + 3$ such that $\text{ft}(m) < \text{ft}(n)$ for all $m \in 4\mathbb{N} + 3$ with $m < n$.*

Table 1: Falling time records up to 2^{35}

$n \equiv 3 \pmod{4}$	$\lfloor \log_2(n) + 1 \rfloor$	$\text{ft}(n)$
3	2	2
7	3	3
27	5	8
60 975	16	9
1 394 431	21	10
6 649 279	23	11
63 728 127	26	13
12 235 060 455	34	14

The list of falling time records up to 2^{35} is given in Table 1. It was built while establishing Proposition 2.1. For instance, we have $\text{ft}(3) = 2$, $\text{ft}(7) = 3$ and $\text{ft}(n) \leq 3$ for all $3 \leq n < 27$ such that $n \equiv 3 \pmod{4}$. The value $\text{ft}(27) = 8$ follows from the fact that 8 jumps are needed from 27 to fall below it, as shown by the orbit of 27 under jumps:

$$\mathcal{O}_{\text{jp}}(27) = (27, 71, 137, 395, 566, 3\,644, 650, 53, 8, 2, 2, \dots). \quad (4)$$

Interestingly, five of the falling time records in Table 1 are also glide records, namely 3, 7, 27, 63 728 127 and 12 235 060 455, as seen by consulting [11].

Table 1 shows that the number 12 and a few smaller ones fail to occur as falling time records. One may then wonder about the smallest $n \in 4\mathbb{N} + 3$ reaching $\text{ft}(n) = 12$.

The answer is to be found in Table 2. Let us define a *new falling time* as an integer $n \in 4\mathbb{N} + 3$ such that $\text{ft}(n)$ is distinct from $\text{ft}(m)$ for all smaller $m \in 4\mathbb{N} + 3$. Of course, every falling time record is a new falling time. The list of new falling times we know so far, which are not already falling time records, is given in Table 2.

Table 2: Some new falling times

n	111	103	71	55	217740015
$\text{ft}(n)$	4	5	6	7	12

2.1 Integers satisfying $\text{ft}(n) > 14$

For $a, b \in \mathbb{Z}$, we denote by $[a, b] = \{n \in \mathbb{Z} \mid a \leq n \leq b\}$ the integer interval they span. Recalling Example 1.4, namely that $\text{ft}(2m) = \text{ft}(m)$ for all $m \geq 3$, a single integer n satisfying $\text{ft}(n) > 14$ yields *infinitely many* integers N satisfying $\text{ft}(N) > 14$, namely $N = 2^r n$ for all $r \geq 1$. However, the latter numbers have stopping time equal to 1, and hence are not particularly interesting.

Only those integers n satisfying $\text{ft}(n) > 14$ and having a reasonably large stopping time are really interesting, for their apparent rarity and their relevance to the Collatz conjecture. Hence, we shall restrict our search to *24-persistent* integers, i.e. to those n satisfying

$$\sigma(n) \geq 24.$$

The property for n of being 24-persistent only depends on its class mod 2^{24} . See [14] for more details on the description of the condition $\sigma(n) \geq k$ by classes mod 2^k . See also [5]. For $k = 24$, the number of 24-persistent classes mod 2^{24} is exactly 286 581.

As shown below, the occurrence of $\text{ft}(n) > 14$ among 24-persistent numbers seems to be very rare. Here is our first computational result.

Proposition 2.3 *The smallest 24-persistent integer n such that $\text{ft}(n) > 14$ is*

$$n_0 = 1\,008\,932\,249\,296\,231.$$

It satisfies $\text{ft}(n_0) = 15$, $\sigma(n_0) = 886$ and $\lfloor \log_2(n_0) + 1 \rfloor = 50$.

Proof. In a few days of computing time with CALCULCO [2]. ■

2.1.1 The neighborhood of g_{30}

It turns out that n_0 is a glide record, and as such is listed under the name

$$n_0 = g_{30}$$

in Table 5 of Section 2.2. We have verified by computer that g_{30} is the smallest 24-persistent integer n satisfying $\text{ft}(n) > 14$. However, because of the restriction $\sigma(n) \geq 24$, we do not know whether g_{30} is an actual falling time record.

In a small neighborhood of g_{30} in the Collatz tree, we found 11 more 24-persistent integers n satisfying $\text{ft}(n) > 14$. By *small neighborhood* of g_{30} , we mean here integers m such that

$$T^{(i)}(m) = T^{(j)}(g_{30})$$

for some $i \in [0, 40]$, $j \in [0, 30]$. It turns out that these 11 integers all satisfy $\text{ft}(n) = 15$, as g_{30} itself. They are displayed in Table 3.

Table 3: 24-persistent integers satisfying $\text{ft}(n) = 15$.

1 513 398 373 944 347, 1 702 573 170 687 391, 2 017 864 498 592 463,
 2 553 859 756 031 087, 3 405 146 341 374 783, 3 830 789 634 046 631,
 5 107 719 512 062 175, 5 746 184 451 069 947, 6 464 457 507 453 691,
 7 272 514 695 885 403, 22 370 169 558 105 279.

2.1.2 The neighborhood of g_{32}

There is another glide record in Table 5 with falling time 15, namely

$$g_{32} = 180\,352\,746\,940\,718\,527.$$

In this case, looking at a somewhat larger neighborhood of g_{32} in the Collatz tree, namely at all m such that

$$T^{(i)}(m) = T^{(j)}(g_{32})$$

for some $i \in [0, 50], j \in [0, 30]$, we found four 24-persistent integers n reaching $\text{ft}(n) = 16$. These four integers are displayed in Table 4.

Table 4: 24-persistent integers satisfying $\text{ft}(n) = 16$.

$$49\,312\,2600\,554\,790\,303, \quad 739\,683\,900\,832\,185\,455, \\ 986\,245\,201\,109\,580\,607, \quad 1\,479\,367\,801\,664\,370\,911.$$

However, these four integers n have a small stopping time. They all satisfy $\sigma(n) \in [35, 48]$, as compared to $\sigma(g_{32}) = 966$. Hence again, they are not particularly interesting.

This leads us to the following conjecture.

Conjecture 2.4 *There exists $B \geq 16$ such that $\text{ft}(n) \leq B$ for all $n \geq 3$.*

An even bolder conjecture, based on the data we currently have, would be to take $B = 16$ above. Anyway, with whatever value of B , Conjecture 2.4 constitutes a strong form of the Collatz conjecture.

2.2 Glide records

Eric Roosendaal maintains the list of all currently known glide records [11], complete up to at least 2^{60} . At the time of writing, there are 34 of them, denoted g_1, \dots, g_{34} below. As noted in [11], only the first 32 ones have been independently checked. The ten biggest are displayed in descending order in Table 5. It turns out that

$$\text{ft}(g_1), \dots, \text{ft}(g_{34}) \leq 15.$$

Moreover, among them, the highest value $\text{ft}(n) = 15$ is only reached by g_{30} and g_{32} . Table 4 displays four 24-persistent integers n satisfying $\text{ft}(n) = 16$ in the neighborhood of g_{32} . We do not know whether $\text{ft}(n) \geq 17$ is at all reachable.

Table 5: Top ten known glide records

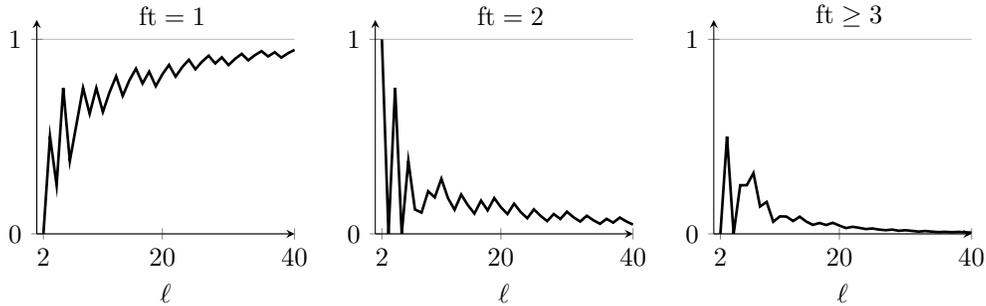
n		$\lfloor \log_2(n) + 1 \rfloor$	glide of n	$\sigma(n)$	$\text{ft}(n)$
g_{34}	2 602 714 556 700 227 743	62	1 639	1005	13
g_{33}	1 236 472 189 813 512 351	61	1 614	990	14
g_{32}	180 352 746 940 718 527	58	1 575	966	15
g_{31}	118 303 688 851 791 519	57	1 471	902	12
g_{30}	1 008 932 249 296 231	50	1 445	886	15
g_{29}	739 448 869 367 967	50	1 187	728	12
g_{28}	70 665 924 117 439	47	1 177	722	13
g_{27}	31 835 572 457 967	45	1 161	712	13
g_{26}	13 179 928 405 231	44	1 122	688	14
g_{25}	2 081 751 768 559	41	988	606	12

2.3 Falling time distribution

In three distinct graphics, we display the distribution of the values taken by the falling time function in large integer intervals. These graphics show that the proportion of the case $\text{ft}(n) \geq 3$ in the integer intervals $[2^\ell, 2^{\ell+1} - 1]$ tends to 0 as ℓ grows.

- Figure 1 displays the proportion of the occurrence of $\text{ft}(n) = 1$, $\text{ft}(n) = 2$ and $\text{ft}(n) \geq 3$, respectively, among all odd integers in the integer intervals $[2^\ell, 2^{\ell+1} - 1]$ for $2 \leq \ell \leq 40$.
- Figure 2 does the same but separates the cases $n \equiv 1 \pmod{4}$ and $n \equiv 3 \pmod{4}$. The purpose is to show that the former case, with stopping time 2, behaves like the more interesting latter case, and so may be safely ignored.
- Finally, Figure 3 is restricted to 24-persistent integers in the integer intervals $[2^\ell, 2^{\ell+1} - 1]$ for $24 \leq \ell \leq 50$.

Figure 1: Proportion of odd integers in $[2^\ell, 2^{\ell+1} - 1]$ with falling time equal to 1, 2 and greater than 2, respectively. The integer ℓ goes from 2 to 40



2.4 A variant of jumps

Let $h \in \mathbb{N}$. For all $n \in \mathbb{N}$, we define

$$\text{jp}_h(n) = T^{(h\ell)}(n)$$

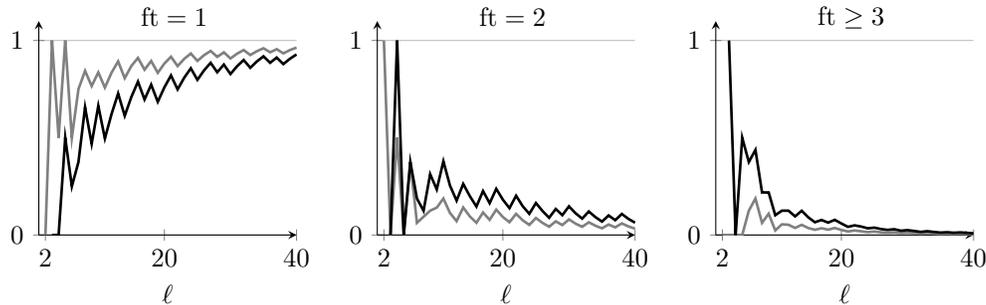
where, as before, ℓ is the number of digits of n in base 2. This is not the same, of course, as the h -iterate $\text{jp}^{(h)}(n)$. Note also that for $h = 1$, we recover jumps, i.e. $\text{jp}_1(n) = \text{jp}(n)$. For $n \geq 3$, the h -falling time $\text{ft}_h(n)$ is defined correspondingly, as the smallest $k \geq 1$, if any, such that $\text{ft}_h^{(k)}(n) < n$.

It turns out that for $h = 18$, and for the glide records g_1, \dots, g_{34} , we have

$$\text{ft}_{18}(g_i) = 1$$

for all $1 \leq i \leq 34$. In view of that fact, is it true that $\text{ft}_{18}(n) = 1$ for all $n \geq 3$? We do not know. But we have verified it up to $n \leq 2^{30}$, and it cannot be outright dismissed, given the conjectural behavior of $\text{ft}(n)$ for very large n as discussed in Section 5. Of course, a positive answer would imply the Collatz conjecture. On the other hand, uncovering any counterexample would be quite a feat.

Figure 2: Same plot as for Figure 1 except that integers are separated with respect to their class 1 or 3 modulo 4. Gray curves are for integers congruent to 1 modulo 4 while black ones are for those congruent to 3 modulo 4.



3 The Syracuse version

Let $\mathbb{O} = 2\mathbb{N} + 1$ denote the set of odd positive integers. Another well-studied version of the $3x + 1$ function is $\text{syr}: \mathbb{O} \rightarrow \mathbb{O}$, defined on any $x \in \mathbb{O}$ by

$$\text{syr}(x) = (3x + 1)/2^\nu,$$

where $\nu \geq 1$ is the largest integer such that 2^ν divides $3x + 1$. That is, $\text{syr}(x)$ is the largest odd factor of $3x + 1$. This specific version is called the *Syracuse function* in [13]. It has been amply investigated in the past, though under different notation or names. For instance in [3], where lower bounds on the length of presumed nontrivial cycles of $\text{syr}(x)$ are given in terms of the convergents p_n/q_n to $\log_2(3)$; or in [6, 7, 12], where statistical properties of $\text{syr}(x)$ and related maps are studied using the Structure theorem of Sinai, which we briefly recall in Section 3.2 below.

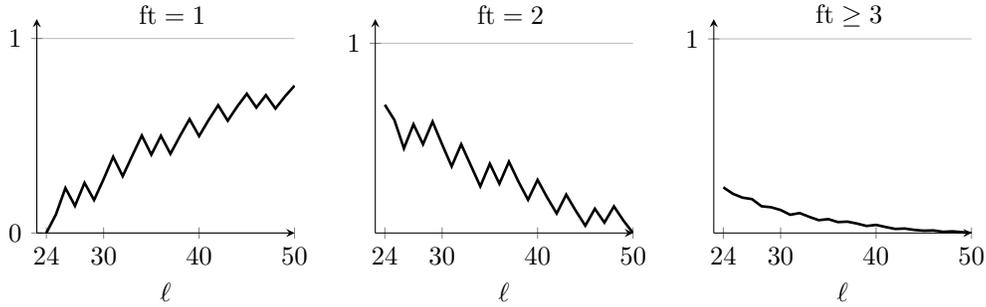
In analogy with the functions $\text{jp}(n)$ and $\text{ft}(n)$ related to the $3x+1$ function $T(x)$, we now introduce the corresponding functions $\text{sjp}(n)$ and $\text{sft}(n)$ related to the Syracuse version $\text{syr}(x)$.

Definition 3.1 We define the Syracuse jump function $\text{sjp}: \mathbb{O} \rightarrow \mathbb{O}$ by

$$\text{sjp}(n) = \text{syr}^{(\ell)}(n), \text{ where } \ell = \lfloor \log_2(n) + 1 \rfloor.$$

Example 3.2 We have $\text{sjp}(1) = 1$, $\text{sjp}(3) = 1$ and $\text{sjp}(27) = \text{syr}^{(5)}(27) = 107$.

Figure 3: Proportion of 24-persistent integers in $[2^\ell, 2^{\ell+1} - 1]$ with falling time equal to 1, 2 and greater than 2, respectively. The integer ℓ goes from 24 to 50



Here is the corresponding Syracuse falling time.

Definition 3.3 Let $n \in \mathbb{O} \setminus \{1\}$. The Syracuse falling time of n , denoted $\text{sft}(n)$, is the least $k \geq 1$ such that $\text{sjp}^{(k)}(n) < n$, or ∞ if there is no such k .

Example 3.4 We have $\text{sft}(27) = 6$, as witnessed by the orbit of 27 under Syracuse jumps, namely

$$\mathcal{O}_{\text{sjp}}(27) = (27, 107, 233, 377, 911, 53, 1, 1, \dots).$$

As one may expect, the inequality $\text{sft}(n) \leq \text{ft}(n)$ holds very often, but not always. For instance, for $n = 199$, we have $\text{ft}(199) = 1$ but $\text{sft}(199) = 5$. The former equality follows from the orbit

$$\mathcal{O}_T(199) = (199, 299, 449, 674, 337, 506, 253, 380, 190, \dots)$$

and the value $\lfloor \log_2(199) + 1 \rfloor = 8$, yielding $\text{jp}(199) = 190$, while the latter one follows from the orbit

$$\mathcal{O}_{\text{syr}}(199) = (199, 323, 395, 479, 577, 1, \dots).$$

Definition 3.5 A Syracuse falling time record is an integer $n \in 4\mathbb{N} + 3$ such that $n \geq 7$ and $\text{sft}(m) < \text{sft}(n)$ for all $m \in 4\mathbb{N} + 3$ with $m < n$.

The complete list of Syracuse falling time records up to 2^{35} is displayed in Table 6. Compared with Table 1, it turns out that all current Syracuse falling time records are also falling time records. The converse does not hold, as shown by the falling time records 60 975 and 1 394 431 in Table 1.

Table 6: Syracuse falling time records up to 2^{35}

$n \equiv 3 \pmod{4}$	$\lfloor \log_2(n) + 1 \rfloor$	$\text{sft}(n)$
7	3	2
27	5	6
6 649 279	23	7
63 728 127	26	9

3.1 Current maximum

The Collatz conjecture is equivalent to the statement $\text{sft}(n) < \infty$ for all $n \in \mathbb{O} \setminus \{1\}$. Again, it is likely that a stronger form holds, namely that $\text{sft}(n)$ is *bounded* on $\mathbb{O} \setminus \{1\}$. Besides the computational evidence above and below, some heuristics point to that possibility in Section 5. Similarly to Proposition 2.1, here is a computational result in that direction.

Proposition 3.6 *We have $\text{sft}(n) \leq 9$ for all $n \in [3, 2^{35} - 1]$ such that $n \equiv 3 \pmod{4}$.*

Proof. By computer with CALCULCO [2]. ■

As yet another hint pointing to the same direction, it turns out that

$$\text{sft}(g_1), \dots, \text{sft}(g_{34}) \leq 10 \tag{5}$$

for the 34 currently known glide records. For definiteness, Table 7 displays the Syracuse falling times of the top ten glide records as listed in Table 5.

Table 7: Syracuse falling times of top ten glide records

n	g_{25}	g_{26}	g_{27}	g_{28}	g_{29}	g_{30}	g_{31}	g_{32}	g_{33}	g_{34}
$\text{sft}(n)$	9	8	8	8	8	10	8	10	9	8

Among the g_i , and as in Section 2.2 for the falling time, only g_{30} and g_{32} reach the current maximum of the Syracuse falling time, namely $\text{sft}(n) = 10$. Interestingly, the biggest currently known glide record, namely $n = g_{34}$, only satisfies $\text{sft}(n) = 8$. With Proposition 3.6 and (5) in the background, here is our formal conjecture.

Conjecture 3.7 *There exists $C \geq 10$ such that $\text{sft}(n) \leq C$ for all $n \equiv 3 \pmod{4}$.*

Again, the truth of this conjecture would yield a strong positive solution of the Collatz conjecture. At the time of writing, no single positive integer $n \equiv 3 \pmod{4}$ is known to satisfy $\text{sft}(n) \geq 11$. Thus, a still bolder conjecture would be to take $C = 10$ in Conjecture 3.7, or $C = 12$ to be on a safer side. Whence the title of this paper.

3.2 A variant of Syracuse jumps

As in Section 2.4 for jumps, we propose here an accelerated variant of Syracuse jumps. Let $h \in \mathbb{N}$. For all $n \in \mathbb{O}$, we define

$$\text{sjp}_h(n) = \text{syr}^{(h\ell)}(n),$$

where ℓ is the number of digits of n in base 2. Of course, $\text{sjp}_1(n) = \text{sjp}(n)$. For $n \geq 3$, the *Syracuse h -falling time* $\text{sft}_h(n)$ is defined correspondingly, as the smallest $k \geq 1$, if any, such that $\text{sft}_h^{(k)}(n) < n$. It turns out that for $h = 12$, and for the glide records g_1, \dots, g_{34} , we have

$$\text{sft}_{12}(g_i) = 1$$

for all $1 \leq i \leq 34$. Again, we may ask whether $\text{sft}_{12}(n) = 1$ holds for all odd $n \geq 3$. A positive answer would imply the Collatz conjecture. We have verified it up to $n \leq 2^{30}$, and our semi-random search did not yield any counterexample.

Anyway, the occurrence of $\text{sft}_{12}(n) = 1$ as n grows to infinity is probably overwhelming; and, just possibly, tools such as Sinai's structure theorem and its applications [6, 7, 12] might help prove that this is indeed the case. But we shall not pursue here this line of investigation.

For convenience, let us briefly recall the statement of that theorem. Given $x \in 6\mathbb{N} + \{1, 5\}$, let $x_i = \text{syr}^{(i)}(x)$ for all $i \geq 0$, and let $k_i \geq 1$ be such that $x_i = (3x_{i-1} + 1)/2^{k_i}$ for all $i \geq 1$. Moreover, for $m \geq 1$, set

$$\gamma_m(x) = (k_1, \dots, k_m).$$

Sinai's structure theorem states that *given any $(k_1, \dots, k_m) \in \mathbb{N}^m$, the set of all $x \in 6\mathbb{N} + \{1, 5\}$ such that $\gamma_m(x) = (k_1, \dots, k_m)$ consists of a unique and full congruence class mod $6 \cdot 2^{k_1 + \dots + k_m}$ in \mathbb{N} .*

4 The case $2^\ell - 1$

In sharp contrast with the stopping time of $2^\ell - 1$, for which $\sigma(2^\ell - 1) \geq \ell$ for all $\ell \geq 2$, the falling time and the Syracuse falling time of $2^\ell - 1$ seem to remain very small as ℓ grows. Here is some strong computational evidence.

Proposition 4.1 *Besides $\text{ft}(2^5 - 1) = \text{ft}(2^6 - 1) = 8$, we have $\text{ft}(2^\ell - 1) \leq 5$ for all $2 \leq \ell \leq 500\,000$ with $\ell \notin \{5, 6\}$.*

Proof. In a few days of computing time with CALCULCO [2]. ■

Moreover, the value $\text{ft}(2^\ell - 1) = 5$ seems to occur finitely many times only, the last one being presumably at $\ell = 132$. In turn, the value $\text{ft}(2^\ell - 1) = 4$ seems to occur infinitely often. Whence the following conjecture, verified by computer up to $\ell = 500\,000$.

Conjecture 4.2 *We have $\text{ft}(2^\ell - 1) \leq 4$ for all $\ell \geq 133$.*

Here are the analogous statement and conjecture for the Syracuse falling time.

Proposition 4.3 *Besides $\text{sft}(2^5 - 1) = \text{sft}(2^6 - 1) = 5$, and $\text{sft}(2^{24} - 1) = 4$, we have*

$$\begin{aligned} \text{sft}(2^\ell - 1) &\in \{2, 3\} \text{ for all } \ell \in [2, 4624] \setminus \{5, 6, 24\}, \\ \text{sft}(2^\ell - 1) &= 2 \text{ for all } \ell \in [4625, 500\,000]. \end{aligned}$$

Proof. In a few days of computing time with CALCULCO [2]. ■

This leads us to the following conjecture, true up to $\ell \leq 500\,000$.

Conjecture 4.4 *We have $\text{sft}(2^\ell - 1) = 2$ for all $\ell \geq 4625$.*

5 For very large n

As hinted by the computational evidence and conjectures of Section 4 on the case $n = 2^\ell - 1$, by intensive semi-random search, and by the heuristics below, it appears to be increasingly difficult for integers n to satisfy $\text{ft}(n) \geq 5$ or $\text{sft}(n) \geq 3$ as they grow very large. Here then are still bolder conjectures.

Conjecture 5.1 *We have $\text{ft}(n) \leq 4$ for all $n \geq 2^{500}$.*

This threshold of 2^{500} is inspired by Conjecture 4.2, of course with a margin for safety. It cannot be significantly lowered, since $\text{ft}(2^{132} - 1) = 5$ as noted before Proposition 4.1. Moreover, intensive random search revealed one integer $n \in [2^{70}, 2^{71} - 1]$ satisfying $\text{ft}(n) = 5$, namely

$$n = 1\ 884\ 032\ 044\ 420\ 885\ 877\ 201\ 579\ 449\ 071\ 924\ 925\ 072\ 300\ 117\ 065\ 411.$$

However, this integer is congruent to 3 mod 16 and hence has stopping time equal to 4 only.

Here is the analogous conjecture for the Syracuse falling time. Its threshold of 2^{5000} is similarly inspired by Proposition 4.3 and Conjecture 4.4.

Conjecture 5.2 *We have $\text{sft}(n) \leq 2$ for all odd $n \geq 2^{5000}$.*

5.1 Heuristics

Besides the computational evidence leading to Conjectures 2.4, 3.7, 4.2, 4.4, 5.1 and 5.2, a heuristic argument would run as follows. It is well known that the Collatz conjecture is equivalent to the statement that, starting with any integer $n \geq 1$, the probability for $T^{(k)}(n)$ to be even or odd tends to 1/2 as k grows to infinity. Thus, even if n written in base 2 is a highly structured binary string, as e.g. for $n = 2^\ell - 1$, one may expect that for $\ell =$ the length of that string, then $T^{(\ell)}(n)$ in base 2 will already look more random. That is, a single jump or Syracuse jump at $n \geq 3$ should already introduce a good dosis of randomness, all the more so as n grows very large. And therefore, a bounded number of jumps or Syracuse jumps at n might well suffice to fall below n .

5.2 A challenge

We hope that the experts in highly efficient computation of the $3x+1$ function will tackle the challenge of probing these conjectures to much higher levels than the ones reported here. For instance, as both a challenge and a request to the reader, and in view of Conjecture 5.1, if you do find any $n \geq 2^{500}$ satisfying $\text{ft}(n) \geq 5$, please e-mail it to the authors. Your solution will be duly recorded on a dedicated webpage.

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