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# Is the Syracuse falling time bounded by 12?

Shalom Eliahou\* and Régnald Simonetto†

## Abstract

Let  $T: \mathbb{N} \rightarrow \mathbb{N}$  denote the  $3x + 1$  function, defined by  $T(n) = n/2$  if  $n$  is even,  $T(n) = (3n+1)/2$  if  $n$  is odd. As an accelerated version of  $T$ , we define the *jump function*  $\text{jp}(n) = T^{(\ell)}(n)$  for  $n \geq 1$ , where  $\ell = \lfloor \log_2(n) \rfloor + 1 =$  the number of digits of  $n$  in base 2. Correspondingly, we define the *falling time*  $\text{ft}(n)$  for  $n \geq 2$  as the least  $k \geq 1$ , if any, such that  $\text{jp}^{(k)}(n) < n$ , or  $\infty$  otherwise. The Collatz conjecture is equivalent to  $\text{ft}(n) < \infty$  for all  $n \geq 2$ . As a stronger form of it, is it conceivable that  $\text{ft}(n)$  be uniformly bounded? So far, the highest value of  $\text{ft}(n)$  we have found is 15.

The related Syracuse function is defined on odd integers  $n \geq 1$  by  $\text{syr}(n) = (3n+1)/2^\nu$ , the largest odd factor of  $3n+1$ . Correspondingly, we define the *Syracuse jump function*  $\text{sjp}(n) = \text{syr}^{(\ell)}(n)$ , where again  $\ell = \lfloor \log_2(n) \rfloor + 1$ , and the *Syracuse falling time*  $\text{sft}(n)$  for odd  $n \geq 3$ , namely the least  $k \geq 1$  such that  $\text{sjp}^{(k)}(n) < n$ , or  $\infty$  otherwise. Again, the Collatz conjecture is equivalent to  $\text{sft}(n) < \infty$  for all odd  $n \geq 3$ . So far, the highest value of  $\text{sft}(n)$  we have found is 10, including among all known glide records and all  $n = 2^\ell - 1$  for  $\ell \geq 2$ .

Based on the computational and heuristic evidence presented here, we conjecture that both  $\text{ft}(n)$  and  $\text{sft}(n)$  are uniformly bounded.

**Keywords.** Collatz conjecture,  $3x + 1$  problem, Syracuse function, stopping time, glide records, jump function

## 1 Introduction

Throughout the paper, we denote by  $\mathbb{N}$  the set of nonnegative integers and by  $\mathbb{N}_+$  the subset of positive integers.

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Let  $T: \mathbb{N} \rightarrow \mathbb{N}$  be the notorious  $3x + 1$  function, defined by  $T(n) = n/2$  if  $n$  is even,  $T(n) = (3n + 1)/2$  if  $n$  is odd. For  $k \geq 0$ , denote by  $T^{(k)}$  the  $k$ th iterate of  $T$ . The *orbit* of  $n$  under  $T$  is the sequence

$$\mathcal{O}_T(n) = (n, T(n), T^{(2)}(n), \dots).$$

The famous Collatz conjecture states that for all  $n \geq 1$ , there exists  $r \geq 1$  such that  $T^{(r)}(n) = 1$ . The least such  $r$  is denoted  $\sigma_\infty(n)$  and called the *total stopping time* of  $n$ . An equivalent version of the Collatz conjecture states that for all  $n \geq 2$ , there exists  $s \geq 1$  such that  $T^{(s)}(n) < n$ . The least such  $s$  is denoted by  $\sigma(n)$  and called the *stopping time* of  $n$ . For instance, we have

$$\sigma(n) = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 2 & \text{if } n \equiv 1 \pmod{4,} \end{cases} \quad (1)$$

as easily checked. A *stopping time record* is an integer  $n \geq 2$  such that  $\sigma(m) < \sigma(n)$  for all  $2 \leq m \leq n - 1$ .

For the original slower version  $C: \mathbb{N} \rightarrow \mathbb{N}$ , where  $C(n) = n/2$  or  $3n + 1$  according as  $n$  is even or odd, the analog of the stopping time is called the *glide* in [4]. The list of all currently known *glide records*, complete up to at least  $2^{61.2}$ , is maintained in [5]. It is quite likely that glide records and stopping time records coincide.

It is well known that  $\sigma(n)$  is unbounded as  $n$  grows. For instance, since

$$T^{(\ell)}(2^\ell - 1) = 3^\ell - 1, \quad (2)$$

as follows from the formula  $T(2^a 3^b - 1) = 2^{a-1} 3^{b+1} - 1$  for  $a \geq 1$ , we have  $\sigma(2^\ell - 1) \geq \ell$  for all  $\ell \geq 2$ .

In this paper, we propose an accelerated version of the function  $T$ . The idea, somewhat as in [6], is to apply an iterate of  $T$  to  $n$  depending on the number of digits of  $n$  in base 2. Accordingly, we introduce the following function.

**Definition 1.1** *The jump function  $\text{jp}: \mathbb{N}_+ \rightarrow \mathbb{N}_+$  is defined for  $n \in \mathbb{N}_+$  by*

$$\text{jp}(n) = T^{(\ell)}(n),$$

where  $\ell = \lfloor \log_2(n) \rfloor + 1 =$  the number of digits of  $n$  in base 2.

For instance, a single jump at  $n = 2^\ell - 1$  yields

$$\text{jp}(2^\ell - 1) = 3^\ell - 1. \tag{3}$$

This follows from the equalities  $\lfloor \log_2(2^\ell - 1) \rfloor + 1 = \ell$  and (2).

In analogy with the stopping time for  $T$ , we now introduce the falling time for jumps.

**Definition 1.2** *Let  $n \geq 2$ . The falling time of  $n$ , denoted  $\text{ft}(n)$ , is the least  $k \geq 1$ , if any, such that  $\text{jp}^{(k)}(n) < n$ , or  $\infty$  otherwise.*

There is no tight comparison between stopping time and falling time. It may happen that  $\sigma(a) < \sigma(b)$  whereas  $\text{ft}(a) > \text{ft}(b)$ . For instance for  $a = 41$  and  $b = 43$ , we have

$$\begin{aligned} \sigma(41) &= 2 < \sigma(43) = 5, \\ \text{ft}(41) &= 8 > \text{ft}(43) = 2. \end{aligned}$$

It may also happen that  $\text{ft}(n) > \sigma(n)$ , as shown by the case  $n = 41$ .

Of course, the Collatz conjecture is equivalent to  $\text{ft}(n) < \infty$  for all  $n \in \mathbb{N}_+$ . In Section 2, we provide computational evidence leading us to a stronger conjecture, namely that  $\text{ft}(n)$  is in fact uniformly bounded. Specifically, all integers  $n$  we have tested so far satisfy  $\text{ft}(n) \leq 15$ . See Conjecture 2.2. In Section 3, in analogy with the falling time, we introduce the *Syracuse falling time*, and a corresponding conjecture, by only considering the odd terms of the orbits  $\mathcal{O}_T(n)$ . In the concluding Section 4, we provide some heuristics towards our stronger conjectures.

## 2 Falling time records

In this section, we only consider those positive integers  $n$  satisfying  $\sigma(n) \geq 3$ , i.e. such that  $n \equiv 3 \pmod{4}$  by (1). Let us denote by  $4\mathbb{N} + 3$  the set of those integers.

**Definition 2.1** *A falling time record is an integer  $n \in 4\mathbb{N} + 3$  such that  $\text{ft}(m) < \text{ft}(n)$  for all  $m \in 4\mathbb{N} + 3$  with  $m < n$ .*

$n \equiv 3 \pmod{4}$	$\lfloor \log_2(n) \rfloor + 1$	$\text{ft}(n)$
3	2	2
7	3	3
27	5	8
60 975	16	9
1 394 431	21	10
6 649 279	23	11
63 728 127	26	13
12 235 060 455	34	14

Table 1: Falling time records up to  $2^{35}$

The list of falling time records up to  $2^{35}$  is given in Table 1. It was built in a few days of home computer time with *Mathematica* 12 [7]. For instance, we have  $\text{ft}(3) = 2$ ,  $\text{ft}(7) = 3$  and  $\text{ft}(n) \leq 3$  for all  $3 \leq n < 27$  such that  $n \equiv 3 \pmod{4}$ . The value  $\text{ft}(27) = 8$  follows from the first 9 terms of the orbit of 27 under jumps, namely

$$\mathcal{O}_{\text{jp}}(27) = (27, 71, 137, 395, 566, 3\,644, 650, 53, 8).$$

Interestingly, five of these falling time records are also glide records, namely 3, 7, 27, 63 728 127 and 12 235 060 455 [5].

As seen in Table 1, the number 12 and a few smaller ones are not reached by falling time records. One may wonder about the smallest  $n \in 4\mathbb{N} + 3$  such that  $\text{ft}(n) = 12$ . More generally, let us define a *new falling time* as an integer  $n \in 4\mathbb{N} + 3$  such that  $\text{ft}(n)$  is distinct from  $\text{ft}(m)$  for all smaller  $m \in 4\mathbb{N} + 3$ . Of course, every falling time record is a new falling time. The list of new falling times we know so far, which are not already falling time records, is given in Table 2.

$\text{ft}(n)$	4	5	6	7	12
$n$	111	103	71	55	217740015

Table 2: Some new falling times

In Section 2.1, we show two large integers  $n$  reaching  $\text{ft}(n) = 15$ . But we do not know whether the smaller one is a falling time record. Now, is

the inequality  $\text{ft}(n) \geq 16$  reachable? We do not know either, but a negative answer would constitute a strong positive solution of the Collatz conjecture.

## 2.1 From Roosendaal's website

Eric Roosendaal maintains the list of all currently known glide records [5], complete up to at least  $2^{61.2}$ . At the time of writing, there are 34 of them. Denoting them by  $g_1, \dots, g_{34}$ , the ten biggest are displayed in descending order in Table 3. As hinted there, it turns out that  $\text{ft}(n) \leq 15$  for all  $n \in \{g_1, \dots, g_{34}\}$ . Moreover, the highest value 15 is reached in only two cases, namely by  $g_{30}$  and  $g_{32}$ .

	$n$	$\lfloor \log_2(n) \rfloor + 1$	glide of $n$	$\sigma(n)$	$\text{ft}(n)$
$g_{34}$	2 602 714 556 700 227 743	61	1 639	1005	13
$g_{33}$	1 236 472 189 813 512 351	60	1 614	990	14
$g_{32}$	180 352 746 940 718 527	57	1 575	966	<b>15</b>
$g_{31}$	118 303 688 851 791 519	56	1 471	902	12
$g_{30}$	1 008 932 249 296 231	49	1 445	886	<b>15</b>
$g_{29}$	739 448 869 367 967	49	1 187	728	12
$g_{28}$	70 665 924 117 439	46	1 177	722	13
$g_{27}$	31 835 572 457 967	44	1 161	712	13
$g_{26}$	13 179 928 405 231	43	1 122	688	14
$g_{25}$	2 081 751 768 559	40	988	606	12

Table 3: Top ten known glide records

So far, the integers  $g_{30}$  and  $g_{32}$  are the only ones we know of satisfying  $\text{ft}(n) \geq 15$ . This leads us to the following bold conjecture, a strong version of the Collatz conjecture.

**Conjecture 2.2** *There exists  $B \geq 15$  such that  $\text{ft}(n) \leq B$  for all  $n \in \mathbb{N}_+$ .*

## 3 The Syracuse version

Let  $\mathbb{O} = 2\mathbb{N} + 1$  denote the set of odd positive integers. Another well-studied version of the  $3x + 1$  function is  $\text{syr}: \mathbb{O} \rightarrow \mathbb{O}$ , defined on any  $n \in \mathbb{O}$  by

$$\text{syr}(n) = (3n + 1)/2^\nu,$$

where  $\nu \geq 1$  is the largest integer such that  $2^\nu$  divides  $3n + 1$ . Thus  $\text{syr}(n)$  is the largest odd factor of  $3n + 1$ . This specific version is called the *Syracuse function* in [6].

In analogy with the functions  $\sigma(n)$ ,  $\text{jp}(n)$  and  $\text{ft}(n)$  related to the  $3x + 1$  function  $T$ , we now introduce the corresponding functions  $\sigma_{\text{syr}}(n)$ ,  $\text{sjp}(n)$  and  $\text{sft}(n)$  related to the Syracuse version  $\text{syr}$ .

**Definition 3.1** For  $n \in \mathbb{O} \setminus \{1\}$ , the Syracuse stopping time, denoted  $\sigma_{\text{syr}}(n)$  is the smallest  $k \geq 1$ , if any, such that  $\text{syr}^{(k)}(n) < n$ , or  $\infty$  otherwise.

Here is the jump analog for the Syracuse function.

**Definition 3.2** We define the Syracuse jump function  $\text{sjp}: \mathbb{O} \rightarrow \mathbb{O}$  by

$$\text{sjp}(n) = \begin{cases} 1 & \text{if } n = 1, \\ \text{syr}^{(\ell)}(n) & \text{if } n \geq 3, \text{ where } \ell = \lfloor \log_2(n) \rfloor + 1. \end{cases}$$

And here is the corresponding falling time.

**Definition 3.3** Let  $n \in \mathbb{O} \setminus \{1\}$ . The Syracuse falling time of  $n$ , denoted  $\text{sft}(n)$ , is the least  $k \geq 1$ , if any, such that  $\text{sjp}^{(k)}(n) < n$ , or  $\infty$  otherwise.

Again, the Collatz conjecture is equivalent to the statement  $\text{sft}(n) < \infty$  for all  $n \in \mathbb{O} \setminus \{1\}$ . As one may expect, the inequality  $\text{sft}(n) \leq \text{ft}(n)$  holds very often, but not always. For instance, for  $n = 199$ , we have  $\text{ft}(199) = 1$  but  $\text{sft}(199) = 5$ . The former equality follows from the orbit

$$\mathcal{O}_T(199) = (199, 299, 449, 674, 337, 506, 253, 380, 190, \dots)$$

and the value  $\lfloor \log_2(199) \rfloor + 1 = 8$ , yielding  $\text{jp}(199) = 190$ , while the latter one follows from the orbit

$$\mathcal{O}_{\text{syr}}(199) = (199, 323, 395, 479, 577, 1, \dots).$$

**Definition 3.4** A Syracuse falling time record is an integer  $n \in 4\mathbb{N} + 3$  such that  $n \geq 7$  and  $\text{sft}(m) < \text{sft}(n)$  for all  $m \in 4\mathbb{N} + 3$  with  $m < n$ .

The list of Syracuse falling time records, complete up to  $2^{35}$ , is displayed in Table 4. Compared with Table 1, it turns out that all current Syracuse falling time records are also falling time records. The converse does not hold, as shown by the falling time records 60 975 and 1 394 431 in Table 1.

$n \equiv 3 \pmod{4}$	$\lfloor \log_2(n) \rfloor + 1$	$\text{sft}(n)$
7	3	2
27	5	6
6 649 279	23	7
63 728 127	26	9

Table 4: Syracuse falling time records up to  $2^{35}$

### 3.1 The case $2^\ell - 1$

In sharp contrast with the stopping time of  $2^\ell - 1$ , for which  $\sigma(2^\ell - 1) \geq \ell$  for all  $\ell \geq 2$ , both the falling time and the Syracuse falling time of  $2^\ell - 1$  seem to remain very small for all  $\ell$ . Indeed, consider the following surprising computational result and conjecture, stated here for  $\text{sft}$  only. For  $a, b \in \mathbb{N}$ , we denote the integer interval they span by  $[a, b] = \{n \in \mathbb{N} \mid a \leq n \leq b\}$ .

**Proposition 3.5** *We have  $\text{sft}(2^\ell - 1) \leq 5$  for all  $2 \leq \ell \leq 50\,000$ . More precisely:*

$$\text{sft}(2^5 - 1) = \text{sft}(2^6 - 1) = 5,$$

$$\text{sft}(2^{24} - 1) = 4,$$

$$\text{sft}(2^\ell - 1) \in \{2, 3\} \text{ for all } \ell \in [2, 4624] \setminus \{5, 6, 24\},$$

$$\text{sft}(2^\ell - 1) = 2 \text{ for all } \ell \in [4625, 50\,000].$$

**Proof.** With *Mathematica* 12 [7], in a few hours of home computer time. ■

This leads to the following conjecture, true up to  $\ell = 50\,000$ .

**Conjecture 3.6** *We have  $\text{sft}(2^\ell - 1) = 2$  for all  $\ell \geq 4625$ .*

### 3.2 Current maximum

As mentioned above, the Collatz conjecture is equivalent to the statement  $\text{sft}(n) < \infty$  for all  $n \in \mathbb{O} \setminus \{1\}$ . However, it is likely that a stronger form holds, namely that  $\text{sft}(n)$  is uniformly bounded. Besides the computational evidence above and below, some heuristics point to that possibility in Section 4. After Proposition 3.5, here is another computational result in that direction.

**Proposition 3.7** *We have  $\text{sft}(n) \leq 9$  for all  $n \in [1, 2^{35} - 1]$  such that  $n \equiv 3 \pmod{4}$ .*



**Proof.** With *Mathematica* 12, in a few days of home computer time. ■

As yet another hint pointing to the same direction, it turns out that

$$\text{sft}(g_1), \dots, \text{sft}(g_{34}) \leq 10 \tag{4}$$

for the 34 currently known glide records. For definiteness, Table 5 displays the Syracuse falling times of the top ten glide records as listed in Table 3.

$n$	$g_{25}$	$g_{26}$	$g_{27}$	$g_{28}$	$g_{29}$	$g_{30}$	$g_{31}$	$g_{32}$	$g_{33}$	$g_{34}$
$\text{sft}(n)$	9	8	8	8	8	<b>10</b>	8	<b>10</b>	9	8

Table 5: Syracuse falling times of top ten glide records

Among the  $g_i$ , and as in Section 2.1 for the falling time, only  $g_{30}$  and  $g_{32}$  reach the current maximum of the Syracuse falling time, namely  $\text{sft}(n) = 10$ . Interestingly, the biggest currently known glide record  $n = g_{34}$  only satisfies  $\text{sft}(n) = 8$ . With Proposition 3.7 and (4) in the background, here is our formal conjecture.

**Conjecture 3.8** *There exists  $C \geq 10$  such that  $\text{sft}(n) \leq C$  for all  $n \equiv 3 \pmod 4$ .*

Again, its truth would yield a strong positive solution of the Collatz conjecture. At the time of writing, no single positive integer  $n \equiv 3 \pmod 4$  is known to satisfy  $\text{sft}(n) \geq 11$ . Thus, a still bolder conjecture would be to take  $C = 10$  in Conjecture 3.8, or  $C = 12$  to be on a safer side.

We hope that the experts in highly efficient computation of the  $3x + 1$  function will tackle the challenge of probing these conjectures to much higher levels.

## 4 Concluding remarks

One may accelerate the jump function  $\text{jp}$  as follows. Given  $h \in \mathbb{N}_+$ , we define

$$\text{jp}_h : \mathbb{N} \rightarrow \mathbb{N}$$

by  $\text{jp}_h(n) = T^{(h\ell)}$  where, as for jumps,  $\ell$  is the number of digits of  $n$  in base 2. This is not the same, of course, as the  $h$ -iterate  $\text{jp}^{(h)}$ . Note also that for

$h = 1$  we recover jumps, i.e.  $jp_1 = jp$ . The  $h$ -falling time  $ft_h$  can be defined correspondingly. It turns out that for  $h = 18$ , all 34 known glide records  $n$  satisfy  $ft_{18}(n) = 1$ .

Besides the computational evidence for Conjectures 2.2, 3.6 and 3.8 provided in this paper, a heuristic argument would run as follows. It is well known that the Collatz conjecture is equivalent to the statement that, starting with any integer  $n \geq 1$ , the probability for  $T^{(k)}(n)$  to be even or odd tends to  $1/2$  as  $k$  grows to infinity. Thus, even if  $n$  written in base 2 is a highly structured binary string, as e.g. for  $n = 2^\ell - 1$ , one may expect that for  $\ell =$  the length of that string, then  $T^{(\ell)}(n)$  in base 2 will already look more random. That is, a single jump or Syracuse jump at  $n$  should already introduce a good dosis of randomness. And therefore, a bounded number of jumps or Syracuse jumps on  $n \geq 3$  might well suffice to fall under  $n$ .

## References

- [1] D. Barina, *Convergence verification of the Collatz problem*, The Journal of Supercomputing 77 (2021) 2681–2688.
- [2] S. Eliahou, *The  $3x + 1$  problem: new lower bounds on nontrivial cycle lengths*, Discrete Math. 11 (1993) 45–56.
- [3] J. C. Lagarias,  *$3x + 1$  problem and related problems*, <https://dept.math.lsa.umich.edu/~lagarias//3x+1.html>
- [4] E. Roosendaal, [www.ericr.nl/wondrous](http://www.ericr.nl/wondrous)
- [5] E. Roosendaal, [www.ericr.nl/wondrous/glidreecs.html](http://www.ericr.nl/wondrous/glidreecs.html)
- [6] T. Tao, *Almost all orbits of the Collatz map attain almost bounded values* (2019) [arXiv:1909.03562](https://arxiv.org/abs/1909.03562)
- [7] Wolfram Research, Inc., Mathematica, Version 12, Champaign, IL (2019).

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