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Drift estimation of the threshold Ornstein-Uhlenbeck process from continuous and discrete observations

Sara Mazzonetto* and Paolo Pigato†

June 5, 2022

Abstract

We refer by *threshold Ornstein-Uhlenbeck* to a continuous-time threshold autoregressive process. It follows the Ornstein-Uhlenbeck dynamics when above or below a fixed level, yet at this level (threshold) its coefficients can be discontinuous. We discuss (quasi)-maximum likelihood estimation of the drift parameters, both assuming continuous and discrete time observations. In the ergodic case, we derive consistency and speed of convergence of these estimators in long time and high frequency. Based on these results, we develop a test for the presence of a threshold in the dynamics. Finally, we apply these statistical tools to short-term US interest rates modeling.

Keywords: Threshold diffusion, maximum likelihood, regime-switching, self-exciting process, interest rates, threshold Vasicek Model, multi-threshold.

AMS 2010: primary: 62M05; secondary: 62F12; 60J60.

JEL Classification: primary: C58; secondary: C22, G12.

1 Introduction

We consider the diffusion process solution to the following stochastic differential equation (SDE)

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t (b(X_s) - a(X_s) X_s) ds, \quad t \geq 0, \quad (1.1)$$

with piecewise constant volatility coefficient, possibly discontinuous at $r \in \mathbb{R}$,

$$\sigma(x) = \sigma_+ \mathbf{1}_{\{x \geq r\}} + \sigma_- \mathbf{1}_{\{x < r\}} > 0 \quad (1.2)$$

and similarly piecewise affine drift coefficient

$$b(x) = b_+ \mathbf{1}_{\{x \geq r\}} + b_- \mathbf{1}_{\{x < r\}} \quad \text{and} \quad a(x) = a_+ \mathbf{1}_{\{x \geq r\}} + a_- \mathbf{1}_{\{x < r\}}. \quad (1.3)$$

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Strong existence of a unique solution to (1.1) follows from the results of Le Gall [1985]. Separately on (r, ∞) and $(-\infty, r)$, the process follows the Ornstein-Uhlenbeck (OU) dynamics which, in the context of interest rates modeling, is referred to as *Vasicek model*. Following this nomenclature, Decamps et al. [2006] refer to (1.1) as *Self Exciting Threshold (SET) Vasicek model*. Su and Chan [2015, 2017] refer to such model as *threshold diffusion* (TD) or first-order continuous-time threshold autoregressive model (see also [Tong, 1990]).

The process is *ergodic* as long as the drift pushes the process up when the process reaches large, negative values, and down when it reaches large, positive values. Note that if $a_- > 0$, the drift points towards b_-/a_- when X_s is below the threshold r , if $a_+ > 0$, towards b_+/a_+ when X_s is above r . We allow here a null linear part, and if $a_- = 0$ we need $b_- > 0$ to push the process up when very negative and if $a_+ = 0$ we need $b_+ < 0$ to push it down when very positive. From these considerations, ergodicity can be easily checked (see explicit condition (2.1) below).

We also consider, in Section B, a *multi-threshold* version of (1.1), where we allow for d discontinuity levels $r_1 < \dots < r_d$. In this case, ergodicity is determined by the same conditions, to be checked on the values of the coefficients on the intervals $(-\infty, r_1)$ and $[r_d, +\infty)$.

In this paper we discuss the asymptotic behavior of maximum likelihood estimators (MLE) and quasi-maximum likelihood estimators (QMLE) for the drift parameters (a_-, a_+, b_-, b_+) , both from continuous and discrete time observations. Let N be the number of observations, T_N the time horizon and Δ_N the largest interval between two consecutive observations. In the ergodic case, if $T_N \rightarrow \infty$ and $T_N \Delta_N \rightarrow 0$ as $N \rightarrow \infty$, we prove a central limit theorem (CLT) giving the convergence of the estimators with speed $\sqrt{T_N}$ to the real parameters, i.e. asymptotic normality (see Theorem 2 below). To the best of our knowledge, this is the first result of this kind for TDs (SDEs with discontinuous -drift and diffusion- coefficients).

The discontinuity in the coefficients makes it difficult to pass from discrete to continuous time observations. Indeed, a precise analysis of the error hinges on the behavior of certain discretizations of the local time of the diffusion at the threshold. We also prove, for fixed time horizon, that the discrete (Q)MLE based on N equally spaced observations converges in high frequency to the continuous (Q)MLE, with speed $N^{1/4}$ (see Theorem 3 below). This slow convergence of the discrete (Q)MLE to the continuous (Q)MLE follows from the slow convergence, with speed $N^{1/4}$, of the discretization of the local time.

Based on these results we provide a test to decide whether a threshold is present in the dynamics. Finally we use these tools to analyze short term US interest rates.

Literature review. Su and Chan [2015, 2017] study the asymptotic behavior of the continuous time QMLE of a TD with drift as in (1.3) and piecewise regular diffusivity. In particular, they construct a hypothesis test to decide whether the drift is affine or piecewise affine.

The estimation of the volatility parameters σ_{\pm} in (1.1) from high-frequency data is studied in [Lejay and Pigato, 2018] and the drift estimation in case $a_{\pm} = 0$ in [Lejay and Pigato, 2020]. In the purely linear drift case $b_{\pm} = 0$, Kutoyants [2012] studies the problem of identifying the threshold parameter r and Dieker and Gao [2013] and the related stream of research consider similar models, with $r = 0$ (so that the drift function is continuous) in a multidimensional setting. The (related) problem of drift estimation in a skew OU process is considered in [Xing et al., 2020].

In this document the coefficients are discontinuous and the behavior at r hard to handle; for high-frequency observations we do so using the discretization results in [Mazonetto, 2019]. The convergence in high frequency and long time for estimators of discretely observed diffusions have been discussed e.g. in [Kessler, 1997, Ben Alaya and Kebaier, 2013, Amorino and Gloter, 2020], but to the best of our knowledge ours is the first such result in the case of discontinuous coefficients.

Yu et al. [2020] study numerically an approximate MLE (AMLE) from discrete time observations simultaneously for threshold, drift and diffusion coefficients of threshold diffusions including OU process or CIR model. They compare their AMLE with QMLE, showing numerical evidence for consistency. The recent work [Hu and Xi, 2022] considers a generalized moment estimator for a TD which is discretely observed, with fixed time lag.

Threshold autoregressive (TAR) models in discrete time were introduced by H. Tong in the early 1980s [Tong, 1983, 2011, 2015]. Within this class, self-exciting TAR (SETAR) models rely on a spatial segmentation, with a change in the dynamics according to the position of the process, below or above a threshold, and can be seen as a discrete analogue to the TD. We refer to [Chan, 1993, Rabemananjara and Zakoian, 1993, Yadav et al., 1994, Brockwell and Williams, 1997, Chen et al., 2011] and references therein for this class of econometric models and related inference problems.

Diffusion processes have been widely used to model interest rate dynamics, celebrated classical examples being [Vasicek, 1977, Cox et al., 1985, Hull and White, 1990, Black and Karasinski, 1991]. These models are designed to capture the fact that interest rates are typically mean reverting, see [Wu and Zhang, 1996]. However, non linear effects (e.g. multi-modality) are not captured by these models. Ait-Sahalia [1996] shows that mean-reversion for interest rates is strong outside a middle region, suggesting the existence of a target band. This is similar to what is observed in exchange rates [Krugman, 1991] and explainable with policy adjustments in response to changes in such rates. There is evidence for a “normal” low-mean regime and an “exceptional” high-mean regime, and in general for bi-modality (or even multi-modality) in interest rate dynamics, that one can model using TD (1.3). In general, non-linearities and regime changes in short-term interest rates have been widely documented, and several threshold models have been proposed both in discrete and continuous time, see [Gray, 1996, Pfann et al., 1996, Ang and Bekaert, 2002a,b, Kalimipalli and Susmel, 2004, Gospodinov, 2005, Ang et al., 2008, Archontakis and Lemke, 2008a,b]. We refer to [Decamps et al., 2006] and bibliography therein for a thorough discussion of SET diffusions in interest rate modeling. In recent years TDs have been used in several aspects of financial modeling, such as option pricing [Lipton and Sepp, 2011, Gairat and Shcherbakov, 2016, Dong and Wong, 2017, Lipton, 2018, Pigato, 2019] and time series modeling [Ang and Timmermann, 2012, Lejay and Pigato, 2019]. TD models for interest rates have been considered in [Pai and Pedersen, 1999, Decamps et al., 2006, Su and Chan, 2015, 2017]. In this paper, we focus on (Q)MLE estimation of such models, and in particular on inference from high frequency observations and their convergence to continuous time estimators, as well as their convergence in long time to real values of the parameters.

Outline. In Section 2 we present our main results on convergence of drift estimators for threshold OU. In Section 3 we implement the estimators, discuss threshold estimation and testing and work with US interest rates data. Proofs are collected in Section A and in Section B we discuss an extension to a multi-threshold setting.

2 (Quasi) maximum likelihood estimation

Let X be the process strong solution to (1.1) where W is a Brownian motion and X_0 is independent of W (e.g., X_0 is deterministic).

We see in next equation (A.1) that X is ergodic if

$$\begin{aligned} & [(a_+ > 0 \text{ and } b_+ \in \mathbb{R}) \text{ or } (a_+ = 0 \text{ and } b_+ < 0)] \\ & \text{and } [(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- > 0)]. \end{aligned} \quad (2.1)$$

2.1 Maximum and quasi-maximum likelihood estimator from continuous time observations

We assume in this section to observe the process on the time interval $[0, T]$, $T \in (0, \infty)$. For $T \in (0, \infty)$ and $m = 0, 1, 2$, we define

$$\mathfrak{M}_T^{\pm, m} := \int_0^T X_s^m \mathbf{1}_{\{\pm(X_s - r) \geq 0\}} dX_s \quad \text{and} \quad \mathfrak{Q}_T^{\pm, m} := \int_0^T X_s^m \mathbf{1}_{\{\pm(X_s - r) \geq 0\}} ds \quad (2.2)$$

and take as likelihood function the Girsanov weight

$$G_T(a_+, b_+, a_-, b_-) = \exp \left(\int_0^T \frac{b(X_s) - a(X_s)X_s}{(\sigma(X_s))^2} dX_s - \frac{1}{2} \int_0^T \frac{(b(X_s) - a(X_s)X_s)^2}{(\sigma(X_s))^2} ds \right). \quad (2.3)$$

We also consider a quasi-likelihood defined as in [Su and Chan, 2015] as

$$\Lambda_T(a_+, b_+, a_-, b_-) = \int_0^T b(X_s) - a(X_s)X_s dX_s - \frac{1}{2} \int_0^T (b(X_s) - a(X_s)X_s)^2 ds. \quad (2.4)$$

Theorem 1. *Let $\pm \in \{+, -\}$.*

i) For every $T \in (0, \infty)$ the MLE and QMLE are given by

$$(\alpha_T^\pm, \beta_T^\pm) = \left(\frac{\mathfrak{M}_T^{\pm, 0} \mathfrak{Q}_T^{\pm, 1} - \mathfrak{Q}_T^{\pm, 0} \mathfrak{M}_T^{\pm, 1}}{\mathfrak{Q}_T^{\pm, 0} \mathfrak{Q}_T^{\pm, 2} - (\mathfrak{Q}_T^{\pm, 1})^2}, \frac{\mathfrak{M}_T^{\pm, 0} \mathfrak{Q}_T^{\pm, 2} - \mathfrak{Q}_T^{\pm, 1} \mathfrak{M}_T^{\pm, 1}}{\mathfrak{Q}_T^{\pm, 0} \mathfrak{Q}_T^{\pm, 2} - (\mathfrak{Q}_T^{\pm, 1})^2} \right). \quad (2.5)$$

Assume now that the process is ergodic.

ii) The following law of large numbers (LLN) holds: $(\alpha_T^\pm - a_\pm, \beta_T^\pm - b_\pm) \xrightarrow[T \rightarrow \infty]{\text{a.s.}} 0$, i.e., the estimator is consistent.

iii) The following CLT holds: $\sqrt{T} (\alpha_T^\pm - a_\pm, \beta_T^\pm - b_\pm) \xrightarrow[T \rightarrow \infty]{\text{stably}} N^\pm = (N^{\pm, \alpha}, N^{\pm, \beta})$ where $(N^{+, \alpha}, N^{+, \beta})$ and $(N^{-, \alpha}, N^{-, \beta})$ are two independent, independent of X , two-dimensional Gaussian random variables with covariance matrices respectively $\sigma_+^2 \Gamma_+^{-1}$ and $\sigma_-^2 \Gamma_-^{-1}$ where

$$\Gamma_\pm := \begin{pmatrix} \overline{\mathfrak{Q}}_\infty^{\pm, 2} & -\overline{\mathfrak{Q}}_\infty^{\pm, 1} \\ -\overline{\mathfrak{Q}}_\infty^{\pm, 1} & \overline{\mathfrak{Q}}_\infty^{\pm, 0} \end{pmatrix}, \quad (2.6)$$

and $\overline{\mathfrak{Q}}_\infty^{\pm, i}$, $i \in \{0, 1, 2\}$ are real constants such that $\lim_{t \rightarrow \infty} \frac{\mathfrak{Q}_t^{\pm, i}}{t} \stackrel{\text{a.s.}}{=} \overline{\mathfrak{Q}}_\infty^{\pm, i}$ (explicit expressions for such constants are given in next Lemma 2; more details on stable convergence can be found in Remark 1 below).

iv) The local asymptotic normality (LAN) property (see Le Cam and Yang [2000]) holds for the likelihood evaluated at the true parameter $\theta := (a_+, b_+, a_-, b_-)$ with rate of convergence $\frac{1}{\sqrt{T}}$ and asymptotic Fisher information

$$\Gamma = \begin{pmatrix} \sigma_+^{-2}\Gamma_+ & 0_{\mathbb{R}^{2 \times 2}} \\ 0_{\mathbb{R}^{2 \times 2}} & \sigma_-^{-2}\Gamma_- \end{pmatrix}.$$

This means that there exists a random vector $A_T \in \mathbb{R}^4$ such that for all small perturbations $\Delta\theta := (\Delta a_+, \Delta b_+, \Delta a_-, \Delta b_-)$ it holds that the quantity

$$\log \frac{G_T(\theta + \frac{1}{\sqrt{T}}\Delta\theta)}{G_T(\theta)} - \left(\Delta\theta \cdot A_T - \frac{1}{2}\Delta\theta \cdot \Gamma \Delta\theta \right) \quad (2.7)$$

converges to 0 in probability as $T \rightarrow \infty$.

Remark 1. The notion of *stable convergence* was introduced by Rényi [1963]. We refer to [Jacod and Shiryaev, 2003] or [Jacod and Protter, 2012] for a detailed exposition. In this document we just mention that this notion of convergence is stronger than convergence in law, but weaker than convergence in probability. We use in this paper the following crucial property: for random variables Y_n, Z_n ($n \geq 1$), Y and Z ,

$$\text{if } Z_n \xrightarrow[n \rightarrow \infty]{\text{stably}} Z \text{ and } Y_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} Y \text{ then } (Y_n, Z_n) \xrightarrow[n \rightarrow \infty]{\text{stably}} (Y, Z). \quad (2.8)$$

2.2 Drift estimation from discrete observations

We assume in this section to observe the process on the discrete time grid $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$, for $N \in \mathbb{N}$, $T \in (0, \infty)$, and set $\Delta_N = \max_{k=1, \dots, N} \{t_k - t_{k-1}\}$. We define $X_i := X_{t_i}$ with $i = 0, \dots, N$.

The discrete versions of (2.2) are defined as follows: for $m = 0, 1, 2$ and $\pm \in \{-, +\}$, let

$$\begin{aligned} \mathfrak{M}_{T,N}^{\pm,m} &:= \sum_{k=0}^{N-1} X_k^m \mathbf{1}_{\{\pm(X_k - r) \geq 0\}} (X_{k+1} - X_k), \quad \text{and} \\ \mathfrak{Q}_{T,N}^{\pm,m} &:= \sum_{k=0}^{N-1} X_k^m \mathbf{1}_{\{\pm(X_k - r) \geq 0\}} (t_{k+1} - t_k). \end{aligned} \quad (2.9)$$

We refer with *discretized likelihood* (corresponding to (2.3)) to

$$\begin{aligned} &G_{T,N}(a_+, b_+, a_-, b_-) \\ &= \exp \left(\sum_{i=0}^{N-1} \left(\frac{b(X_i) - a(X_i)X_i}{\sigma(X_i)^2} (X_{i+1} - X_i) - \frac{t_{i+1} - t_i}{2} \frac{(b(X_i) - a(X_i)X_i)^2}{\sigma(X_i)^2} \right) \right) \end{aligned}$$

and with *discretized quasi-likelihood* (corresponding to (2.4)) to

$$\begin{aligned} &\Lambda_{T,N}(a_+, b_+, a_-, b_-) \\ &= \sum_{i=0}^{N-1} \left((b(X_i) - a(X_i)X_i)(X_{i+1} - X_i) - \frac{t_{i+1} - t_i}{2} (b(X_i) - a(X_i)X_i)^2 \right). \end{aligned}$$

For $N \in \mathbb{N}$, $T \in (0, \infty)$, let

$$\left(\widehat{a}_{T,N}^{\pm}, \widehat{b}_{T,N}^{\pm} \right) = \left(\frac{\mathfrak{M}_{T,N}^{\pm,0} \mathfrak{Q}_{T,N}^{\pm,1} - \mathfrak{Q}_{T,N}^{\pm,0} \mathfrak{M}_{T,N}^{\pm,1}}{\mathfrak{Q}_{T,N}^{\pm,0} \mathfrak{Q}_{T,N}^{\pm,2} - (\mathfrak{Q}_{T,N}^{\pm,1})^2}, \frac{\mathfrak{M}_{T,N}^{\pm,0} \mathfrak{Q}_{T,N}^{\pm,2} - \mathfrak{Q}_{T,N}^{\pm,1} \mathfrak{M}_{T,N}^{\pm,1}}{\mathfrak{Q}_{T,N}^{\pm,0} \mathfrak{Q}_{T,N}^{\pm,2} - (\mathfrak{Q}_{T,N}^{\pm,1})^2} \right). \quad (2.10)$$

Theorem 2. Let $(T_N)_{N \in \mathbb{N}}$ be a sequence in $(0, \infty)$. For all $N \in \mathbb{N}$, let Δ_N above be T_N/N , let $\widehat{a}_{T_N, N}^\pm, \widehat{b}_{T_N, N}^\pm$ be defined as in (2.10).

- i) For every $N \in \mathbb{N}$ the vector $(\widehat{a}_{T_N, N}^+, \widehat{b}_{T_N, N}^+, \widehat{a}_{T_N, N}^-, \widehat{b}_{T_N, N}^-)$ maximizes both the likelihood $G_{T_N, N}(a_+, b_+, a_- b_-)$ and the quasi-likelihood $\Lambda_{T_N, N}(a_+, b_+, a_- b_-)$.

Assume now that the process is ergodic, i.e., (2.1) is satisfied, and that X is the stationary solution to (1.1), i.e., X_0 follows the stationary distribution (cf. (A.2)). Moreover, assume

$$\lim_{N \rightarrow \infty} T_N = \infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \Delta_N = 0.$$

- ii) The following LLN holds: $(\widehat{a}_{T_N, N}^\pm, \widehat{b}_{T_N, N}^\pm) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} (a_\pm, b_\pm)$ (the estimator is consistent).

- iii) If $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$, the following CLT jointly holds for the positive and negative sides:

$$\sqrt{T_N} \left(\widehat{a}_{T_N, N}^\pm - a_\pm, \widehat{b}_{T_N, N}^\pm - b_\pm \right) \xrightarrow[N \rightarrow \infty]{\text{stably}} N^\pm = (N^{\pm, \alpha}, N^{\pm, \beta})$$

where $(N^{+, \alpha}, N^{+, \beta})$ and $(N^{-, \alpha}, N^{-, \beta})$ are as in Theorem 1.

- iv) If $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$, the discretized likelihood satisfies (2.7) (with $T = T_N$).

Remark 2. If the largest time lag $\Delta_N = O(T_N/N)$ the conditions in Theorem 2 become $\lim_{N \rightarrow \infty} T_N = \infty$ and $\lim_{N \rightarrow \infty} T_N/N = 0$ for consistency and $\lim_{N \rightarrow \infty} T_N = \infty$ and $\lim_{N \rightarrow \infty} T_N^2/N = 0$ for asymptotic normality.

The next result states that, for fixed time horizon, in high frequency, the estimator from discrete observations converges, with an ‘‘anomalous’’ speed, towards the estimator from continuous observations. Let $Y: \Omega \times [0, \infty) \rightarrow \mathbb{R}$ be a semi-martingale, let $r \in \mathbb{R}$, and let $T \in [0, \infty)$. Then we recall that

$$L_T^r(Y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^T \mathbf{1}_{\{-\varepsilon \leq Y_s - r \leq \varepsilon\}} d\langle Y \rangle_s \quad (2.11)$$

is the symmetric local time of Y at r , up to time T .

Theorem 3. Let $T \in (0, \infty)$ be fixed.

- i) For every $N \in \mathbb{N}$, the likelihood $G_{T, N}(a_+, b_+, a_- b_-)$ and the quasi-likelihood $\Lambda_{T, N}(a_+, b_+, a_- b_-)$ are both maximal at $(\widehat{a}_{T, N}^+, \widehat{b}_{T, N}^+, \widehat{a}_{T, N}^-, \widehat{b}_{T, N}^-)$ given in (2.10).

- ii) It holds that $(\widehat{a}_{T, N}^\pm, \widehat{b}_{T, N}^\pm) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} (\alpha_T^\pm, \beta_T^\pm)$ and

$$\begin{aligned} & N^{1/4} \left((\widehat{a}_{T, N}^+, \widehat{b}_{T, N}^+, \widehat{a}_{T, N}^-, \widehat{b}_{T, N}^-) - (\alpha_T^+, \beta_T^+, \alpha_T^-, \beta_T^-) \right) \\ & \xrightarrow[N \rightarrow \infty]{\text{stably}} \sqrt{\frac{4\sqrt{T}}{3\sqrt{2\pi}} \frac{\sigma_-^2 + \sigma_+^2}{\sigma_- + \sigma_+}} \left(\frac{\mathfrak{Q}_T^{+,1} - r\mathfrak{Q}_T^{+,0}}{\mathfrak{Q}_T^{+,0}\mathfrak{Q}_T^{+,2} - (\mathfrak{Q}_T^{+,1})^2}, \frac{\mathfrak{Q}_T^{+,2} - r\mathfrak{Q}_T^{+,1}}{\mathfrak{Q}_T^{+,0}\mathfrak{Q}_T^{+,2} - (\mathfrak{Q}_T^{+,1})^2}, \right. \\ & \quad \left. - \frac{\mathfrak{Q}_T^{-,1} - r\mathfrak{Q}_T^{-,0}}{\mathfrak{Q}_T^{-,0}\mathfrak{Q}_T^{-,2} - (\mathfrak{Q}_T^{-,1})^2}, - \frac{\mathfrak{Q}_T^{-,2} - r\mathfrak{Q}_T^{-,1}}{\mathfrak{Q}_T^{-,0}\mathfrak{Q}_T^{-,2} - (\mathfrak{Q}_T^{-,1})^2} \right) B_{L_T^r(X)} \end{aligned} \quad (2.12)$$

with B Brownian motion independent of X and $L_T^r(X)$ symmetric local time of X at r , up to time T (see (2.11)).

Remark 3. The right hand side of (2.12) has the same law as

$$\sqrt{\frac{4\sqrt{T} \sigma_-^2 + \sigma_+^2}{3\sqrt{2\pi} \sigma_- + \sigma_+}} \begin{pmatrix} \left(\begin{array}{cc} \mathfrak{Q}_T^{+,2} & -\mathfrak{Q}_T^{+,1} \\ -\mathfrak{Q}_T^{+,1} & \mathfrak{Q}_T^{+,0} \end{array} \right)^{-1} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \left(\begin{array}{cc} \mathfrak{Q}_T^{-,2} & -\mathfrak{Q}_T^{-,1} \\ -\mathfrak{Q}_T^{-,1} & \mathfrak{Q}_T^{-,0} \end{array} \right)^{-1} \end{pmatrix} \begin{pmatrix} -r \\ 1 \\ r \\ -1 \end{pmatrix} \sqrt{L_T^r(X)} B_1.$$

Remark 4. One usually expects such discretizations to converge with speed \sqrt{N} . In this case, the lower speed of convergence is due to the discontinuity in the coefficients, and appears in connection with the local time. Indeed, the asymptotic behavior of the estimators is intrinsically related to the one of the local time of the process at the threshold. More precisely the difference $\mathfrak{M}_{T,N}^{\pm,m} - \mathfrak{M}_T^{\pm,m}$, $m = 0, 1$ can be rewritten involving terms $L_{T,N}^r - L_T^r(X)$, where $L_{T,N}^r$ is the following approximation of the local time from discrete time observations

$$L_{T,N}^r := 2 \sum_{i=0}^{N-1} \mathbf{1}_{\{(X_{iT/N-r})(X_{(i+1)T/N-r}) < 0\}} |X_{(i+1)T/N} - r| \quad (2.13)$$

for $N \in \mathbb{N}$ (see equation (A.13) for a more precise statement).

Remark 5 (The skew OU process). Let us consider the solution to the following SDE involving the local time

$$Y_t = Y_0 + \int_0^t \bar{\sigma}(Y_s) dW_s + \int_0^t (\bar{b}(Y_s) - \bar{a}(Y_s) Y_s) ds + \bar{\beta} L_t^{\bar{r}}(Y), \quad t \geq 0, \quad (2.14)$$

with $\bar{\beta} \in (-1, 1)$ and piecewise constant functions $\bar{\sigma}$, \bar{a} , \bar{b} possibly discontinuous at the threshold $\bar{r} \in \mathbb{R}$, as in (1.2) and (1.3).

Xing et al. [2020] assume $\bar{\beta}$ and $\bar{\sigma}$ known and consider drift parameters estimation for Y , based on discrete observations, in the case of constant $\bar{\sigma}$, \bar{a} , \bar{b} coefficients and local time at 0. In this setting, Y is referred to as ‘‘skew OU process’’ (see also [Feng, 2016]).

Consider now the more general case of $\bar{\sigma}$, \bar{a} , \bar{b} as in (1.2) and (1.3). If we assume that only $\bar{\beta}$ is known, all the results in Section 2 on drift estimation for X hold similarly for drift estimation of Y . This follows from the fact that a simple transformation allows us to reduce the skew OU to a threshold OU with threshold at 0, getting rid of the local time in the dynamics.

Remark 6 (The threshold CIR process). Su and Chan [2015, 2017] and Yu et al. [2020] consider diffusion processes with drift as in (1.3), with more flexibility on the diffusion coefficient $\sigma(\cdot)$, so that their analysis also applies to the process as in (1.1) with

$$\sigma(x) = \sigma_+ \sqrt{x} \mathbf{1}_{\{x \geq r\}} + \sigma_- \sqrt{x} \mathbf{1}_{\{x < r\}}. \quad (2.15)$$

We refer here to this process by threshold CIR (Cox-Ingersoll-Ross). In these works, the proposed estimators are always QMLE, maximising (2.4), that does not depend on the diffusivity $\sigma(\cdot)$. Here, with our (more restrictive) piecewise-constant choice for the diffusivity, we are able to show that the considered estimator is a genuine MLE. In our setting, we expect a result analogous to Theorem 1 to apply to the QMLE for the threshold CIR as well, but with a less explicit limit Gaussian law in the CLT, cf. also [Su and Chan, 2015]. On the other hand, the proof of the discrete time Theorem 2 makes use of bounds on hitting times for the OU process. The corresponding result for the threshold CIR process does not seem to be a trivial extension.

3 Threshold estimation, testing and interest rates

We simulate the threshold OU process using the Euler scheme [Bokil et al., 2020] (an alternative approach for simulation consists in discretizing space instead of time, cf. [Ding et al., 2020]) and use the estimator based on discrete observations. The implementation has been done using R. Parameters are as in Table 1. In Figure 1 we see an example of

| r | b_- | b_+ | a_- | a_+ | σ_- | σ_+ |
|------|--------|-------|-------|-------|------------|------------|
| 0.01 | -0.002 | 0.003 | 0.1 | 0.11 | 0.011 | 0.01 |

Table 1: Simulations parameters.

the CLT in Theorem 2. In Table 2 we show mean, standard deviation and mean squared

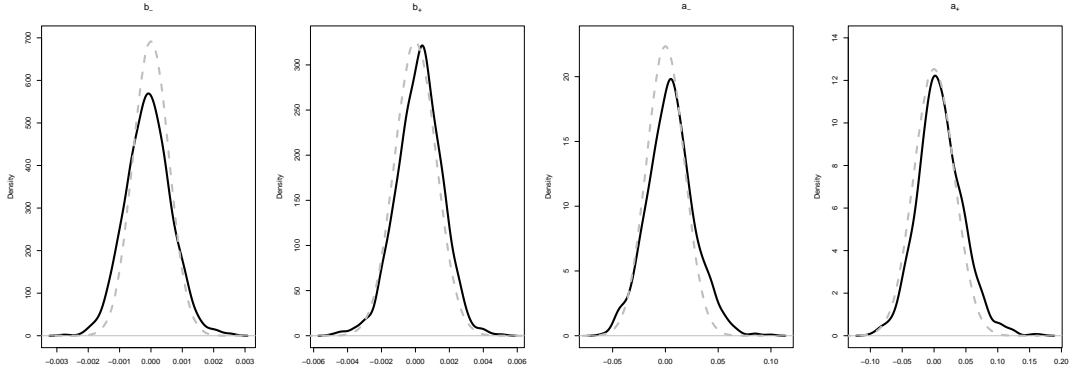


Figure 1: CLT in Theorem 2.(iii), with parameters as in Table 1. We plot the theoretical distribution of the estimation error (dashed line) and compare with the distribution of the error on $n = 10^3$ trajectories, with $T = 10^3$ and $N = 10^6$ observations on each trajectory.

error of the estimators on simulated trajectories. In Figure 2 we see an example of the

| parameter | b_- | b_+ | a_- | a_+ |
|--------------|-----------------------|-----------------------|----------|---------|
| mean | -0.00204 | 0.00318 | 0.105 | 0.119 |
| MSE | 5.10×10^{-7} | 1.62×10^{-6} | 0.000526 | 0.00130 |
| simulated sd | 0.000713 | 0.00126 | 0.0223 | 0.0349 |
| predicted sd | 0.000575 | 0.00122 | 0.0178 | 0.0317 |

Table 2: Mean, mean squared error (MSE) and standard deviation (simulated sd) of the (Q)MLE estimators in (2.10), with parameters as in Table 1, on $n = 10^3$ trajectories, with $T = 10^3$ and $N = 10^6$ observations on each trajectory. The “predicted sd” is the sd predicted by the Gaussian CLT in Theorem 2.

convergence in Theorem 3, using that (2.12) can be rewritten as

$$N^{1/4} \left(\frac{\hat{a}_{T,N}^\pm - \alpha_T^\pm}{\mathfrak{Q}_T^{\pm,1} - r\mathfrak{Q}_T^{\pm,0}}, \frac{\hat{b}_{T,N}^\pm - \beta_T^\pm}{\mathfrak{Q}_T^{\pm,2} - r\mathfrak{Q}_T^{\pm,1}} \right) \sqrt{\frac{3\sqrt{2\pi}}{4\sqrt{T}L_T^r(X)} \frac{\sigma_- + \sigma_+}{\sigma_-^2 + \sigma_+^2} (\mathfrak{Q}_T^{\pm,0}\mathfrak{Q}_T^{\pm,2} - (\mathfrak{Q}_T^{\pm,1})^2)} \xrightarrow[N \rightarrow \infty]{\text{stably}} \pm B_1, \text{ for } \pm \in \{-, +\}. \quad (3.1)$$

To estimate the local time $L_T^r(X)$ and the occupation times $\mathfrak{Q}_T^{\pm,i}$, we use the discrete time approximations in (2.13) and (2.9). If we want to simulate a stationary version of

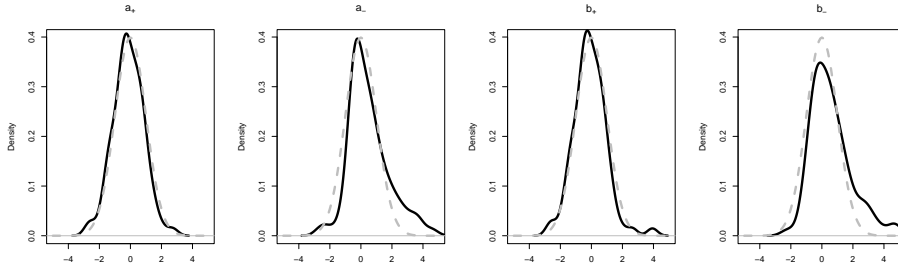


Figure 2: Convergence in Theorem 3, with parameters as in Table 1. We compare on $n = 100$ trajectories the distribution of the left hand side of (3.1), where $T = 1$ and $N = 500$ discrete observations, with a standard Gaussian (dashed line).

process (1.1), we can simulate X_0 using explicit stationary density (A.2) or running the process until large time T and then using the r.v. X_T as initial condition. In Figure 3 we compare the empirical distribution obtained in this way with the theoretical stationary density. This constitutes an example of bi-modal stationary distribution (density) with two peaks, corresponding to the two different mean reversion levels.

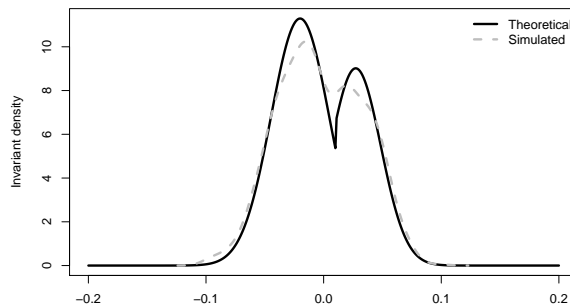


Figure 3: Theoretical invariant density in (A.2) vs empirical distribution of X_T , with $T = 10^3$, with $N = 10^6$ discretization steps in Euler scheme, on $n = 10^3$ trajectories. Parameters are as in Table 1.

3.1 On threshold estimation

The estimation results in Section 2 suppose the previous knowledge of the threshold. In practice, this assumption is not realistic and the threshold r has to be estimated as well. In [Su and Chan, 2015], threshold QMLE from continuous observations is shown to be T -consistent. We implement here also the analogous threshold MLE, and we directly consider discrete observations starting from the convergence results in Theorem 2.

Given N discrete observations of one trajectory up to time T_N , we proceed as follows. First, for a given threshold r , we compute (Q)MLE $(\hat{a}^\pm, \hat{b}^\pm)_{T_N, N}$, and denote this estimator $(\hat{a}^\pm, \hat{b}^\pm)_{T_N, N}^r$. For each fixed r , we can then compute the quasi-likelihood function $\Lambda_{T_N, N}$. We can also compute the likelihood function $G_{T_N, N}$, after estimating σ_\pm using the quadratic variation estimators in [Lejay and Pigato, 2018]. We take c to be the δ -percentile and d the $1 - \delta$ percentile of the observed data (in the implementation we always take

$\delta = 0.15$ and vary r on a discrete grid). Maximizing now the (quasi-)likelihood function over $r \in [c, d]$ we obtain the (Q)MLE of the threshold, \hat{r} . The estimator of all the drift parameters is then $(\hat{r}, (\hat{a}^\pm, \hat{b}^\pm)_{T_N, N}^\hat{r})$.

We display a sample trajectory in Figure 4, together with the threshold estimated on that trajectory and mean reversion levels. Estimated parameters are in Table 3 (cf. with

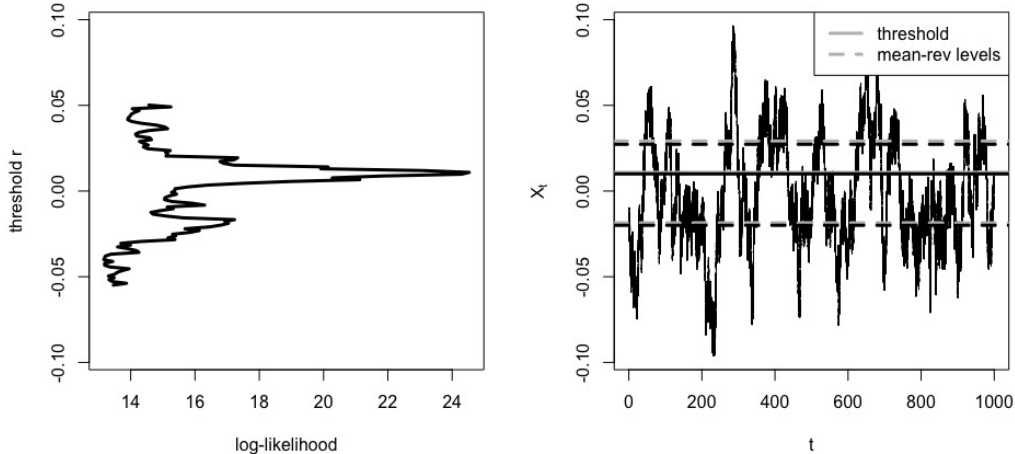


Figure 4: A sample trajectory with parameters as in Table 1, $T = 10^3$ and $N = 10^6$ time steps, and the results of estimation of both threshold and parameters, using MLE. On the left, we show the log-likelihood (on the x-axis) as a function of the threshold (on the y-axis), in order to visualize the procedure for threshold estimation described in Section 3.1. On the right, we show estimated vs actual threshold level and mean reversion levels b_-/a_- and b_+/a_+ .

simulation parameters in Table 1). Note that MLE and QMLE give the same parameter

| parameter | r | b_- | b_+ | a_- | a_+ | b_-/a_- | b_+/a_+ |
|-----------|--------|----------|---------|--------|--------|-----------|-----------|
| (Q)MLE | 0.0109 | -0.00222 | 0.00403 | 0.119 | 0.138 | -0.0186 | 0.0292 |
| sd | | 0.000649 | 0.00129 | 0.0218 | 0.0342 | | |

Table 3: Estimated drift parameters corresponding to Figure 4. Note that in this case, the threshold maximizing MLE and QMLE is the same (on the discrete grid we consider), but this is not necessarily the case. With the same threshold, also estimates for a_\pm, b_\pm are the same, from Theorem 2.(i). We also show the standard deviation of such estimators according to the Gaussian CLT in Theorem 2.

estimates once the threshold is fixed (Theorem 2.(i)). However, when maximizing also over the choice of the threshold, the MLE can also account of a possible change in the volatility, and this may give a different choice of the threshold. The model with different volatilities (SET Vasicek) is used by Decamps et al. [2006]. Su and Chan [2015, 2017] use the QMLE, so their drift estimator does not account of possible changes in the volatility.

3.2 Testing for threshold

We aim to test the presence of a threshold in the diffusion dynamics. Su and Chan [2017] propose a test for the presence of a threshold based on quasi-likelihood ratio.

Here, we derive a test from the CLT in Theorem 2.(iii); therefore, we suppose that its assumptions are satisfied. Moreover, we assume that the threshold parameter r is given. In applications, a natural choice for r will be the (Q)MLE \hat{r} . With fixed threshold, we can estimate the drift parameters obtaining $(\hat{a}_{T_N,N}^\pm, \hat{b}_{T_N,N}^\pm)$. From Theorem 2.(iii), if T_N^2/N goes to 0 as $N \rightarrow \infty$,

$$\begin{aligned} & \sqrt{T_N} \left((\hat{a}_{T_N,N}^+ - \hat{a}_{T_N,N}^-) - (a_+ - a_-), (\hat{b}_{T_N,N}^+ - \hat{b}_{T_N,N}^-) - (b_+ - b_-) \right) \\ & \xrightarrow[N \rightarrow \infty]{\text{stably}} N^+ - N^- \end{aligned} \quad (3.2)$$

which is a centered Gaussian vector with covariance matrix given by $\Sigma := \sigma_+^2 \Gamma_+^{-1} + \sigma_-^2 \Gamma_-^{-1}$, invertible from Cauchy-Schwarz inequality. The inverse matrix Σ^{-1} can be expressed as a function of σ_\pm and $\mathfrak{Q}_\infty^{\pm,i}$. Note that σ_\pm can be estimated from one observed trajectory using quadratic variation as in [Lejay and Pigato, 2018] and $\mathfrak{Q}_\infty^{\pm,i}$ can be estimated computing $\frac{1}{T_N} \mathfrak{Q}_{T_N,N}^{\pm,i}$ as Riemann sums on the observed trajectory, from (2.6). We denote $\hat{\Sigma}^{-1}$ the estimate of Σ^{-1} obtained from these estimations.

To test for the presence of a threshold in the drift we consider hypothesis

$$\begin{cases} H_0: \text{Null hypothesis} & (a_+, b_+) = (a_-, b_-) \\ H_1: \text{Alternative hypothesis} & (a_+, b_+) \neq (a_-, b_-). \end{cases}$$

Under the null hypothesis the statistics

$$T_N \left((\hat{a}_{T_N,N}^+ - \hat{a}_{T_N,N}^-), (\hat{b}_{T_N,N}^+ - \hat{b}_{T_N,N}^-) \right) \hat{\Sigma}^{-1} \begin{pmatrix} (\hat{a}_{T_N,N}^+ - \hat{a}_{T_N,N}^-) \\ (\hat{b}_{T_N,N}^+ - \hat{b}_{T_N,N}^-) \end{pmatrix}$$

converges to a χ^2 distribution with 2 degrees of freedom. We reject H_0 if the statistics is larger than q_α , where q_α is the quantile of a χ^2 distribution with two degrees of freedom such that $\mathbb{P}(\chi_2^2 \geq q_\alpha) = \alpha$.

To conclude, note that (3.2) similarly allows to test separately the presence of a threshold in $a(\cdot)$ or in $b(\cdot)$, i.e. testing for the presence of a discontinuity in the piecewise linear or in the piecewise constant part of the drift.

3.3 Interest rate analysis

We consider the 3 months US Treasury Bill rate, time series of daily closing rate on period Jan 04, 1960 - Apr 29, 2020 (source: Yahoo Finance). We perform quasi-maximum and maximum likelihood estimation using (2.10), adopting the convention that the ‘‘daily’’ time interval is $dt = 0.046$ months, while one month is the time unit. The number of observations is $N = 15057$, whereas $T \approx 60$ years. We choose as percentile for the search of the threshold $\delta = 0.15$. We report both our MLE and QMLE parameters.

We see in Figure 5 (bottom) that in the case of QMLE our result is consistent with the one in [Su and Chan, 2015], so that the estimation identifies two regimes. One is low rates, with negligible drift, so that in this regime the process is almost a martingale. In the high regime, a stronger reversion to lower rates is ensured by the drift when the rates are very high. When - in Figure 5 (top) - we use MLE (with σ_\pm estimated using quadratic variation), the estimation identifies a low regime corresponding to the period of extremely low rates, with minimal fluctuations, that followed the 2007-2008 financial crisis, whereas almost all the rest of the time series is in the high regime. Volatilities thus estimated

are $\sigma_- = 0.186$ in the low regime, $\sigma_+ = 0.453$ in the high regime. To compute standard deviations of these estimates we apply [Lejay and Pigato, 2018, Corollary 3.8.], in the form of [Lejay and Pigato, 2019, Proposition 3.1], with the same justification as after [Lejay and Pigato, 2019, Proposition 3.1]. We get that the standard deviation is 0.00472 for σ_-^2 and 0.0120 for σ_+^2 . With this MLE for the drift, the mean reverting effect looks non-negligible both above and below the threshold. We note that parameter estimates obtained through MLE and QMLE are substantially different. This is due to the different choice of the threshold, that in the QMLE does not depend on the behaviour of the volatility, while the MLE is influenced by the volatility as well. For this reason, when using the MLE, one of the two regimes isolated by the threshold only consists of the period of extremely low and stable rates that followed the 2008 financial crisis.

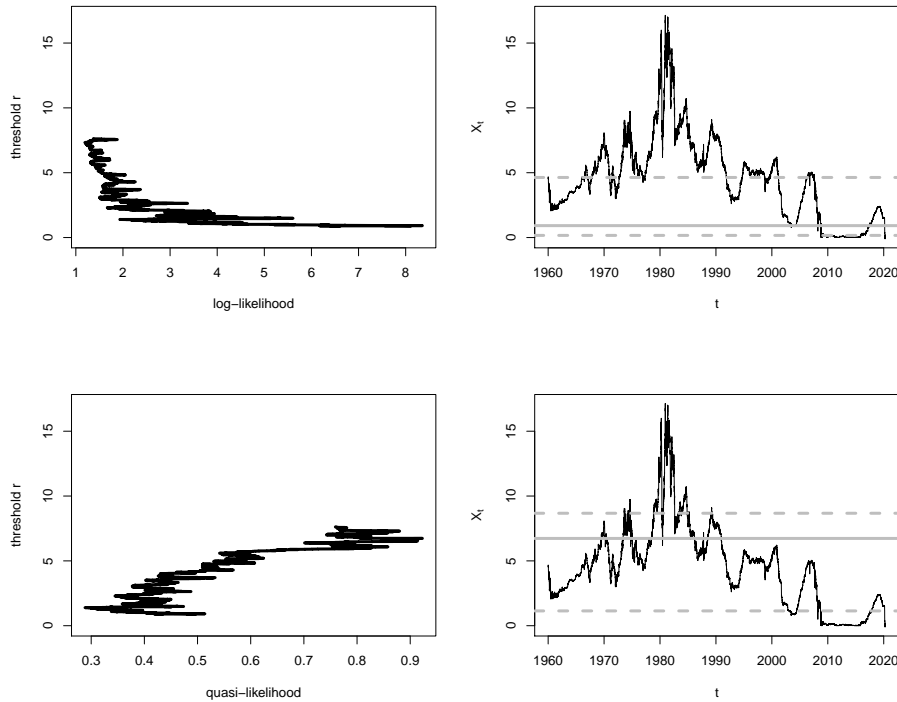


Figure 5: 3 months US Treasury Bill rate, time series of daily closing rate on period Jan 04, 1960 - Apr 29, 2020. In the top figure we use the MLE, in the bottom figure the QMLE. On the right hand side we show log-likelihood and quasi-likelihood as a function of the threshold. On the right hand side we show estimated threshold levels (solid grey line) and mean reversion levels b_-/a_- and b_+/a_+ (dashed grey line).

| parameter | r | b_- | b_+ | a_- | a_+ | b_-/a_- | b_+/a_+ |
|-----------|-------|---------|--------|---------|---------|-----------|-----------|
| MLE | 0.919 | 0.0469 | 0.0492 | 0.284 | 0.0106 | 0.165 | 4.63 |
| MLE sd | | 0.0223 | 0.0402 | 0.0757 | 0.00672 | | |
| QMLE | 6.73 | 0.00131 | 0.417 | 0.00115 | 0.0481 | 1.14 | 8.67 |
| QMLE sd | | 0.0341 | 0.144 | 0.00877 | 0.0153 | | |

Table 4: Estimated drift parameters corresponding to Figure 5. and corresponding standard deviation for b_{\pm} and a_{\pm} according to Theorem 2

In this analysis, following Su and Chan [2015, 2017], we estimated our model param-

eters on the whole period 1960-2020. From an econometric perspective, it is natural to wonder whether it is reasonable to assume the stationarity of the process on such a long time interval. To address this issue, we consider within the period 2010-2020 five two-years time windows, with daily observations as before. With this choice, $T^2 \approx N$, and therefore we expect from Theorem 2 that the discretization error should be negligible, assuming that $T \approx 2$ years is large enough for the theorem to apply.

With 1% significance level, only in the subperiod Jan 2018-Dec 2019 the H_0 hypothesis (absence of a threshold in the parameters) is not rejected. In all other time windows (2010-2011, 2012-2013, 2014-2015, 2016-2017) we conclude that a threshold is present. In Figure 6 we see three examples of estimation in such windows. In order to check whether such test is reliable on time series with such sample sizes, we tried the same procedure (selection of threshold and successive test with 1% significance level) on simulated time series with parameters and sample sizes of the same order as the estimated ones. We found that when there is no threshold present (constant parameters) H_0 is rejected 14% of the times, when the threshold is present (non-constant parameters) H_0 is rejected 96% of the times, which seems to confirm the validity of the procedure, even in these smaller time windows.

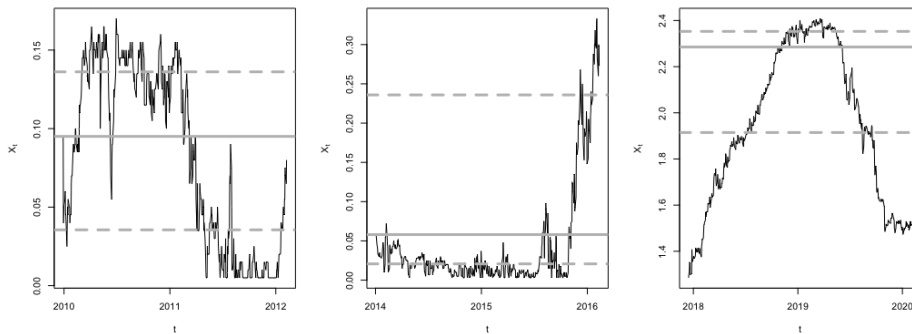


Figure 6: 3 months US Treasury Bill rate, time series of daily closing rate on periods Jan, 2010 - Dec 2012, Jan, 2014 - Dec 2016, Jan, 2018 - Dec 2020. Estimated threshold level and mean reversion levels given by QMLE. Each time window consists in 24 months, with 22 observations per month. Our test concludes that a threshold in the dynamics is present in every time window, except Jan, 2018 - Dec 2020.

A Proofs

A.1 The regimes of the process

In this section, we establish for which values of the coefficients (a_{\pm}, b_{\pm}) the process X is (positively or null) recurrent or transient. Since X is a one-dimensional diffusion it is characterized by two quantities: *scale function* S and *speed measure*.

X is *recurrent* if and only if $\lim_{x \rightarrow +\infty} S(x) = +\infty$ and $\lim_{x \rightarrow -\infty} S(x) = -\infty$, otherwise it is transient. Moreover a recurrent process is *positive recurrent* if the speed measure is a finite measure, otherwise null recurrent.

The scale density is continuous, unique up to a multiplicative constant, and its derivative satisfies $S'(x) = \exp\left(-\int_r^x \frac{2(b(y)-a(y)y)}{(\sigma(y))^2} dy\right)$. It follows that X is recurrent if and only

if $[(a_+ > 0 \text{ and } b_+ \in \mathbb{R}) \text{ or } (a_+ = 0 \text{ and } b_+ \leq 0)]$ and $[(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- \geq 0)]$. The complementary leads to transience.

The density of the *speed measure* with respect to the Lebesgue measure is given by $m(x) := \frac{2}{(\sigma(x))^2 S'(x)}$. It is *discontinuous* if and only if σ^2 is so. Assume X is recurrent. The speed measure is a finite measure, and so X is positive recurrent, if and only if

$$\begin{aligned} & [(a_+ > 0 \text{ and } b_+ \in \mathbb{R}) \text{ or } (a_+ = 0 \text{ and } b_+ < 0)] \\ & \text{and } [(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- > 0)]. \end{aligned} \quad (\text{A.1})$$

See Lemma 1 below. In these cases, the process is actually *ergodic* and the stationary distribution μ is equal to the renormalized speed measure:

$$\mu(dx) = \frac{m(x)}{\int_{-\infty}^{\infty} m(y) dy} dx. \quad (\text{A.2})$$

Lemma 1. *Let $\pm \in \{-, +\}$ and let*

$$\mathbf{m}_{\pm} := \frac{\sqrt{\pi}}{\sigma_{\pm} \sqrt{a_{\pm}}} \exp\left(\frac{a_{\pm}}{\sigma_{\pm}^2} \left(\frac{b_{\pm}}{a_{\pm}} - r\right)^2\right) \operatorname{erfc}\left(\mp \frac{\sqrt{a_{\pm}}}{\sigma_{\pm}} \left(\frac{b_{\pm}}{a_{\pm}} - r\right)\right). \quad (\text{A.3})$$

Then

$$\int_{-\infty}^{\infty} \mathbf{1}_{\{\pm(y-r) \geq 0\}} m(y) dy = \begin{cases} +\infty & \text{if } a_{\pm} = 0 \text{ and } b_{\pm} = 0, \\ \frac{1}{|b_{\pm}|} & \text{if } a_{\pm} = 0 \text{ and } \mp b_{\pm} > 0, \\ \mathbf{m}_{\pm} & \text{if } a_{\pm} > 0 \text{ and } b_{\pm} \in \mathbb{R}. \end{cases}$$

Lemma 2. *Assume the process is ergodic. Let $\mathbf{m}_-, \mathbf{m}_+$, given by (A.3), $\mathbf{b}_{\pm} = 1/|b_{\pm}|$, $\pm \in \{-, +\}$, and μ be the stationary distribution. For all $i \in \{0, 1, 2\}$ let $\overline{\mathfrak{Q}}_{\infty}^{\pm, i}$ be the constant such that $\overline{\mathfrak{Q}}_{\infty}^{\pm, i, \text{a.s.}} = \lim_{t \rightarrow \infty} \frac{\mathfrak{Q}_t^{\pm, i}}{t} \in \mathbb{R}$. We have the following explicit formulas:*

- if $a_+ > 0, a_- > 0, b_-, b_+ \in \mathbb{R}$ then

$$\begin{aligned} \overline{\mathfrak{Q}}_{\infty}^{\pm, 0} &= \frac{\mathbf{m}_{\pm}}{\mathbf{m}_+ + \mathbf{m}_-}, \quad \overline{\mathfrak{Q}}_{\infty}^{\pm, 1} = \frac{1}{\mathbf{m}_+ + \mathbf{m}_-} \left(\frac{b_{\pm}}{a_{\pm}} \mathbf{m}_{\pm} \pm \frac{1}{a_{\pm}} \right), \quad \text{and} \\ \overline{\mathfrak{Q}}_{\infty}^{\pm, 2} &= \frac{1}{\mathbf{m}_+ + \mathbf{m}_-} \left(\left(\frac{b_{\pm}^2}{a_{\pm}^2} + \frac{\sigma_{\pm}^2}{2a_{\pm}} \right) \mathbf{m}_{\pm} \pm \left(\frac{b_{\pm}}{a_{\pm}} + r \right) \frac{1}{a_{\pm}} \right); \end{aligned}$$

- if $a_+ = 0, a_- = 0, b_+ < 0, b_- > 0$ then

$$\begin{aligned} \overline{\mathfrak{Q}}_{\infty}^{\pm, 0} &= \frac{\mathbf{b}_{\pm}}{\mathbf{b}_+ + \mathbf{b}_-}, \quad \overline{\mathfrak{Q}}_{\infty}^{\pm, 1} = \frac{\mathbf{b}_{\pm}}{\mathbf{b}_+ + \mathbf{b}_-} \left(r \pm \frac{\sigma_{\pm}^2}{2} \mathbf{b}_{\pm} \right), \quad \text{and} \\ \overline{\mathfrak{Q}}_{\infty}^{\pm, 2} &= \frac{\mathbf{b}_{\pm}}{\mathbf{b}_+ + \mathbf{b}_-} \left(r^2 \pm r \sigma_{\pm}^2 \mathbf{b}_{\pm} + \frac{\sigma_{\pm}^4}{2} (\mathbf{b}_{\pm})^2 \right); \end{aligned}$$

- $a_+ > 0, b_+ \in \mathbb{R}, a_- = 0, b_- > 0$ then

$$\begin{aligned} \overline{\mathfrak{Q}}_{\infty}^{+, 0} &= \frac{\mathbf{m}_+}{\mathbf{m}_+ + \mathbf{b}_-}, \quad \overline{\mathfrak{Q}}_{\infty}^{-, 0} = \frac{\mathbf{b}_-}{\mathbf{m}_+ + \mathbf{b}_-} \\ \overline{\mathfrak{Q}}_{\infty}^{+, 1} &= \frac{1}{\mathbf{m}_+ + \mathbf{b}_-} \left(\frac{b_+}{a_+} \mathbf{m}_+ + \frac{1}{a_+} \right), \quad \overline{\mathfrak{Q}}_{\infty}^{-, 1} = \frac{\mathbf{b}_-}{\mathbf{m}_+ + \mathbf{b}_-} \left(r - \frac{\sigma_-^2}{2} \mathbf{b}_- \right), \\ \overline{\mathfrak{Q}}_{\infty}^{+, 2} &= \frac{1}{\mathbf{m}_+ + \mathbf{b}_-} \left(\left(\frac{b_+^2}{a_+^2} + \frac{\sigma_+^2}{2a_+} \right) \mathbf{m}_+ + \left(\frac{b_+}{a_+} + r \right) \frac{1}{a_+} \right), \quad \text{and} \\ \overline{\mathfrak{Q}}_{\infty}^{-, 2} &= \frac{\mathbf{b}_-}{\mathbf{m}_+ + \mathbf{b}_-} \left(r^2 - r \sigma_-^2 \mathbf{b}_- + \frac{\sigma_-^4}{2} (\mathbf{b}_-)^2 \right); \end{aligned}$$

- $a_+ = 0, b_+ < 0, a_- > 0, b_- \in \mathbb{R}$ then

$$\begin{aligned}\bar{\mathfrak{Q}}_\infty^{+,0} &= \frac{\mathfrak{b}_+}{\mathfrak{b}_+ + \mathfrak{m}_-}, & \bar{\mathfrak{Q}}_\infty^{-,0} &= \frac{\mathfrak{m}_-}{\mathfrak{b}_+ + \mathfrak{m}_-} \\ \bar{\mathfrak{Q}}_\infty^{+,1} &= \frac{\mathfrak{b}_+}{\mathfrak{b}_+ + \mathfrak{m}_-} \left(r + \frac{\sigma_+^2}{2} \mathfrak{b}_+ \right), & \bar{\mathfrak{Q}}_\infty^{-,1} &= \frac{1}{\mathfrak{b}_+ + \mathfrak{m}_-} \left(\frac{b_-}{a_-} \mathfrak{m}_- - \frac{1}{a_-} \right), \\ \bar{\mathfrak{Q}}_\infty^{+,2} &= \frac{\mathfrak{b}_+}{\mathfrak{b}_+ + \mathfrak{m}_-} \left(r^2 + r\sigma_+^2 \mathfrak{b}_+ + \frac{\sigma_+^4}{2} (\mathfrak{b}_+)^2 \right), & \text{and} \\ \bar{\mathfrak{Q}}_\infty^{-,2} &= \frac{1}{\mathfrak{b}_+ + \mathfrak{m}_-} \left(\left(\frac{b_-^2}{a_-^2} + \frac{\sigma_-^2}{2a_-} \right) \mathfrak{m}_- - \left(\frac{b_-}{a_-} + r \right) \frac{1}{a_-} \right).\end{aligned}$$

A.2 Proof of Theorem 1

Proof of Item (i) of Theorem 1. Let $\theta := (a_+, b_+, a_-, b_-)$. It holds that

$$\Lambda_T(\theta) = \sum_{\pm \in \{-, +\}} \left(b_\pm \mathfrak{M}_T^{\pm,0} - a_\pm \mathfrak{M}_T^{\pm,1} - \frac{1}{2} \left(b_\pm^2 \mathfrak{Q}_T^{\pm,0} + a_\pm^2 \mathfrak{Q}_T^{\pm,2} - 2a_\pm b_\pm \mathfrak{Q}_T^{\pm,1} \right) \right). \quad (\text{A.4})$$

To find the maximum we compute the derivatives with respect to a_\pm, b_\pm and observe that the gradient has a unique singular point given by (2.5) and the Hessian is negative definite.

Moreover the fact that $\partial_{a_\pm} \Lambda_T = \sigma_\pm^2 \partial_{a_\pm} \log G_T$ and $\partial_{b_\pm} \Lambda_T = \sigma_\pm^2 \partial_{b_\pm} \log G_T$ shows that the MLE for the drift parameters will be the same as the QMLE, i.e. (2.5). \square

In order to study the asymptotic behavior of the estimator we introduce a different expression for the estimators in (2.5) based on the following notation. Given $T \in (0, \infty)$, let

$$Q_T^{\pm,i} := \int_0^T |X_s - r|^i \mathbf{1}_{\{\pm(X_s - r) \geq 0\}} ds \quad \text{and} \quad M_T^{\pm,j} := \sigma_\pm \int_0^T |X_s - r|^{j-1} \mathbf{1}_{\{\pm(X_s - r) \geq 0\}} dW_s$$

with $i \in \{0, 1, 2\}, j \in \{1, 2\}$. Observe that (1.1) yields for $i \in \{0, 1\}$:

$$\mathfrak{M}_T^{\pm,i} = r^i M^{\pm,1} \pm i M^{\pm,2} + b_\pm \mathfrak{Q}_T^{\pm,i} - a_\pm \mathfrak{Q}_T^{\pm,i+1}. \quad (\text{A.5})$$

Moreover, $\mathfrak{Q}_T^{\pm,0} = Q_T^{\pm,0}$, $\mathfrak{Q}_T^{\pm,1} = \pm (Q_T^{\pm,1} \pm r Q_T^{\pm,0})$, and $\mathfrak{Q}_T^{\pm,2} = Q_T^{\pm,2} \pm 2r Q_T^{\pm,1} + r^2 Q_T^{\pm,0}$.

Lemma 3. *Let $T \in (0, \infty)$. The MLE and QMLE can be expressed as*

$$\begin{cases} \alpha_T^\pm = a_\pm \pm \frac{Q_T^{\pm,1} M_T^{\pm,1} - Q_T^{\pm,0} M_T^{\pm,2}}{Q_T^{\pm,2} Q_T^{\pm,0} - (Q_T^{\pm,1})^2} \\ \beta_T^\pm = b_\pm + \frac{(Q_T^{\pm,2} \pm r Q_T^{\pm,1}) M_T^{\pm,1} - (Q_T^{\pm,1} \pm r Q_T^{\pm,0}) M_T^{\pm,2}}{Q_T^{\pm,2} Q_T^{\pm,0} - (Q_T^{\pm,1})^2}, \end{cases} \quad (\text{A.6})$$

that can be rewritten as

$$\begin{pmatrix} \alpha_T^\pm \\ \beta_T^\pm \end{pmatrix} = \begin{pmatrix} a_\pm \\ b_\pm \end{pmatrix} + \begin{pmatrix} 0 & \mp 1 \\ 1 & \mp r \end{pmatrix} \begin{pmatrix} Q_T^{\pm,0} & Q_T^{\pm,1} \\ Q_T^{\pm,1} & Q_T^{\pm,2} \end{pmatrix}^{-1} \begin{pmatrix} M_T^{\pm,1} \\ M_T^{\pm,2} \end{pmatrix}. \quad (\text{A.7})$$

Proof. Note that $\mathfrak{Q}_T^{\pm,0} \mathfrak{Q}_T^{\pm,2} - (\mathfrak{Q}_T^{\pm,1})^2 = Q_T^{\pm,0} Q_T^{\pm,2} - (Q_T^{\pm,1})^2$ which is \mathbb{P} -a.s. positive by Cauchy-Schwarz. This and replacing the equalities (A.5) in (2.5) completes the proof. \square

Proof of Item (ii) of Theorem 1. Follows from combining equation (A.6) in Lemma 3 with [Lépingle, 1995, Theorem 1, p.150] and Lemma 2. \square

Proof of Item (iii) of Theorem 1. Follows from Lemma 2, (2.6), and Theorem 2.2 in [Crimaldi and Pratelli, 2005]. \square

Proof of Item (iv) of Theorem 1. By (A.5) it holds

$$\begin{aligned} & \log \frac{G_T(a_+ + \frac{1}{\sqrt{T}}\Delta a_+, b_+ + \frac{1}{\sqrt{T}}\Delta b_+, a_- + \frac{1}{\sqrt{T}}\Delta a_-, b_- + \frac{1}{\sqrt{T}}\Delta b_-)}{G_T(a_+, b_+, a_-, b_-)} \\ &= \sum_{\pm \in \{+, -\}} \left(\frac{1}{\sqrt{T}\sigma_{\pm}} \begin{pmatrix} \Delta a_{\pm} \\ \Delta b_{\pm} \end{pmatrix} \cdot A_T^{\pm} - \frac{1}{2T\sigma_{\pm}^2} \begin{pmatrix} \Delta a_{\pm} \\ \Delta b_{\pm} \end{pmatrix} \cdot \langle A^{\pm}, A^{\pm} \rangle_T \begin{pmatrix} \Delta a_{\pm} \\ \Delta b_{\pm} \end{pmatrix} \right). \end{aligned}$$

where $A_T^{\pm} := \begin{pmatrix} -M_T^{\pm,1} \\ M_T^{\pm,0} \end{pmatrix}$. Note that $\langle A^{\pm}, A^{\pm} \rangle_T = \begin{pmatrix} \mathfrak{Q}_T^{\pm,2} & -\mathfrak{Q}_T^{\pm,1} \\ -\mathfrak{Q}_T^{\pm,1} & \mathfrak{Q}_T^{\pm,0} \end{pmatrix}$ and $\langle A^+, A^- \rangle_T = 0$. Lemma 2 ensures that $\frac{1}{T} \langle A^{\pm}, A^{\pm} \rangle_T \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \Gamma_{\pm}$. This and the same argument as in the proof of Theorem 1.(iii) show that

$$A_T := \frac{1}{\sqrt{T}} \begin{pmatrix} \sigma_+^{-1} A_T^+ \\ \sigma_-^{-1} A_T^- \end{pmatrix} \xrightarrow[T \rightarrow \infty]{\text{law}} \mathcal{N}(0, \Gamma)$$

and $\langle A_T, A_T \rangle_T = \frac{1}{T} \begin{pmatrix} \sigma_+^{-2} \langle A^+, A^+ \rangle_T & 0 \\ 0 & \sigma_-^{-2} \langle A^-, A^- \rangle_T \end{pmatrix} \xrightarrow[T \rightarrow \infty]{\text{a.s.}} \Gamma$. \square

A.3 Proof of Theorem 2

The proof of Item (i) of Theorem 2 is analogous to the proof of Item (i) of Theorem 1, therefore omitted. The proof of Items (ii)-(iii) of Theorem 2 follows from Lemma 4 below. Let us be more precise. For all $N \in \mathbb{N}$ it holds

$$\left(\widehat{a}_{T_N, N}^{\pm} - a_{\pm}, \widehat{b}_{T_N, N}^{\pm} - b_{\pm} \right) = \left(\widehat{a}_{T_N, N}^{\pm} - \alpha_{T_N}^{\pm}, \widehat{b}_{T_N, N}^{\pm} - \beta_{T_N}^{\pm} \right) + \left(\alpha_{T_N}^{\pm} - a_{\pm}, \beta_{T_N}^{\pm} - b_{\pm} \right).$$

The second term of the sum is handled with Theorem 1 (more precisely Item (iii)) providing the desired limit distribution. The first instead can be rewritten, using equations (2.5) and (2.10), as an expression which involves only terms of the kind

$$\left(\frac{\mathfrak{Q}_{T_N, N}^{\pm, i}}{\mathfrak{Q}_{T_N, N}^{\pm, 0} \mathfrak{Q}_{T_N, N}^{\pm, 2} - (\mathfrak{Q}_{T_N, N}^{\pm, 1})^2} - \frac{\mathfrak{Q}_{T_N}^{\pm, i}}{\mathfrak{Q}_{T_N}^{\pm, 0} \mathfrak{Q}_{T_N}^{\pm, 2} - (\mathfrak{Q}_{T_N}^{\pm, 1})^2} \right) \mathfrak{M}_{T_N}^{\pm, j} + \frac{\mathfrak{Q}_{T_N, N}^{\pm, i} (\mathfrak{M}_{T_N, N}^{\pm, j} - \mathfrak{M}_{T_N}^{\pm, j})}{\mathfrak{Q}_{T_N, N}^{\pm, 0} \mathfrak{Q}_{T_N, N}^{\pm, 2} - (\mathfrak{Q}_{T_N, N}^{\pm, 1})^2}$$

for $j \in \{0, 1\}$, $i \in \{0, 1, 2\}$,

Combining Lemma 4 with Lemma 2 and Theorem 2.2 in [Crimaldi and Pratelli, 2005] ensures, the consistency of the estimator if $T_N/N \rightarrow 0$ as $N \rightarrow \infty$, and if $T_N^2/N \rightarrow 0$ as $N \rightarrow \infty$ then it implies also that

$$\sqrt{T_N} \left(\widehat{a}_{T_N, N}^{\pm} - \alpha_{T_N}^{\pm}, \widehat{b}_{T_N, N}^{\pm} - \beta_{T_N}^{\pm} \right) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Lemma 4. *Assume the process is ergodic. Let X be the solution to (1.1), with X_0 distributed as the stationary distribution μ in (A.2), let $\lambda \in \{1, 2\}$ be fixed, and let $(T_N)_{N \in \mathbb{N}} \subset$*

$(0, \infty)$ be a sequence satisfying, as $N \rightarrow \infty$, that $T_N \rightarrow \infty$ and $\lim_{N \rightarrow \infty} T_N^{-1/\lambda} \sqrt{\Delta_N} = 0$ where $\Delta_N := \sup_{k=1, \dots, N} (t_k - t_{k-1})$. Then for all $m \in \{0, 1, 2\}$, $j \in \{0, 1\}$ it holds

$$\limsup_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} [|\mathfrak{Q}_{T_N}^{\pm, m} - \mathfrak{Q}_{T_N, N}^{\pm, m}|] = 0 \text{ and } \limsup_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} [|\mathfrak{M}_{T_N}^{\pm, j} - \mathfrak{M}_{T_N, N}^{\pm, j}|] = 0$$

where $\mathfrak{Q}_{T_N}^{\pm, m}$, $\mathfrak{Q}_{T_N, N}^{\pm, m}$, $\mathfrak{M}_{T_N}^{\pm, j}$, $\mathfrak{M}_{T_N, N}^{\pm, j}$ are defined in (2.2) and (2.9).

Proof. Without loss of generality, we reduce to prove the statement for threshold $r = 0$. Indeed the quantities $\mathfrak{Q}_{T_N}^{\pm, m} - \mathfrak{Q}_{T_N, N}^{\pm, m}$ and $\mathfrak{M}_{T_N}^{\pm, m} - \mathfrak{M}_{T_N, N}^{\pm, m}$ for the process X (with threshold r) can be written as linear combination (coefficients depending on m and r) of the same quantities for the process $X - r$ (which solves (1.1) with threshold at 0 and new drift coefficients $b_{\pm} - a_{\pm}r$ and a_{\pm}). We keep denoting as b_{\pm} (instead of $b_{\pm} - a_{\pm}r$) and a_{\pm} the drift coefficients. In this proof we use the round ground notation $\lfloor t \rfloor_{\Delta_N} := t_k$ for $t \in [t_k, t_{k+1}) \subseteq [t_k, t_k + \Delta_N]$. Moreover, without loss of generality, we assume $T_N \leq N$ for all $N \in \mathbb{N}$.

Let us first note that for $m = 0, 1, 2$:

$$\begin{aligned} \mathfrak{Q}_{T_N}^{\pm, m} - \mathfrak{Q}_{T_N, N}^{\pm, m} &= - \sum_{k=1}^N J_{k, N}^{(m)} \\ &= \mp \int_0^{T_N} \text{sgn}(X_{\lfloor t \rfloor_{\Delta_N}}) X_{\lfloor t \rfloor_{\Delta_N}}^m \mathbf{1}_{\{X_{\lfloor t \rfloor_{\Delta_N}} X_t < 0\}} dt + \int_0^{T_N} (X_t^m - X_{\lfloor t \rfloor_{\Delta_N}}^m) \mathbf{1}_{\{\pm X_t > 0\}} dt \end{aligned}$$

therefore

$$\mathbb{E} [|\mathfrak{Q}_{T_N}^{\pm, m} - \mathfrak{Q}_{T_N, N}^{\pm, m}|] \leq \int_0^{T_N} \mathbb{E} [|X_{\lfloor t \rfloor_{\Delta_N}}|^m \mathbf{1}_{\{X_{\lfloor t \rfloor_{\Delta_N}} X_t < 0\}}] + \mathbb{E} [|X_t^m - X_{\lfloor t \rfloor_{\Delta_N}}^m|] dt.$$

Analogously, observe that for $m = 0, 1$ it holds

$$\begin{aligned} \mathfrak{M}_{T_N}^{\pm, m} - \mathfrak{M}_{T_N, N}^{\pm, m} &= \int_0^{T_N} (X_t^m \mathbf{1}_{\{\pm X_t > 0\}} - X_{\lfloor t \rfloor_{\Delta_N}}^m \mathbf{1}_{\{\pm X_{\lfloor t \rfloor_{\Delta_N}} > 0\}}) (b(X_t) - a(X_t) X_t) dt \\ &\quad + \int_0^{T_N} (X_t^m \mathbf{1}_{\{\pm X_t > 0\}} - X_{\lfloor t \rfloor_{\Delta_N}}^m \mathbf{1}_{\{\pm X_{\lfloor t \rfloor_{\Delta_N}} > 0\}}) \sigma(X_t) dW_t. \end{aligned}$$

Let us rewrite the integrand as

$$\begin{aligned} &X_t^m \mathbf{1}_{\{\pm X_t > 0\}} - X_{\lfloor t \rfloor_{\Delta_N}}^m \mathbf{1}_{\{\pm X_{\lfloor t \rfloor_{\Delta_N}} > 0\}} \\ &= (X_t^m - X_{\lfloor t \rfloor_{\Delta_N}}^m) \mathbf{1}_{\{\pm X_t > 0\}} - \text{sgn}(X_{\lfloor t \rfloor_{\Delta_N}}) X_{\lfloor t \rfloor_{\Delta_N}}^m \mathbf{1}_{\{X_t X_{\lfloor t \rfloor_{\Delta_N}} < 0\}}. \end{aligned}$$

Triangular inequality, Hölder's inequality, and Itô-isometry imply that

$$\begin{aligned} &\mathbb{E} [|\mathfrak{M}_{T_N}^{\pm, m} - \mathfrak{M}_{T_N, N}^{\pm, m}|] \\ &\leq \int_0^{T_N} \mathbb{E} [|X_t^m - X_{\lfloor t \rfloor_{\Delta_N}}^m| (|b_{\pm}| + a_{\pm} |X_{\lfloor t \rfloor_{\Delta_N}}| + a_{\pm} |X_t - X_{\lfloor t \rfloor_{\Delta_N}}|)] dt \\ &\quad + \int_0^{T_N} \mathbb{E} [|X_{\lfloor t \rfloor_{\Delta_N}}|^m \mathbf{1}_{\{X_t X_{\lfloor t \rfloor_{\Delta_N}} < 0\}} (|b_-| \vee |b_+| + (a_- \vee a_+)) \\ &\quad \quad \times (|X_{\lfloor t \rfloor_{\Delta_N}}| + |X_t - X_{\lfloor t \rfloor_{\Delta_N}}|)] dt \\ &\quad + \sqrt{2} (\sigma_- \vee \sigma_+) \left(\int_0^{T_N} \mathbb{E} [(X_t^m - X_{\lfloor t \rfloor_{\Delta_N}}^m)^2 + X_{\lfloor t \rfloor_{\Delta_N}}^{2m} \mathbf{1}_{\{X_t X_{\lfloor t \rfloor_{\Delta_N}} < 0\}}] dt \right)^{1/2}. \end{aligned}$$

Hence, the proof of Lemma 4, reduces to prove two inequalities:

$$\int_0^{T_N} \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^j |X_{[t]_{\Delta_N}}|^m \right] dt \text{ is } o(T_N^{1/\lambda}) \quad (\text{A.8})$$

for all $j \in \{1, 2, 4\}$, $m \in \{0, 1, 2\}$, and

$$\int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m \mathbf{1}_{\{X_{[t]_{\Delta_N}} < 0\}} \right] dt \text{ is } o(T_N^{1/\lambda}) \text{ for } m \in \{0, 1, 2, 3, 4\}. \quad (\text{A.9})$$

Step 1. Given $s \in [0, \infty)$ and $t \in [0, \Delta_N]$ we show that for every $j \in \{1, 2, 4\}$ there exists a constant $C \in (0, \infty)$ depending only on $j, a_{\pm}, b_{\pm}, \sigma_{\pm}$ such that

$$\mathbb{E} [|X_{t+s} - X_s|^j |X_s] \leq C t^{j/2} (1 + |X_s|^j). \quad (\text{A.10})$$

Let $\xi_t := X_{t+s} - X_s$ then

$$\xi_t = \int_0^t (b(\xi_u + X_s) - a(\xi_u + X_s)X_s) - a(\xi_u + X_s)\xi_u du + \int_0^t \sigma(\xi_u + X_s) dW_u^s$$

where W^s a Wiener process independent of $\sigma(X_u, u \in [0, s])$. So, given X_s , ξ is an OU with threshold $-X_s$ (since X has threshold 0). Now, e.g. [Hudde et al., 2021, Corollary 2.5] applied to ξ implies (A.10).

Step 2. (Proof of (A.8)). Since X_0 is distributed as the stationary distribution μ then $\sup_{u \in [0, \infty)} \mathbb{E}[|X_u|^m] = \mathbb{E}[|X_0|^m] = \int_{-\infty}^{\infty} |x|^m \mu(dx) < \infty$. This, the tower property, and (A.10) imply that there exists $C \in (0, \infty)$ depending only on $m, j, a_{\pm}, b_{\pm}, \sigma_{\pm}$ such that

$$\begin{aligned} & \frac{1}{T_N^{1/\lambda}} \int_0^{T_N} \mathbb{E} \left[|X_t - X_{[t]_{\Delta_N}}|^j |X_{[t]_{\Delta_N}}|^m \right] dt \\ & \leq C \sqrt{\Delta_N^j T_N^{2(1-\lambda^{-1})}} = C \sqrt{\Delta_N^j T_N^{(\lambda-1)}} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

Step 3. (Proof of (A.9)). Let $s, t \in [0, \infty)$ be fixed such that $t - s \in [0, \Delta_N]$. Let us first note that we just need to consider $\mathbb{E} [\mathbf{1}_{\{\pm X_t < 0\}} \mathbf{1}_{\{\pm X_s > 0\}} |X_s]$. This, given X_s , is bounded by $\mathbb{P}(\tau_{s, \pm} \leq t - s) \leq \mathbb{P}(\tau_{s, \pm} \leq \Delta_N)$ where τ_s is the first hitting time of the level 0 of the OU process solution to the following SDE: $\xi_u = X_s + \int_0^u b_{\pm} - a_{\pm} \xi_v dv + \sigma_{\pm} W_u^s$ with W^s a Brownian motion independent of $\sigma(X_v, v \in [0, s])$. If $a_{\pm} \neq 0$, [Lipton and Kaushansky, 2020, Section 6.2.1] with $b = -\frac{b_{\pm}}{\sqrt{a_{\pm} \sigma_{\pm}}}$, $z = \frac{\sqrt{a_{\pm}}}{\sigma_{\pm}} X_s - \frac{b_{\pm}}{\sqrt{a_{\pm} \sigma_{\pm}}}$, and $t = a_{\pm} \Delta_N$, prove that

$$\mathbb{P}(\tau_{s, \pm} \leq \Delta_N) = 2e^{-\frac{b_{\pm}}{\sigma_{\pm}} X_s} \Phi\left(-\frac{\sqrt{a_{\pm}}}{\sigma_{\pm}} |X_s| \gamma_N\right) \text{ with } \gamma_N := \frac{e^{-\frac{a_{\pm} \Delta_N}{2}}}{\sqrt{\sinh(a_{\pm} \Delta_N)}}.$$

If $a_{\pm} = 0$ and $\pm b_{\pm} < 0$ then

$$\begin{aligned} \mathbb{P}(\tau_{s, \pm} \leq \Delta_N) &= \int_0^{\Delta_N} \frac{|X_s|}{\sigma_{\pm} \sqrt{2\pi u^3}} \exp\left(-\frac{(X_s - b_{\pm} u)^2}{2\sigma_{\pm}^2 u}\right) du \\ &\leq \left(1 + e^{\frac{2|X_s||b_{\pm}|}{\sigma_{\pm}^2}}\right) \Phi\left(-\frac{|X_s| - |b_{\pm}| \Delta_N}{\sqrt{2\sigma_{\pm}^2 \Delta_N}}\right). \end{aligned}$$

Therefore, using the stationary distribution (A.2), to establish Step 3 it suffices to prove that the following quantity vanishes as $N \rightarrow \infty$:

$$\frac{1}{T_N^{1/\lambda}} \int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor \Delta_N}|^m \mathbb{P} \left(\tau_{\lfloor t \rfloor \Delta_N, \pm} \leq \Delta_N \right) \right] dt.$$

Let us first consider the case $a_{\pm} = 0$ and $\pm b_{\pm} < 0$. The desired quantity is bounded by

$$\begin{aligned} & \frac{T_N}{T_N^{1/\lambda}} \frac{1}{\int_{-\infty}^{\infty} m(y) dy} \int_{\mathbb{R}^{\pm}} \frac{2|y|^m}{(\sigma_{\pm})^2} \exp\left(\frac{2yb_{\pm}}{(\sigma_{\pm})^2}\right) \left(1 + e^{\frac{2|y||b_{\pm}|}{\sigma_{\pm}^2}}\right) \Phi\left(-\frac{|y| - |b_{\pm}|\Delta_N}{\sqrt{2\sigma_{\pm}^2\Delta_N}}\right) dy \\ & \leq C_1 \frac{T_N}{T_N^{1/\lambda}} \frac{1}{\int_{-\infty}^{\infty} m(y) dy} \int_{\mathbb{R}^{\pm}} \exp(-C_2|y|^2/\Delta_N) dy \leq C_3 T_N^{1-\frac{1}{\lambda}} \sqrt{\Delta_N} \xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

for constants $C_1, C_2, C_3 \in (0, \infty)$ depending on $a_{\pm}, b_{\pm}, \sigma_{\pm}$. Let us now consider the case $a_{\pm} > 0$ and $b_{\pm} \in \mathbb{R}$. The desired quantity is bounded by

$$\begin{aligned} & \frac{T_N}{T_N^{1/\lambda}} \frac{1}{\int_{-\infty}^{\infty} m(y) dy} \int_{\mathbb{R}^{\pm}} \frac{2|y|^m}{(\sigma_{\pm})^2} \exp\left(-\frac{y(a_{\pm}y - 2b_{\pm})}{(\sigma_{\pm})^2}\right) 2e^{-\frac{b_{\pm}}{\sigma_{\pm}^2}y} \Phi\left(-\frac{\sqrt{a_{\pm}}}{\sigma_{\pm}}|y|\gamma_N\right) dy \\ & \leq C_1 \frac{T_N}{T_N^{1/\lambda}} \frac{1}{\int_{-\infty}^{\infty} m(y) dy} \int_{\mathbb{R}^{\pm}} \exp(-C_2|y|^2\gamma_N^2) dy \leq C_3 \frac{T_N^{1-\frac{1}{\lambda}}}{\gamma_N} \end{aligned}$$

for constants $C_1, C_2, C_3 \in (0, \infty)$ depending on $a_{\pm}, b_{\pm}, \sigma_{\pm}$. The latter term vanishes since $\lim_{N \rightarrow \infty} \frac{T_N^{1-\frac{1}{\lambda}}}{\gamma_N} \leq \lim_{N \rightarrow \infty} T_N^{1-\frac{1}{\lambda}} \sqrt{\Delta_N} = 0$. The proof is thus completed. \square

Proof of Item (iv) of Theorem 2. Similarly to Item (iv) of Theorem 1. The analogous of A_T^{\pm} is $A_{T_N}^{N, \pm} := (-\mathfrak{M}_{T_N, N}^{\pm, 1} - \mathfrak{a}_{\pm} \mathfrak{Q}_{T_N, N}^{\pm, 2} + \mathfrak{b}_{\pm} \mathfrak{Q}_{T_N, N}^{\pm, 1}, \mathfrak{M}_{T_N, N}^{\pm, 0} - \mathfrak{b}_{\pm} \mathfrak{Q}_{T_N, N}^{\pm, 0} + \mathfrak{a}_{\pm} \mathfrak{Q}_{T_N, N}^{\pm, 1})$ and Lemma 4 ensures that the asymptotic behavior of the latter quantity is the same as the one of the continuous time analogue, Theorem 2.(iv). \square

A.4 Proof of Theorem 3

The proof of Item (i) of Theorem 3 is along the lines of the one of Theorem 2.(i).

The proof of Item (ii) of Theorem 3 is based on Lemma 5 and Lemma 6 below.

Let us be more precise. For all $T \in (0, \infty)$ and $N \in \mathbb{N}$ the difference $(\widehat{a}_{T, N}^{\pm} - \alpha_T^{\pm}, \widehat{b}_{T, N}^{\pm} - \beta_T^{\pm})$ can be rewritten, using Theorem 1.(i) and Theorem 3.(i), as an expression involving only terms of the kind

$$\left(\frac{\mathfrak{Q}_{T, N}^{\pm, i}}{\mathfrak{Q}_{T, N}^{\pm, 0} \mathfrak{Q}_{T, N}^{\pm, 2} - (\mathfrak{Q}_{T, N}^{\pm, 1})^2} - \frac{\mathfrak{Q}_T^{\pm, i}}{\mathfrak{Q}_T^{\pm, 0} \mathfrak{Q}_T^{\pm, 2} - (\mathfrak{Q}_T^{\pm, 1})^2} \right) \mathfrak{M}_T^{\pm, j} + \frac{\mathfrak{Q}_{T, N}^{\pm, i} (\mathfrak{M}_{T, N}^{\pm, j} - \mathfrak{M}_T^{\pm, j})}{\mathfrak{Q}_{T, N}^{\pm, 0} \mathfrak{Q}_{T, N}^{\pm, 2} - (\mathfrak{Q}_{T, N}^{\pm, 1})^2}$$

for $j \in \{0, 1\}$, $i \in \{0, 1, 2\}$. The convergence $(\widehat{a}_{T, N}^{\pm}, \widehat{b}_{T, N}^{\pm}, \widehat{a}_{T, N}^{\mp}, \widehat{b}_{T, N}^{\mp}) \xrightarrow{N \rightarrow \infty} (\alpha_T^{\pm}, \beta_T^{\pm}, \alpha_T^{\mp}, \beta_T^{\mp})$ is obtained combining Lemma 2 and Theorem 2.2 in [Crimaldi and Pratelli, 2005] with the convergences in probability in Lemma 5 and Lemma 6 below. The proof of (2.12) relies on Lemma 2, Theorem 2.2 in [Crimaldi and Pratelli, 2005] Lemma 6 and equation (A.11) in Lemma 5.

Lemma 5. *Let $m = 0, 1$. Then $\mathfrak{M}_{T,N}^{\pm,m} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \mathfrak{M}_T^{\pm,m}$ and*

$$N^{1/4}(\mathfrak{M}_{T,N}^{\pm,m} - \mathfrak{M}_T^{\pm,m}) \xrightarrow[N \rightarrow \infty]{\text{stably}} \pm r^m \sqrt{\frac{4\sqrt{T} \sigma_-^2 + \sigma_+^2}{3\sqrt{2\pi} \sigma_- + \sigma_+}} B_{L_T^r(X)} \quad (\text{A.11})$$

where B is a Brownian motion independent of X .

Lemma 6. *Let $m \in \mathbb{N}$. Then $\sqrt{N}(\mathfrak{Q}_{T,N}^{\pm,m} - \mathfrak{Q}_T^{\pm,m}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$.*

Moreover, if the threshold $r = 0$, then $\sqrt{N^{1+m\varepsilon}}(\mathfrak{Q}_{T,N}^{\pm,m} - \mathfrak{Q}_T^{\pm,m}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$ for every $\varepsilon < 1$.

The latter convergence result extends [Lejay and Pigato, 2018, Theorem 4.14], where only $m = 0$ is considered. The proof strategies are the same.

In the remainder of the section we prove Lemma 5 and Lemma 6 below for X solution to (1.1).

A.4.1 Proof of Lemma 5

Without loss of generality we can assume X_0 deterministic and also reduce ourselves to prove all results of the section in the case of null drift, i.e. X is an oscillating Brownian motion (OBM). Indeed all statements are about convergence in probability or stable convergence, and, once these convergences have been proved for the null drift case, they can be extended to the drifted case (piecewise linear drift) using the fact that Girsanov weight is an exponential martingale and dominated convergence theorem. In the case of convergence in probability one proves that for every sub-sequence there exists a sub-sub-sequence converging a.s., instead stable convergence follows by property (2.8) and Skorokhod representation theorem.

Therefore, in the remainder of the section, let X be an OBM with deterministic starting point X_0 and let $T \in (0, \infty)$ be fixed.

Lemma 7. *It holds that*

$$N^{\frac{1}{4}} \left(\sum_{k=0}^{N-1} (X_{(k+1)T/N} - X_{kT/N})^2 \mathbf{1}_{\{\pm(X_{kT/N}-r)>0\}} - \sigma_{\pm}^2 \int_0^T \mathbf{1}_{\{\pm(X_s-r)>0\}} ds \right) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Proof of Lemma 7. We write $X^{r,\pm} := (X - r)\mathbf{1}_{\{\pm(X-r)>0\}}$, $X_k := X_{k\frac{T}{N}}$, and $\Delta_k X := X_{k+1} - X_k$. Moreover we can assume $r = 0$ (just note that given X with threshold 0, $\eta = X + r$ has threshold r and $\Delta_i X = \Delta_i \eta$, $\Delta_i X^{0,\pm} = \Delta_i \eta^{r,\pm}$).

We observe that

$$\sum_{i=0}^{N-1} (\Delta_i X)^2 \mathbf{1}_{\{\pm X_i > 0\}} = \sum_{i=0}^{N-1} \Delta_i X \Delta_i X^{0,\pm} \mp \sum_{i=0}^{N-1} (\Delta_i X) |X_{i+1}| \mathbf{1}_{\{X_i X_{i+1} < 0\}}.$$

Proposition 2 in Mazzone [2019] (or [Lejay et al., 2019, Proposition 2]) ensures that

$$\frac{1}{N^{1/4}} \cdot N^{1/2} \sum_{i=0}^{N-1} (\Delta_i X) |X_{i+1}| \mathbf{1}_{\{X_i X_{i+1} < 0\}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0 \cdot \frac{2\sqrt{2}(\sigma_+^2 + \sigma_-^2)}{3\sqrt{\pi}(\sigma_+ + \sigma_-)} L_T^0(X) = 0.$$

Now consider the remaining term. In [Lejay and Pigato, 2018] (cf. proof of Theorem 3.5, page 3594) it is shown that there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} \sqrt{N} \left(\sum_{k=0}^{N-1} (\Delta_k X)^2 \mathbf{1}_{\{\pm X_k > 0\}} - \sigma_{\pm}^2 \int_0^T \mathbf{1}_{\{\pm X_s > 0\}} ds \right) &\xrightarrow[N \rightarrow \infty]{\text{stably}} \\ \sqrt{2T} \sigma_{\pm}^2 \int_0^T \mathbf{1}_{\{\pm X_s > 0\}} dB_s \mp CL_T^0(X) \end{aligned}$$

where B is a Brownian motion independent of X . Using (2.8) completes the proof. \square

We are now ready to prove Lemma 5.

Proof of Lemma 5. Let $\{\cdot\}^{\pm}$ denote positive and negative part. Note that for all $t \in [0, T]$ it holds \mathbb{P} -a.s. that $(X_t - r) \mathbf{1}_{\{\pm(X_t - r) > 0\}} = \pm \{X_t - r\}^{\pm}$. Applying Itô-Tanaka formula establishes that the following equalities hold \mathbb{P} -a.s.:

$$(X_T + r)^i \{X_T - r\}^{\pm} - (X_0 + r)^i \{X_0 - r\}^{\pm} = \pm 2^i \mathfrak{M}_T^{\pm, i} \pm i \sigma_{\pm}^2 \mathfrak{Q}_T^{\pm, 0} + \frac{r^i}{2^{1-i}} L_T^r(X), \quad (\text{A.12})$$

for $i = 0, 1$. Next note that it holds \mathbb{P} -a.s. for all $i \in \{0, \dots, N-1\}$ that

$$\mathbf{1}_{\{\pm(X_i - r) > 0\}} \Delta_i X = \pm \{X_{i+1} - r\}^{\pm} \mp \{X_i - r\}^{\pm} \mp \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} |X_{i+1} - r|$$

and

$$\begin{aligned} 2X_i \mathbf{1}_{\{\pm(X_i - r) > 0\}} \Delta_i X &= (X_{i+1}^2 - r^2) \mathbf{1}_{\{\pm(X_{i+1} - r) > 0\}} - (X_i^2 - r^2) \mathbf{1}_{\{\pm(X_i - r) > 0\}} - (X_{i+1} - X_i)^2 \mathbf{1}_{\{\pm(X_i - r) > 0\}} \\ &\mp 2r \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} |X_{i+1} - r| \mp \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} |X_{i+1} - r| (X_{i+1} - r). \end{aligned}$$

Combining this with (2.13) and (A.12) imply that it holds \mathbb{P} -a.s. that

$$\mp 2 (\mathfrak{M}_{T,N}^{\pm, 0} - \mathfrak{M}_T^{\pm, 0}) = L_{T,N}^r - L_T^r(X) \quad \text{and} \quad (\text{A.13})$$

$$\begin{aligned} 2 (\mathfrak{M}_{T,N}^{\pm, 1} - \mathfrak{M}_T^{\pm, 1}) &= \mp r (L_{T,N}^r - L_T^r(X)) - \sum_{k=0}^{N-1} (X_{k+1} - X_k)^2 \mathbf{1}_{\{\pm(X_k - r) > 0\}} + \sigma_{\pm}^2 \mathfrak{Q}_T^{\pm, 0} \\ &\mp \sum_{k=0}^{N-1} (X_{k+1} - r) |X_{k+1} - r| \mathbf{1}_{\{(X_k - r)(X_{k+1} - r) < 0\}}. \end{aligned}$$

The result follows from [Mazonetto, 2019, Proposition 7], Lemma 7, and from the following convergence:

$$\sqrt{N} \sum_{i=0}^{N-1} (X_{i+1} - r) |X_{i+1} - r| \mathbf{1}_{\{(X_i - r)(X_{i+1} - r) < 0\}} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \frac{2\sqrt{2}}{3\sqrt{\pi}} (\sigma_+ - \sigma_-) L_T^r(X)$$

which follows from [Mazonetto, 2019, Proposition 2] or [Lejay et al., 2019, Proposition 2]. \square

A.4.2 Proof of Lemma 6

As in the previous section, let X be an OBM with deterministic starting point X_0 and let $T \in (0, \infty)$ be fixed. We reduce to consider threshold $r = 0$ because for $t \in (0, \infty)$ it holds that $\mathfrak{Q}_t^{\pm, m} := \mathfrak{Q}_t^{\pm, m}(X, r) = \sum_{k=0}^m \binom{m}{k} r^{m-k} \mathfrak{Q}_t^{\pm, k}(X - r, 0)$ and the same holds for $\mathfrak{Q}_{N, t}^{\pm, m}$.

The following result follows from Lemma 4.3 in [Lejay and Pigato, 2018] and the scaling property for OBM.

Lemma 8. *Let f be a bounded function such that $\int |x|^k |f(x)| dx < \infty$ for $k = 0, 1, 2$. Then for all $T \in (0, \infty)$*

$$(N/T)^{-1/2} \sum_{i=0}^{N-1} f(\sqrt{N/T} X_{iT/N}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} \lambda_\sigma(f) L_T^0(X) \quad (\text{A.14})$$

where $\lambda_\sigma(f) := \left(\frac{1}{\sigma_+^2} \int_0^\infty f(x) dx + \frac{1}{\sigma_-^2} \int_{-\infty}^0 f(x) dx \right)$.

Let us denote by \mathcal{G} the natural filtration associated to the process X (or equivalently to its driving BM). Let $m \in \mathbb{N}$ be fixed. For $i = 1, \dots, N$, we consider $X_{i-1, N} := X_{(i-1)T/N}$, $\mathcal{G}_{i-1, N} := \mathcal{G}_{(i-1)T/N}$, and

$$\begin{aligned} J_{i, N}^{(m)} &= \left(\frac{T}{N} X_{i-1, N}^m \mathbf{1}_{\{\pm X_{i-1, N} \geq 0\}} - \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} X_s^m \mathbf{1}_{\{\pm X_s \geq 0\}} ds \right) \\ &= \pm \operatorname{sgn}(X_{i-1, N}) X_{i-1, N}^m \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} \mathbf{1}_{\{X_{i-1, N} X_s < 0\}} ds + \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} (X_{i-1, N}^m - X_s^m) \mathbf{1}_{\{\pm X_s > 0\}} ds, \\ U_{i, N}^{(m)} &= J_{i, N}^{(m)} - \mathbb{E}[J_{i, N}^{(m)} | \mathcal{G}_{i-1, N}]. \end{aligned} \quad (\text{A.15})$$

Observe that $U_{i, N}^{(m)}$ are martingale increments and

$$\mathfrak{Q}_{T, N}^{\pm, m} - \mathfrak{Q}_T^{\pm, m} = \sum_{i=1}^N \mathbb{E}[J_{i, N}^{(m)} | \mathcal{G}_{i-1, N}] + \sum_{i=1}^N U_{i, N}^{(m)}.$$

The following lemma proves the convergence of the two terms.

Lemma 9. *Let $\varepsilon \in [0, 1)$, $m \in \mathbb{N}$, and let \mathcal{G} , $J_{i, N}^{(m)}$ and $U_{i, N}^{(m)}$, $i \in \{1, \dots, N\}$ defined by (A.15). Then*

- i) $N^{\frac{1+m\varepsilon}{2}} \sum_{i=1}^N \mathbb{E}[J_{i, N}^{(m)} | \mathcal{G}_{i-1, N}] \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$ and
- ii) $N^{\frac{1+m\varepsilon}{2}} \sum_{i=1}^N U_{i, N}^{(m)} \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0$.

Proof of Lemma 9. In this proof we use the following notation: For every $q \in [0, \infty)$ let $f_m, g_{m, q}, h_q$ be the real functions satisfying

$$f_m(x) = \begin{cases} \frac{2\sigma_+}{\sigma_- + \sigma_+} \int_0^1 x^m \Phi(-x/(\sigma_+ \sqrt{t})) dt & \text{if } x \geq 0 \\ \frac{-2\sigma_-}{\sigma_- + \sigma_+} \int_0^1 x^m \Phi(x/(\sigma_- \sqrt{t})) dt & \text{if } x < 0 \end{cases}$$

with $\Phi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{y^2}{2}} e^{-\frac{y^2}{2}} dy$, $h_q(x) = |x|^q |f_0(x)|$, and

$$g_{m,q}(y) := \begin{cases} \int_0^1 \int_0^\infty |y^m - x^m|^q \frac{1}{\sqrt{2\pi t}} \frac{1}{\sigma_+} e^{-\frac{1}{2t} \left(\frac{x}{\sigma_+} - \frac{y}{\sigma_+} \right)^2} \left(1 + \frac{\sigma_- - \sigma_+}{\sigma_- + \sigma_+} e^{-\frac{4xy}{2t\sigma_+^2}} \right) dx dt & \text{if } y \geq 0 \\ \int_0^1 \int_0^\infty |y^m - x^m|^q \frac{1}{\sqrt{2\pi t}} \frac{2\sigma_-}{\sigma_- + \sigma_+} e^{-\frac{1}{2t} \left(\frac{x}{\sigma_+} - \frac{y}{\sigma_-} \right)^2} dx dt & \text{if } y < 0. \end{cases}$$

Let us show that the functions above satisfy the assumptions of Lemma 8. Indeed, since $\Phi(-x) \mathbf{1}_{\{x \geq 0\}} \leq \frac{1}{2} e^{-x^2/2}$, it holds

$$|f_0(x)| \leq \frac{\sigma(x)}{\sigma_- + \sigma_+} \int_0^1 e^{-\frac{x^2}{2t\sigma(x)^2}} dt \leq \frac{\sigma(x)}{\sigma_- + \sigma_+} e^{-\frac{x^2}{2\sigma(x)^2}}.$$

This ensures that the coefficients (defined in (A.14)) $\lambda_\sigma(f_m)$, $\lambda_\sigma(h_q)$ are finite. Moreover it can be shown that $\lambda_\sigma(g_{m,q}) < \infty$ for $q \in [0, \infty)$. In particular note that

$$\lambda_\sigma(f_m) = \frac{2(\sigma_+^m - (-\sigma_-)^m)}{\sigma_- + \sigma_+} \int_0^\infty x^m \int_0^1 \Phi(-x/\sqrt{s}) ds dx.$$

Hence Lemma 8 shows for all $q \in [0, \infty)$, $t \in (0, \infty)$, $f \in \{f_m, h_q, g_{m,q}\}$ that

$$\frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} f(\sqrt{N/T} X_{iT/N}) \xrightarrow[N \rightarrow \infty]{\text{law}} \frac{\lambda_\sigma(f)}{\sqrt{T}} L_T(X). \quad (\text{A.16})$$

Let us first show an easy useful equality, which uses the explicit expression of the transition density of the OBM

$$q_\sigma(t, \sigma(x)x, \sigma(y)y) = \frac{1}{\sqrt{2\pi t\sigma(y)}} \left(e^{-\frac{(x-y)^2}{2t}} + \frac{(\sigma_- - \sigma_+)}{(\sigma_- + \sigma_+)} \text{sgn}(y) e^{-\frac{(|x|+|y|)^2}{2t}} \right). \quad (\text{A.17})$$

Let Y be an OBM with starting point Y_0 and threshold $r = 0$, then for all $c \in \{-1, +1\}$, $q \in [0, \infty)$, Fubini, (A.17), a change of variable yield

$$\begin{aligned} \mathbf{1}_{\{cY_0 > 0\}} \mathbb{E} \left[\int_0^{\frac{T}{N}} |Y_0|^q \mathbf{1}_{\{cY_s < 0\}} ds | Y_0 \right] &= \mathbf{1}_{\{cY_0 > 0\}} \int_0^{\frac{T}{N}} |Y_0|^q \mathbb{E} [\mathbf{1}_{\{cY_s < 0\}} | Y_0] ds \\ &= \mathbf{1}_{\{cY_0 > 0\}} (T/N)^{\frac{q}{2}+1} \int_0^1 \frac{2\sigma(Y_0) |\sqrt{N/T} Y_0|^q}{\sigma_- + \sigma_+} \Phi(-|\sqrt{N/T} Y_0| / (\sigma(Y_0)\sqrt{t})) dt. \end{aligned} \quad (\text{A.18})$$

Let us now prove Item (i).

First step. Let $i \in \{1, \dots, N\}$ be fixed.

We prove in this step that

$$N^{\frac{1+m\varepsilon}{2}} \sum_{k=0}^{N-1} \mathbb{E} \left[\int_{\frac{kT}{N}}^{\frac{(k+1)T}{N}} \pm \text{sgn}(X_{k,N}) X_{k,N}^m \mathbf{1}_{\{X_{k,N} X_s < 0\}} ds | \mathcal{G}_{k,N} \right] \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0. \quad (\text{A.19})$$

Note that the Markov's property and (A.18) ensure that

$$\begin{aligned} \sqrt{N^{1+m\varepsilon}} \sum_{i=1}^N \mathbb{E} \left[\int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} \pm \text{sgn}(X_{(i-1)T/N}) X_{(i-1)T/N}^m \mathbf{1}_{\{X_{(i-1)T/N} X_s < 0\}} ds | \mathcal{G}_{i-1,N} \right] \\ = \pm T^{1+\frac{m}{2}} \sum_{i=0}^{N-1} N^{-\frac{1+m(1-\varepsilon)}{2}} f_m(\sqrt{N/T} X_{iT/N}). \end{aligned} \quad (\text{A.20})$$

This vanishes because equation (A.16) holds and $\lambda_\sigma(f_0) = 0$ and when $m \neq 0$ it holds $\varepsilon < 1$. The proof of (A.19) is thus completed.

Second step. Let $j \in \{1, 2\}$. We prove now that

$$N^{\frac{j(1+m\varepsilon)}{2}} \sum_{i=1}^N \mathbb{E} \left[\int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} (X_{i-1,N}^m - X_s^m)^j \mathbf{1}_{\{X_s > 0\}} ds | \mathcal{G}_{i-1,N} \right] \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0. \quad (\text{A.21})$$

By the Markov property, a simple change of variable, Fubini, and the explicit expression of the transition density of the OBM in (A.17) we obtain for all $i \in \{1, \dots, N\}$

$$\begin{aligned} & \mathbb{E} \left[\int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} |X_{\frac{(i-1)T}{N}}^m - X_s^m|^j \mathbf{1}_{\{X_s > 0\}} ds | \mathcal{G}_{i-1,N} \right] \\ &= \frac{T}{N} \int_0^1 \mathbb{E} \left[|X_{\frac{(i-1)T}{N}}^m - X_{t\frac{T}{N}}^m|^j \mathbf{1}_{\{X_{t\frac{T}{N}} > 0\}} | X_{\frac{(i-1)T}{N}} \right] dt = \left(\frac{T}{N} \right)^{\frac{mj}{2}+1} g_{m,j}(\sqrt{N/T} X_{\frac{(i-1)T}{N}}). \end{aligned}$$

Combining (A.16) with the fact that $\lambda_\sigma(g_{0,j}) = 0$ and $\varepsilon < 1$ it follows that the latter quantity converges in probability to 0 with the speed which proves (A.21). Taking $j = 1$ establishes Item (i).

Third step. (Proof of Item (ii)). Note that Jensen's inequality implies that

$$\begin{aligned} & \mathbb{E}[(U_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] = \mathbb{E}[(J_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] - \left(\mathbb{E}[J_{i,N}^{(m)} | \mathcal{G}_{i-1,N}] \right)^2 \leq \mathbb{E}[(J_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] \\ & \leq \mathbb{E} \left[\frac{2T}{N} X_{i-1,N}^{2m} \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} \mathbf{1}_{\{X_{i-1,N} X_s < 0\}} ds + \frac{2T}{N} \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} (X_{i-1,N}^m - X_s^m)^2 \mathbf{1}_{\{X_s > 0\}} ds | \mathcal{G}_{i-1,N} \right]. \end{aligned}$$

This and (A.21) with $j = 2$ ensure that it suffices to prove

$$N^{\frac{1}{2}+m\varepsilon} \mathbb{E} \left[\frac{2T}{N} X_{i-1,N}^{2m} \int_{\frac{(i-1)T}{N}}^{\frac{iT}{N}} \mathbf{1}_{\{X_{i-1,N} X_s < 0\}} ds | \mathcal{G}_{i-1,N} \right] \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

By the Markov's property and (A.18) we reduce to study the convergence of

$$2N^{-\frac{1}{2}+m(\varepsilon-1)} \sum_{k=0}^{N-1} N^{-\frac{1}{2}} h_{2m}(\sqrt{N/T} X_{kT/N}).$$

It follows from (A.16) that the latter quantity converges to 0 in probability as $N \rightarrow \infty$. We have therefore obtained that

$$N^{1+m\varepsilon} \sum_{i=1}^N \mathbb{E}[(U_{i,N}^{(m)})^2 | \mathcal{G}_{i-1,N}] \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Applying Theorem 4.4 in [Lejay and Pigato, 2018] completes the proof. \square

B The multi-threshold Ornstein-Uhlenbeck process

Let us consider in this section the multi-threshold version of the threshold OU process, by which we mean the solution to

$$X_t = X_0 + \int_0^t \sigma(X_s) dW_s + \int_0^t (b(X_s) - a(X_s) X_s) ds, \quad t \geq 0, \quad (\text{B.1})$$

with piecewise constant coefficients σ , a , and b possibly discontinuous at levels $-\infty = r_0 < r_1 < \dots < r_d < r_{d+1} = +\infty$, $d \in \mathbb{N}$. Let $I_0 = (-\infty, r_1)$ and $I_j := [r_j, r_{j+1})$, for $j \in \{1, \dots, d\}$. The volatility coefficient is given by

$$\sigma(x) = \sum_{j=0}^d \sigma_j \mathbf{1}_{I_j}(x) > 0 \quad (\text{B.2})$$

and similarly the drift coefficients are given by

$$b(x) = \sum_{j=0}^d b_j \mathbf{1}_{I_j}(x) \quad \text{and} \quad a(x) = \sum_{j=0}^d a_j \mathbf{1}_{I_j}(x). \quad (\text{B.3})$$

In analogy to the result for $d = 1$ we also denote a_0, b_0, σ_0 by a_-, b_-, σ_- and a_d, b_d, σ_d by a_+, b_+, σ_+ .

B.1 The regimes of the process

In this section, we establish for which values of the coefficients the process X is (positively or null) recurrent or transient. Recall that we denote *scale function* and *speed measure* respectively by S and m . The derivative (up to a multiplicative constant) of the scale function satisfies for all $j = 0, 1$ and $k = 2, \dots, d$

$$S'(x) \mathbf{1}_{I_j}(x) = \frac{1}{s_j(x, r_1)} \quad \text{and} \quad S'(x) \mathbf{1}_{I_k}(x) = \frac{1}{s_k(x, r_k) \prod_{i=1}^{k-1} s_i(r_{i+1}, r_i)}$$

where for all $k \in \{0, 1, \dots, d\}$, $x, r \in \mathbb{R}$ we define the functions

$$s_k(x, r) := \exp\left(\frac{2b_k(x-r) - a_k(x^2 - r^2)}{\sigma_k^2}\right).$$

The speed measure is $m(x) dx = \frac{2}{\sigma(x)^2 S'(x)} dx$.

Recall that X is recurrent if and only if $\lim_{x \rightarrow +\infty} S(x) = +\infty$ and $\lim_{x \rightarrow -\infty} S(x) = -\infty$, which happens if and only if $[(a_+ > 0 \text{ and } b_+ \in \mathbb{R}) \text{ or } (a_+ = 0 \text{ and } b_+ \leq 0)]$ and $[(a_- > 0 \text{ and } b_- \in \mathbb{R}) \text{ or } (a_- = 0 \text{ and } b_- \geq 0)]$. The complementary leads to transience. If X is recurrent and the speed measure is a finite measure, then X is positive recurrent and ergodic. It admits a stationary distribution, denoted by μ , which is the renormalized speed measure. Hence we have the ergodicity condition (A.1) (the one for the single threshold case $d = 1$).

Lemma 10 (Multi-threshold version of Lemma 1). *The speed measure is finite if and only if condition (A.1) holds. More precisely, let $C_0 = C_1 = 1$, $C_j = \prod_{k=1}^{j-1} s_k(r_{k+1}, r_k)$, $j = 2, \dots, d$, let*

$$\mathbf{m}_{i,j,k} := \frac{\sqrt{\pi}}{\sigma_i \sqrt{a_i}} \exp\left(\frac{a_i}{\sigma_i^2} \left(\frac{b_i}{a_i} - r_j\right)^2\right) \operatorname{erfc}\left(-\frac{\sqrt{a_i}}{\sigma_i} \left(\frac{b_i}{a_i} - r_k\right)\right)$$

with $i \in \{0, \dots, d\}$ and $j, k \in \{1, \dots, d\}$ and let $\mathbf{m}_+ = -\mathbf{m}_{0,1,1}$ and $\mathbf{m}_- = \mathbf{m}_{d,d,d}$. Then if $j \in \{0, d\}$

$$\int_{-\infty}^{\infty} \mathbf{1}_{I_j}(y) m(y) dy = \begin{cases} +\infty & \text{if } a_{\pm} = 0 \text{ and } b_{\pm} = 0, \\ \frac{C_j}{|b_{\pm}|} & \text{if } a_{\pm} = 0 \text{ and } (-1)^{\pm} b_{\pm} > 0, \\ C_j \mathbf{m}_{\pm} & \text{if } a_{\pm} > 0 \text{ and } b_{\pm} \in \mathbb{R} \end{cases}$$

and if $j \in \{1, \dots, d-1\}$

$$\int_{-\infty}^{\infty} \mathbf{1}_{I_j}(y)m(y) dy = \begin{cases} \frac{r_{j+1}-r_j}{\sigma_j^2} & \text{if } a_j = 0 \text{ and } b_j = 0, \\ \frac{C_j}{b_j} \exp\left(\frac{2b_j(r_{j+1}-r_j)}{\sigma_j^2} - 1\right) & \text{if } a_j = 0 \text{ and } b_j \neq 0, \\ C_j(\mathbf{m}_{j,j,j} - \mathbf{m}_{j,j,j+1}) & \text{if } a_j > 0 \text{ and } b_j \in \mathbb{R}, \\ C_j \int_{r_j}^{r_{j+1}} s_j(x, r_j) dx < \infty & \text{if } a_j < 0 \text{ and } b_j \in \mathbb{R}. \end{cases}$$

B.2 MLE and QMLE from continuous time observations

We assume in this section to observe the process on the time interval $[0, T]$, $T \in (0, \infty)$. For $T \in (0, \infty)$, $m = 0, 1, 2$, and $j = 0, 1, \dots, d$ we define

$$\mathfrak{M}_T^{j,m} := \int_0^T X_s^m \mathbf{1}_{I_j}(X_s) dX_s \quad \text{and} \quad \mathfrak{Q}_T^{j,m} := \int_0^T X_s^m \mathbf{1}_{I_j}(X_s) ds \quad (\text{B.4})$$

and take as likelihood function $G_T(a, b)$ and as quasi-likelihood $\Lambda_T(a, b)$ defined as in Section 2.

Lemma 11 (Multi-threshold version of Lemma 2). *Assume the process is ergodic: condition (A.1) holds. Then, for all $i \in \{0, 1, 2\}$, $j \in \{0, 1, \dots, d\}$, the quantities $\bar{\mathfrak{Q}}_\infty^{j,i}$ defined as follows are finite constants:*

$$\bar{\mathfrak{Q}}_\infty^{j,i} \stackrel{\text{a.s.}}{=} \lim_{t \rightarrow \infty} \frac{\mathfrak{Q}_t^{j,i}}{t} = \int_{I_j} x^i \mu(dx) \in \mathbb{R}.$$

Theorem 4. *i) For every $T \in (0, \infty)$ the MLE and QMLE are given by $(\alpha_T, \beta_T) = (\alpha_T^{(j)}, \beta_T^{(j)})_{j=0}^d$ with*

$$(\alpha_T^{(j)}, \beta_T^{(j)}) = \left(\frac{\mathfrak{M}_T^{j,0} \mathfrak{Q}_T^{j,1} - \mathfrak{Q}_T^{j,0} \mathfrak{M}_T^{j,1}}{\mathfrak{Q}_T^{j,0} \mathfrak{Q}_T^{j,2} - (\mathfrak{Q}_T^{j,1})^2}, \frac{\mathfrak{M}_T^{j,0} \mathfrak{Q}_T^{j,2} - \mathfrak{Q}_T^{j,1} \mathfrak{M}_T^{j,1}}{\mathfrak{Q}_T^{j,0} \mathfrak{Q}_T^{j,2} - (\mathfrak{Q}_T^{j,1})^2} \right). \quad (\text{B.5})$$

Assume now that the process is ergodic (i.e., (A.1) is satisfied).

ii) *The following LLN holds, i.e., the estimator is (strongly) consistent: $(\alpha_T, \beta_T) - (a, b) \xrightarrow[T \rightarrow \infty]{\text{a.s.}} 0$.*

iii) *The following CLT holds:*

$$\sqrt{T} \left((\alpha_T^{(j)}, \beta_T^{(j)}) - (a_j, b_j) \right)_{j=0}^d \xrightarrow[T \rightarrow \infty]{\text{stably}} (\mathcal{N}_j)_{j=0}^d$$

where $\mathcal{N}_j = (N^{j,\alpha}, N^{j,\beta})$, $j = 0, 1, \dots, d$ are independent, independent of X , two-dimensional Gaussian random variables with covariance matrices respectively $\sigma_j^2 \Gamma_j^{-1}$ and $\sigma_j^2 \Gamma_j^{-1}$ where

$$\Gamma_j := \begin{pmatrix} \bar{\mathfrak{Q}}_\infty^{j,2} & -\bar{\mathfrak{Q}}_\infty^{j,1} \\ -\bar{\mathfrak{Q}}_\infty^{j,1} & \bar{\mathfrak{Q}}_\infty^{j,0} \end{pmatrix}, \quad (\text{B.6})$$

and $\bar{\mathfrak{Q}}_\infty^{j,i}$, $i \in \{0, 1, 2\}$ are real constants such that $\lim_{t \rightarrow \infty} \frac{\mathfrak{Q}_t^{j,i}}{t} \stackrel{\text{a.s.}}{=} \bar{\mathfrak{Q}}_\infty^{j,i}$.

iv) The LAN property holds for the likelihood evaluated at the true parameters (a, b) with rate of convergence $\frac{1}{\sqrt{T}}$: there exists a random vector A_T such that for small perturbations $\Delta(a, b) := (\Delta a_j, \Delta b_j)_{j=0}^d$ it holds that

$$\log \frac{G_T((a, b) + \frac{1}{\sqrt{T}}\Delta(a, b))}{G_T(a, b)} - (A_T \cdot \Delta(a, b) - \Delta(a, b) \cdot \Gamma \Delta(a, b))$$

converges to 0 in probability as $T \rightarrow \infty$. The matrix Γ is the asymptotic Fisher information

$$\Gamma = \begin{pmatrix} \sigma_+^{-2}\Gamma_+ & 0_{\mathbb{R}^{2 \times 2}} & \cdots & 0_{\mathbb{R}^{2 \times 2}} \\ 0_{\mathbb{R}^{2 \times 2}} & \sigma_{d-1}^{-2}\Gamma_{d-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0_{\mathbb{R}^{2 \times 2}} \\ 0_{\mathbb{R}^{2 \times 2}} & \cdots & 0_{\mathbb{R}^{2 \times 2}} & \sigma_-^{-2}\Gamma_- \end{pmatrix}.$$

In order to study the asymptotic behavior of the estimator, we introduce a different expression for the estimators in (B.5) based on the following notation. Given $T \in (0, \infty)$, $j \in \{0, 1, \dots, d\}$, and $i \in \{0, 1\}$ let

$$M_T^{j,i} := \int_0^T (X_s)^i \mathbf{1}_{I_j}(X_s) dW_s.$$

Observe that (B.1) yields for $i \in \{0, 1\}$, $j \in \{0, 1, \dots, d\}$:

$$\mathfrak{M}_T^{j,i} = \sigma_j M_T^{j,i} + b_j \mathfrak{Q}_T^{j,i} - a_j \mathfrak{Q}_T^{j,i+1}. \quad (\text{B.7})$$

Note that $\mathfrak{Q}_T^{\pm,0} \mathfrak{Q}_T^{\pm,2} - (\mathfrak{Q}_T^{\pm,1})^2$ is \mathbb{P} -a.s. positive by Cauchy-Schwarz. By (B.7), we have the following reformulation of (B.5).

Lemma 12 (Multi-threshold version of Lemma 3). *Let $T \in (0, \infty)$ and $j \in \{0, 1, \dots, d\}$. Equation (B.5) can be expressed as*

$$(\alpha_T, \beta_T)_j = (a, b)_j + \sigma_j \left(\frac{M_T^{j,0} \mathfrak{Q}_T^{j,1} - \mathfrak{Q}_T^{j,0} M_T^{j,1}}{\mathfrak{Q}_T^{j,0} \mathfrak{Q}_T^{j,2} - (\mathfrak{Q}_T^{j,1})^2}, \frac{M_T^{j,0} \mathfrak{Q}_T^{j,2} - \mathfrak{Q}_T^{j,1} M_T^{j,1}}{\mathfrak{Q}_T^{j,0} \mathfrak{Q}_T^{j,2} - (\mathfrak{Q}_T^{j,1})^2} \right), \quad (\text{B.8})$$

that can be rewritten as

$$\begin{pmatrix} \alpha_T^{(j)} \\ \beta_T^{(j)} \end{pmatrix} = \begin{pmatrix} a_j \\ b_j \end{pmatrix} + \sigma_j \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} Q_T^{j,2} & Q_T^{j,1} \\ Q_T^{j,1} & Q_T^{j,0} \end{pmatrix}^{-1} \begin{pmatrix} M_T^{j,1} \\ M_T^{j,0} \end{pmatrix}. \quad (\text{B.9})$$

The proofs of Lemma 12 and of Theorem 4 are easily adapted from the case of a unique threshold ($d = 1$) to the multi-threshold case and therefore omitted.

B.3 Drift estimation from discrete observations

We assume now to observe the process on the discrete time grid $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = T$, for $N \in \mathbb{N}$, $T \in (0, \infty)$, and set $\Delta_N = \max_{k=1, \dots, N} \{t_k - t_{k-1}\}$. We define $X_i := X_{t_i}$ with $i = 0, \dots, N$.

The discrete versions of (B.4) are defined as follows: for $m = 0, 1, 2$, $j = 0, \dots, d$ let

$$\begin{aligned} \mathfrak{M}_{T,N}^{j,m} &:= \sum_{k=0}^{N-1} (X_{k+1} - X_k) X_k^m \mathbf{1}_{I_j}(X_k), \quad \text{and} \\ \mathfrak{Q}_{T,N}^{j,m} &:= \sum_{k=0}^{N-1} (t_{k+1} - t_k) X_k^m \mathbf{1}_{I_j}(X_k). \end{aligned} \quad (\text{B.10})$$

The *discretized likelihood* $G_{T,N}(a, b)$ and the *discretized quasi-likelihood* $\Lambda_{T,N}(a, b)$ are defined as in Section 2.

For $N \in \mathbb{N}$ and $T \in (0, \infty)$ let $(\widehat{a}_{T,N}, \widehat{b}_{T,N}) = (\widehat{a}_{T,N,N}^{(j)}, \widehat{b}_{T,N,N}^{(j)})_{j=0}^d$ with

$$(\widehat{a}_{T,N}^{(j)}, \widehat{b}_{T,N}^{(j)}) = \left(\frac{\mathfrak{m}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,1} - \mathfrak{Q}_{T,N}^{j,0} \mathfrak{m}_{T,N}^{j,1}}{\mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,2} - (\mathfrak{Q}_{T,N}^{j,1})^2}, \frac{\mathfrak{m}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,2} - \mathfrak{Q}_{T,N}^{j,1} \mathfrak{m}_{T,N}^{j,1}}{\mathfrak{Q}_{T,N}^{j,0} \mathfrak{Q}_{T,N}^{j,2} - (\mathfrak{Q}_{T,N}^{j,1})^2} \right). \quad (\text{B.11})$$

Theorem 5. *Let $(T_N)_{N \in \mathbb{N}}$ be a sequence in $(0, \infty)$. For all $N \in \mathbb{N}$ let $(\widehat{a}_{T_N,N}, \widehat{b}_{T_N,N})$ be defined as in (B.11).*

i) *For every $N \in \mathbb{N}$ the vector $(\widehat{a}_{T_N,N}, \widehat{b}_{T_N,N})$ maximizes both the likelihood $G_{T_N,N}(a, b)$ and the quasi-likelihood $\Lambda_{T_N,N}(a, b)$.*

Assume that the process is ergodic and that X_0 follows the stationary distribution μ . Moreover, assume

$$\lim_{N \rightarrow \infty} T_N = \infty \quad \text{and} \quad \lim_{N \rightarrow \infty} \Delta_N = 0.$$

ii) *The following LLN holds: $(\widehat{a}_{T_N,N}, \widehat{b}_{T_N,N}) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} (a, b)$ (the estimator is consistent).*

iii) *If $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$, the following CLT holds:*

$$\sqrt{T_N} ((\widehat{a}_{T_N,N}^{(j)}, \widehat{b}_{T_N,N}^{(j)}) - (a_j, b_j))_{j=0}^d \xrightarrow[N \rightarrow \infty]{\text{stably}} (\mathcal{N})_{j=0}^d$$

where \mathcal{N} is as in Theorem 4.

iv) *If $\lim_{N \rightarrow \infty} T_N \Delta_N = 0$, the analogous of the LAN property holds for the discretized likelihood evaluated at the true parameters with rate of convergence $\frac{1}{\sqrt{T_N}}$ and asymptotic Fisher information Γ as in Theorem 4.*

B.4 Proof of Theorem 5

Analogously to the case $d = 1$, the main ingredient of the proof is the following lemma.

Lemma 13 (Multi-threshold version of Lemma 4). *Assume the process is ergodic. Let X be the solution to (B.1), with X_0 distributed as the stationary distribution μ , let $\lambda \in \{1, 2\}$ be fixed, and let $(T_N)_{N \in \mathbb{N}} \subset (0, \infty)$ be a sequence satisfying, as $N \rightarrow \infty$, that $T_N \rightarrow \infty$ and $\lim_{N \rightarrow \infty} T_N^{1-1/\lambda} \sqrt{\Delta_N} = 0$ where $\Delta_N := \sup_{k=1, \dots, N} (t_k - t_{k-1})$. Then for all $m \in \{0, 1, 2\}$, $j \in \{0, 1\}$, $k \in \{0, \dots, d\}$ it holds*

$$\limsup_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} \left[|\mathfrak{Q}_{T_N}^{k,m} - \mathfrak{Q}_{T_N,N}^{k,m}| \right] = 0 \quad \text{and} \quad \limsup_{N \rightarrow \infty} T_N^{-1/\lambda} \mathbb{E} \left[|\mathfrak{M}_{T_N}^{k,j} - \mathfrak{M}_{T_N,N}^{k,j}| \right] = 0$$

where $\mathfrak{Q}_{T_N}^{k,m}$, $\mathfrak{Q}_{T_N,N}^{k,m}$, $\mathfrak{M}_{T_N}^{k,j}$, $\mathfrak{M}_{T_N,N}^{k,j}$ are defined in (B.4) and (B.10).

Before providing the proof of Lemma 13, we show how it intervenes in the proof of Items (ii)-(iii) of Theorem 5. For all $N \in \mathbb{N}$

$$(\widehat{a}_{T_N,N}, \widehat{b}_{T_N,N}) - (a, b) = (\widehat{a}_{T_N,N}, \widehat{b}_{T_N,N}) - (\alpha_{T_N}, \beta_{T_N}) + (\alpha_{T_N}, \beta_{T_N}) - (a, b)$$

The second term of the sum is handled with Theorem 4 (more precisely Item (iii)) providing the desired limit distribution. The first instead can be rewritten, using equations (B.5) and (B.11), as an expression which involves only terms of the kind

$$\left(\frac{\mathfrak{Q}_{T_N, N}^{j,i}}{\mathfrak{Q}_{T_N, N}^{j,0} \mathfrak{Q}_{T_N, N}^{j,2} - (\mathfrak{Q}_{T_N, N}^{j,1})^2} - \frac{\mathfrak{Q}_{T_N}^{j,i}}{\mathfrak{Q}_{T_N}^{j,0} \mathfrak{Q}_{T_N}^{j,2} - (\mathfrak{Q}_{T_N}^{j,1})^2} \right) \mathfrak{M}_{T_N}^{j,k} + \frac{\mathfrak{Q}_{T_N, N}^{j,i} (\mathfrak{M}_{T_N, N}^{j,k} - \mathfrak{M}_{T_N}^{j,k})}{\mathfrak{Q}_{T_N, N}^{j,0} \mathfrak{Q}_{T_N, N}^{j,2} - (\mathfrak{Q}_{T_N, N}^{j,1})^2}$$

for $i \in \{0, 1, 2\}$, $j \in \{0, \dots, d\}$, $k \in \{0, 1\}$.

Combining Lemma 13 with Lemma 11 and Theorem 2.2 in [Crimaldi and Pratelli, 2005] ensures the consistency of the estimator if $\Delta_N \rightarrow 0$ as $N \rightarrow \infty$, and if $T_N \Delta_N \rightarrow 0$ as $N \rightarrow \infty$ then it implies also that

$$\sqrt{T_N} \left((\widehat{a}_{T_N, N}, \widehat{b}_{T_N, N}) - (\alpha_{T_N}, \beta_{T_N}) \right) \xrightarrow[N \rightarrow \infty]{\mathbb{P}} 0.$$

Proof of Lemma 13. The proof is similar to the one of Lemma 4. We provide here the main idea of the key step, that is the proof of the analogous of (A.9): for all $j \in \{0, 1, \dots, d\}$, $m \in \{0, 1, 2, 3, 4\}$

$$\int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor_{\Delta_N}}|^m \mathbf{1}_{\{X_{\lfloor t \rfloor_{\Delta_N}} \in I_j, X_t \notin I_j\}} \right] dt \text{ is } o(T_N^{1/\lambda}). \quad (\text{B.12})$$

We reduce to compute, given $X_{\lfloor t \rfloor_{\Delta_N}}$ for $\lfloor t \rfloor_{\Delta_N} = t_k$, the probability that the first exit time of a standard Brownian motion from a suitable symmetric interval is smaller than $t - t_k$: $p_t := \mathbb{P}(\tau_{X_{\lfloor t \rfloor_{\Delta_N}}, I_j} \leq t - \lfloor t \rfloor_{\Delta_N})$. Indeed starting from $X_{t_k} \in I_j$ for some $k \in \{0, \dots, N\}$, $j \in \{0, \dots, d\}$ at a suitable distance $R_{X_{t_k}}$ from the boundary of I_j , if the Brownian motion driving the OU process does not exit in small time a suitable interval then the OU of parameters (a_j, b_j, σ_j) stays in $(X_{t_k} - R_{X_{t_k}}, X_{t_k} + R_{X_{t_k}}) \subset I_j$ because the drift is small. More precisely if $X_{t_k} \in I_j$ let $R_{X_{t_k}} := \min\{X_{t_k} - r_j, r_{j+1} - X_{t_k}\}$, let $\tilde{r}_j \in \{r_j, r_{j+1}\}$ such that $R_{X_{t_k}} = |X_{t_k} - \tilde{r}_j|$, and let

$$B_{X_{t_k}} := (e^{-|a_j|(t_{k+1} - t_k)} - |a_j|(t_{k+1} - t_k)) R_{X_{t_k}} - \frac{|b_j - a_j \tilde{r}_j|}{\sigma_j} (t_{k+1} - t_k).$$

Note that $\sup_{j=0, \dots, d} |\tilde{r}_j| < \infty$. If $B_{X_{t_k}} > 0$, then $\tau_{X_{t_k}, I_j}$ is the first exit time of a Brownian motion from the interval $[-B_{X_{t_k}}, B_{X_{t_k}}]$. Moreover for $t \in (t_k, t_{k+1}]$ it holds that $t \mapsto p_t$ is increasing, and

$$p_t \leq 2 \sum_{n=0}^{\infty} \operatorname{erfc} \left(\frac{B_{X_{t_k}} (1+2n)}{\sqrt{t-t_k}} \right) \leq 2 \sum_{n=0}^{\infty} \exp \left(-\frac{B_{X_{t_k}}^2 (1+2n)^2}{t-t_k} \right)$$

and so $p_t \leq 2 \exp \left(-\frac{B_{X_{t_k}}^2}{t-t_k} \right) \left(1 - \exp \left(-8 \frac{B_{X_{t_k}}^2}{t-t_k} \right) \right)^{-1}$.

Then using that for every $t > 0$, $N \in \mathbb{N}$ it holds $1 = \mathbf{1}_{\{B_{X_{\lfloor t \rfloor_{\Delta_N}}} > \sqrt{t - \lfloor t \rfloor_{\Delta_N}}\}} + \mathbf{1}_{\{B_{X_{\lfloor t \rfloor_{\Delta_N}}} \leq \sqrt{t - \lfloor t \rfloor_{\Delta_N}}\}}$, we split the integrals into two parts to deal with in two different ways.

Note that

$$\begin{aligned}
& \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m \mathbf{1}_{\{X_{[t]_{\Delta_N}} \in I_j, X_t \notin I_j\}} \mathbf{1}_{\{B_{X_{[t]_{\Delta_N}}} \leq \sqrt{t-[t]_{\Delta_N}}\}} \right] dt \\
& \leq \sum_{k=0}^{N-1} (t_{k+1} - t_k) \mathbb{E} \left[|X_{t_k}|^m \mathbf{1}_{\{X_{t_k} \in I_j\}} \mathbf{1}_{\{B_{X_{t_k}} \leq \sqrt{t_{k+1}-t_k}\}} \right] \\
& \leq 2 \left(\sup_{x \in I_j} \mu(x) \right) \sum_{k=0}^{N-1} (t_{k+1} - t_k)^{3/2} \frac{1 + \frac{|b_j - a_j \tilde{r}_j|}{\sigma_j} \sqrt{t_{k+1} - t_k}}{e^{-|a_j|(t_{k+1}-t_k)} - |a_j|(t_{k+1} - t_k)} \\
& \quad \cdot \left(|\tilde{r}_j| + \frac{\sqrt{t_{k+1} - t_k} + \frac{|b_j - a_j \tilde{r}_j|}{\sigma_j} (t_{k+1} - t_k)}{e^{-|a_j|(t_{k+1}-t_k)} - |a_j|(t_{k+1} - t_k)} \right)^m.
\end{aligned}$$

Therefore for N big, since Δ_N is small, the latter quantity goes like $\sum_{k=0}^{N-1} (t_{k+1} - t_k)^{3/2} \leq T_N \sqrt{\Delta_N}$ and it is $o(T_N^{1/\lambda})$.

The other integral satisfies:

$$\begin{aligned}
& \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m \mathbf{1}_{\{X_{[t]_{\Delta_N}} \in I_j, X_t \notin I_j\}} \mathbf{1}_{\{B_{X_{[t]_{\Delta_N}}} > \sqrt{t-[t]_{\Delta_N}}\}} \right] dt \\
& \leq \int_0^{T_N} \mathbb{E} \left[|X_{[t]_{\Delta_N}}|^m p_{X_{[t]_{\Delta_N}}} \mathbf{1}_{\{X_{[t]_{\Delta_N}} \in I_j\}} \mathbf{1}_{\{B_{X_{[t]_{\Delta_N}}} > \sqrt{t-[t]_{\Delta_N}}\}} \right] dt \\
& \leq \frac{2}{1 - e^{-8}} \sum_{k=0}^{N-1} \int_0^{t_{k+1}-t_k} \mathbb{E} \left[|X_{t_k}|^m e^{-\frac{B_{X_k}^2}{t}} \mathbf{1}_{\{X_{t_k} \in I_j\}} \mathbf{1}_{\{B_{X_k} > \sqrt{t}\}} \right] dt.
\end{aligned}$$

Let $g_{j,k} := e^{-|a_j|(t_{k+1}-t_k)} - |a_j|(t_{k+1} - t_k)$, $f_{j,k} := \frac{|b_j - a_j \tilde{r}_j| (t_{k+1}-t_k)}{g_{j,k}}$, and let μ denote the invariant measure, then

$$\begin{aligned}
& \mathbb{E} \left[|X_{t_k}|^m e^{-\frac{B_{X_k}^2}{t}} \mathbf{1}_{\{X_{t_k} \in I_j\}} \mathbf{1}_{\{B_{X_k} > \sqrt{t}\}} \right] \\
& = \int_{r_j}^{r_{j+1}} |x|^m e^{-\frac{g_{j,k}^2 (|x - \tilde{r}_j| - f_{j,k})^2}{t}} \mathbf{1}_{\{|x - \tilde{r}_j| > f_{j,k} + \frac{\sqrt{t}}{g_{j,k}}\}} \mu(dx) \\
& \leq C_j \int_{r_j - \tilde{r}_j}^{r_{j+1} - \tilde{r}_j} \left(|x| + |\tilde{r}_j| \right)^m e^{-\frac{g_{j,k}^2 (|x| - f_{j,k})^2}{t} - \frac{a_j x^2 - 2(b_j - a_j \tilde{r}_j)x}{\sigma_j^2}} \mathbf{1}_{\{|x| > f_{j,k} + \frac{\sqrt{t}}{g_{j,k}}\}} dx \\
& \leq C_j \int_{r_j - \tilde{r}_j}^{r_{j+1} - \tilde{r}_j} \left(|x| + |\tilde{r}_j| \right)^m e^{-\frac{g_{j,k}^2 (|x| - f_{j,k})^2}{t} - \frac{a_j x^2 - 2|b_j - a_j \tilde{r}_j||x|}{\sigma_j^2}} \mathbf{1}_{\{|x| > f_{j,k} + \frac{\sqrt{t}}{g_{j,k}}\}} dx \\
& \leq C_j \int_0^{|I_j|} \left(|x| + |\tilde{r}_j| \right)^m e^{-\frac{g_{j,k}^2 (x - f_{j,k})^2}{t} - \frac{a_j x^2 - 2|b_j - a_j \tilde{r}_j|x}{\sigma_j^2}} \mathbf{1}_{\{x > f_{j,k} + \frac{\sqrt{t}}{g_{j,k}}\}} dx
\end{aligned}$$

where the constant C_j may change from line to line and $|I_j|$ is the length of the interval. For N big enough $\Delta_N \ll 1$, $f_{j,k} + \frac{t_{k+1}-t_k}{g_{j,k}} < |I_j|$, $g_{j,k} \in [1/2, 1]$, $f_{j,k} \leq 2|b_j - a_j \tilde{r}_j|/\sigma_j$

hence the latter integral is bounded from above by

$$\begin{aligned}
& C_j \int_{f_{j,k} + \frac{\sqrt{t}}{g_{j,k}}}^{|I_j|} |x + |\tilde{r}_j||^m e^{-\frac{g_{j,k}^2(x-f_{j,k})^2}{t} - \frac{a_j x^2 - 2|b_j - a_j \tilde{r}_j| x}{\sigma_j^2}} dx \\
&= C_j \int_{\frac{\sqrt{t}}{g_{j,k}}}^{|I_j|} |x + |\tilde{r}_j||^m e^{-\frac{g_{j,k}^2 x^2}{t} - \frac{a_j(x+f_{j,k})^2 - 2|b_j - a_j \tilde{r}_j|(x+f_{j,k})}{\sigma_j^2}} dx \\
&\leq C_j \sqrt{t} \int_0^{+\infty} |x + |\tilde{r}_j||^m e^{-x^2 + \frac{8x}{\sigma_j}} dx
\end{aligned}$$

where the constant C_j may change from line to line. Thus, for some positive constant C_j it holds that

$$\begin{aligned}
& \int_0^{T_N} \mathbb{E} \left[|X_{\lfloor t \rfloor \Delta_N}|^m \mathbf{1}_{\{X_{\lfloor t \rfloor \Delta_N} \in I_j, X_t \notin I_j\}} \mathbf{1}_{\{B_{X_{\lfloor t \rfloor \Delta_N}} > \sqrt{t - \lfloor t \rfloor \Delta_N}\}} \right] dt \\
&\leq C_j \sum_{k=0}^{N-1} \int_0^{t_{k+1} - t_k} \sqrt{t} dt = \frac{2}{3} C_j \sum_{k=0}^{N-1} (t_{k+1} - t_k)^{3/2} \leq \frac{2}{3} C_j T_N \sqrt{\Delta_N}
\end{aligned}$$

and it is $o(T_N^{1/\lambda})$. The proof is thus completed. \square

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