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# SYMMETRY BREAKING OPERATORS FOR DUAL PAIRS WITH ONE MEMBER COMPACT

M. MCKEE, A. PASQUALE, AND T. PRZEBINDA

ABSTRACT. We consider a dual pair  $(G, G')$ , in the sense of Howe, with  $G$  compact acting on  $L^2(\mathbb{R}^n)$ , for an appropriate  $n$ , via the Weil representation  $\omega$ . Let  $\tilde{G}$  be the preimage of  $G$  in the metaplectic group. Given a genuine irreducible unitary representation  $\Pi$  of  $\tilde{G}$ , let  $\Pi'$  be the corresponding irreducible unitary representation of  $\tilde{G}'$  in Howe's correspondence. The orthogonal projection onto the  $\Pi$ -isotypic component  $L^2(\mathbb{R}^n)_\Pi$  is, up to a constant multiple, the unique symmetry breaking operator in  $\text{Hom}_{\tilde{G}\tilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ . We study this operator by computing its Weyl symbol. Our results allow us to compute the wavefront set of  $\Pi'$  by elementary means.

## CONTENTS

1. <b>Notation and preliminaries.</b>	4
2. <b>The center of the metaplectic group.</b>	6
3. <b>Dual pairs as Lie supergroups.</b>	8
4. <b>Main results.</b>	12
5. <b>A few examples</b>	21
5.1. $(G, G') = (O_2, \text{Sp}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R}))$	21
5.2. $(G, G') = (U_l, U_{p,p})$ and $\Pi = \text{triv}$	23
6. <b>The integral over <math>-G^0</math> as an integral over <math>\mathfrak{g}</math>.</b>	25
7. <b>The invariant integral over <math>\mathfrak{g}</math> as an integral over <math>\mathfrak{h}</math>.</b>	26
8. <b>An intertwining distribution in terms of orbital integrals on the symplectic space.</b>	31
9. <b>The special case for the pair <math>O_{2l}, \text{Sp}_{2l'}(\mathbb{R})</math>.</b>	49
10. <b>The special case for the pair <math>O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})</math> with <math>1 \leq l \leq l'</math>.</b>	53
11. <b>The special case for the pair <math>O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})</math> with <math>l &gt; l'</math>.</b>	55
12. <b>A sketch of a computation of the wave front of <math>\Pi'</math>.</b>	57
Appendix A. <b>The Jacobian of the Cayley transform</b>	59
Appendix B. <b>Proof of formula (112)</b>	60
Appendix C. <b>The special functions <math>P_{a,b}</math> and <math>Q_{a,b}</math></b>	61

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Appendix D. <b>The covering <math>\widetilde{G} \rightarrow G</math></b>	69
Appendix E. <b>Highest weights of irreducible genuine representations of <math>\widetilde{G}</math></b>	71
References	72

Let  $W$  be a finite dimensional vector space over  $\mathbb{R}$  equipped with a non-degenerate symplectic form  $\langle \cdot, \cdot \rangle$  and let  $\mathrm{Sp}(W)$  denote the corresponding symplectic group. Write  $\widetilde{\mathrm{Sp}}(W)$  for the metaplectic group. Let us fix the character  $\chi$  of  $\mathbb{R}$  given by  $\chi(r) = e^{2\pi ir}$ ,  $r \in \mathbb{R}$ . Then the Weil representation of  $\widetilde{\mathrm{Sp}}(W)$  associated to  $\chi$  is denoted by  $(\omega, \mathcal{H}_\omega)$ .

For  $G, G' \subseteq \mathrm{Sp}(W)$  forming a reductive dual pair in the sense of Howe, let  $\widetilde{G}, \widetilde{G}'$  denote their preimages in  $\widetilde{\mathrm{Sp}}(W)$ . Howe's correspondence (or local  $\theta$ -correspondence) for  $\widetilde{G}, \widetilde{G}'$  is a bijection  $\Pi \leftrightarrow \Pi'$  between the irreducible admissible representations of  $\widetilde{G}$  and  $\widetilde{G}'$  which occur as smooth quotients of  $\omega$ , [How89b]. It can be formulated as follows. Assume that  $\mathrm{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty) \neq 0$ . Then  $\mathrm{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty)$  is a  $\widetilde{G}'$ -module under the action via  $\omega$ . Howe proved that it has a unique irreducible quotient, which is an irreducible admissible representation  $(\Pi', \mathcal{H}_{\Pi'})$  of  $\widetilde{G}'$ . Conversely,  $\mathrm{Hom}_{\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty)$  is a  $\widetilde{G}$ -module which has a unique irreducible admissible quotient, infinitesimally equivalent to  $(\Pi, \mathcal{H}_\Pi)$ . Furthermore,  $\Pi \otimes \Pi'$  occurs as a quotient of  $\omega^\infty$  in a unique way, i.e.

$$\dim \mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = 1. \quad (1)$$

In [Kob15], the elements of

$$\mathrm{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty), \quad \mathrm{Hom}_{\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty) \quad \text{and} \quad \mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$$

are called symmetry breaking operators. Their construction is part of Stage C of Kobayashi's program for branching problems in the representation theory of real reductive groups.

Since the last space is one dimensional, it deserves a closer look. The explicit construction of the (essentially unique) symmetry breaking operator in  $\mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$  provides an alternative and direct approach to Howe's correspondence. To do this is the aim of the present paper.

Our basic assumption is that  $(G, G')$  is an irreducible dual pair with  $G$  compact. As shown by Howe [How79], up to an isomorphism,  $(G, G')$  is one of the pairs

$$(O_d, \mathrm{Sp}_{2m}(\mathbb{R})), \quad (U_d, U_{p,q}), \quad (\mathrm{Sp}_d, O_{2m}^*). \quad (2)$$

Then the representations  $\Pi$  and  $\Pi'$  together with their contragredients are arbitrary irreducible unitary highest weight representations. They have been defined by Harish-Chandra in [Har55], were classified in [EHW83] and have been studied in terms of Zuckerman functors in [Wal84], [Ada83] and [Ada87].

The crucial fact for constructing the symmetry breaking operator in  $\mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$  is that, up to a non-zero constant multiple, there is a unique  $GG'$ -invariant tempered distribution  $f_{\Pi \otimes \Pi'}$  on  $W$  such that

$$\mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = \mathbb{C}(\mathrm{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'}), \quad (3)$$

where  $\text{Op}$  and  $\mathcal{K}$  are classical transformations which we shall review in section 1. In [Prz93],  $f_{\Pi \otimes \Pi'}$  is called the intertwining distribution associated to  $\Pi \otimes \Pi'$ . In fact, if we work in a Schrödinger model of  $\omega$ , then  $f_{\Pi \otimes \Pi'}$  happens to be the Weyl symbol, [Hör83], of the operator  $(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$ .

Suppose that the group  $G$  is compact. Let  $\Theta_{\Pi}$  and  $d_{\Pi}$  respectively denote the character and the degree of  $\Pi$ . Then the projection on the  $\Pi$ -isotypic component of  $\omega$  is equal to  $d_{\Pi}/2$  times

$$\int_{\tilde{G}} \omega(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) d\tilde{g} = \omega(\check{\Theta}_{\Pi}), \quad (4)$$

where  $\check{\Theta}_{\Pi}(\tilde{g}) = \Theta_{\Pi}(\tilde{g}^{-1})$  and we normalize the Haar measure  $d\tilde{g}$  of  $\tilde{G}$  to have the total mass 2. (This explains the constant multiple  $\frac{1}{2}$  needed for the projection. In this way, the mass of  $G$  is equal to 1.) By Howe's correspondence with  $G$  compact, the projection onto the  $\Pi$ -isotypic component of  $\omega$  is a symmetry breaking operator for  $\Pi \otimes \Pi'$ . The intertwining distribution for  $\Pi \otimes \Pi'$  is therefore determined by the equation

$$(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'}) = \frac{1}{2} \omega(\check{\Theta}_{\Pi}). \quad (5)$$

There are more cases when  $f_{\Pi \otimes \Pi'}$  may be computed via the formula (5), see [Prz93]. However, if the group  $G$  is compact then the distribution character  $\Theta_{\Pi'}$  may also be recovered from  $f_{\Pi \otimes \Pi'}$  via an explicit formula, see [Prz91]. Thus, in this case, we have a diagram

$$\Theta_{\Pi} \longrightarrow f_{\Pi \otimes \Pi'} \longrightarrow \Theta_{\Pi'}. \quad (6)$$

In general, the asymptotic properties of  $f_{\Pi \otimes \Pi'}$  relate the associated varieties of the primitive ideals of  $\Pi$  and  $\Pi'$  and, under some more assumptions, the wave front sets of these representations, see [Prz93] and [Prz91].

The usual, often very successful, approach to Howe's correspondence avoids any work with distributions on the symplectic space. Instead, one finds Langlands parameters (see [Moe89], [AB95], [Pau98], [Pau00], [Pau05], [LPTZ03]), character formulas (see [Ada98], [Ren98], [DP96]), or candidates for character formulas (as in [BP14]), or one establishes preservation of unitarity (as in [Li89], [He03], [Prz93] or [ABP<sup>+</sup>07]). However, in the background (explicit or not), there is the orbit correspondence induced by the unnormalized moment maps

$$\mathfrak{g}^* \longleftarrow W \longrightarrow \mathfrak{g}'^*,$$

where  $\mathfrak{g}$  and  $\mathfrak{g}'$  denote the Lie algebras of  $G$  and  $G'$ , respectively, and  $\mathfrak{g}^*$  and  $\mathfrak{g}'^*$  are their duals. This correspondence of orbits has been studied in [DKP97], [DKP05] and [Pan10]. Furthermore, in their recent work, [LM15], H. Y. Loke and J. J. Ma computed the associated variety of the representations for the dual pairs in the stable range in terms of the orbit correspondence. The  $p$ -adic case was studied in detail in [Moe98].

Working with the  $GG'$ -invariant distributions on  $W$  is a more direct approach than relying on the orbit correspondence and provides different insights and results. As a complementary contribution to all work mentioned above, we compute the intertwining distributions  $f_{\Pi \otimes \Pi'}$  explicitly, see section 4. As an application, we obtain the wave front set of  $\Pi'$  by elementary means. The computation will be sketched in section 12, but the detailed proofs will appear in a forthcoming paper, [MPP21]. Another application of the methods presented in this paper leads to the explicit formula for the character

of the corresponding irreducible unitary representation  $\Pi'$  of  $\widetilde{G}'$ . This can be found in [Mer17, Mer20].

The explicit formulas for the intertwining distribution provide important information on the nature of the symmetry breaking operators. Namely, they show that none of the symmetry breaking operators of the form  $(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$  is a differential operator. For the present situation, this answers in the negative the question on the existence of differential symmetry breaking operators, addressed in different contexts by several authors (see for instance [KP16a, KP16b, KS15] and the references given there). This property is the content of Corollary 9.

Finally, observe that our computations leading to the intertwining distributions apply to any genuine irreducible representation  $\Pi$  of the compact member  $\widetilde{G}$  of a dual pair. They provide an explicit formula for the Weyl symbol of the projection of  $\omega|_{\widetilde{G}}$  onto the  $\Pi$ -isotypic component. According to Howe's duality theorem, this projection is non-zero if and only if there is a unitary highest weight representation  $\Pi'$  of  $\widetilde{G}'$  such that  $\Pi \otimes \Pi'$  occurs in  $\omega|_{\widetilde{G}\widetilde{G}'}$ , i.e.  $\Pi$  occurs in Howe's correspondence. When  $\mathbb{D} = \mathbb{C}$  or  $\mathbb{H}$  and the rank of  $G$  is strictly bigger than that of  $G'$ , we obtain the necessary and sufficient conditions on the highest weights of  $\Pi$  so that it occurs in Howe's correspondence. See Corollary 8.

The paper is organized as follows. In section 1, we introduce some notation and review the construction of the intertwining distributions. Section 2 computes the intertwining distribution for the dual pair  $(Z, \text{Sp}(W))$ , where  $Z = O_1$  is the center of the symplectic group  $\text{Sp}(W)$ , and introduces some properties needed in the sequel. Section 3 recalls how to realize the dual pairs with one member compact as Lie supergroups. Section 4 collects the main results of this paper. In the next section, section 5, we present a few examples illustrating them. The proofs of the main results are in sections 6, 7 and 8. We treat the special cases concerning the non-identity connected components of the orthogonal groups in sections 9, 10 and 11. Finally, in section 12, we outline how the results of this paper lead, for each representation  $\Pi$  of  $\widetilde{G}$  occurring in Howe's duality, to the computation of the wave front set of the representation  $\Pi'$  dual to  $\Pi$ . The five appendices collect and prove some auxiliary results.

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## 1. Notation and preliminaries.

Let us first recall the construction of the metaplectic group  $\widetilde{\text{Sp}}(W)$  and the Weil representation  $\omega$ . We are using the approach of [AP14, Section 4], to which we refer the reader for more details.

Let  $\mathfrak{sp}$  denote the Lie algebra of  $\text{Sp}(W)$ . Fix a positive definite compatible complex structure  $J$  on  $W$ , that is an element  $J \in \mathfrak{sp}$  such that  $J^2 = -1$  (minus the identity on  $W$ ) and the symmetric bilinear form  $\langle J \cdot, \cdot \rangle$  is positive definite on  $W$ . For an element  $g \in \text{Sp}(W)$ , let  $J_g = J^{-1}(g - 1)$ . The adjoint of  $J_g$  with respect to the form  $\langle J \cdot, \cdot \rangle$  is  $J_g^* = Jg^{-1}(1 - g)$ . In particular,  $J_g$  and  $J_g^*$  have the same kernel. Hence the image of  $J_g$

is

$$J_g W = (\text{Ker } J_g^*)^\perp = (\text{Ker } J_g)^\perp,$$

where  $\perp$  denotes the orthogonal complement with respect to  $\langle J \cdot, \cdot \rangle$ . Therefore, the restriction of  $J_g$  to  $J_g W$  defines an invertible element. Thus for every  $g \neq 1$ , it makes sense to talk about  $\det(J_g)_{J_g W}^{-1}$ , the reciprocal of the determinant of the restriction of  $J_g$  to  $J_g W$ . With this notation, we have

$$\widetilde{\text{Sp}}(W) = \{\tilde{g} = (g; \xi) \in \text{Sp}(W) \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)W} \det(J_g)_{J_g W}^{-1}\}, \quad (7)$$

with the convention that  $\det(J_g)_{J_g W}^{-1} = 1$  if  $g = 1$ . There exists a 2-cocycle  $C : \text{Sp}(W) \times \text{Sp}(W) \rightarrow \mathbb{C}$ , explicitly described in [AP14, Proposition 4.13], such that  $\widetilde{\text{Sp}}(W)$  is a group with respect to the multiplication

$$(g_1; \xi_1)(g_2; \xi_2) = (g_1 g_2; \xi_1 \xi_2 C(g_1, g_2)) \quad (8)$$

and the homomorphism

$$\widetilde{\text{Sp}}(W) \ni (g; \xi) \rightarrow g \in \text{Sp}(W) \quad (9)$$

does not split.

Let  $\mu_W$  (or simply  $dw$ ) be the Lebesgue measure on  $W$  normalized by the condition that the volume of the unit cube with respect to the form  $\langle J \cdot, \cdot \rangle$  is 1. (Since all positive complex structures are conjugate by elements of  $\text{Sp}$ , this normalization does not depend on the particular choice of  $J$ .) Let  $W = X \oplus Y$  be a complete polarization. Similar normalizations are fixed for the Lebesgue measures on every vector subspace of  $W$ , for instance on  $X$  and on  $Y$ . Furthermore, for every vector space  $V$ , we write  $\mathcal{S}(V)$  for the Schwartz space on  $V$  and  $\mathcal{S}'(V)$  for the space of tempered distributions on  $V$ . We use the notation  $G'$  for the second member of a dual pair because it is the centralizer of  $G$  in  $\text{Sp}(W)$ . We also use the notation  $'$  for all the objects associated with  $G'$ , such as  $\mathfrak{g}'$ ,  $\Pi'$ , ... Unfortunately, this collides with the usual notation for the dual of a linear topological space in functional analysis, also used in this paper, such as  $\mathcal{D}'(\mathbb{R}^n)$ ,  $\mathcal{S}'(\mathbb{R}^n)$ , ... We hope the reader will guess from the context the correct meaning of the notation.

Each element  $K \in \mathcal{S}'(X \times X)$  defines an operator  $\text{Op}(K) \in \text{Hom}(\mathcal{S}(X), \mathcal{S}'(X))$  by

$$\text{Op}(K)v(x) = \int_X K(x, x')v(x') dx'. \quad (10)$$

The map

$$\text{Op} : \mathcal{S}'(X \times X) \rightarrow \text{Hom}(\mathcal{S}(X), \mathcal{S}'(X)) \quad (11)$$

is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [Hör83, Theorem 5.2.1].

The Weyl transform is the linear isomorphism  $\mathcal{K} : \mathcal{S}'(W) \rightarrow \mathcal{S}'(X \times X)$  defined for  $f \in \mathcal{S}(W)$  by

$$\mathcal{K}(f)(x, x') = \int_Y f(x - x' + y) \chi\left(\frac{1}{2}\langle y, x + x' \rangle\right) dy, \quad (12)$$

(Recall that  $\chi$  is the character of  $\mathbb{R}$  we fixed at the beginning of the introduction.)

For  $g \in \text{Sp}(W)$ , let

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u \rangle\right) \quad (u = (g-1)w, w \in W). \quad (13)$$

Notice that, if  $g - 1$  is invertible on  $W$ , then

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle c(g)u, u \rangle\right),$$

where  $c(g) = (g + 1)(g - 1)^{-1}$  is the usual Cayley transform.

Following [AP14, Definition 4.23 and (114)], we define

$$T : \widetilde{\mathrm{Sp}}(W) \ni \tilde{g} = (g; \xi) \longrightarrow \xi \chi_{c(g)} \mu_{(g-1)W} \in \mathcal{S}'(W), \quad (14)$$

where  $\mu_{(g-1)W}$  is the Lebesgue measure on the subspace  $(g - 1)W$  normalized as above. Set

$$\omega = \mathrm{Op} \circ \mathcal{K} \circ T. \quad (15)$$

As proved in [AP14, Theorem 4.27],  $\omega$  is a unitary representation of  $\widetilde{\mathrm{Sp}}$  on  $L^2(X)$ . In fact,  $(\omega, L^2(X))$  is the Schrödinger model of Weil representation of  $\widetilde{\mathrm{Sp}}$  attached to the character  $\chi$  and the polarization  $W = X \oplus Y$ . In this realization,  $\mathcal{H}_\omega = L^2(X)$  and  $\mathcal{H}_\omega^\infty = \mathcal{S}(X)$ .

The distribution character of the Weil representation turns out to be the function

$$\Theta : \widetilde{\mathrm{Sp}}(W) \ni (g; \xi) \rightarrow \xi \in \mathbb{C}^\times, \quad (16)$$

[AP14, Proposition 4.27]. Hence for  $\tilde{g} \in \widetilde{\mathrm{Sp}}(W)$  in the preimage of  $g \in \mathrm{Sp}(W)$  under the double covering map (9), we have

$$T(\tilde{g}) = \Theta(\tilde{g}) \chi_{c(g)} \mu_{(g-1)W} \quad (\tilde{g} \in \widetilde{\mathrm{Sp}}(W)). \quad (17)$$

Suppose now that  $G, G' \subseteq \mathrm{Sp}(W)$  is a dual pair. Every irreducible admissible representation  $\Pi \otimes \Pi'$  of  $\widetilde{G} \times \widetilde{G}'$  occurring in Howe's correspondence may be realized, up to infinitesimal equivalence, as a subspace of  $\mathcal{H}_\omega^{\infty'} = \mathcal{S}'(X)$ . Hence

$$\mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) \subseteq \mathrm{Hom}(\mathcal{S}(X), \mathcal{S}'(X)).$$

The existence of the intertwining distribution  $f_{\Pi \otimes \Pi'} \in \mathcal{S}'(W)$  defined (up to a multiplicative constant) by (3) is thus a consequence of (1), (11) and (12).

Finally, because of (15), equation (4) and (5) lead to the equality

$$f_{\Pi \otimes \Pi'} = \frac{1}{2} T(\check{\Theta}_\Pi) = \int_G T(g) \check{\Theta}_\Pi(g) dg. \quad (18)$$

The problem of finding an explicit expression for  $f_{\Pi \otimes \Pi'}$  is hence transformed into the task of computing the right-hand side of (18).

## 2. The center of the metaplectic group.

Let  $Z = \{1, -1\}$  be the center of the symplectic group  $\mathrm{Sp}(W)$ . Then  $(Z, \mathrm{Sp}(W))$  is a dual pair in  $\mathrm{Sp}(W)$  with compact member  $Z$ . Let  $(\widetilde{Z}, \widetilde{\mathrm{Sp}}(W))$  be the corresponding dual pair in the metaplectic group  $\widetilde{\mathrm{Sp}}(W)$ . Then  $\widetilde{Z}$  coincides with the center of  $\widetilde{\mathrm{Sp}}(W)$  and is equal to

$$\widetilde{Z} = \{(1; 1), (1; -1), (-1; \zeta), (-1; -\zeta)\}, \quad (19)$$

where  $\zeta = \left(\frac{i}{2}\right)^{\frac{1}{2} \dim W}$ .

In this section we illustrate how to compute the intertwining distributions for the pair  $(Z, \mathrm{Sp}(W))$ . At the same time, we introduce some facts that will be needed in the rest of the paper.

The formula for the cocycle in (8) is particularly simple over  $Z$ :

$$C(1, \pm 1) = C(-1, 1) = 1 \quad \text{and} \quad C(-1, -1) = 2^{\dim W}.$$

Also,  $C(g, 1) = C(1, g) = 1$  for all  $g \in \text{Sp}(W)$  by [AP14, Proposition 4.13]. Notice that

$$(-1; \pm \zeta)^2 = (1; \zeta^2 C(-1, -1)) = (1; (-1)^{\frac{1}{2} \dim W}).$$

Hence the covering (9) restricted to  $\tilde{Z}$ ,

$$\tilde{Z} \ni \tilde{z} \rightarrow z \in Z \tag{20}$$

splits if and only if  $\frac{1}{2} \dim W$  is even.

By (14) we have

$$\begin{aligned} T(1; 1) &= \delta, & T(1; -1) &= -\delta, \\ T(-1; \zeta) &= \zeta \mu_W, & T(-1; -\zeta) &= -\zeta \mu_W. \end{aligned}$$

Moreover, [AP14, Proposition 4.28] shows that for  $v \in L^2(X)$  and  $x \in X$ ,

$$\begin{aligned} \omega(1; 1)v(x) &= v(x), & \omega(1; -1)v(x) &= -v(x), \\ \omega(-1; \zeta)v(x) &= \frac{\zeta}{|\zeta|}v(-x), & \omega(-1; -\zeta)v(x) &= -\frac{\zeta}{|\zeta|}v(-x). \end{aligned}$$

Since  $T(\tilde{z}) = \Theta(\tilde{z})\chi_{e(z)}\mu_{(z-1)W}$  for  $\tilde{z} \in \tilde{Z}$ , it follows that

$$\omega(\tilde{z})v(x) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|}v(zx) \quad (\tilde{z} \in \tilde{Z}). \tag{21}$$

The fraction

$$\chi_+(\tilde{z}) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad (\tilde{z} \in \tilde{Z}) \tag{22}$$

defines an irreducible character  $\chi_+$  of the group  $\tilde{Z}$ . Let  $\varepsilon$  be the unique non-trivial irreducible character of the two element group  $Z$ . Then

$$\chi_-(\tilde{z}) = \varepsilon(z) \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad (\tilde{z} \in \tilde{Z}) \tag{23}$$

is also an irreducible character of  $\tilde{Z}$ .

Let  $L^2(X)_+ \subseteq L^2(X)$  denote the subspace of the even functions and let  $L^2(X)_- \subseteq L^2(X)$  denote the subspace of the odd functions. Then, as is well known, [KV78, (6.9)], the restriction  $\omega_{\pm}$  of  $\omega$  to  $L^2(X)_{\pm}$  is irreducible. As we have seen above, the center  $\tilde{Z}$  acts on  $L^2(X)_{\pm}$  via the character  $\chi_{\pm}$ . Thus  $\chi_{\pm}$  is the central character of  $\omega_{\pm}$ .

Hence, in the case of the dual pair  $(Z, \text{Sp}(W))$ , Howe's correspondence looks as follows

$$(\chi_+, \mathbb{C}) \leftrightarrow (\omega_+, L^2(X)_+) \quad \text{and} \quad (\chi_-, \mathbb{C}) \leftrightarrow (\omega_-, L^2(X)_-). \tag{24}$$

The projections

$$L^2(X) \rightarrow L^2(X)_+ \quad \text{and} \quad L^2(X) \rightarrow L^2(X)_-$$

are respectively given by

$$\frac{1}{2}\omega(\check{\chi}_+) = \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_+(\tilde{z})\omega(\tilde{z}) \quad \text{and} \quad \frac{1}{2}\omega(\check{\chi}_-) = \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_-(\tilde{z})\omega(\tilde{z}).$$



The corresponding intertwining distributions are

$$\begin{aligned} f_{\chi_+ \otimes \omega_+} &= \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_+(\tilde{z}) T(\tilde{z}) = \frac{1}{2} (\delta + 2^{-\frac{1}{2} \dim W} \mu_W), \\ f_{\chi_- \otimes \omega_-} &= \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_-(\tilde{z}) T(\tilde{z}) = \frac{1}{2} (\delta - 2^{-\frac{1}{2} \dim W} \mu_W), \end{aligned} \tag{25}$$

where we normalize the total mass of  $Z$  to be 1, as we did for a general dual pair  $(G, G')$  with  $G$  compact.

The right-hand side of (25) is a sum of two homogenous distributions of different homogeneity degrees. So, asymptotically, they can be isolated. This allows us to recover  $\mu_W$ , and hence  $\tau_{\text{sp}(W)}(W)$ , the wave front of  $\omega_{\pm}$ , out of the intertwining distribution.

### 3. Dual pairs as Lie supergroups.

To present the main results of this paper, we need the realization of dual pairs with one member compact as Lie supergroups. The content of this section is taken from [Prz06] and [MPP15]. We recall the relevant material for making our exposition self-contained.

For a dual pair  $(G, G')$  as in (2), there is a division algebra  $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  with an involution over  $\mathbb{R}$ , a finite dimensional right  $\mathbb{D}$ -vector space  $V$  with a positive definite hermitian form  $(\cdot, \cdot)$  and a finite dimensional right  $\mathbb{D}$ -vector space  $V'$  with a non-degenerate skew-hermitian form  $(\cdot, \cdot)'$  such that  $G$  coincides with the isometry group of  $(\cdot, \cdot)$  and  $G'$  coincides with the isometry group of  $(\cdot, \cdot)'$ . We assume that  $G$  centralizes the complex structure  $J$  and that  $J$  normalizes  $G'$ . Then the conjugation by  $J$  is a Cartan involution on  $G'$ , which we denote by  $\theta$ .

Let  $V_{\bar{0}} = V$ ,  $d = \dim_{\mathbb{D}} V_{\bar{0}}$ ,  $V_{\bar{1}} = V'$  and  $d' = \dim_{\mathbb{D}} V_{\bar{1}}$ . We assume that both  $V_{\bar{0}}$  and  $V_{\bar{1}}$  are right vector spaces over  $\mathbb{D}$ . Set  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  and define an element  $S \in \text{End}(V)$  by

$$S(v_0 + v_1) = v_0 - v_1 \quad (v_0 \in V_{\bar{0}}, v_1 \in V_{\bar{1}}).$$

Let

$$\begin{aligned} \text{End}(V)_{\bar{0}} &= \{x \in \text{End}(V); Sx = xS\}, \\ \text{End}(V)_{\bar{1}} &= \{x \in \text{End}(V); Sx = -xS\}, \\ \text{GL}(V)_{\bar{0}} &= \text{End}(V)_{\bar{0}} \cap \text{GL}(V). \end{aligned}$$

Denote by  $(\cdot, \cdot)''$  the direct sum of the two forms  $(\cdot, \cdot)$  and  $(\cdot, \cdot)'$ . Let

$$\begin{aligned} \mathfrak{s}_{\bar{0}} &= \{x \in \text{End}(V)_{\bar{0}}; (xu, v)'' = -(u, xv)'', u, v \in V\}, \\ \mathfrak{s}_{\bar{1}} &= \{x \in \text{End}(V)_{\bar{1}}; (xu, v)'' = (u, Sxv)'', u, v \in V\}, \\ \mathfrak{s} &= \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}, \\ S &= \{s \in \text{GL}(V)_{\bar{0}}; (su, sv)'' = (u, v)'', u, v \in V\}, \\ \langle x, y \rangle &= \text{tr}_{\mathbb{D}/\mathbb{R}}(Sxy). \end{aligned} \tag{26}$$

(Here  $\text{tr}_{\mathbb{D}/\mathbb{R}}(x)$  denotes the trace of  $x$  considered as a real endomorphism of  $V$ .) Then  $(S, \mathfrak{s})$  is a real Lie supergroup, i.e. a real Lie group  $S$  together with a real Lie superalgebra  $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ , whose even component  $\mathfrak{s}_{\bar{0}}$  is the Lie algebra of  $S$ . (In terms of [DM99, §3.8],  $(S, \mathfrak{s})$  is a Harish-Chandra pair.) We shall write  $\mathfrak{s}(V)$  instead of  $\mathfrak{s}$  whenever we want to

specify the Lie superalgebra  $\mathfrak{s}$  constructed as above from  $\mathbf{V}$  and  $(\cdot, \cdot)''$ . By restriction, we have the identification

$$\mathfrak{s}_{\bar{1}} = \text{Hom}_{\mathbb{D}}(\mathbf{V}_{\bar{1}}, \mathbf{V}_{\bar{0}}). \quad (28)$$

The group  $S$  acts on  $\mathfrak{s}$  by conjugation and  $\langle \cdot, \cdot \rangle$  is a non-degenerate  $S$ -invariant form on the real vector space  $\mathfrak{s}$ , whose restriction to  $\mathfrak{s}_{\bar{0}}$  is symmetric and restriction to  $\mathfrak{s}_{\bar{1}}$  is skew-symmetric. We shall employ the notation

$$s.x = \text{Ad}(s)x = sxs^{-1} \quad (s \in S, x \in \mathfrak{s}), \quad (29)$$

$$x(w) = \text{ad}(x)(w) = xw - wx \quad (x \in \mathfrak{s}_{\bar{0}}, w \in \mathfrak{s}_{\bar{1}}). \quad (30)$$

In terms of the notation introduced at the beginning of this section,

$$\mathfrak{g} = \mathfrak{s}_{\bar{0}}|_{\mathbf{V}_{\bar{0}}}, \quad \mathfrak{g}' = \mathfrak{s}_{\bar{0}}|_{\mathbf{V}_{\bar{1}}}, \quad G = S|_{\mathbf{V}_{\bar{0}}}, \quad G' = S|_{\mathbf{V}_{\bar{1}}}.$$

Define  $W = \text{Hom}_{\mathbb{D}}(\mathbf{V}_{\bar{1}}, \mathbf{V}_{\bar{0}})$ . Then, by restriction, we have the identification

$$W = \mathfrak{s}_{\bar{1}}. \quad (31)$$

Under this identification, the adjoint action of  $G$  on  $\mathfrak{s}_{\bar{1}}$  becomes the action on  $W$  by the left (post)multiplication. Similarly, the adjoint action of  $G'$  on  $\mathfrak{s}_{\bar{1}}$  becomes the action of  $G'$  on  $W$  via the right (pre)multiplication by the inverse. Also, we have the unnormalized moment maps

$$\tau : W \ni w \rightarrow w^2|_{\mathbf{V}_{\bar{0}}} \in \mathfrak{g}, \quad \tau' : W \ni w \rightarrow w^2|_{\mathbf{V}_{\bar{1}}} \in \mathfrak{g}'. \quad (32)$$

An element  $x \in \mathfrak{s}$  is called semisimple (resp., nilpotent) if  $x$  is semisimple (resp., nilpotent) as an endomorphism of  $\mathbf{V}$ . We say that a semisimple element  $x \in \mathfrak{s}_{\bar{1}}$  is regular if it is nonzero and  $\dim(S.x) \geq \dim(S.y)$  for all semisimple  $y \in \mathfrak{s}_{\bar{1}}$ . Let  $x \in \mathfrak{s}_{\bar{1}}$  be fixed. For  $x, y \in \mathfrak{s}_{\bar{1}}$  let  $\{x, y\} = xy + yx \in \mathfrak{s}_{\bar{0}}$  be their anticommutator.

The anticommutator and the double anticommutator of  $x$  in  $\mathfrak{s}_{\bar{1}}$  are

$$\begin{aligned} {}^x\mathfrak{s}_{\bar{1}} &= \{y \in \mathfrak{s}_{\bar{1}} : \{x, y\} = 0\}, \\ {}^{x\mathfrak{s}_{\bar{1}}}\mathfrak{s}_{\bar{1}} &= \bigcap_{y \in {}^x\mathfrak{s}_{\bar{1}}} {}^y\mathfrak{s}_{\bar{1}}, \end{aligned}$$

respectively. A Cartan subspace  $\mathfrak{h}_{\bar{1}}$  of  $\mathfrak{s}_{\bar{1}}$  is defined as the double anticommutator of a regular semisimple element  $x \in \mathfrak{s}_{\bar{1}}$ . We denote by  $\mathfrak{h}_{\bar{1}}^{reg}$  the set of regular elements in  $\mathfrak{h}_{\bar{1}}$ .

Next we describe the Cartan subspaces  $\mathfrak{h}_{\bar{1}} \subseteq \mathfrak{s}_{\bar{1}}$ . We refer to [Prz06, §6] and [MPP15, §4] for the proofs omitted here. Let  $l$  be the rank of  $\mathfrak{g}$ ,  $l'$  the rank of  $\mathfrak{g}'$ , and set

$$l'' = \min\{l, l'\}. \quad (33)$$

Given a Cartan subspace  $\mathfrak{h}_{\bar{1}}$ , there are  $\mathbb{Z}/2\mathbb{Z}$ -graded subspaces  $\mathbf{V}^j \subseteq \mathbf{V}$  such that the restriction of the form  $(\cdot, \cdot)''$  to each  $\mathbf{V}^j$  is non-degenerate,  $\mathbf{V}^j$  is orthogonal to  $\mathbf{V}^k$  for  $j \neq k$  and

$$\mathbf{V} = \mathbf{V}^0 \oplus \mathbf{V}^1 \oplus \mathbf{V}^2 \oplus \cdots \oplus \mathbf{V}^{l''}. \quad (34)$$

The subspace  $\mathbf{V}^0$  coincides with the intersection of the kernels of the elements of  $\mathfrak{h}_{\bar{1}}$  (equivalently,  $\mathbf{V}^0 = \text{Ker}(x)$  if  $\mathfrak{h}_{\bar{1}} = {}^{x\mathfrak{s}_{\bar{1}}}\mathfrak{s}_{\bar{1}}$ ). For  $1 \leq j \leq l''$ , the subspaces  $\mathbf{V}^j = \mathbf{V}_{\bar{0}}^j \oplus \mathbf{V}_{\bar{1}}^j$  are

described as follows. Suppose  $\mathbb{D} = \mathbb{R}$ . Then there is a basis  $v_0, v'_0$  of  $\mathbf{V}_0^j$  and basis  $v_1, v'_1$  of  $\mathbf{V}_1^j$  such that

$$\begin{aligned} (v_0, v_0)'' &= (v'_0, v'_0)'' = 1, & (v_0, v'_0)'' &= 0, \\ (v_1, v_1)'' &= (v'_1, v'_1)'' = 0, & (v_1, v'_1)'' &= 1. \end{aligned} \quad (35)$$

The following formulas define an element  $u_j \in \mathfrak{s}_{\overline{1}}(\mathbf{V}^j)$ ,

$$\begin{aligned} u_j(v_0) &= \frac{1}{\sqrt{2}}(v_1 - v'_1), & u_j(v_1) &= \frac{1}{\sqrt{2}}(v_0 - v'_0), \\ u_j(v'_0) &= \frac{1}{\sqrt{2}}(v_1 + v'_1), & u_j(v'_1) &= \frac{1}{\sqrt{2}}(v_0 + v'_0). \end{aligned}$$

Suppose  $\mathbb{D} = \mathbb{C}$ . Then  $\mathbf{V}_0^j = \mathbb{C}v_0$ ,  $\mathbf{V}_1^j = \mathbb{C}v_1$ , where  $(v_0, v_0)'' = 1$  and  $(v_1, v_1)'' = \delta_j i$ , with  $\delta_j = \pm 1$ . The following formulas define an element  $u_j \in \mathfrak{s}_{\overline{1}}(\mathbf{V}^j)$ ,

$$u_j(v_0) = e^{-i\delta_j \frac{\pi}{4}} v_1, \quad u_j(v_1) = e^{-i\delta_j \frac{\pi}{4}} v_0. \quad (36)$$

Suppose  $\mathbb{D} = \mathbb{H}$ . Then  $\mathbf{V}_0^j = \mathbb{H}v_0$ ,  $\mathbf{V}_1^j = \mathbb{H}v_1$ , where  $(v_0, v_0)'' = 1$  and  $(v_1, v_1)'' = i$ . The following formulas define an element  $u_j \in \mathfrak{s}_{\overline{1}}(\mathbf{V}^j)$ ,

$$u_j(v_0) = e^{-i\frac{\pi}{4}} v_1, \quad u_j(v_1) = e^{-i\frac{\pi}{4}} v_0.$$

In any case, by extending each  $u_j$  by zero outside  $\mathbf{V}^j$ , we have

$$\mathfrak{h}_{\overline{1}} = \sum_{j=1}^{l''} \mathbb{R}u_j. \quad (37)$$

The formula (37) describes a maximal family of mutually non-conjugate Cartan subspaces of  $\mathfrak{s}_{\overline{1}}$ . Notice that there is only one such subspace unless the dual pair  $(G, G')$  is isomorphic to  $(U_l, U_{p,q})$  with  $l'' = l < p+q$ . In the last case there are  $\min(l, p) - \max(l-q, 0) + 1$  such subspaces, assuming  $p \leq q$ . For each  $m$  such that  $\max(l-q, 0) \leq m \leq \min(p, l)$  there is a Cartan subspace  $\mathfrak{h}_{\overline{1}, m}$  determined by the condition that  $m$  is the number of positive  $\delta_j$ 's in (36). We may assume that  $\delta_1 = \dots = \delta_m = 1$  and  $\delta_{m+1} = \dots = \delta_l = -1$ . The choice of the spaces  $\mathbf{V}_0^j$  may be done independently of  $m$ . The spaces  $\mathbf{V}_1^j$  depend on  $m$ . If  $(G, G')$  is isomorphic to  $(U_l, U_{p,q})$  with  $l \geq l'' = p+q$ , then there is a unique Cartan subspace of  $\mathfrak{s}_{\overline{1}}$  up to conjugation. It is determined by the condition that in (36) there are  $p$  positive and  $q$  negative  $\delta_j$ 's. We may assume that the positive  $\delta_j$ 's are the first  $p$ .

The Weyl group  $W(S, \mathfrak{h}_{\overline{1}})$  is the quotient of the stabilizer of  $\mathfrak{h}_{\overline{1}}$  in  $S$  by the subgroup  $S^{\mathfrak{h}_{\overline{1}}}$  fixing each element of  $\mathfrak{h}_{\overline{1}}$ . If  $\mathbb{D} \neq \mathbb{C}$ , then  $W(S, \mathfrak{h}_{\overline{1}})$  acts by all sign changes and all permutations of the  $u_j$ 's. If  $\mathbb{D} = \mathbb{C}$ , the Weyl group acts by all sign changes and all permutations of the  $u_j$ 's which preserve  $(\delta_1, \dots, \delta_{l''})$ , see [Prz06, (6.3)].

Set  $\delta_j = 1$  for all  $1 \leq j \leq l''$ , if  $\mathbb{D} \neq \mathbb{C}$ , and define

$$J_j = \delta_j \tau(u_j), \quad J'_j = \delta_j \tau'(u_j) \quad (1 \leq j \leq l''). \quad (38)$$

Then  $J_j, J'_j$  are complex structures on  $\mathbf{V}_0^j$  and  $\mathbf{V}_1^j$  respectively. Explicitly,

$$\begin{aligned} J_j(v_0) &= -v'_0, & J_j(v'_0) &= v_0, & J'_j(v_1) &= -v'_1, & J'_j(v'_1) &= v_1, & \text{if } \mathbb{D} = \mathbb{R}, \\ J_j(v_0) &= -iv_0, & J'_j(v_1) &= -iv_1, & & & & & \text{if } \mathbb{D} = \mathbb{C} \text{ or } \mathbb{H}. \end{aligned} \quad (39)$$

(The point of the multiplication by the  $\delta_j$  in (38) is that the complex structures  $J_j, J'_j$  do not depend on the Cartan subspace  $\mathfrak{h}_{\bar{1}}$ .) In particular, if  $w = \sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\bar{1}}$ , then

$$\tau(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J_j \quad \text{and} \quad \tau'(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J'_j. \quad (40)$$

Let  $\mathfrak{h}_{\bar{1}}^2 \subseteq \mathfrak{s}_{\bar{0}}$  be the subspace spanned by all the squares  $w^2$ ,  $w \in \mathfrak{h}_{\bar{1}}$ . Then

$$\mathfrak{h}_{\bar{1}}^2 = \sum_{j=1}^{l''} \mathbb{R}(J_j + J'_j). \quad (41)$$

We shall use the following identification

$$\mathfrak{h}_{\bar{1}}^2|_{\mathfrak{v}_{\bar{0}}} \ni \sum_{j=1}^{l''} y_j J_j = \sum_{j=1}^{l''} y_j J'_j \in \mathfrak{h}_{\bar{1}}^2|_{\mathfrak{v}_{\bar{1}}}. \quad (42)$$

Recall from (33) that  $l'' = \min\{l, l'\}$ . If  $l'' = l$ , then  $\mathfrak{h}_{\bar{1}}^2|_{\mathfrak{v}_{\bar{0}}}$  is an elliptic Cartan subalgebra of  $\mathfrak{g}$  which we denote by  $\mathfrak{h}$ . (This means that all the roots of  $\mathfrak{h}$  in  $\mathfrak{g}_{\mathbb{C}}$  are purely imaginary.) The identification (42) embeds  $\mathfrak{h}$  diagonally in  $\mathfrak{g}$  and in  $\mathfrak{g}'$ . It is contained in an elliptic Cartan subalgebra of  $\mathfrak{g}'$ , say  $\mathfrak{h}'$ . Similarly, if  $l'' = l'$ , then  $\mathfrak{h}_{\bar{1}}^2|_{\mathfrak{v}_{\bar{1}}}$  is an elliptic Cartan subalgebra of  $\mathfrak{g}'$  which we denote by  $\mathfrak{h}'$ . If  $l \leq l'$  we denote by  $\mathfrak{z}' \subseteq \mathfrak{g}'$  the centralizer of  $\mathfrak{h}$ . Similarly, if  $l' \leq l$  we denote by  $\mathfrak{z} \subseteq \mathfrak{g}$  the centralizer of  $\mathfrak{h}'$ . In particular, if  $l' = l$ , then  $\mathfrak{z}' = \mathfrak{h}' = \mathfrak{h} = \mathfrak{z}$ , where the first equality is in  $\mathfrak{g}$ , the second is (42) and the last is in  $\mathfrak{g}'$ .

Let  $\mathfrak{s}_{\bar{0}\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} \oplus \mathfrak{g}'_{\mathbb{C}}$  be the complexification of  $\mathfrak{s}_{\bar{0}}$ . Fix a system of positive roots for the adjoint action of  $\mathfrak{h}_{\bar{1}}^2$  on  $\mathfrak{s}_{\bar{0}\mathbb{C}}$ . Suppose first that  $l \leq l'$ . By the identification (42),  $\mathfrak{h}$  preserves both  $\mathfrak{g}_{\mathbb{C}}$  and  $\mathfrak{g}'_{\mathbb{C}}$ . So our choice of positive roots for  $(\mathfrak{h}_{\bar{1}}^2, \mathfrak{s}_{\bar{0}\mathbb{C}})$  fixes a positive root system of  $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$  and extends to a compatible positive root system for  $(\mathfrak{h}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ . Let  $\pi_{\mathfrak{g}/\mathfrak{h}}$  be the product of positive roots of  $(\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})$  and let  $\pi_{\mathfrak{g}'/\mathfrak{z}'}$  be the product of positive roots of  $(\mathfrak{h}'_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$  such that the corresponding root spaces do not occur in  $\mathfrak{z}'_{\mathbb{C}}$ . If  $l > l'$ , then  $\pi_{\mathfrak{g}'/\mathfrak{h}'}$  and  $\pi_{\mathfrak{g}/\mathfrak{z}}$  can be similarly defined.

Suppose  $l' < l$ . Then  $\mathfrak{V}_{\bar{1}}^0 = 0$ ,  $\mathfrak{V}_{\bar{0}}^0 \neq 0$  and

$$\mathfrak{V} = \mathfrak{V}_{\bar{0}} = \mathfrak{V}_{\bar{0}}^0 \oplus \mathfrak{V}_{\bar{0}}^1 \oplus \mathfrak{V}_{\bar{0}}^2 \oplus \cdots \oplus \mathfrak{V}_{\bar{0}}^{l''} \quad (43)$$

is a direct sum orthogonal decomposition with respect to the positive definite hermitian form  $(\cdot, \cdot)$ . We extend  $\mathfrak{h} \subseteq \mathfrak{g}$  to a Cartan subalgebra  $\mathfrak{h}(\mathfrak{g}) \subseteq \mathfrak{g}$  as follows. The restriction of  $\mathfrak{h}(\mathfrak{g})$  to  $\mathfrak{V}_{\bar{0}}^1 \oplus \mathfrak{V}_{\bar{0}}^2 \oplus \cdots \oplus \mathfrak{V}_{\bar{0}}^{l''}$  coincides with  $\mathfrak{h}$ . Pick an orthogonal direct sum decomposition

$$\mathfrak{V}_{\bar{0}}^0 = \mathfrak{V}_{\bar{0}}^{0,0} \oplus \mathfrak{V}_{\bar{0}}^{0,l''+1} \oplus \mathfrak{V}_{\bar{0}}^{0,l''+2} \oplus \cdots \oplus \mathfrak{V}_{\bar{0}}^{0,l} \quad (44)$$

where for  $j > l''$ ,  $\dim_{\mathbb{D}} \mathfrak{V}_{\bar{0}}^{0,j} = 2$  if  $\mathbb{D} = \mathbb{R}$  and  $\dim_{\mathbb{D}} \mathfrak{V}_{\bar{0}}^{0,j} = 1$  if  $\mathbb{D} \neq \mathbb{R}$ . Also  $\mathfrak{V}_{\bar{0}}^{0,0} = 0$  unless  $G = O_{2l+1}$ , in which case  $\dim_{\mathbb{D}} \mathfrak{V}_{\bar{0}}^{0,0} = 1$ . In each space  $\mathfrak{V}_{\bar{0}}^{0,j}$ , with  $j > l''$ , we pick an orthonormal basis and define  $J_j$  as in (39). Then

$$\mathfrak{h}(\mathfrak{g}) = \sum_{j=1}^l \mathbb{R}J_j. \quad (45)$$

If  $l \leq l'$ , then we set  $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}$ .

Let  $J_j^*$ ,  $1 \leq j \leq l$ , be the basis of the space  $\mathfrak{h}(\mathfrak{g})^*$  which is dual to  $J_1, \dots, J_l$ , and set

$$e_j = -iJ_j^*, \quad 1 \leq j \leq l. \quad (46)$$

If  $\mu \in i\mathfrak{h}(\mathfrak{g})^*$ , then  $\mu = \sum_{j=1}^l \mu_j e_j$  with  $\mu_j \in \mathbb{R}$ . We say that  $\mu$  is strictly dominant if  $\mu_1 > \mu_2 > \dots > \mu_l$ .

#### 4. Main results.

Suppose an irreducible representation  $\Pi$  of  $\tilde{G}$  occurs in Howe's correspondence. This means that there is a subspace  $\mathcal{H}_\Pi \subseteq L^2(X)$  on which the restriction of  $\omega$  coincides with  $\Pi$ . Since  $\tilde{Z} \subseteq \tilde{G} \cap \tilde{G}'$ , then either  $\mathcal{H}_\Pi \subseteq L^2(X)_+$  or  $\mathcal{H}_\Pi \subseteq L^2(X)_-$ . In the first case the restriction of the central character  $\chi_\Pi$  of  $\Pi$  to  $\tilde{Z}$  is equal to  $\chi_+$  and in the second case to  $\chi_-$ . Thus for  $\tilde{z} \in \tilde{Z}$  and  $\tilde{g} \in \tilde{G}$ ,

$$\begin{aligned} \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_+(\tilde{z})\Theta_\Pi(\tilde{g}) & \text{if } \mathcal{H}_\Pi \subseteq L^2(X)_+, \\ \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_-(\tilde{z})\Theta_\Pi(\tilde{g}) & \text{if } \mathcal{H}_\Pi \subseteq L^2(X)_-. \end{aligned} \quad (47)$$

We see from equations (17), (22), (23) and (47) that the function

$$\tilde{G} \ni \tilde{g} \rightarrow T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) \in \mathcal{S}'(W)$$

is constant on the fibers of the covering map (9). The following lemma is a restatement of (18). Our main results will be the explicit expressions of the various integrals appearing on the right-hand sides of the equations below.

**Lemma 1.** *Let  $G^0 \subseteq G$  denote the connected identity component. Suppose  $(G, G') = (U_d, U_{p,q})$  or  $(Sp_d, O_{2m}^*)$ . Then  $G = G^0 = -G^0$  and*

$$f_{\Pi \otimes \Pi'} = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg. \quad (48)$$

Formula (48) holds also if  $(G, G') = (O_d, Sp_{2m}(\mathbb{R}))$  with  $d$  even and  $\Theta_\Pi$  supported in  $\tilde{G}^0$ , because  $G^0 = SO_d = -SO_d = -G^0$ . In the remaining cases

$$f_{\Pi \otimes \Pi'} = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg + \int_{G \setminus (-G^0)} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg. \quad (49)$$

The integrals over  $-G^0$  in (48) and (49) are given in Theorems 2 and 3 below, proved in section 8. The integrals over the other connected component, in the cases where (49) holds, are computed in Theorems 4, 6 and 7, respectively, and proved in sections 9, 10, and 11. Theorem 5, proved in this section, will furthermore show that the second integral on the right-hand side of (49) vanishes when  $(G, G') = (O_{2l}, Sp_{2l'}(\mathbb{R}))$  with  $l > l'$ .

**Remark 1.** Notice that, since the character  $\Theta_\Pi$  is conjugation invariant,

$$\int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi^G) dg$$

where  $\phi^G$  is the projection of  $\phi$  onto the space of the  $G$ -invariants in  $\mathcal{S}(W)$ ,

$$\phi^G(w) = \int_G \phi(g.w) dg \quad (w \in W) \quad (50)$$

(Recall that we have normalized the Haar measure on  $G$  so that its mass is 1.)

Let

$$\iota = \begin{cases} 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C} \\ \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{H}. \end{cases} \quad (51)$$

and let

$$r = \frac{2 \dim \mathfrak{g}}{\dim V_{\mathbb{R}}}, \quad (52)$$

where the subscript  $\mathbb{R}$  indicates that we are viewing  $V$  as a vector space over  $\mathbb{R}$ . Explicitly,

$$r = \begin{cases} 2l - 1 & \text{if } G = O_{2l}, \\ 2l & \text{if } G = O_{2l+1}, \\ l & \text{if } G = U_l, \\ l + \frac{1}{2} & \text{if } G = Sp_l. \end{cases} \quad (53)$$

Let

$$\delta = \frac{1}{2l}(d' - r + \iota) \quad \text{and} \quad \beta = \frac{2\pi}{l}. \quad (54)$$

Fix an irreducible representation  $\Pi$  of  $\tilde{G}$  that occurs in the restriction of the Weil representation  $\omega$  to  $\tilde{G}$ . Let  $\mu \in i\mathfrak{h}(\mathfrak{g})^*$  be the Harish-Chandra parameter of  $\Pi$  with  $\mu_1 > \mu_2 > \dots$ . This means that  $\mu = \lambda + \rho$ , where  $\lambda$  is the highest weight of  $\Pi$  and  $\rho$  is one half times the sum of the positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . See Appendix E.

Let  $P_{a,b}$  and  $Q_{a,b}$  be the piecewise polynomial functions defined in (C.8) and (C.9). Define

$$a_j = -\mu_j - \delta + 1, \quad b_j = \mu_j - \delta + 1 \quad (55)$$

$$p_j(y_j) = P_{a_j, b_j}(\beta y_j) e^{-\beta|y_j|}, \quad q_j(y_j) = \beta^{-1} Q_{a_j, b_j}(\beta^{-1} y_j) \quad (1 \leq j \leq l''), \quad (56)$$

where  $\delta$  and  $\beta$  are as in (54). Notice that  $a_j$  and  $b_j$  are integers (see Lemma 14).

**Theorem 2.** *Let  $l \leq l'$ . Then there is a non-zero constant  $C$  which depends only of the dual pair  $(G, G')$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) dg = C \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h} \cap \tau(W)} \left( \prod_{j=1}^{l'} (p_j(y_j) + q_j(-\partial_{y_j}) \delta_0(y_j)) \right) \cdot F_{\phi}(y) dy, \quad (57)$$

where  $\chi_{\Pi}$  is the central character of  $\Pi$  (see (47)),  $\tilde{c}$  is a real analytic lift of the Cayley transform (see (95)),  $\delta_0$  is the Dirac delta at 0, and  $F_{\phi}(y)$  is the image of  $\phi$  under the Harish-Chandra regular almost-elliptic orbital integral on  $W$  (see [MPP20, Definition 3.2] and Lemma 17 below). The term

$$\prod_{j=1}^{l'} (p_j(y_j) + q_j(-\partial_{y_j}) \delta_0(y_j)) \quad (58)$$

is:

- (1) a function in  $y$  if and only if all the  $q_j$ 's are zero, and this happens if and only if  $l = l'$  and  $(G, G') \neq (O_{2l}, Sp_{2l'}(\mathbb{R}))$ ;
- (2) a linear combination of products of functions and Dirac delta's at 0 in some coordinates  $y_j$  if and only if all the  $q_j$ 's are of degree zero. This happens if and only if either  $(G, G') = (O_{2l}, Sp_{2l}(\mathbb{R}))$ , or  $l' = l + 1$  and  $\mathbb{D} = \mathbb{C}$  or  $\mathbb{H}$ .

In the remaining cases, (58) is a distribution, but not a measure.

**Remark 2.** The integration domain  $\mathfrak{h} \cap \tau(W)$  appearing in Theorem 2 was explicitly determined in [MPP20, Lemma 3.5]. It is equal to  $\mathfrak{h}$  if  $\mathbb{D} \neq \mathbb{C}$  or if  $\mathbb{D} = \mathbb{C}$  and  $l = l'$ . By (141), (142) and Appendix E, we see that  $a_j \leq 0$  for all  $1 \leq j \leq l$  when  $l \leq l'$ . Hence each  $P_{a_j, b_j}(\beta y_j)$  vanishes for  $y_j < 0$ . In cases (1) and (2) of Theorem 2 with  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , we can therefore replace the domain of integration  $\mathfrak{h}$  with the smaller domain  $\tau(\mathfrak{h}_{\overline{1}})$ . If  $\mathbb{D} = \mathbb{C}$  and  $l = l' = p + q$ , then  $\mathfrak{h} \cap \tau(W) = \mathfrak{h}$  contains the support of  $\prod_{j=1}^l p(y_j)$ , which is  $\sum_{j=1}^l \mathbb{R}^+ J_j$ . However,  $\tau(\mathfrak{h}_{\overline{1}}) = (\sum_{j=1}^p \mathbb{R}^+ J_j) \oplus (\sum_{j=p+1}^l \mathbb{R}^- J_j)$  and is therefore different.

In the case  $l > l'$ , up to conjugation, there is a unique Cartan subspace  $\mathfrak{h}_{\overline{1}}$  in  $W$ . Recall that for  $\mathbb{D} = \mathbb{C}$  we are supposing that  $p \leq q$ .

Define  $s_0 \in W(G, \mathfrak{h}(\mathfrak{g}))$  by

$$s_0(J_j) = J_j \quad (1 \leq j \leq l) \quad \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H}, \quad (59)$$

$$s_0(J_j) = \begin{cases} J_j & (1 \leq j \leq p) \\ J_{q+j} & (p+1 \leq j \leq l-q) \\ J_{j-l+p} & (l-q+1 \leq j \leq l) \end{cases} \quad \text{if } \mathbb{D} = \mathbb{C}. \quad (60)$$

**Theorem 3.** *Let  $l > l'$ . Then*

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g}) dg \neq 0 \quad (61)$$

if and only if the highest weight  $\lambda = \sum_{j=1}^l \lambda_j e_j$  of  $\Pi$  is of the form

$$(a) \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{l'} \geq 0 \text{ and } \lambda_j = 0 \text{ for } l'+1 \leq j \leq l \quad \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H},$$

$$(b) \quad \lambda_j = \frac{q-p}{2} + \nu_j \quad \text{where}$$

$$\nu_1 \geq \cdots \geq \nu_p \geq 0, \nu_j = 0 \text{ for } p+1 \leq j \leq l-q, 0 \geq \nu_{l-q+1} \geq \cdots \geq \nu_l \quad \text{if } \mathbb{D} = \mathbb{C}.$$

Suppose that (a) and (b) are satisfied. Then there is a non-zero constant  $C$  which depends only of the dual pair  $(G, G')$  such that for all  $\phi \in \mathcal{S}(W)$

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g})(\phi) dg = C \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\tau'(\mathfrak{h}_{\overline{1}}^{reg})} \left( \prod_{j=1}^{l'} p_j((s_0^{-1}y)_j) \right) \cdot F_{\phi}(y) dy, \quad (62)$$

where, explicitly,

$$\prod_{j=1}^{l'} p_j((s_0^{-1}y)_j) = \begin{cases} \prod_{j=1}^{l'} p_j(y_j) & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H} \\ \left( \prod_{j=1}^p p_j(y_j) \right) \left( \prod_{j=l-q+1}^l p_j(y_j) \right) & \text{if } \mathbb{D} = \mathbb{C}. \end{cases}$$

Moreover, (62) is defined by the locally integrable  $S$ -invariant function on  $W$  whose restriction to  $\mathfrak{h}_{\overline{1}}^{reg}$  is a non-zero constant multiple of

$$\frac{\sum_{s' \in W(G', \mathfrak{h}')} \text{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0, j}, b_{s_0, j}, 2\delta_j}(\beta(s'y)_j)}{e^{-\beta \sum_{j=1}^{l'} |y_j|}} \pi_{\mathfrak{g}/\mathfrak{h}}(y) \quad (y_j = J_j^* y, y = \tau(w) = \tau'(w), w \in \mathfrak{h}_{\overline{1}}^{reg}), \quad (63)$$

where  $\mu$  is the Harish-Chandra parameter of  $\Pi$ ,

$$a_{s,j} = -(s\mu)_j - \delta + 1, \quad b_{s,j} = (s\mu)_j - \delta + 1 \quad (s \in W(G, \mathfrak{h}), 1 \leq j \leq l), \quad (64)$$

$P_{a,b,\pm 2}$  is the polynomial defined in (C.5) or (C.6), and the  $\delta_j$ 's are as in (36). (See (42) for the identifications  $y = \tau(w) = \tau'(w)$  in (63).) The fraction in (63) is the pullback by  $\tau'$  of a  $G'$ -invariant polynomial on  $\mathfrak{g}'$ . The exponential in (63) extends to a  $GG'$ -invariant function. It is a Gaussian on  $W$  if and only if  $G'$  is compact.

**Remark 3.** The Cartan subalgebra  $\mathfrak{h}'$  is  $\theta$ -stable, where  $\theta$  is the fixed Cartan involution of  $\mathfrak{g}'$ . Let  $H' \subseteq G'$  be the Cartan subgroup which is the centralizer of  $\mathfrak{h}'$  in  $G'$ , and let  $K'$  be the maximal compact subgroup of  $G'$  which is fixed by  $\theta$ . Then, by [Har56, Lemma 10], the Weyl group  $W(G', \mathfrak{h}')$  coincides with the normalizer of  $H'$  in  $K'$  modulo the centralizer of  $H'$  in  $K'$ . Explicitly,  $K'$  is  $U_{l'}$  if  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , and  $U_p \times U_q$  if  $\mathbb{D} = \mathbb{C}$ . Hence  $W(G', \mathfrak{h}')$  acts on  $\mathfrak{h}'$  by permuting the  $J'_j$ , (38), if  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , and by separately permuting the first  $p$  and the last  $q$  elements  $J'_j$  if  $\mathbb{D} = \mathbb{C}$ . Since  $\delta_j = 1$  for all  $j = 1, \dots, l'$  if  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , and  $\delta_j = 1$  for  $j = 1, \dots, p$  and  $\delta_j = -1$  for  $j = p+1, \dots, p+q$  if  $\mathbb{D} = \mathbb{C}$ , it follows from (40) that the domain of integration  $\tau'(\mathfrak{h}'^{reg})$  appearing in Theorem 3 is  $W(G', \mathfrak{h}')$ -invariant.

**Remark 4.** Let  $\Pi$  be a genuine irreducible representation of  $\tilde{G}$  of highest weight  $\lambda$ . Conditions (a) and (b) in Theorem 3 are precisely those ensuring that  $\Pi$  occurs in Howe's correspondence. See [Prz96, Appendix] and Corollary 8 below.

Now we consider the integrals over  $G \setminus (-G^0)$  in the cases where (49) holds.

Consider the case  $(G, G') = (O_{2l}, Sp_{2l'}(\mathbb{R}))$  and the character  $\Theta_\Pi$  is not supported on the preimage  $\tilde{G}^0$  of the connected identity component  $G^0 \subseteq G$ . As before, we have to distinguish the cases  $l \leq l'$  and  $l > l'$ .

Suppose first that  $l \leq l'$  and  $(l, l') \neq (1, 1)$ . Then the graded vector space (34) is equal to

$$V = V_{\bar{1}}^0 \oplus V^1 \oplus V^2 \oplus \dots \oplus V^l.$$

Recall from (35) that in each  $V_{\bar{0}}^j$  we selected an orthonormal basis  $v_0, v'_0$ . For convenience, we introduce the index  $j$  in the notation and we write  $v_{2j-1} = v_0$  and  $v_{2j} = v'_0$ , for  $1 \leq j \leq l$ . Then  $v_1, v_2, \dots, v_{2l}$  is an orthonormal basis of  $V_{\bar{0}}$  and

$$J_j v_{2j-1} = v_{2j}, \quad J_j v_{2j} = -v_{2j-1} \quad (1 \leq j \leq l).$$

In terms of the dual basis (46) of  $\mathfrak{h}_{\mathbb{C}}^*$ , the positive roots are

$$e_j \pm e_k \quad (1 \leq j < k \leq l).$$

Define an element  $s \in G$  by

$$sv_1 = v_1, \quad sv_2 = v_2, \quad \dots, \quad sv_{2l-1} = v_{2l-1}, \quad sv_{2l} = -v_{2l}. \quad (65)$$

Then  $G^0 = -G^0$  and  $G = G^0 \cup G^0 s$  is the disjoint union of two connected components. Set

$$V_{\bar{0},s} = V_{\bar{0}}^1 \oplus V_{\bar{0}}^2 \oplus \dots \oplus V_{\bar{0}}^{l-1} \oplus \mathbb{R}v_{2l}, \quad V_s = V_{\bar{0},s} \oplus V_{\bar{1}}.$$

The dual pair corresponding to  $(V_{\bar{0},s}, V_{\bar{1}})$  is  $(G_s, G'_s) = (O_{2l-1}, Sp_{2l'}(\mathbb{R}))$  acting on the symplectic space  $W_s = \text{Hom}(V_{\bar{1}}, V_{\bar{0},s})$ . Let  $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R}J_j$  be the centralizer of  $s$  in  $\mathfrak{h}$ . This is a Cartan subalgebra of  $\mathfrak{g}_s$ , the Lie algebra of  $G_s$ .



Denote by

$$\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_{l-1} e_{l-1} \quad (66)$$

the highest weight of  $\Pi$ . (Here  $\lambda_l = 0$  because the character  $\Theta_\Pi$  is not supported on  $\widetilde{G^0}$ .) See [GW09, p. 277] and [Wen01, Theorem 2.6].

**Theorem 4.** *Let  $(G, G') = (O_{2l}, Sp_{2l'}(\mathbb{R}))$ . Assume that the character  $\Theta_\Pi$  is not supported on  $\widetilde{G^0}$ . Suppose that  $l \leq l'$  and the pair  $(O_2, Sp_2(\mathbb{R}))$  is excluded. Then for all  $\phi \in \mathcal{S}(W)$*

$$\int_{G^0_s} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi) dg = C(\Pi) \int_{-G^0_s} \check{\Theta}_{\Pi_{\lambda_s} \otimes \sigma}(\tilde{g}) T_s(\tilde{g})(\phi^G|_{W_s}) dg, \quad (67)$$

where  $C(\Pi)$  is a constant equal to  $\pm 1$ , see (197),  $\sigma$  is the spin representation of  $Spin_{2l-1}$ ,  $\Pi_{\lambda_s}$  is the irreducible representation of  $Spin_{2l-1}$  of highest weight  $\lambda_s = \lambda + \frac{1}{2} \sum_{j=1}^{l-1} e_j$ , and  $T_s$  is the map  $T$ , see (14), corresponding to the symplectic space  $W_s$ .

The integral on the right-hand side of (67) was computed in Theorem 2. The excluded case  $(O_2, Sp_2(\mathbb{R}))$  is treated in the following remark.

**Remark 5.** We consider here the case  $(G, G') = (O_2, Sp_2(\mathbb{R}))$ . By (D.39) and Proposition D.8, we can identify

$$\widetilde{O}_2 = \{(g; \zeta) \in O_2 \times \mathbb{C}^\times; \zeta^2 = \det g\}$$

and the  $\det^{1/2}$ -covering  $\pi_1 : \widetilde{O}_2 \ni (g; \zeta) \rightarrow g \in O_2$  does not split. Let  $\Pi \in \widetilde{O}_2$  occur in Howe's correspondence and let  $\nu : \widetilde{O}_2 \rightarrow \mathbb{C}^\times$  be the character of  $\widetilde{O}_2$  defined by  $\nu(g; \zeta) = \zeta$ . Then  $\text{supp}(\Theta_\Pi) = \text{supp}(\nu\Theta_\Pi) = \text{supp}(\Theta_{\Pi \otimes \nu})$ . By Remark 13,  $(\omega \otimes \nu)(g; \zeta) = \zeta \omega(g; \zeta)$  and  $(\Pi \otimes \nu)(g; \zeta) = \zeta \Pi(g; \zeta)$  are independent of  $\zeta$ . Hence there are a unitary representation  $\omega_0$  of  $O_2$  and  $\Xi \in \widetilde{O}_2$  such that  $\Xi(g) = (\Pi \otimes \nu)(g; \zeta)$  for all  $\zeta$ . Accordingly,

$$\int_{O_2} \check{\Theta}_\Pi(\tilde{g}) \omega(\tilde{g}) dg = \int_{O_2} \check{\Theta}_\Xi(g) \omega_0(g) dg.$$

Explicitly,  $\nu(g; \zeta) = \det(g)^{1/2}$  and, in the Schrödinger model,

$$\omega_0(g)v(x) = v(g^{-1}x) \quad (g \in O_2, v \in \mathcal{S}(\mathbb{R}^2), x \in \mathbb{R}^2).$$

Observe that  $\pi_1(\text{supp}(\Theta_\Pi)) = \text{supp}(\Theta_\Xi)$ . Since  $\pi_1 : \widetilde{SO}_2 \rightarrow SO_2$  splits by (D.43), we conclude that  $\Theta_\Pi$  is supported on  $\widetilde{G^0} = \widetilde{SO}_2$  if and only if  $\Theta_\Xi$  is supported on  $SO_2$ .

For  $n \in \mathbb{Z}$ , let  $\rho_n$  be the character of  $SO_2$  defined by  $\rho_n(e^{i\theta}) = e^{in\theta}$ . Up to equivalence, the irreducible representations of  $O_2$  are of the form  $\Xi_n = \text{Ind}_{SO_2}^{O_2}(\rho_n)$  with  $n > 0$ , together with the trivial representation 1 and  $\det$ . (Moreover,  $\Xi_n \simeq \Xi_{-n}$  and  $\Xi_0 = 1 \oplus \det$ .) Hence  $\Theta_\Xi$  does not have support contained in  $\widetilde{SO}_2$  if and only if  $\Xi|_{SO_2} = 1$ . Also,  $\Xi = \det$  is excluded because  $\det$  does not occur in  $\omega_0|_{O_2}$  by [Prz96, page 99]. So,

$$\int_{O_2} \Theta_{\det}(g) \omega_0(g) dg = 0.$$

Suppose then that  $\Xi = 1$ , i.e.  $\Pi = \nu^{-1}$ . Since  $(\Theta_1 + \Theta_{\det})|_{\mathrm{O}_2 \setminus \mathrm{SO}_2} = 0$  and  $\Theta_1|_{\mathrm{SO}_2} = \Theta_{\det}|_{\mathrm{SO}_2} = 1$ , we conclude that

$$\begin{aligned} \int_{\mathrm{O}_2} \Theta_1(g) \omega_0(g) dg &= \int_{\mathrm{O}_2} (\Theta_1 + \Theta_{\det})(g) \omega_0(g) dg = \int_{\mathrm{SO}_2} (\Theta_1 + \Theta_{\det})(g) \omega_0(g) dg \\ &= 2 \int_{\mathrm{SO}_2} \Theta_1(g) \omega_0(g) dg = 2 \int_{\mathrm{SO}_2} \nu(\tilde{g}) \omega(\tilde{g}) dg. \end{aligned}$$

Thus

$$f_{\nu^{-1} \otimes (\nu^{-1})'} = 2 \int_{\mathrm{SO}_2} \nu(\tilde{g}) T(\tilde{g}) dg = 2 \int_{\mathrm{SO}_2} T(\tilde{g}) dg,$$

which is computed by Theorem 2; see subsection 5.1. (Recall that  $(\nu^{-1})'$  denotes the representation of  $\widetilde{\mathrm{Sp}}_2(\mathbb{R})$  corresponding to the representation  $\nu^{-1}$  of  $\widetilde{\mathrm{O}}_2$ .)

**Theorem 5.** *Suppose now that  $l > l'$ . Let  $\Pi$  be a representation of  $\widetilde{\mathrm{O}}_{2l}$  occurring in Howe's correspondence. Then*

$$\int_{\mathrm{G}} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg = 2 \int_{\mathrm{G}^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg. \quad (68)$$

*Proof.* Let  $\lambda$  be the highest weight of  $\Pi$ . By [How92, Proposition 3.6.3], the depth  $k$  of  $\lambda$  must be  $\leq l'$ . Hence case (ii) in [How92, (3.6.2.8)], i.e.  $k > l$ , cannot occur. This means that the second representation of  $\widetilde{\mathrm{O}}_{2l}$  which has the same restriction as  $\Pi$  to  $\mathrm{G}^0 = \mathrm{SO}_{2l}$  does not occur. Therefore the  $\Pi|_{\widetilde{\mathrm{SO}}_{2l}}$ -isotypic component of  $\omega$  coincides with the  $\Pi$ -isotypic component of  $\omega$ .  $\square$

Since  $\mathrm{G}^0 = -\mathrm{G}^0$ , the integral on the right-hand side of (68) was computed in Theorem 3.

Now we consider the case  $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$  with  $1 \leq l \leq l'$ . Recall from (34) the graded vector space  $\mathbf{V}$ . In the case we consider,  $\dim \mathbf{V}_0^0 = 1$ ,  $\dim \mathbf{V}_1^0 = 2(l' - l)$  and for  $1 \leq j \leq l$ ,  $\dim \mathbf{V}_0^j = \dim \mathbf{V}_1^j = 2$ . Let

$$\mathbf{W}_s = \mathrm{Hom}(\mathbf{V}_1, \mathbf{V}_0^1 \oplus \cdots \oplus \mathbf{V}_0^l) \quad \text{and} \quad \mathbf{W}_s^\perp = \mathrm{Hom}(\mathbf{V}_1, \mathbf{V}_0^0).$$

Then

$$\mathbf{W} = \mathbf{W}_s \oplus \mathbf{W}_s^\perp \quad (69)$$

is a direct sum orthogonal decomposition. Let  $\mathrm{G}_s \subseteq \mathrm{G}$  be the subgroup acting trivially on the space  $\mathbf{V}_0^0$  and let  $\mathrm{G}'_s = \mathrm{G}'$ . Then the dual pair corresponding to  $\mathbf{W}_s$  is  $(\mathrm{G}_s, \mathrm{G}'_s) = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$  and dual pair corresponding to  $\mathbf{W}_s^\perp$  is  $(\mathrm{O}_1, \mathrm{Sp}_{2l'}(\mathbb{R}))$ . If  $\mathrm{H}$  is a Cartan subgroup of  $\mathrm{G}$ , then  $\mathrm{H}^0 = \mathrm{H}_s$  is a Cartan subgroup of  $\mathrm{G}_s$ , and the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}_s$  share the same Cartan subalgebra  $\mathfrak{h} = \mathfrak{h}_s$ .

Define

$$p_j^\flat(y_j) = p_j^{a_j, b_j}(y_j) - p_j^{a_j-1, b_j}(y_j) \quad \text{and} \quad q_j^\flat(y_j) = q_j^{a_j, b_j}(y_j) - q_j^{a_j-1, b_j}(y_j), \quad (70)$$

where  $p_j^{a_j, b_j}$  and  $q_j^{a_j, b_j}$  respectively denote the functions  $p_j$  and  $q_j$  from (56) corresponding to the integers  $a_j$  and  $b_j$ .

**Theorem 6.** *Let  $(G, G') = (O_{2l+1}, Sp_{2l'}(\mathbb{R}))$  with  $1 \leq l \leq l'$ . There is a non-zero constant  $C$  depending only on the dual pair  $(G, G')$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1-g)T_s(\tilde{g})(\phi^G|_{W_s}) dg \quad (71)$$

$$= C \int_{\mathfrak{h}} \prod_{j=1}^l (p_j^b(y_j) + q_j^b(-\partial_{y_j})\delta_0(y_j)) F_{\phi^G|_{W_s}}(y) dy, \quad (72)$$

where  $T_s$  is the operator  $T$ , see (14), corresponding to the symplectic space  $W_s$ .

The term

$$\prod_{j=1}^l (p_j^b(y_j) + q_j^b(-\partial_{y_j})\delta_0(y_j)) \quad (73)$$

is a polynomial if and only if  $l = l'$ . In this case, all the terms  $q_j^b$  vanish. It is never a linear combinations of products of polynomials and Dirac delta's at zero in some coordinates  $y_j$ .

**Remark 6.** Let  $\mu = \sum_{j=1}^l \mu_j e_j$  be the Harish-Chandra parameter of  $\Pi$ . Since  $l \leq l'$ ,  $a_{j-1} < a_j = -\mu_j - \delta + 1 = -\lambda_j - \rho_j + l - l' + \frac{1}{2} \leq 0$  because  $\rho_j \geq \frac{1}{2}$ ; see (141), (142) and Appendix E. So  $P_{a_j, b_j, -2} = P_{a_{j-1}, b_j, -2} = 0$ , see (C.5), i.e.  $p_j^b(y_j)$  vanishes for  $y_j < 0$  for all  $1 \leq j \leq l$ . Thus, (73) is actually supported in  $\tau(\mathfrak{h}_\Gamma)$ .

**Remark 7.** The pair  $(O_1, Sp_{2l'}(\mathbb{R}))$  was studied in detail in section 2.

Now we consider the case  $(G, G') = (O_{2l+1}, Sp_{2l'}(\mathbb{R}))$  with  $l > l'$ . Recall the graded vector space  $V$ , (34) and (43),

$$V = V^0 \oplus V^1 \oplus \cdots \oplus V^{l'},$$

where, as in (44),

$$\begin{aligned} V_0^0 &= V_0^{0,0} \oplus (V_0^{0,0})^\perp, \\ V_1^0 &= 0, \end{aligned}$$

with  $\dim V_0^{0,0} = 1$  and  $\dim (V_0^{0,0})^\perp = 2(l - l')$ . Let

$$W_s = \text{Hom}(V_1^1 \oplus \cdots \oplus V_{l'}^{l'}, (V_0^{0,0})^\perp \oplus V_0^1 \oplus \cdots \oplus V_0^{l'}), \quad W_s^\perp = \text{Hom}(V_1^1, V_0^{0,0}).$$

Then

$$W = W_s \oplus W_s^\perp \quad (74)$$

is a direct sum orthogonal decomposition. Let  $G_s \subseteq G$  be the subgroup acting trivially on the space  $V_0^{0,0}$  and let  $G'_s = G'$ . The dual pair corresponding to  $W_s$  is  $(G_s, G'_s) = (O_{2l}, Sp_{2l'}(\mathbb{R}))$  and dual pair corresponding to  $W_s^\perp$  is  $(O_1, Sp_{2l'}(\mathbb{R}))$ .

**Theorem 7.** *Let  $(G, G') = (O_{2l+1}, Sp_{2l'}(\mathbb{R}))$  with  $l > l'$ . Then*

$$\int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg \neq 0 \quad (75)$$

if and only if the highest weight  $\lambda = \sum_{j=1}^l \lambda_j e_j$  of  $\Pi$  satisfies condition (a) of Theorem 3 for  $\mathbb{D} = \mathbb{R}$ . Suppose that this condition is satisfied. Then there is a non-zero constant  $C$  which depends only of the dual pair  $(G, G')$  such that for all  $\phi \in \mathcal{S}(W)$

$$\begin{aligned} \int_{G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g})(\phi) dg &= \int_{G_s^0} \check{\Theta}_{\Pi}(\tilde{g}) \det(1 - g)T_s(\tilde{g})(\phi^G|_{W_s}) dg \\ &= C \int_{\tau'(\mathfrak{h}_1^{reg})} \prod_{j=1}^{l'} p_j^b(y_j) \cdot F_{\phi^G|_{W_s}}(y) dy, \end{aligned} \quad (76)$$

where  $p_j^b$  is as in (70).

The proofs of the theorems stated above are included in the following sections.

**Remark 8.** The term  $\det(1 - g)$  admits a representation theoretical interpretation. Indeed, let  $\sigma$  be the spin representation of the spin cover of  $G_s^0$ . Then the tensor product  $\sigma \otimes \sigma^c$  is a representation of  $G_s^0$  and, by [Lit06, Ch. XI, III., p. 254]

$$\Theta_{\sigma \otimes \sigma^c}(g) = |\Theta_{\sigma}(g)|^2 = \det(1 + g) \quad (g \in G_s^0). \quad (77)$$

So  $\det(1 - g) = \Theta_{\sigma \otimes \sigma^c}(-g)$ .

**Remark 9.** Suppose that  $G = O_d$ . Then, for each highest weight  $\lambda$  of a unitary irreducible representation of  $G^0$  there are one or two irreducible unitary genuine representations of  $\tilde{G}$  having highest weight  $\lambda$ . There are two if and only if  $\lambda_l = 0$ . Suppose then that  $\lambda_l = 0$  and let  $\Pi_+$  and  $\Pi_-$  be these representations. Set  $\chi_+(\tilde{g}) = \frac{\Theta(\tilde{g})}{|\Theta(\tilde{g})|}$ , where  $\Theta$  is defined in (16). Then  $\chi_+$  is a character of  $\tilde{G}$  with the property that  $\Pi_{\pm} \otimes \chi_+^{-1}$  descends to a representation of  $G$ . If  $(\Pi_+ \otimes \chi_+^{-1})|_G = \Pi_0$ , then  $(\Pi_- \otimes \chi_+^{-1})|_G = \Pi_0 \otimes \det$ .

The values of  $\lambda$  for which only one of  $\Pi_+$  and  $\Pi_-$  occurs in Howe's correspondence can be read off from the list classifying the pairs of representations in duality; see e.g. [Prz96, Appendix]. For example, for each possible  $\lambda$ , exactly one of the representations  $\Pi_+$  and  $\Pi_-$  of  $\tilde{G}$  with highest weight  $\lambda$  occurs when  $l > l'$  or when  $(G, G') = (O_{2l+1}, \text{Sp}_{2l}(\mathbb{R}))$  (these are the cases when the condition  $c - n \leq l$  in [Prz96, (A.4.2.2)] fails.)

The intertwining distributions associated with  $\Pi_+$  and  $\Pi_-$  distinguish these representations because

$$\int_{G^0} \check{\Theta}_+(\tilde{g})T(\tilde{g}) dg = \int_{G^0} \check{\Theta}_-(\tilde{g})T(\tilde{g}) dg \quad (78)$$

$$\int_{G \setminus G^0} \check{\Theta}_+(\tilde{g})T(\tilde{g}) dg = - \int_{G \setminus G^0} \check{\Theta}_-(\tilde{g})T(\tilde{g}) dg. \quad (79)$$

The equalities (78) and (79) can be explained as follows. Notice first that, by (47), for  $\tilde{z} \in \tilde{Z}$ ,

$$\frac{\check{\chi}_{\Pi_-}(\tilde{z})}{\check{\chi}_{\Pi_+}(\tilde{z})} = \frac{\check{\Theta}_{\Pi_-}(\tilde{z})}{\check{\Theta}_{\Pi_+}(\tilde{z})} = \frac{\check{\Theta}_{\Pi_- \otimes \chi_+^{-1}}(\tilde{z})}{\check{\Theta}_{\Pi_+ \otimes \chi_+^{-1}}(\tilde{z})} = \det(z). \quad (80)$$

Suppose first  $l \leq l'$ . If  $G = O_{2l}$ , then  $-G^0 = G^0$  and (78) is a consequence of (57) in Theorem 2 with  $z = c(0) = -1$  in (80) because  $\det(-1) = 1$  in  $O_{2l}$ . Since  $\det(s) = -1$ , the equality (79) follows from (67) in Theorem 4, the explicit value of the constants  $C(\Pi_{\pm})$

in (197) and  $z = s$  in (80). If  $G = O_{2l+1}$ , then (78) can be seen from (72) in Theorem 71, whereas (79) follows from (57) in Theorem 2 and  $z = -1$  in (80), since  $\det(-1) = -1$  in  $O_{2l+1}$ .

Let now  $l > l'$ . If  $G = O_{2l+1}$ , then (78) follows from Theorem 7, whereas (79) is a consequence of the right-hand side of Theorem 3 because  $\det(-1) = -1$  in this case. If  $G = O_{2l}$ , then (62) in Theorem 3 implies (78) by setting  $z = c(0) = -1$  in (80) because  $\det(-1) = 1$  in  $O_{2l}$ .

We did not compute explicitly the integrals appearing in (79) but used the fact that the projection onto the  $\Pi$ -isotypic component is equal to the projection onto the  $\Pi|_{\widetilde{G}^0}$ -isotypic component, which implies that the integral over  $G^0$  agrees with that over  $G \setminus G^0$ . The equality of isotypic components is a consequence of the property, mentioned above, that only one of  $\Pi_+$  and  $\Pi_-$  occurs in the correspondence. We could have used the same property and avoid the computation of the integral over  $G \setminus G^0$  also when  $G = O_{2l+1}$ . Nevertheless, for  $G = O_{2l+1}$ , our computation allows to determine (without using the known classification of highest weights) that a representation with weight not satisfying condition (a) of Theorem 3 does not occur in Howe's correspondence.

It would be interesting to avoid classification completely and obtain the equalities between the integrals over  $G^0$  and  $G \setminus G^0$  when  $l > l'$  directly from our formulas. This would have also allowed us to use the intertwining distributions to check that only one of  $\Pi_+$  and  $\Pi_-$  occurs. The main difficulty in comparing these integrals is that in one the orbital integral  $F_\phi$  on  $W$  appears, in the other one the corresponding factor is  $F_{\phi|_{W_s}}$ . At the moment, we are not able to relate them.

The difficulties in comparing the contributions to the intertwining distributions coming from the two connected components of  $G$  obviously do not play any role when  $\mathbb{D} = \mathbb{C}$  or  $\mathbb{H}$ . In these cases, as a byproduct of our calculations of the intertwining distributions, we obtain the list of highest weights of the genuine irreducible representations  $\Pi$  of  $\widetilde{G}$  that occur in Howe's correspondence when  $l > l'$ . This list was first determined (without any restrictions on  $\mathbb{D}$  or the ranks  $l$  and  $l'$ ) in [KV78]. As it will be seen in the proof of Theorems 3 and 7, our list is deduced from an observation about the support of the distributions appearing in the integral formulas for the intertwining distributions. Unfortunately, this method is not refined enough to apply when  $\mathbb{D} = \mathbb{R}$  or  $l \leq l'$ .

**Corollary 8.** *Suppose that  $l > l'$  and  $\mathbb{D} = \mathbb{C}$  or  $\mathbb{H}$ . A genuine representation  $\Pi \in \widetilde{G}^\wedge$  occurs in Howe's correspondence if and only if its highest weight satisfies conditions (a) or (b) of Theorem 3.*

*Proof.* Our computations of the intertwining distribution  $\int_G \widetilde{\Theta}_\Pi(\widetilde{g})T(\widetilde{g}) dg$  can be applied to any genuine irreducible representation  $\Pi \in \widetilde{G}^\wedge$ . This distribution is nonzero if and only if  $\omega|_{\widetilde{G}}$  has a nonzero  $\Pi$ -isotypic component. This is equivalent to the fact that there is a unitary highest weight representation  $\Pi'$  of  $\widetilde{G}'$  such that  $\Pi \otimes \Pi'$  occurs in  $\omega|_{\widetilde{G}\widetilde{G}'}$ . The nonvanishing of the intertwining distributions leads to conditions (a) or (b) of Theorem 3 when  $G = U_l$  or  $Sp_l$ .  $\square$

We conclude this section with a corollary on the non-differential operator nature of the symmetry breaking operators in  $\text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ .

**Corollary 9.** *Let  $(G, G')$  be a real reductive dual pair with one member compact. Then the essentially unique non-zero symmetry breaking operator in*

$$\mathrm{Hom}_{\widetilde{GG'}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$$

*is not a differential operator.*

*Proof.* We are going to show that  $(\mathrm{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$  is not a differential operator.

Let  $f \in \mathcal{S}'(W)$  and recall the definition of  $\mathcal{K}(f)$  in (12). According to [Hör83, Theorems 5.2.1 (the Schwartz kernel theorem) and 5.2.3], the continuous linear map  $\mathrm{Op} \circ \mathcal{K}(f)$  is a distribution-valued differential operator if and only if  $\mathcal{K}(f) \in \mathcal{S}'(X \times X)$  is supported by the diagonal  $\Delta = \{(x, x); x \in X\}$ . This implies that  $f$  is supported in  $Y$ . Indeed, given  $\varphi \in \mathcal{S}(X \times X)$ , let  $\psi \in \mathcal{S}(X \times X)$  be defined by  $\varphi(x, x') = \psi(x - x', x + x')$  for all  $x, x' \in X$ . Furthermore, let  $\psi(\cdot, \widehat{\cdot}) \in \mathcal{S}(X \times Y)$  denote the partial Fourier transform of  $\psi$  with respect to its second variable, defined by

$$\psi(a, \widehat{y}) = \int_X \chi\left(\frac{1}{2}\langle y, b \rangle\right) \psi(a, b) db \quad ((a, y) \in X \times Y).$$

Then

$$\mathrm{supp} \varphi \cap \Delta = \emptyset \quad \text{if and only if} \quad \mathrm{supp} \psi(\cdot, \widehat{\cdot}) \cap (\{0\} \times Y) = \emptyset.$$

Since  $\mathcal{K}(f)(\varphi) = f(\psi(\cdot, \widehat{\cdot}))$  by (12), we obtain the claim.

Notice that this cannot happen in our case. Indeed, the support of  $f_{\Pi \otimes \Pi'}$  is  $GG'$ -invariant. Since the pair is of type I,  $GG'$  acts irreducibly on  $W$ . Therefore the inclusion  $\mathrm{supp} f_{\Pi \otimes \Pi'} \subseteq Y$  would imply  $\mathrm{supp} f_{\Pi \otimes \Pi'} = \{0\}$ . This would mean that the wavefront set of  $\Pi'$  is 0, i.e.  $\Pi'$  is finite dimensional. By classification, see e.g. [Prz96, Appendix] all highest weight representations occurring in Howe's correspondence are infinite dimensional unless  $G' = U_{\nu'}$ , which is compact. In this case, the intertwining distributions is a smooth function; see [MPP23]. In particular, its support is not 0. Hence the intertwining operator is not a differential operator.  $\square$

## 5. A few examples

5.1.  $(G, G') = (O_2, \mathrm{Sp}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}))$ . In this case,  $H = \mathrm{SO}_2$  and  $\mathfrak{g} = \mathfrak{h} = \mathbb{R}J_1$ , where  $J_1 = \mathbb{R} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Moreover,  $\tau(\mathfrak{h}_{\overline{1}}) = \mathbb{R}^+ J_1$  and  $\mathfrak{h} \cap \tau(W) = \mathfrak{h}$ . The Harish-Chandra parameter of  $\Pi \in \widetilde{O}_2^\wedge$  (which coincides with its highest weight since  $\rho = 0$ ) is of the form  $\mu e_1$ , where  $\mu \geq 0$  is an integer. Hence, in the notation of Theorem 2,  $a = -b = -\mu$  and  $\beta = 2\pi$ .

If  $\mu = 0$ , then  $P_{-\mu, \mu} = 0$ . If  $\mu > 0$ , then the function  $P_{-\mu, \mu}$  is supported in  $[0, +\infty)$  and, by (C.8) and Remark 11,

$$P_{-\mu, \mu, 2}(2\pi y_1) = 2(-1)^{\mu-1} L_{\mu-1}^1(4\pi y_1) = 2(-1)^{\mu-1} \sum_{h=0}^{\mu-1} \binom{\mu}{\mu-1-h} \frac{(-4\pi y_1)^h}{h!}, \quad (81)$$

where  $L_{\mu-1}^1$  is a Laguerre polynomial. Moreover, by (C.9),  $Q_{-\mu, \mu}(y) = 2\pi(-1)^\mu$  for all  $\mu \geq 0$ .

Suppose first  $\mu > 0$ . Then, by Lemma 1, Theorem 2 and Remark 5, for every  $\phi \in \mathcal{S}(W)$ ,

$$\begin{aligned} f_{\Pi \otimes \Pi'}(\phi) &= \int_{\text{SO}_2} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g}) dg \\ &= 2\pi C(-1)^\mu \int_0^{+\infty} P_{-\mu, \mu, 2}(2\pi y_1) e^{-2\pi y_1} F_\phi(y_1 J_1) dy_1 + C \int_{\mathfrak{h}} \delta_0(y) F_\phi(y) dy, \end{aligned} \quad (82)$$

where  $C$  is the constant appearing in Theorem 2. To make formula (82) explicit, we need to calculate terms involving  $F(y)$ , the Harish-Chandra regular almost-elliptic orbital integral on  $W$ .

Recall that  $S^{\mathfrak{h}_{\bar{1}}} = H \times H'$  when  $l = l'$ . Hence  $S/S^{\mathfrak{h}_{\bar{1}}} = \{\pm 1\} \times G'/H'$ . By [MPP20, Definition 3.1, (39) and (27)] for all  $y = y_1 J_1 = \tau(w) \in \tau(\mathfrak{h}_{\bar{1}})$

$$F_\phi(y) = C_{\mathfrak{h}_{\bar{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y') \int_{S/S^{\mathfrak{h}_{\bar{1}}}} \phi(s.w) d(sS_{\bar{1}}^{\mathfrak{h}}) = C'_{\mathfrak{h}_{\bar{1}}} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y') \int_{G'/H'} \psi(g'.y') d(g'H') \quad (83)$$

where  $C_{\mathfrak{h}_{\bar{1}}}$  and  $C'_{\mathfrak{h}_{\bar{1}}}$  are suitable constants,  $y' = y_1 J'_1 = y_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tau'(w)$ , and  $\psi = \tau'_*(\phi) \in \mathcal{S}(\mathfrak{g}')$  is uniquely determined by the condition that  $\psi \circ \tau' = \phi^G$ . The right-hand side of (83) is Harish-Chandra's orbital integral for the orbit  $G'.y'$ .

Notice that, for  $G = O_2$  and  $l = 1 \leq l'$ , the extension of  $F(y)$  from  $y \in \mathfrak{h}^+ = \tau(\mathfrak{h}_{\bar{1}})$  to  $-\tau(\mathfrak{h}_{\bar{1}})$  is even in  $y$ ; see [MPP20, Theorem 3.6]. Hence,

$$\int_{\mathfrak{h}} \delta_0(y) F_\phi(y) dy = \lim_{y_1 \rightarrow 0^+} F_\phi(y_1 J_1) \quad (\phi \in \mathcal{S}(W)).$$

Write  $x \in \mathfrak{g}'$  as

$$x = x_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x_3 J'_1 = \begin{pmatrix} x_1 & x_2 + x_3 \\ x_2 - x_3 & -x_1 \end{pmatrix} = A(x_1, x_2, x_3),$$

where  $(x_1, x_2, x_3) \in \mathbb{R}^3$ . Then the map  $A : \mathbb{R}^3 \rightarrow \mathfrak{g}'$  is a linear isomorphism. It transfers the adjoint action of  $G'$  on  $\mathfrak{g}'$  to the natural action on  $\mathbb{R}^3$  by  $\text{SO}(2, 1)^0$ , the identity component of  $\text{SO}(2, 1)$ , i.e. the group of isometries of  $x_1^2 + x_2^2 - x_3^2 = -\det(A(x_1, x_2, x_3))$  preserving the positive light cone

$$X^{0+} = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 = x_3^2, x_3 > 0\}.$$

See [HT92, Chapter IV, §5.1]. Under the map  $A$ , the orbit  $G'.y'$  with  $y' = y_1 J'_1$  and  $y_1 > 0$  is the image of the hyperboloid's upper sheet

$$O_{y_1}^- = \{(x_1, x_2, x_3) \in \mathbb{R}^3; x_1^2 + x_2^2 - x_3^2 = -y_1^2, x_3 > 0\}.$$

Under  $A$ , the positive light cone  $X^{0+}$  corresponds to the  $G'$ -orbit of  $x_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Moreover  $G'.x_0 \simeq G'/MN$ , where  $M = \{\pm 1\}$  and  $N = \exp(\mathbb{R}x_0) = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; t \in \mathbb{R} \right\}$ . As the geometry suggests, for suitable normalizations of the  $\text{SO}(2, 1)^0$ -invariant orbital measures,

$$\lim_{y_1 \rightarrow 0^+} \int_{O_{y_1}^-} f d\mu_{O_{y_1}^-} = \int_{X^{0+}} f d\mu_{X^{0+}} \quad (f \in \mathcal{S}(\mathbb{R}^3)).$$

Thus, for a suitable positive constant  $C''_{\mathfrak{h}\Gamma}$

$$\int_{\mathfrak{h}} \delta_0(y) F_\phi(y) dy = C''_{\mathfrak{h}\Gamma} \int_{G'/MN} \psi(g'.x_0) d(g'MN) \quad (\phi \in \mathcal{S}(W), \psi \in \mathcal{S}(\mathfrak{g}')^G, \psi \circ \tau' = \phi^G). \quad (84)$$

Suppose now that  $\mu = 0$ . By Remark 5,  $\Pi = \nu^{-1}$  where  $\nu((g, \xi)) = \det(g)^{1/2}$  for all  $\tilde{g} = (g, \xi) \in \tilde{O}_2$ . Moreover, for all  $\phi \in \mathcal{S}(W)$ ,

$$f_{\nu^{-1} \otimes (\nu^{-1})'}(\phi) = 2 \int_{\text{SO}_2} \nu(\tilde{g}) T(\tilde{g})(\phi) dg = 2 \int_{\text{SO}_2} \nu(\tilde{g}) T(\tilde{g})(\phi) dg = 2C \int_{\mathfrak{h}} \delta_0(y) F_\phi(y) dy,$$

where  $C$  is the constant appearing in Theorem 2 and the last integral is computed in (84).

5.2.  $(G, G') = (U_l, U_{p,p})$  **and**  $\Pi = \widetilde{\text{triv}}$ . Let  $(G, G') = (U_l, U_{p,p})$ . Hence  $l' = 2p$ . Consider the trivial representation  $\text{triv}$  of  $U_l$ . In the Schrödinger model, with a polarization  $W = X \oplus Y$  preserved by  $G$ , we have

$$\omega(\tilde{g})v(x) = \chi_+(\tilde{g})v(g^{-1}x) \quad (\tilde{g} \in \tilde{G}, v \in \mathcal{S}(X), x \in X) \quad (85)$$

where  $\chi_+ : \widetilde{\text{Sp}}(W) \rightarrow \widetilde{U}_1$  is a function whose restriction to  $\tilde{G}$  is a character. See [AP14, Proposition 4.28]. Let  $\widetilde{\text{triv}}$  denote this restriction. Then  $\widetilde{\text{triv}}$  is the lift to  $\tilde{U}_l$  of  $\text{triv}$  that occurs in Howe's correspondence. Moreover, (85) implies that

$$\omega(\check{\Theta}_{\widetilde{\text{triv}}})v(x) = \int_G v(g^{-1}x) dg \quad (v \in \mathcal{S}(X), x \in X).$$

Let  $\widetilde{\text{triv}}'$  be the representation of  $\tilde{U}_{p,p}$  which corresponds to  $\widetilde{\text{triv}}$ . If  $l = 1$ , then  $\widetilde{\text{triv}}'$  is a minimal representation of  $U_{p,p}$ , called the Wallach representation.

In this section we are computing  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}'}$ , which is the Weyl symbol of the operator  $\omega(\check{\Theta}_{\widetilde{\text{triv}}})$ . As in our main theorems, we distinguish the cases  $l \leq l'$  and  $l > l'$ . Notice first that the parameters appearing in (54) are

$$\beta = 2\pi \quad \text{and} \quad \delta = p + \frac{1-l}{2} = \frac{1+l'-l}{2}.$$

Moreover,  $\rho = \sum_{j=1}^l \binom{l+1}{2} e_j$  for  $G = U_l$ .

5.2.1. *The case  $l \leq l'$ .* The parameters (55) corresponding to  $\Pi = \widetilde{\text{triv}}$  are

$$a_j = -\frac{l'}{2} + j \quad \text{and} \quad b_j = -\frac{l'}{2} + l + 1 - j, \quad (86)$$

where  $1 \leq j \leq l$ . Observe that the  $a_j$ 's and the  $b_j$ 's describe the same set

$$\{-l'/2 + 1, \dots, -l'/2 + l - 1, -l'/2 + l\}$$

and  $b_{l+1-j} = a_j$  for all  $1 \leq j \leq l$ . Hence, by (C.10),

$$P_{a_{l+1-j}, b_{l+1-j}}(\xi) = P_{b_j, a_j}(\xi) = P_{a_j, b_j}(-\xi), \quad (87)$$

$$Q_{a_{l+1-j}, b_{l+1-j}}(\xi) = Q_{b_j, a_j}(\xi) = Q_{a_j, b_j}(-\xi). \quad (88)$$

Since  $a_j = b_{l+1-j} \leq 0$  for all  $1 \leq j \leq \min\{l, l'/2\}$ , by (C.7),

$$P_{a_j, b_j, -2}(\xi) = P_{a_{l+1-j}, b_{l+1-j}, 2}(\xi) = 0 \quad (1 \leq j \leq \min\{l, l'/2\}). \quad (89)$$



In particular,  $a_j \leq 0$  for all  $j$  (and hence  $b_j \leq 0$  for all  $j$ ) if and only if  $l \leq l'/2$ . Furthermore,  $a_j + b_j = l - l' + 1$ , which is independent of  $j$ , is  $\geq 1$  if and only if  $l = l' = 2p$ . As a consequence,

$$\begin{aligned} P_{a_j, b_j} &= 0 \text{ for all } 1 \leq j \leq l \quad \text{if and only if} \quad l \leq \frac{l'}{2}, \\ Q_{a_j, b_j} &\neq 0 \text{ for all } 1 \leq j \leq l \quad \text{if } l < l', \\ Q_{a_j, b_j} &= 0 \text{ for all } 1 \leq j \leq l \quad \text{if } l = l' = 2p, . \end{aligned}$$

We now examine more precisely the formula for  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'$  when  $l \leq l'/2$ . This is the stable range case. As remarked above,  $P_{a_j, b_j} = 0$  for all  $1 \leq j \leq l$ , whereas

$$Q_{a_j, b_j}(y_j) = 2\pi(1 + y_j)^{-a_j}(1 - y_j)^{-b_j}.$$

Hence  $p_j = 0$  for all  $1 \leq j \leq l$ , whereas

$$q_j(-\partial_{y_j})^* = q_j(\partial_{y_j}) = \left(1 + \frac{1}{2\pi}\partial_{y_j}\right)^{\frac{l'}{2}-j} \left(1 - \frac{1}{2\pi}\partial_{y_j}\right)^{\frac{l'}{2}-(l-j+1)},$$

where  $*$  denotes the formal adjoint. Theorem 2 yields for  $\phi \in \mathcal{S}(\mathbb{W})$

$$\begin{aligned} f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'(\phi) &= \int_{\mathbb{U}_l} \check{\Theta}_{\widetilde{\text{tr}}}(\widetilde{g})T(\widetilde{g})(\phi) dg \\ &= C \int_{\mathfrak{h}} \left[ \prod_{l=1}^l q_j(-\partial_{y_j}) \delta_0(y_j) \right] F_\phi(y) dy \\ &= C \left[ \left( \prod_{l=1}^l q_j(\partial_{y_j}) \right) F_\phi \right](0), \end{aligned} \tag{90}$$

where  $C$  is a nonzero constant. Hence  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'$  has support inside the nilpotent cone in  $\mathbb{W}$ .

Another case where the formula for  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'$  simplifies is when  $l = l' = 2p$  because  $Q_{a_j, b_j} = 0$  for all  $j$ . Since  $a_j = b_{2p+1-j} \leq 0$  for  $1 \leq j \leq p$ , we have

$$P_{a_j, b_j}(\xi) = \begin{cases} 2\pi P_{a_j, b_j, 2}(\xi) \mathbb{I}_{\mathbb{R}^+}(\xi) & \text{if } 1 \leq j \leq p, \\ 2\pi P_{a_j, b_j, -2}(\xi) \mathbb{I}_{\mathbb{R}^-}(\xi) & \text{if } p+1 \leq j \leq 2p. \end{cases}$$

In particular, in this case, we can replace in (57) the domain of integration  $\mathfrak{h} \cap \tau(\mathbb{W})$  with  $\tau(\mathfrak{h}_{\overline{1}})$ , where  $\mathfrak{h}_{\overline{1}}$  is the unique Cartan subspace of  $\mathbb{W}$  and  $\tau(\mathfrak{h}_{\overline{1}})$  determined by the condition that the first  $p$  values  $\delta_j$  in (36) are equal to 1 and the last  $p$  are equal to  $-1$ . The explicit expression for  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'$  can be easily computed using (57), (C.5) and (C.6). For instance, if  $p = 1$ , i.e.  $(G, G') = (\mathbb{U}_2, \mathbb{U}_{1,1})$ , then

$$f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}'(\phi) = C \int_0^\infty \int_{-\infty}^0 e^{2\pi(y_2 - y_1)} F_\phi(y_1, y_2) dy_1 dy_2 \quad (\phi \in \mathcal{S}(\mathbb{W})),$$

where  $C$  is a nonzero constant.

5.2.2. *The case  $l > l'$ .* In this case,  $Q_{a_j, b_j} = 0$  for all  $j$  and, according to Theorem 3, the distribution  $f_{\widetilde{\text{triv}} \otimes \widetilde{\text{triv}}}$  is the locally integrable  $U_l \times U_{p,p}$ -invariant function on  $W$  whose restriction to  $\mathfrak{h}_1^{\text{reg}}$  is equal to the function given by (63). The Weyl group  $W(U_{p,p}, \mathfrak{h}')$  acts on  $\mathfrak{h}'$  by permuting the first  $p$  coordinates and the last  $p$  coordinates (see Remark 3). The parameter  $a_{s,j}$  and  $b_{s,j}$  appearing in (63) are therefore obtained by separately permuting the first  $p = l'/2$  and the last  $p$  terms appearing in (86), but the index  $j$  now ranges between 1 and  $l'$ . Notice that

$$\begin{aligned} a_j \leq 0 & \quad \text{if and only if} \quad 1 \leq j \leq \frac{l'}{2}, \\ b_j \leq 0 & \quad \text{if and only if} \quad l \leq \frac{3l'}{2} - 1 \text{ and } l + 1 - \frac{l'}{2} \leq j \leq l'. \end{aligned}$$

In particular, since  $l > l'$ , for each  $j$ , at most one between  $a_j$  and  $b_j$  can be  $\leq 0$ . Moreover, there is at least one index  $j$  for which both  $a_j$  and  $b_j$  are positive, namely  $j = \frac{l'}{2} + 1$ .

When  $G = U_{1,1}$  (and hence  $l' = 2$ ), then  $W(U_{1,1}, \mathfrak{h}')$  is trivial,  $s_0 = 1$  and (63) simplifies to a nonzero constant multiple of

$$\frac{P_{a_1, b_1, 2}(2\pi y_1) P_{a_2, b_2, -2}(2\pi y_2)}{(y_2 - y_1)(y_1 y_2)^{l-2}} e^{-2\pi(y_1 - y_2)} \quad (y = \tau'(w), w \in \mathfrak{h}_1^{\text{reg}}),$$

where  $a_j, b_j$  are as in (86) and the denominator is the root product (176).

## 6. The integral over $-G^0$ as an integral over $\mathfrak{g}$ .

Let  $\mathfrak{sp}(W)$  be the Lie algebra of  $\text{Sp}(W)$ . Set

$$\mathfrak{sp}(W)^c = \{x \in \mathfrak{sp}(W); x - 1 \text{ is invertible}\}, \quad (91)$$

$$\text{Sp}(W)^c = \{g \in \text{Sp}(W); g - 1 \text{ is invertible}\}. \quad (92)$$

The Cayley transform  $c : \mathfrak{sp}(W)^c \rightarrow \text{Sp}(W)^c$  is the bijective rational map defined by  $c(x) = (x+1)(x-1)^{-1}$ . Its inverse  $c^{-1} : \text{Sp}(W)^c \rightarrow \mathfrak{sp}(W)^c$  is given by the same formula,  $c^{-1}(g) = (g+1)(g-1)^{-1}$ .

Since all eigenvalues of  $x \in \mathfrak{g} \subseteq \text{End}(W)$  are purely imaginary,  $x - 1$  is invertible. Therefore  $\mathfrak{g} \subseteq \mathfrak{sp}(W)^c$ . Moreover,  $c(\mathfrak{g}) \subseteq G$ . Since the map  $c$  is continuous, the range  $c(\mathfrak{g})$  is connected. Also,  $-1 = c(0)$  is in  $c(\mathfrak{g})$ . Furthermore, for  $x \in \mathfrak{g}$ ,

$$c(x) - 1 = (x+1)(x-1)^{-1} - (x-1)(x-1)^{-1} = 2(x-1)^{-1}$$

is invertible. Hence  $c(\mathfrak{g}) \subseteq G \cap \text{Sp}(W)^c$ . This is an equality since  $c(c(y)) = y$  and  $c(G) \subseteq \mathfrak{g}$ . Thus

$$c(\mathfrak{g}) = \{g \in G; \det(g-1) \neq 0\}.$$

This is an open dense subset of  $G$  if  $G = U_d$  or  $G = \text{Sp}_d$ . If  $G = \text{O}_{2l}$  then  $c(\mathfrak{g})$  is dense in  $\text{SO}_{2l}$ . If  $G = \text{O}_{2l+1}$  then  $c(\mathfrak{g})$  is dense in the connected component of  $-1$ , which coincides with  $-\text{SO}_{2l+1}$ . Therefore,

$$\int_{-G^0} T(\tilde{g}) \check{\Theta}_\Pi(\tilde{g}) dg = \int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_\Pi(\tilde{g}) dg, \quad (93)$$

if  $G \neq O_{2l+1}$  and

$$\begin{aligned} \int_{G^0} T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) dg &= \int_{-c(\mathfrak{g})} T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) dg, \\ \int_{-G^0} T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) dg &= \int_{c(\mathfrak{g})} T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) dg, \end{aligned} \quad (94)$$

if  $G = O_{2l+1}$ . Let

$$\tilde{c} : \mathfrak{g} \rightarrow \tilde{G} \quad (95)$$

be a real analytic lift of  $c$ . Set  $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$ . Then  $\tilde{c}_-(0)$  is the identity of the group  $\widetilde{\text{Sp}}(\mathbb{W})$ .

By (14), for  $x \in \mathfrak{g}$  we have an explicit formula,

$$t(c(x)) = \chi_x \mu_{\mathbb{W}},$$

where

$$\chi_x(w) = \chi\left(\frac{1}{4}\langle x(w), w \rangle\right) \quad (w \in \mathbb{W}). \quad (96)$$

Hence

$$T(\tilde{c}(x)) = \Theta(\tilde{c}(x)) \chi_x \mu_{\mathbb{W}}. \quad (97)$$

Therefore, for a suitable normalization of the Lebesgue measure on  $\mathfrak{g}$ ,

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_x \mu_{\mathbb{W}} dx, \quad (98)$$

where  $j_{\mathfrak{g}}(x)$  is the Jacobian of the map  $c : \mathfrak{g} \rightarrow c(\mathfrak{g})$  (see Appendix A for its explicit expression). Also, since  $\tilde{c}(0)$  is in the center of the metaplectic group,

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_x \mu_{\mathbb{W}} dx, \quad (99)$$

where  $\chi_\Pi$  is the central character of  $\Pi$ ; see (47). In the rest of this paper we shall write  $dw = d\mu_{\mathbb{W}}(w)$ , when convenient.

## 7. The invariant integral over $\mathfrak{g}$ as an integral over $\mathfrak{h}$ .

We now apply the Weyl integration formula to reduce the integral on  $\mathfrak{g}$  in (99) to an integral on a Cartan subalgebra of  $\mathfrak{g}$ . In section 3, this Cartan subalgebra was denoted by  $\mathfrak{h}(\mathfrak{g})$ , see (45). To make our notation lighter, in this section we will write  $\mathfrak{h}$  instead of  $\mathfrak{h}(\mathfrak{g})$ . Let  $H \subseteq G$  be the corresponding Cartan subgroup. Fix a system of positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . For any positive root  $\alpha$  let  $\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$  be the corresponding  $\text{ad}(\mathfrak{h}_{\mathbb{C}})$ -eigenspace and let  $X_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$  be a non-zero vector. Then there is a character (group homomorphism)  $\xi_\alpha : H \rightarrow \mathbb{C}^\times$  such that

$$\text{Ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha \quad (h \in H).$$

The derivative of  $\xi_\alpha$  at the identity coincides with  $\alpha$ . Let  $\rho \in \mathfrak{h}_{\mathbb{C}}^*$  denote one half times the sum of all the positive roots. Then in all cases except when  $G = O_{2l+1}$  or  $G = U_l$  with  $l$  even, there is a character  $\xi_\rho : H \rightarrow \mathbb{C}^\times$  whose derivative at the identity is equal to  $\rho$ , see [GW09, (2.21) and p. 145]. When  $G = O_{2l+1}$  or  $G = U_l$  with  $l$  even, the character  $\xi_\rho$  exists as a map defined on a non-trivial double cover

$$\widehat{H} \ni \widehat{h} \rightarrow h \in H. \quad (100)$$

of  $H$ . In particular the Weyl denominator

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha > 0} (1 - \xi_{-\alpha}(h)) \quad (101)$$

is defined for  $h \in H$  or  $h \in \widehat{H}$  according to the cases described above. The sign representation  $\text{sgn}_{\mathfrak{g}/\mathfrak{h}}$  of the Weyl group  $W(G, \mathfrak{h})$  is defined by

$$\Delta(sh) = \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\Delta(h) \quad (s \in W(G, \mathfrak{h})), \quad (102)$$

where either  $h \in H$  or  $h \in \widehat{H}$ . Notice that  $W(G, \mathfrak{h}) = W(\widetilde{G}, \mathfrak{h})$  because  $\widetilde{G}$  is a central extension of  $G$ .

We now determine the preimage  $\widetilde{H}$  of  $H$  in  $\widetilde{G}$  and, where needed, a double cover  $\widehat{\widetilde{H}}$  of  $\widetilde{H}$  on which all functions  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  can be defined. At the same time, we define a lift to  $\widetilde{H}$ , or to  $\widehat{\widetilde{H}}$ , of the modified Cayley transform defined on  $\mathfrak{h}$  by

$$c_-(x) = (1+x)(1-x)^{-1} = -c(x).$$

Suppose first that  $G = O_{2l+1}$ . Then  $H = H^0 \cdot Z$  is the direct product of the connected identity component  $H^0$  of  $H$  and the center  $Z$  of  $\text{Sp}(W)$ . According to Proposition D.8,  $\widetilde{G} \rightarrow G$  is isomorphic to the  $\det^{1/2}$ -covering (D.45). Hence  $\widetilde{H}^0 \rightarrow H^0$  is isomorphic to the  $\det^{1/2}$ -covering  $\sqrt{H^0} = \{(h; \zeta) \in H^0 \times \mathbb{C}^\times : \zeta^2 = 1\} = \{(h; \pm 1) : h \in H^0\}$ , whose identity component is isomorphic to  $H^0$ . This connected component will be again denoted by  $H^0$ . In this notation,  $\widetilde{H} = H^0 \cdot \widetilde{Z}$ .

We have a chain of double covering homomorphisms

$$\begin{aligned} \widehat{\widetilde{H}} \times \widetilde{Z} &\longrightarrow H^0 \times \widetilde{Z} \longrightarrow H^0 \times Z \longrightarrow H^0 \\ (\widehat{h}, \widetilde{z}) &\longrightarrow (h, \widetilde{z}) \longrightarrow (h, z) \longrightarrow h, \end{aligned} \quad (103)$$

where  $\widehat{\widetilde{H}}^0$  is the preimage of  $H^0$  in  $\widehat{\widetilde{H}}$ . We set  $\widehat{\widetilde{H}} = \widehat{\widetilde{H}}^0 \times \widetilde{Z}$ . We lift  $\Delta$  and  $\xi_\mu$  to functions on  $\widehat{\widetilde{H}}$  constant on the fibers of composition of all the maps (103) and the character  $\check{\Theta}_\Pi$  to a function on  $\widehat{\widetilde{H}}^0 \times \widetilde{Z}$  constant on the fibers of the leftmost map in (103).

The subset  $c_-(\mathfrak{h}) \subseteq H^0$  consists of all  $h \in H^0$  such that  $h+1$  is invertible. In particular, it is dense in  $H^0$ . We choose a real analytic section

$$\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\widetilde{H}}^0$$

to the covering map (100) and set

$$\widehat{c}_- : \mathfrak{h} \ni x \rightarrow (\sigma(c_-(x)), 1) \in \widehat{\widetilde{H}} = \widehat{\widetilde{H}}^0 \times \widetilde{Z}. \quad (104)$$

Suppose now that  $G = U_l$  with  $l$  even. If  $G' = U_{p,q}$  with  $p+q$  odd, then the covering  $\widetilde{H} \rightarrow H$  does not split (see the proof of Proposition D.8). Hence  $\widehat{\widetilde{H}} = \widetilde{H}$ . Therefore  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  are defined on  $\widehat{\widetilde{H}}$ . We have the Cayley transform  $c_- : \mathfrak{h} \rightarrow H$ , an analytic section  $\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\widetilde{H}}$  and the map

$$\widehat{c}_- : \mathfrak{h} \ni x \rightarrow \sigma(c_-(x)) \in \widehat{\widetilde{H}}. \quad (105)$$

If  $G' = U_{p,q}$  with  $p + q$  even, then  $\tilde{H} = H \times \{1, \tilde{1}\}$  splits by Proposition D.8, and we have maps

$$\begin{aligned} \widehat{H} &\longrightarrow H \longrightarrow \tilde{H} \longrightarrow H, \\ \widehat{h} &\rightarrow h \rightarrow (h; 1) \rightarrow h. \end{aligned} \quad (106)$$

Again  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  are defined on  $\widehat{H}$  and (105) defines the lift of the Cayley transform we shall use. In this case, we set then  $\widehat{\tilde{H}} = \widehat{H}$ .

For the remaining dual pairs,  $\widehat{H} = H$  and we lift  $\Delta$  and  $\xi_\mu$  to functions on  $\tilde{H}$  constant on the fibers of the covering map  $\tilde{H} \rightarrow H$  and write  $\widehat{c}_-$  for  $\tilde{c}_-$ , which was defined under the equation (95).

**Lemma 10.** *Let  $\mu \in i\mathfrak{h}^*$ . Then*

$$\xi_{-\mu}(\widehat{c}_-(x)) = \prod_{j=1}^l \left( \frac{1 + ix_j}{1 - ix_j} \right)^{\mu_j} = \prod_{j=1}^l (1 + ix_j)^{\mu_j} (1 - ix_j)^{-\mu_j} \quad (x \in \mathfrak{h}). \quad (107)$$

*Proof.* By (34), it is enough to verify this formula when  $l = 1$ . In this case,  $x = x_1 J_1$  and  $\mu = \mu_1 e_1 = -i\mu_1 J_1^*$ . Let  $\log$  denote the local inverse of the exponential map near 1. Then, for  $x$  sufficiently close to 0,

$$\log(c_-(x)) = \log((1+x)(1-x)^{-1}) = \log(1+x) - \log(1-x)$$

is a real analytic odd function of  $x$ . Hence it admits a Taylor series development

$$\sum_{n \geq 0} a_n x^{2n+1} = \sum_{n \geq 0} a_n (-1)^n x_1^{2n+1} J_1.$$

Thus

$$\mu(\log(c_-(x))) = - \sum_{n \geq 0} a_n (-1)^n x_1^{2n+1} i\mu_1 = - \sum_{n \geq 0} a_n (ix_1)^{2n+1} \mu_1 = \ln \left( \frac{1 - ix_j}{1 + ix_j} \right) \mu_1.$$

By taking exponentials, we obtain

$$\xi_{-\mu}(\widehat{c}_-(x)) = e^{-\mu(\log(c_-(x)))} = \left( \frac{1 + ix_j}{1 - ix_j} \right)^{\mu_1},$$

and the result extends to all  $x \in \mathfrak{h}$  by real analyticity.  $\square$

Let  $\Pi$  be an irreducible representation of  $\widetilde{G}$ , and let  $\mu \in i\mathfrak{h}^*$  represent the infinitesimal character of  $\Pi$ . When  $\mu$  is dominant, then we will refer to it as the Harish-Chandra parameter of  $\Pi$ . This is consistent with the usual terminology; see e.g. [Kna86, Theorem 9.20]. Then the corresponding character  $\xi_\mu$  is defined as  $\xi_\mu = \xi_\rho \xi_{\mu-\rho}$ , where  $\xi_{\mu-\rho}$  is one of the extremal H-weights of  $\Pi$ . In these terms, Weyl's character formula looks as follows,

$$\Theta_\Pi(h) \Delta(h) = \kappa_0 \sum_{s \in W(G, \mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{s\mu}(h), \quad (108)$$

where  $h \in \widetilde{H}^0$  or  $h \in \widetilde{\widetilde{H}}^0$ , according to the cases above, and

$$\kappa_0 = \begin{cases} 1/2 & \text{if } G = O_{2l} \text{ and } \lambda_l = \mu_l = 0 \\ 1 & \text{otherwise.} \end{cases} \quad (109)$$

**Lemma 11.** *Let  $\pi_{\mathfrak{g}/\mathfrak{h}}$  be the product of the positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  and let*

$$\kappa(x) = \kappa_0 \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \quad (x \in \mathfrak{h}),$$

*Then, for a suitable normalization of the Lebesgue measure on  $\mathfrak{h}$  and any  $\phi \in \mathcal{S}(W)$ ,*

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) dg \\ &= \frac{\check{\chi}_{\Pi}(\tilde{c}(0))}{|W(G, \mathfrak{h})|} \int_{\mathfrak{h}} (\check{\Theta}_{\Pi} \Delta)(\widehat{c}_-(x)^{-1}) \cdot \kappa(x) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx \\ &= \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \cdot \kappa(x) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx, \end{aligned}$$

*where  $\phi^G$  is as in (50) and each consecutive integral is absolutely convergent.*

*Proof.* Applied to a test function  $\phi \in \mathcal{S}(W)$ , the integral (94) over  $c(\mathfrak{g})$  is absolutely convergent because both, the character and the function  $T(\tilde{g})(\phi)$  are continuous and bounded (see for example [Prz93, Proposition 1.13]) and the group  $G$  is compact. Hence, each consecutive integral in the formula (99) applied to  $\phi$ ,

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) dg = \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_{\Pi}(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_W \chi_x(w) \phi(w) dw dx, \quad (110)$$

is absolutely convergent. Since

$$\chi_{g \cdot x}(w) = \chi_x(g^{-1} \cdot w)$$

and the Lebesgue measure  $dw$  is  $G$ -invariant,

$$\int_G \int_W \chi_{g \cdot x}(w) \phi(w) dw dg = \int_W \chi_x(w) \phi^G(w) dw.$$

Observe also that  $\widetilde{\text{Ad}}(\tilde{g}) = \text{Ad}(g)$  and the characters  $\check{\Theta}_{\Pi}$  and  $\Theta$  are  $\widetilde{G}$ -invariant. Moreover, by (101) and (107),

$$\overline{\Delta(\widehat{c}_-(x))} = \Delta(\widehat{c}_-(x)^{-1}) = (-1)^m \Delta(\widehat{c}_-(x)) \quad (x \in \mathfrak{h}),$$

where  $m$  is the number of positive roots, and

$$\overline{\pi_{\mathfrak{g}/\mathfrak{h}}(x)} = (-1)^m \pi_{\mathfrak{g}/\mathfrak{h}}(x) \quad (x \in \mathfrak{h}).$$

Therefore the Weyl integration formula on  $\mathfrak{g}$  shows that (110) is equal to  $\frac{\check{\chi}_{\Pi}(\tilde{c}(0))}{|W(G, \mathfrak{h})|}$  times

$$\begin{aligned} & \int_{\mathfrak{h}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2 \check{\Theta}_{\Pi}(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_W \chi_x(w) \phi^G(w) dw dx \\ &= \int_{\mathfrak{h}} \check{\Theta}_{\Pi}(\widehat{c}_-(x)) \overline{\Delta(\widehat{c}_-(x))} \left( \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \right) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx \\ &= \int_{\mathfrak{h}} \Theta_{\Pi}(\widehat{c}_-(x)^{-1}) \Delta(\widehat{c}_-(x)^{-1}) \kappa(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx. \end{aligned}$$

(Here, we suppose that the Haar measure on  $H$  is normalized to have total mass 1.) This verifies the first equality and the absolute convergence. Since

$$\begin{aligned}\Theta_{\Pi}(\widehat{c}_-(x)^{-1})\Delta(\widehat{c}_-(x)^{-1}) &= \kappa_0 \sum_{s \in W(G, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{s\mu}(\widehat{c}_-(x)^{-1}) \\ &= \kappa_0 \sum_{s \in W(G, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{-s\mu}(\widehat{c}_-(x)),\end{aligned}$$

the second equality follows via a straightforward computation.  $\square$

Since any element  $x \in \mathfrak{g}$ , viewed as an endomorphism of  $V$  over  $\mathbb{R}$ , has imaginary eigenvalues which come in complex conjugate pairs, we have  $\det(1 - x)_{V_{\mathbb{R}}} \geq 1$ . Define

$$\operatorname{ch}(x) = \det(1 - x)_{V_{\mathbb{R}}}^{1/2} \quad (x \in \mathfrak{g}). \quad (111)$$

Recall the symbols  $r$  and  $\iota$  from (52) and (51).

**Lemma 12.** *There is a constant  $C$  which depends only on the dual pair  $(G, G')$  such that*

$$\kappa(x) = C \operatorname{ch}^{d'-r-\iota}(x) \quad (x \in \mathfrak{h}).$$

*Proof.* Recall [Prz93, Lemma 5.7] that  $\pi_{\mathfrak{g}/\mathfrak{h}}(x)$  is a constant multiple of  $\Delta(\widehat{c}_-(x)) \operatorname{ch}^{r-\iota}(x)$ ,

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) = C \Delta(\widehat{c}_-(x)) \operatorname{ch}^{r-\iota}(x). \quad (112)$$

(For the reader's convenience, this is verified in Appendix B.) As is well known, [AP14, Definition 4.16],

$$\Theta(\tilde{c}(x))^2 = i^{\dim W} \det(2^{-1}(x - 1))_{\mathbb{W}} \quad (x \in \mathfrak{sp}(W), \det(x - 1) \neq 0). \quad (113)$$

Hence there is a choice of  $\tilde{c}$  so that

$$\Theta(\tilde{c}(x)) = \left(\frac{i}{2}\right)^{\frac{1}{2} \dim W} \det(1 - x)_{\mathbb{W}}^{\frac{1}{2}} \quad (x \in \mathfrak{g}). \quad (114)$$

Furthermore, since the symplectic space may be realized as  $W = \operatorname{Hom}_{\mathbb{D}}(V', V)$ , see (31), we obtain that

$$\det(1 - x)_{\mathbb{W}} = \det(1 - x)_{V_{\mathbb{R}}}^{d'} \quad (x \in \mathfrak{g}). \quad (115)$$

Also, as checked in [Prz91, (3.11)], the Jacobian of  $\tilde{c}_- : \mathfrak{g} \rightarrow G$  is a constant multiple of  $\operatorname{ch}^{-2r}(x)$ . (Again, for reader's convenience a slightly different proof is included in Appendix A.) Hence the claim follows.  $\square$

**Corollary 13.** *For any  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g})(\phi) dg = C \check{\chi}_{\Pi}(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathbb{W}} \chi_x(w) \phi^G(w) dw dx,$$

where  $C$  is a constant which depends only on the dual pair  $(G, G')$ ,  $\phi^G$  is as in (50), and each consecutive integral is absolutely convergent.

### 8. An intertwining distribution in terms of orbital integrals on the symplectic space.

We keep the notation introduced in section 3. Let

$$W(G, \mathfrak{h}(\mathfrak{g})) = \begin{cases} \Sigma_l & \text{if } \mathbb{D} = \mathbb{C}, \\ \Sigma_l \ltimes \{\pm 1\}^l & \text{otherwise.} \end{cases} \quad (116)$$

Denote the elements of  $\Sigma_l$  by  $\eta$  and the elements of  $\{\pm 1\}^l$  by  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_l)$ , so that an arbitrary element of the group (116) is of the form  $t = \epsilon\eta$ , with  $\epsilon = (1, 1, \dots, 1)$ , if  $\mathbb{D} = \mathbb{C}$ . This group acts on  $\mathfrak{h}(\mathfrak{g})$ , see (45), as follows: for  $t = \epsilon\eta$ ,

$$t \left( \sum_{j=1}^l y_j J_j \right) = \sum_{j=1}^l \epsilon_j y_{\eta^{-1}(j)} J_j. \quad (117)$$

As the indicated by the notation,  $W(G, \mathfrak{h}(\mathfrak{g}))$  coincides with the Weyl group, equal to the quotient of the normalizer of  $\mathfrak{h}(\mathfrak{g})$  in  $G$  by the centralizer of  $\mathfrak{h}(\mathfrak{g})$  in  $G$ .

The action of  $W(G, \mathfrak{h}(\mathfrak{g}))$  on  $\mathfrak{h}(\mathfrak{g})$  extends by duality to  $i\mathfrak{h}(\mathfrak{g})^*$ . More precisely, let  $e_j$  be as in (46). If  $\mu \in i\mathfrak{h}(\mathfrak{g})^*$ , then  $\mu = \sum_{j=1}^l \mu_j e_j$  with all  $\mu_j \in \mathbb{R}$ . If  $t = \epsilon\eta \in W(G, \mathfrak{h}(\mathfrak{g}))$ , then

$$t \left( \sum_{j=1}^l \mu_j e_j \right) = \sum_{j=1}^l \epsilon_j \mu_{\eta^{-1}(j)} e_j. \quad (118)$$

Recall the notation of Lemma 11 and the symbol  $\delta$  from (54).

**Lemma 14.** *The following formulas hold for any  $y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}(\mathfrak{g})$ ,*

$$\xi_{-\mu}(\widehat{c}_-(ty)) = \xi_{-t^{-1}\mu}(\widehat{c}_-(y)) \quad (t \in W(G, \mathfrak{h})) \quad (119)$$

and

$$\xi_{-\mu}(\widehat{c}_-(y)) \operatorname{ch}^{d-r-\iota}(y) = \prod_{j=1}^l (1 + iy_j)^{\mu_j + \delta - 1} (1 - iy_j)^{-\mu_j + \delta - 1}, \quad (120)$$

where all the exponents are integers:

$$\pm \mu_j + \delta \in \mathbb{Z} \quad (1 \leq j \leq l). \quad (121)$$

In particular, (120) is a rational function in the variables  $y_1, y_2, \dots, y_l$ .

*Proof.* By (107),

$$\xi_{-\mu}(\widehat{c}_-(y)) = \prod_{j=1}^l \left( \frac{1 + iy_j}{1 - iy_j} \right)^{\mu_j} = \prod_{j=1}^l (1 + iy_j)^{\mu_j} (1 - iy_j)^{-\mu_j}.$$

Hence (119) and (120) follow from the definition of the action of  $W(G, \mathfrak{h})$ , the definition of  $\operatorname{ch}$  in (111), and the following easy-to-check formula:

$$\operatorname{ch}(y) = \prod_{j=1}^l (1 + y_j^2)^{\frac{1}{2i}} = \prod_{j=1}^l (1 + iy_j)^{\frac{1}{2i}} (1 - iy_j)^{\frac{1}{2i}}. \quad (122)$$

Let  $\lambda = \sum_{j=1}^l \lambda_j e_j$  be the highest weight of the representation  $\Pi$  and let  $\rho = \sum_{j=1}^l \rho_j e_j$  be one half times the sum of the positive roots of  $\mathfrak{h}(\mathfrak{g})$  in  $\mathfrak{g}_{\mathbb{C}}$ . If  $\mu$  is the Harish-Chandra



parameter of  $\Pi$ , then  $\lambda + \rho = \mu = \sum_{j=1}^l \mu_j e_j$ ; see Appendix E. Hence, the statement (121) is equivalent to

$$\lambda_j + \rho_j + \frac{1}{2l}(d' - r + \iota) \in \mathbb{Z}. \quad (123)$$

Indeed, if  $G = O_d$ , then with the standard choice of the positive root system,  $\rho_j = \frac{d}{2} - j$ . Also,  $\lambda_j \in \mathbb{Z}$ ,  $\iota = 1$ ,  $r = d-1$ . Hence, (123) follows. Similarly, if  $G = U_d$ , then  $\rho_j = \frac{d+1}{2} - j$ ,  $\lambda_j + \frac{d'}{2} \in \mathbb{Z}$ ,  $\iota = 1$ ,  $r = d$ , which implies (123). If  $G = Sp_d$ , then  $\rho_j = d + 1 - j$ ,  $\lambda_j \in \mathbb{Z}$ ,  $\iota = \frac{1}{2}$ ,  $r = d + \frac{1}{2}$ , and (123) follows.  $\square$

Our next goal is to understand the integral

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw,$$

occurring in the formula for  $\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg$  in Lemma 11 and Corollary 13, in terms of orbital integrals on the symplectic space  $W$ . The results depend on whether  $l \leq l'$  or  $l > l'$  and will be given in Lemmas 17 and 18. We first need two other lemmas.

**Lemma 15.** *Fix an element  $z \in \mathfrak{h}(\mathfrak{g})$ . Let  $\mathfrak{z} \subseteq \mathfrak{g}$  and  $Z \subseteq G$  denote the centralizer of  $z$ . (Then  $Z$  is a real reductive group with Lie algebra  $\mathfrak{z}$ .) Denote by  $\mathfrak{c}$  the center of  $\mathfrak{z}$  and by  $\pi_{\mathfrak{g}/\mathfrak{z}}$  the product of the positive roots for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}(\mathfrak{g})_{\mathbb{C}})$  which do not vanish on  $z$ . Let  $B(\cdot, \cdot)$  be any non-degenerate symmetric  $G$ -invariant real bilinear form on  $\mathfrak{g}$ . Then there is a constant  $C_{\mathfrak{z}}$  such that for  $x \in \mathfrak{h}(\mathfrak{g})$  and  $x' \in \mathfrak{c}$ ,*

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \pi_{\mathfrak{g}/\mathfrak{z}}(x') & \int_G e^{iB(g.x, x')} dg \\ & = C_{\mathfrak{z}} \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}x) e^{iB(x, t(x'))}. \end{aligned} \quad (124)$$

(Here  $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})} = 1$  if  $\mathfrak{z} = \mathfrak{h}$ . Recall also the notation  $g.x = gxg^{-1}$ .)

*Proof.* The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57, Theorem 2, page 104]. A more general, and by now classical, result is [DV90, Proposition 34, p. 49].  $\square$

The symplectic form  $\langle \cdot, \cdot \rangle$  on  $W$  according to the Lie superalgebra structure introduced in (27) is

$$\langle w', w \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}w'w) \quad (w', w \in W). \quad (125)$$

Hence

$$\langle xw, w \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) \quad (x \in \mathfrak{g} \oplus \mathfrak{g}', w \in W). \quad (126)$$

Set

$$B(x, y) = \pi \text{tr}_{\mathbb{D}/\mathbb{R}}(xy) \quad (x, y \in \mathfrak{g}). \quad (127)$$

**Lemma 16.** *Recall the Gaussian  $\chi_x$  from (96). Then*

$$\chi_x(w) = e^{iB(x, \tau(w))} \quad (x \in \mathfrak{g}, w \in W). \quad (128)$$

*Proof.* Notice that

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_0}) - \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_1}),$$

where

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_0}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x|_{\mathfrak{V}_1}w|_{\mathfrak{V}_1}w|_{\mathfrak{V}_0}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w|_{\mathfrak{V}_0}x|_{\mathfrak{V}_0}w|_{\mathfrak{V}_1}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathfrak{V}_1})$$

and similarly

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_1}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathfrak{V}_0}).$$

Hence

$$\langle xw, w \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) = -\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}wxw) = -\langle wx, w \rangle.$$

Therefore

$$\langle x(w), w \rangle = 2 \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) \quad (x \in \mathfrak{g} \oplus \mathfrak{g}', w \in \mathbf{W}). \quad (129)$$

Then (126) and (32) show that

$$\frac{\pi}{2} \langle x(w), w \rangle = B(x, \tau(w)) \quad (x \in \mathfrak{g}, w \in \mathbf{W}),$$

which completes the proof.  $\square$

The Harish-Chandra regular almost semisimple orbital integral  $F(y)$ ,  $y \in \mathfrak{h}$ , was defined in [MPP20, Definition 3.2 and Theorems 3.4 and 3.6]. In particular, [MPP20, Theorem 3.6] implies that, in the statements below, all the integrals over  $\mathfrak{h}$  involving  $F(y)$  are absolutely convergent. Recall the notation  $F_\phi(y)$  for  $F(y)(\phi)$ .

**Lemma 17.** *Suppose  $l \leq l'$ . Then, with the notation of Lemma 11,*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathbf{W}} \chi_x(w) \phi^G(w) dw = C \int_{\mathfrak{h} \cap \tau(\mathbf{W})} e^{iB(x,y)} F_\phi(y) dy.$$

where  $C$  is a non-zero constant which depends only of the dual pair  $(G, G')$ .

*Proof.* Here we use the notation introduced in [MPP20, Definition 3.2]. The Weyl integration formula on  $\mathbf{W}$  (see [MPP15, Theorem 21] and [MPP20, (16) and (23)]) shows that

$$\int_{\mathbf{W}} \chi_x(w) \phi^G(w) dw = \sum_{\mathfrak{h}_1^-} \int_{\tau(\mathfrak{h}_1^+)} \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w)) \pi_{\mathfrak{g}'/\mathfrak{g}'}(\tau(w)) C(\mathfrak{h}_1^-) \mu_{\mathcal{O}(w)}(\chi_x \phi^G) d\tau(w), \quad (130)$$

where  $\mathfrak{h}_1^+ \subseteq \mathfrak{h}_1^{reg}$  is an open fundamental domain for the action of the Weyl group  $W(\mathbf{S}, \mathfrak{h}_1^-)$  and  $C(\mathfrak{h}_1^-)$  is a constant, determined in [MPP20, Lemma 2.1]. Let us consider first the case of a semisimple orbital integral

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{\mathbf{S}/\mathbf{S}^{\mathfrak{h}_1^-}} (\chi_x \phi^G)(s.w) d(\mathbf{S}^{\mathfrak{h}_1^-}),$$

where  $\mathbf{S}^{\mathfrak{h}_1^-}$  is the centralizer of  $\mathfrak{h}_1^-$  in  $\mathbf{S}$ . Recall the identification  $y = \tau(w) = \tau'(w)$  and let us write  $s = gg'$ , where  $g \in G$  and  $g' \in G'$ . Then

$$\chi_x(s.w) = e^{i\frac{\pi}{2}\langle x(s.w), s.w \rangle} = e^{iB(x, \tau(s.w))} = e^{iB(x, g.\tau(w))} = e^{iB(x, g.y)} \quad (131)$$

and

$$\phi^G(s.w) = \phi^G(g'.w). \quad (132)$$

Since  $l \leq l'$ , we have  $S^{\mathfrak{h}_\Gamma} = H \times Z'$ , where  $H$  is the Cartan subgroup of  $G$  corresponding to  $\mathfrak{h}$  and  $Z' \subseteq G'$  is the centralizer of  $\mathfrak{h}$ . Hence  $S/S^{\mathfrak{h}_\Gamma} = G/H \cdot G'/Z'$  and there is a positive constant  $C_1$  such that

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_1 \int_G e^{iB(x,g,y)} dg \int_{G'/Z'} \phi^G(g'.w) d(g'Z').$$

However we know from Harish-Chandra (Lemma 15) that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \left( \int_G e^{iB(x,g,y)} dg \right) \pi_{\mathfrak{g}/\mathfrak{h}}(y) = C_2 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) e^{iB(x,t,y)}.$$

Hence, using (130) and [MPP20, Definition 3.2 and Lemma 3.4], we obtain for some suitable positive constants  $C_k$ ,

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw & \tag{133} \\ &= C_3 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \sum_{\mathfrak{h}_\Gamma} \int_{\tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} C(\mathfrak{h}_\Gamma) \pi_{\mathfrak{g}'/\mathfrak{z}'}(y) \int_{G'/Z'} \phi^G(g'.w) d(g'Z') dy \\ &= C_4 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \int_{\bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} F_{\phi^G}(y) dy \\ &= C_4 \sum_{t \in W(G,\mathfrak{h})} \int_{\bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} F_{\phi^G}(t,y) dy \\ &= C_4 \int_{W(G,\mathfrak{h}) \cup \bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,y)} F_{\phi^G}(y) dy \\ &= C_4 \int_{\mathfrak{h} \cap \tau(W)} e^{iB(x,y)} F_{\phi^G}(y) dy. \end{aligned}$$

Since  $F_{\phi^G} = \text{vol}(G) F_\phi = F_\phi$ , the formula follows.

Next we consider the case  $G = O_{2l+1}$ ,  $G' = Sp_{2l'}(\mathbb{R})$ ,  $l < l'$ . Then

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{S/S^{\mathfrak{h}_\Gamma + w_0}} (\chi_x \phi^G)(s.(w + w_0)) d(sS^{\mathfrak{h}_\Gamma + w_0}),$$

where  $w_0 \in \mathfrak{s}(V^0)$  is a nonzero element. Since the Cartan subspace  $\mathfrak{h}_\Gamma$  preserves the decomposition (34),  $(w + w_0)^2 = w^2 + w_0^2$ . Hence,  $(s.(w + w_0))^2 = s.(w^2 + w_0^2)$ . The element  $x \in \mathfrak{h}$  acts by zero on  $\mathfrak{g}'$ . Therefore  $x(s.(w + w_0))^2 = x(s.(w + w_0))^2|_{\mathfrak{v}_\Gamma}$ . Since  $S(V^0) = O_1 \times Sp_{2(l-l)}(\mathbb{R})$  we see that  $w_0^2|_{\mathfrak{v}_\Gamma} = 0$ . Thus  $x s.w_0^2|_{\mathfrak{v}_\Gamma} = 0$ . Therefore, by (26),

$$\langle x(s.(w + w_0)), s.(w + w_0) \rangle = \text{tr}(x(s.(w + w_0))^2) = \text{tr}(x s.w^2|_{\mathfrak{v}_\Gamma}) = \text{tr}(xg.\tau(w)),$$

because  $s = gg'$ . Hence,

$$\chi_x(s.(w + w_0)) = e^{i\frac{\pi}{2} \langle x(s.(w+w_0)), s.(w+w_0) \rangle} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)}$$

and

$$\phi^G(s.(w + w_0)) = \phi^G(g'.(w + w_0)).$$

Therefore, with  $n = \tau'(w_0)$ ,

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_1 \int_G e^{iB(x,g,y)} dg \int_{G'/Z'^n} \phi^G(g'.w) d(g'Z'^n).$$

Thus, the computation (133) holds again, and we are done.  $\square$

**Lemma 18.** *Suppose  $l > l'$ . Let  $\mathfrak{z} \subseteq \mathfrak{g}$  and  $Z \subseteq G$  be the centralizers of  $\tau(\mathfrak{h}_\Gamma)$ . Then for  $\phi \in \mathcal{S}(W)$*

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw \\ = C \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} e^{iB(x,t,y)} F_\phi(y) dy, \end{aligned}$$

where  $C$  is a non-zero constant which depends only on the dual pair  $(G, G')$ .

*Proof.* By the Weyl integration formula with the roles of  $G$  and  $G'$  reversed, see [MPP20, (23)],

$$\int_W \chi_x(w) \phi^G(w) dw = C_1 \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} \pi_{\mathfrak{g}/\mathfrak{z}}(\tau'(w)) \pi_{\mathfrak{g}'/\mathfrak{h}'(\tau'(w))} \mu_{\mathcal{O}(w)}(\chi_x \phi^G) d\tau'(w),$$

where

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{S/S^{\mathfrak{h}_\Gamma}} (\chi_x \phi^G)(s.w) d(sS^{\mathfrak{h}_\Gamma}).$$

Recall the identification  $y = \tau(w) = \tau'(w)$  and let us write  $s = gg'$ , where  $g \in G$  and  $g' \in G'$ . Then, as in (131) and (132),

$$\chi_x(s.w) = e^{iB(x,g,y)} \quad \text{and} \quad \phi^G(s.w) = \phi^G(g'.w).$$

Since  $l > l'$ , according the notation introduced in (42),  $\mathfrak{h}$  is a Cartan subalgebra of  $G'$ , with corresponding Cartan subgroup  $H'$ . Then  $S^{\mathfrak{h}_\Gamma} = Z \times H'$ . Hence  $S/S^{\mathfrak{h}_\Gamma} = G/Z \cdot G'/H'$  and there is a constant  $C_2$  such that

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_2 \int_G e^{iB(x,g,y)} dg \int_{G'/H'} \phi^G(g'.w) d(g'H').$$

By (124) in Lemma 15 and [MPP20, (34)], we obtain for some constants  $C_k$

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw & \tag{134} \\ = C_3 \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} e^{iB(x,t,y)} \pi_{\mathfrak{g}'/\mathfrak{h}}(y) \\ & \times \int_{G'/H'} \phi^G(g'.w) d(g'H') dy \\ = C_4 \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} e^{iB(x,t,y)} F_{\phi^G}(y) dy. \end{aligned}$$

Since  $F_{\phi^G} = \text{vol}(G)F_\phi = F_\phi$ , the formula follows.  $\square$

**Lemma 19.** *Suppose  $l \leq l'$ . Then there is a seminorm  $q$  on  $\mathcal{S}(W)$  such that*

$$\left| \int_{\mathfrak{h} \cap \tau(W)} F_\phi(y) e^{iB(x,y)} dy \right| \leq q(\phi) \operatorname{ch}(x)^{-d'+r-1} \quad (x \in \mathfrak{h}, \phi \in \mathcal{S}(W)).$$

*Proof.* The boundedness of the function  $T(\tilde{g})(\phi)$ ,  $\tilde{g} \in \tilde{G}$ , means that there is a seminorm  $q(\phi)$  on  $\mathcal{S}(\mathfrak{g})$  such that

$$\left| \Theta(\tilde{c}(x)) \int_W \chi_x(w) \phi(w) dw \right| \leq q(\phi) \quad (x \in \mathfrak{g}). \quad (135)$$

Equivalently, replacing  $q(\phi)$  with a constant multiple of  $q(\phi)$ , and using (111), (114) and (115), we see that

$$\left| \int_W \chi_x(w) \phi(w) dw \right| \leq q(\phi) \operatorname{ch}^{-d'}(x) \quad (x \in \mathfrak{g}). \quad (136)$$

Since  $l \leq l'$ , Lemma 17 together with (136) prove that (again up to a multiplicative constant that can be absorbed by  $q(\phi)$ ),

$$\left| \int_{\mathfrak{h} \cap \tau(W)} F_\phi(y) e^{iB(x,y)} dy \right| \leq q(\phi) |\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \operatorname{ch}(x)^{-d'}.$$

One may choose the system of the positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  so that for every  $y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}$

$$\pi_{\mathfrak{g}/\mathfrak{h}}(y) = \begin{cases} \prod_{1 \leq j < k \leq l} i(-y_j + y_k) & \text{if } \mathbb{D} = \mathbb{C}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^l 2iy_j & \text{if } \mathbb{D} = \mathbb{H}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^l iy_j & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1}. \end{cases} \quad (137)$$

Recall the constants  $r$  and  $\iota$  from (53) and (51). Then

$$\max\{\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}}; 1 \leq j \leq l\} = \frac{1}{\iota}(r-1), \quad (138)$$

where  $\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}}$  denotes the degree of  $\pi_{\mathfrak{g}/\mathfrak{h}}(y)$  with respect to the variable  $y_j$ .

Also, (138) and (122) imply that

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \leq C_5 \operatorname{ch}^{r-1}(x) \leq C_5 \operatorname{ch}^{r-\iota}(x) \quad (x \in \mathfrak{h}),$$

where  $C_5$  is a constant. Thus, the claim follows.  $\square$

Lemmas 17 and 18 allows us to restate Corollary 13 in terms of orbital integrals on the symplectic space  $W$ .

**Corollary 20.** *Suppose  $l \leq l'$ . Then for any  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi) dg = C \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\mathfrak{h} \cap \tau(W)} e^{iB(x,y)} F_\phi(y) dy dx,$$

where  $C$  is a constant that depends only on the dual pair  $(G, G')$  and each consecutive integral is absolutely convergent.

*Proof.* The equality is immediate from Corollary 13 and Lemma 17. The absolute convergence of the outer integral over  $\mathfrak{h}$  follows from Lemma 19.  $\square$

**Corollary 21.** *Suppose  $l > l'$ . Then for any  $\phi \in \mathcal{S}(W)$ ,*

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg \\ &= C \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-s\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx, \end{aligned}$$

where  $C$  is a constant that depends only on the dual pair  $(G, G')$  and each consecutive integral is absolutely convergent.

*Proof.* The formula is immediate from Corollary 13, Lemma 18 and formula (119):

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg(\phi) \\ &= C_1 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \left( \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw \right) dx \\ &= C_2 \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ & \quad \times \left( \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,t,y)} F_\phi(y) dy \right) dx \\ &= \frac{C_2 \check{\chi}_\Pi(\tilde{c}(0))}{|W(Z, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ & \quad \times \left( \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,t,y)} F_\phi(y) dy \right) dx \\ &= C_3 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(t \cdot x)) \text{ch}^{d'-r-\iota}(t \cdot x) \\ & \quad \times \left( \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(t \cdot x, t \cdot y)} F_\phi(y) dy \right) dx \\ &= C_3 \check{\chi}_\Pi(\tilde{c}(0)) \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-t^{-1}\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ & \quad \times \left( \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy \right) dx. \end{aligned}$$

Let  $G''$  be the isometry group of the restriction of the form  $(\cdot, \cdot)$  to  $\mathbf{V}_0^{0,0}$  and let  $\mathfrak{h}'' = \sum_{j=l'+1}^l \mathbb{R}J_j$ . Then, as in (138), we check that

$$\max\{\deg_{x_j} \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}; 1 \leq j \leq l\} = \max\{\deg_{x_j} \pi_{\mathfrak{z}''/\mathfrak{h}''}; l' + 1 \leq j \leq l\} = \frac{1}{l}(r'' - 1),$$

where  $r'' = \frac{2 \dim \mathfrak{g}'_{\mathbb{R}}}{\dim \mathbb{V}_{\mathbb{R}}^{0,0}}$  is defined as in (52). Since  $r - r'' = d'$ , we see that

$$\mathrm{ch}^{d'-r-1}(x) |\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x)| \leq \mathrm{const} \mathrm{ch}^{d'-r-1+r''-1}(x) = \mathrm{const} \mathrm{ch}^{-2}(x).$$

Furthermore,  $F_\phi$  is absolutely integrable. Therefore, the absolute convergence of the last integral over  $\mathfrak{h}(\mathfrak{g})$  follows from the fact that  $\mathrm{ch}^{-2\iota}$  is absolutely integrable.  $\square$

To prove Theorem 2 (and Theorem 3), we still need the following explicit formula for the form  $B(x, y)$ . Let  $\beta = \frac{2\pi}{\iota}$ , where  $\iota$  is as in (51). Then

$$B(x, y) = -\beta \sum_{j=1}^l x_j y_j \quad \left( x = \sum_{j=1}^l x_j J_j, y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}(\mathfrak{g}) \right). \quad (139)$$

Indeed, the definition of the form  $B$ , (127), shows that

$$\begin{aligned} B(x, y) &= \pi \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) = \pi \sum_{j,k} \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(J_j J_k) x_j y_k \\ &= \pi \sum_j \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(-1_{\mathbb{V}_0^j}) x_j y_j = -\frac{2\pi}{\iota} \sum_j x_j y_j. \end{aligned} \quad (140)$$

*Proof of Theorem 2.* Notice that the degree of the polynomial  $Q_{a_j, b_j}$  is  $-a_j - b_j = 2\delta - 2$  and is independent of  $\mu$  and  $j$ . Explicitly,

$$2\delta - 2 = \frac{1}{\iota}(d' - r - \iota) \quad (141)$$

(where  $\iota = 1/2$  if  $\mathbb{D} = \mathbb{H}$  and 1 otherwise). Hence, by [MPP20, Theorem 3.6], the function  $F_\phi$  has the required number of continuous derivatives for the formula (57) to make sense. The operators appearing in the integrand of (57) act on different variables and therefore commute. Also, the constants  $a_j, b_j$  are integers by (121). Hence, equation (57) follows from Corollary 20, Lemma 14, formula (139), and Proposition C.5.

For the last statement about (58), we have

$$d' - r - \iota = \begin{cases} 2l' - 2l & \text{if } (G, G') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}), \\ 2l' - 2l - 1 & \text{if } (G, G') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}), \\ l' - l - 1 & \text{if } (G, G') = (\mathrm{U}_l, \mathrm{U}_{p,q}), p + q = l', \\ l' - l - 1 & \text{if } (G, G') = (\mathrm{Sp}_l, \mathrm{O}_{2l}^*). \end{cases} \quad (142)$$

Thus, since we assume  $l \leq l'$ , the product (58) is a function if and only if  $d' - r - \iota < 0$ , i.e. if and only if  $l = l'$  and  $(G, G') \neq (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'})$ . Furthermore, (58) contains no derivatives (but terms involving  $\delta_0$  are allowed) if and only if  $d' - r - \iota = 0$ , which corresponds to either  $l = l'$  and  $(G, G') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ , or  $l' = l + 1$  and  $\mathbb{D} = \mathbb{C}$  or  $\mathbb{H}$ . This completes the proof.  $\square$

Suppose now  $l > l'$ . Let  $\mathfrak{h}'' = \sum_{j=l'+1}^l \mathbb{R} J_j$ , so that

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{h} \oplus \mathfrak{h}''. \quad (143)$$

Then the centralizer of  $\tau(\mathfrak{h}_{\overline{1}})$  coincides with the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  and is equal to  $\mathfrak{z} = \mathfrak{h} \oplus \mathfrak{g}''$ , where  $\mathfrak{g}''$  is the Lie algebra of the group  $G''$  of the isometries of the restriction

of the form  $(\cdot, \cdot)$  to  $V_0^0$ . Furthermore, the derived Lie algebras of  $\mathfrak{z}$  and  $\mathfrak{g}''$  coincide (i.e.  $[\mathfrak{z}, \mathfrak{z}] = [\mathfrak{g}'', \mathfrak{g}'']$ ) and  $\mathfrak{h}''$  is a Cartan subalgebra of  $\mathfrak{g}''$ . We shall identify  $\mathfrak{h}$  and  $\mathfrak{h}'$  by means of (42). This justifies writing  $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}' \oplus \mathfrak{h}''$  when we need to emphasize the role of  $\mathfrak{g}'$ .

**Lemma 22.** *Suppose  $l > l'$ . In terms of Corollary 21 and the decomposition (143)*

$$\begin{aligned} & \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \\ &= \left( \xi_{-s\mu}(\widehat{c}_-(x')) \operatorname{ch}^{d'-r-\iota}(x') \right) \left( \xi_{-s\mu}(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') \right), \end{aligned} \quad (144)$$

where  $x = x' + x'' \in \mathfrak{h}(\mathfrak{g})$ , with  $x' \in \mathfrak{h}'$  and  $x'' \in \mathfrak{h}''$ . Moreover,

$$\begin{aligned} & \int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') dx'' \\ &= C \sum_{s'' \in W(\mathfrak{G}'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \mathbb{I}_{\{0\}}(-s\mu|_{\mathfrak{h}''} + s''\rho''), \end{aligned} \quad (145)$$

where  $C$  is a constant,  $\rho''$  is one half times the sum of the positive roots for  $(\mathfrak{g}'', \mathfrak{h}'')$  and  $\mathbb{I}_{\{0\}}$  is the indicator function of zero.

*Proof.* Formula (144) is obvious, because  $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x' + x'') = \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')$ . We shall verify (145). Let  $r''$  denote the number defined in (52) for the Lie algebra  $\mathfrak{g}''$ . A straightforward computation verifies the following table:

$\mathfrak{g}$	$r$	$r''$	$d' - r + r''$
$\mathfrak{u}_d$	$d$	$d - d'$	0
$\mathfrak{o}_d$	$d - 1$	$d - d' - 1$	0
$\mathfrak{sp}_d$	$d + \frac{1}{2}$	$d - d' + \frac{1}{2}$	0

By (112) applied to  $G'' \supseteq H''$  and  $\mathfrak{g}'' \supseteq \mathfrak{h}''$ ,

$$\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') = C_1'' \Delta''(\widehat{c}_-(x'')) \operatorname{ch}^{r''-\iota}(x'') \quad (x'' \in \mathfrak{h}''),$$

where  $\Delta''$  is the Weyl denominator for  $G''$ ,

$$\Delta'' = \kappa_0 \sum_{s'' \in W(\mathfrak{G}'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \xi_{s''\rho''} \quad (146)$$

and  $\kappa_0$  is as in (109). Hence, by (122), the integral (145) is a constant multiple of

$$\int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_-(x'')) \Delta''(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r+r''}(x'') \operatorname{ch}^{-2\iota}(x'') dx'' = 2^{\dim \mathfrak{h}''} \int_{\widehat{c}_-(\mathfrak{h}'')} \xi_{-s\mu}(h) \Delta''(h) dh, \quad (147)$$

where  $\widehat{c}_-(\mathfrak{h}'') \subseteq \widehat{H}''^0$ .

Notice that the function

$$\widehat{H}''^0 \ni h \rightarrow \xi_{-s\mu}(h) \Delta''(h) \in \mathbb{C}$$

is constant on the fibers of the covering map

$$\widehat{H}''^0 \rightarrow H''^0. \quad (148)$$



Indeed, the covering (148) is non-trivial only in two cases, namely  $G'' = O_{2l''+1}$  and  $G'' = U_{l''}$  with  $l''$  even; see (100). In these cases, (146) shows that this claim is true provided that the weight  $-s\mu + s''\rho''$  is integral for the Cartan subgroup  $H''$  (i.e. it is equal to the derivative of a character of  $H''$ ).

Suppose  $G'' = O_{2l''+1}$ . Then  $G = O_{2l+1}$ ,  $\lambda_j \in \mathbb{Z}$  and  $\rho_j \in \mathbb{Z} + \frac{1}{2}$ . Hence,  $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$ . Since,  $\rho''_j \in \mathbb{Z} + \frac{1}{2}$ , we see that  $(-s\mu)_j + \rho''_j \in \mathbb{Z}$ .

Suppose now that  $G'' = U_{l''}$  with  $l''$  even. Then  $G = U_l$  and  $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$ . In fact, if  $l'$  is even, i.e.  $l = l' + l''$  is even, then  $\lambda_j \in \mathbb{Z}$  and  $\rho_j \in \mathbb{Z} + \frac{1}{2}$ . If  $l'$  is odd, i.e.  $l = l' + l''$  is odd, then  $\lambda_j \in \mathbb{Z} + \frac{1}{2}$  and  $\rho_j \in \mathbb{Z}$ . Since  $\rho''_j \in \mathbb{Z} + \frac{1}{2}$ , in both cases, we conclude that  $(-s\mu)_j + \rho''_j \in \mathbb{Z}$ .

Therefore, (147) is a constant multiple of

$$\begin{aligned} \sum_{s'' \in W(G'', \mathfrak{h}'')} \text{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \int_{H''^0} \xi_{-s\mu}(h) \xi_{s''\rho''}(h) dh & \quad (149) \\ &= \begin{cases} \text{vol}(H''^0) \text{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') & \text{if } (s\mu)|_{\mathfrak{h}''} = s''\rho'', \\ 0 & \text{if } (s\mu)|_{\mathfrak{h}''} \notin W(G'', \mathfrak{h}'')\rho'', \end{cases} \\ &= \text{vol}(H''^0) \sum_{s'' \in W(G'', \mathfrak{h}'')} \text{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''). \end{aligned}$$

□

**Corollary 23.** *Suppose  $l > l'$  and keep the notation of Lemma 22 . Then*

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg = 0$$

unless there is  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  such that

$$(s\mu)|_{\mathfrak{h}''} = \rho''. \quad (150)$$

If  $G = O_{2l+1}$  or  $Sp_l$ , then (150) is equivalent to

$$\mu|_{\mathfrak{h}''} = \rho'' \quad \text{and} \quad s|_{\mathfrak{h}''} = 1. \quad (151)$$

Suppose  $G = O_{2l}$  and write  $\mathfrak{h}'' = \mathfrak{h}''_0 \oplus \mathbb{R}J_l$ , where  $\mathfrak{h}''_0 = \sum_{j=l'+1}^{l-1} \mathbb{R}J_j$ . Then (150) is equivalent to

$$\mu|_{\mathfrak{h}''} = \rho'', \quad s|_{\mathfrak{h}''_0} = 1, \quad \text{and} \quad s|_{\mathbb{R}J_l} = \pm 1. \quad (152)$$

Finally, if  $G = U_l$ , then (150) holds if and only if there is  $j_0 \in \{0, 1, \dots, l'\}$  such that

$$\mu_{j_0+j} = \rho''_{l'+j} \quad \text{and} \quad s(J_{j_0+j}) = J_{l'+j} \quad (1 \leq j \leq l - l'). \quad (153)$$

Suppose that (150) holds. Then for any  $\phi \in \mathcal{S}(W)$

$$\begin{aligned} \int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg(\phi) &= C \check{\chi}_{\Pi}(\tilde{c}(0)) \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g})), (s\mu)|_{\mathfrak{h}''} = \rho''} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \\ &\quad \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \text{ch}^{d-r-l}(x) \int_{\tau'(\mathfrak{h}_1^{\text{reg}})} e^{iB(x,y)} F_{\phi}(y) dy dx. \end{aligned} \quad (154)$$

where  $C$  is a non-zero constant which depends on the dual pair  $(G, G')$ , and each consecutive integral is absolutely convergent.

*Proof.* Observe that  $B(x' + x'', y) = B(x', y)$  for  $x' \in \mathfrak{h}'$ ,  $x'' \in \mathfrak{h}''$  and  $y \in \tau'(\mathfrak{h}_1^{reg}) \subseteq \mathfrak{h}'$ . We see therefore from Corollary 21 and Lemma 22 that

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg(\phi) \\ &= C \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}(\mathfrak{h})/\mathfrak{h}}(s) \operatorname{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') \\ & \quad \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx. \end{aligned} \quad (155)$$

Notice that for  $x \in \mathfrak{h}'$  and  $s'' \in W(G'', \mathfrak{h}'')$ , we have  $s''x = x$ . Thus  $\xi_{-s\mu}(\widehat{c}_-(x)) = \xi_{-s''s\mu}(\widehat{c}_-(x))$  by (119). Notice also that  $W(G'', \mathfrak{h}'') \subseteq W(G, \mathfrak{h})$  and  $\operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(s'') = \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s'')$ . Moreover,  $\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') = \mathbb{I}_{\{0\}}(-(s''^{-1}s\mu)|_{\mathfrak{h}''} + \rho'')$ . Hence, replacing  $s$  by  $s''s$  in (155), we see that this expression is equal to

$$\begin{aligned} & C \check{\chi}_\Pi(\tilde{c}(0)) \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}(\mathfrak{h})/\mathfrak{h}}(s) \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + \rho'') \\ & \quad \times \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx, \end{aligned} \quad (156)$$

which yields (154), with a new non-zero constant  $C$ , equal to  $C|W(G'', \mathfrak{h}'')|$ . Clearly (156) is zero if there is no  $s$  such that  $(s\mu)|_{\mathfrak{h}''} = \rho''$ . The absolute convergence of the integrals was checked in the proof of Corollary 21.

Recall that  $\mathfrak{h}'' = \sum_{j=l'+1}^l \mathbb{R}J_j$  and  $\mu = \lambda + \rho$  where  $\lambda$  is the highest weight of the genuine representation  $\Pi$ ; see Appendix E. If  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , then  $\rho|_{\mathfrak{h}''} = \rho''$ . All coefficients of  $\rho$  are positive and strictly decreasing by 1 except when  $G = O_{2l}$ , where  $\rho_l = 0$ . Hence  $s|_{\mathfrak{h}''}$  cannot contain sign changes when  $G = O_{2l+1}$  or  $Sp_l$ , whereas  $s|_{\mathfrak{h}_0''}$  cannot contain sign changes when  $G = O_{2l}$ . Using the form of the coefficients of  $\lambda$ , one easily sees that (150) is equivalent to (151) or (152). If  $G = U_l$ , then  $\lambda = \frac{q-p}{2} + \nu$ , where  $\nu_1 \geq \nu_2 \geq \dots \geq \nu_l$  are integers. Moreover,

$$\frac{q-p}{2} + \rho_{q+1+j} = \frac{l-p-q+1}{2} - j = \rho''_{l'+j} \quad (1 \leq j \leq l-l'). \quad (157)$$

The Weyl group  $W(G, \mathfrak{h}(\mathfrak{g}))$  consists of permutations of the  $J_j$ 's. Hence a genuine Harish-Chandra parameter  $\mu$  satisfies (150) if and only if among its coefficients  $\mu_1, \dots, \mu_l$  we can find a string of  $l-l'$  successive coefficients  $\mu_j$  equal to  $\rho''_{l'+1}, \dots, \rho''_l$  and the permutation  $s$  translates the corresponding string of  $J_j$ 's onto  $J_{l'+1}, \dots, J_l$ . This proves (153).  $\square$

In the next lemmas we study the integrals appearing on the right-hand side of (154).

**Lemma 24.** *For  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  and  $y \in \tau'(\mathfrak{h}_1^{reg})$ , in the sense of distributions on  $\tau'(\mathfrak{h}_1^{reg})$ ,*

$$\int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) e^{iB(x,y)} dx = \left( \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta y_j) \right) e^{-\beta \sum_{j=1}^{l'} |y_j|}, \quad (158)$$

where  $a_{s,j}$ ,  $b_{s,j}$  and  $\beta$  are as in (64) and (54), and  $P_{a,b}$  is defined in (C.8).

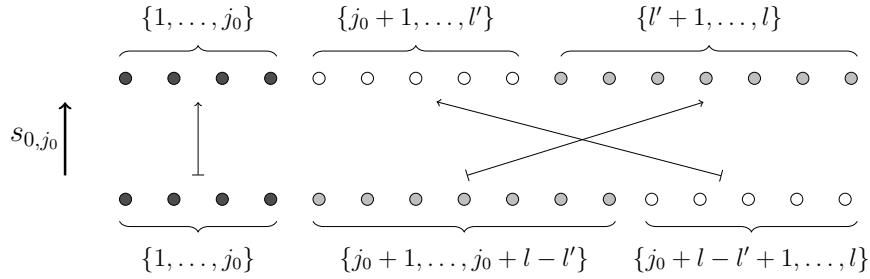
*Proof.* This follows immediately from Lemma 14, (C.9), and Proposition C.5, since  $a_{s,j} + b_{s,j} = -2\delta + 2 \geq 1$  for  $l > l'$ .  $\square$

Suppose that  $\mu$  satisfies (150) for some  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$ . The integral corresponding to  $s$  in (154) vanishes when the intersection of the support of the right-hand side of (158) and  $\tau'(\mathfrak{h}_{\bar{1}}^{reg})$  has an empty interior. We first study this intersection for some specific elements in  $W(G, \mathfrak{h}(\mathfrak{g}))$ .

If  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , define  $s_0 = 1$  as in (59). Then clearly  $s_0\mu|_{\mathfrak{h}''} = \rho''$  by (151). If  $\mathbb{D} = \mathbb{C}$ , fix  $j_0 \in \{0, 1, \dots, l'\}$  as in (153) and define  $s_{0,j_0}$  as the permutation in  $W(G, \mathfrak{h}(\mathfrak{g}))$  given by

$$s_{0,j_0}(J_j) = \begin{cases} J_j & (1 \leq j \leq j_0) \\ J_{l'-j_0+j} & (j_0 + 1 \leq j \leq j_0 + l - l') \\ J_{j-l+l'} & (j_0 + l - l' + 1 \leq j \leq l), \end{cases} \quad (159)$$

i.e.



Equivalently,

$$(s_{0,j_0}\mu)_j = \mu_{s_{0,j_0}^{-1}(j)} = \begin{cases} \mu_j & (1 \leq j \leq j_0) \\ \mu_{l+j_0-l'+j} & (j_0 + 1 \leq j \leq l') \\ \mu_{j_0-l'+j} & (l' + 1 \leq j \leq l). \end{cases} \quad (160)$$

Hence  $(s_{0,j_0}\mu)|_{\mathfrak{h}''} = \rho''$ . Notice that  $s_{0,p}$  is the element  $s_0$  defined in (60).

**Lemma 25.** *Let  $l > l'$  and suppose that  $\mu$  satisfies (150). Let  $s_0 = 1$ , as in (59), if  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , and let  $s_{0,j_0}$  be as in (159) if  $\mathbb{D} = \mathbb{C}$ .*

*If  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ , then*

$$\prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta y_j) = (2\pi)^{l'} \prod_{j=1}^{l'} P_{a_j, b_j, 2}(\beta y_j) \mathbb{I}_{\mathbb{R}^+}(y_j) \quad (y = \sum_{j=1}^{l'} y_j J'_j \in \mathfrak{h}') \quad (161)$$

*has support equal to  $\tau'(\mathfrak{h}_{\bar{1}})$ .*

*If  $\mathbb{D} = \mathbb{C}$ , then*

$$\prod_{j=1}^{l'} P_{a_{s_{0,j_0,j}}, b_{s_{0,j_0,j}}}(\beta y_j) = (2\pi)^{l'} \left( \prod_{j=1}^{j_0} P_{a_j, b_j, 2}(\beta y_j) \mathbb{I}_{\mathbb{R}^+}(y_j) \right) \left( \prod_{j=j_0+1}^{l'} P_{a_{j+l-l'}, b_{j+l-l', -2}}(\beta y_j) \mathbb{I}_{\mathbb{R}^-}(y_j) \right) \quad (y = \sum_{j=1}^{l'} y_j J'_j \in \mathfrak{h}') \quad (162)$$

*has support equal to  $(\sum_{j=1}^{j_0} \mathbb{R}^+ J'_j) \oplus (\sum_{j=j_0+1}^{l'} \mathbb{R}^- J'_j)$ . This support is equal to  $\tau'(\mathfrak{h}_{\bar{1}})$  if  $j_0 = p$ , whereas its intersection with  $\tau'(\mathfrak{h}_{\bar{1}})$  has empty interior if  $j_0 \neq p$ .*

*Proof.* Let  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ . By (54) and (141) and since  $\mu|_{\mathfrak{h}''} = \rho'' = \rho|_{\mathfrak{h}''}$ , we see that

$$\mu_1 > \cdots > \mu_{l'} > \mu_{l'+1} = \rho''_{l'+1} = -\delta,$$

These inequalities are equivalent to

$$a_1 = -\mu_1 - \delta + 1 < a_2 = -\mu_2 - \delta + 1 < \cdots < a_{l'} = -\mu_{l'} - \delta + 1 \leq 0 \quad (163)$$

because the  $\mu_j$ 's and  $\delta$  are either all in  $\mathbb{Z}$  or all in  $\mathbb{Z} + \frac{1}{2}$ . Hence  $P_{a_j, b_j, -2} = 0$  for all  $1 \leq j \leq l'$  by (C.6). Moreover, since  $a_j + b_j = -2\delta + 2 > 2$ , the polynomial  $P_{a_j, b_j, 2}$  is nonzero for all  $1 \leq j \leq l'$ . Hence the function on the right-hand side of (161) has support equal  $\sum_{j=1}^{l'} \mathbb{R}^+ J'_j = \tau'(\mathfrak{h}_{\overline{1}})$ .

Let now  $\mathbb{D} = \mathbb{C}$ . By (153), (157), (54) and (141),

$$\begin{aligned} \mu_1 > \mu_2 > \cdots > \mu_{j_0} > \mu_{j_0+1} = \rho''_{l'+1} = \frac{l-l'-1}{2} = -\delta (< 0), \\ (0 >) \delta = -\frac{l-l'-1}{2} = \rho''_l = \mu_{j_0+l-l'} > \mu_{j_0+l-l'+1} > \cdots > \mu_l. \end{aligned}$$

Since the  $\mu_j$ 's and  $\delta$  are either all in  $\mathbb{Z}$  or all in  $\mathbb{Z} + \frac{1}{2}$ , these inequalities are equivalent to

$$\begin{aligned} a_1 = -\mu_1 - \delta + 1 < a_2 = -\mu_2 - \delta + 1 < \cdots < a_{j_0} = -\mu_{j_0} - \delta + 1 \leq 0 \\ 0 \geq b_{j_0+l-l'+1} = \mu_{j_0+l-l'+1} - \delta + 1 > \cdots > b_l = \mu_l - \delta + 1. \end{aligned} \quad (164)$$

Hence

$$\begin{aligned} P_{a_j, b_j, -2} = 0 \quad \text{i.e.} \quad P_{a_j, b_j}(y_j) = 2\pi P_{a_j, b_j, 2}(y_j) \mathbb{I}_{\mathbb{R}^+}(y_j) \quad (1 \leq j \leq j_0), \\ P_{a_j, b_j, 2} = 0 \quad \text{i.e.} \quad P_{a_j, b_j}(y_j) = 2\pi P_{a_j, b_j, -2}(y_j) \mathbb{I}_{\mathbb{R}^-}(y_j) \quad (j_0 + l - l' + 1 \leq j \leq l). \end{aligned}$$

The polynomials appearing in these expressions of  $P_{a_j, b_j}$  are nonzero because  $a_j + b_j = -2\delta + 2 > 0$  for all  $j$ . By (40) and the convention on the symbols  $\delta_j$ 's for the dual pair  $(U_l, U_{p,q})$  with  $l > l' = p + q$ , the claims on the support of the right-hand side of (162) follow.  $\square$

Let  $\mathbb{D} = \mathbb{C}$ . Suppose that there is  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  such that  $(s\mu)|_{\mathfrak{h}''} = \rho''$  and that the string of coefficients of  $\mu$  equal to those of  $\rho''$ , see (153), starts at  $j_0 + 1$ , where  $j_0 \in \{0, 1, \dots, l'\}$ . Then  $s = s_{0, j_0}$  satisfies  $(s\mu)|_{\mathfrak{h}''} = \rho''$ . Lemma 25 shows that if  $j_0 \neq p$  then the intersection of the support of  $\prod_{j=1}^{l'} P_{a_{s_{0, j_0, j}}, b_{s_{0, j_0, j}}}$  with  $\tau'(\mathfrak{h}_{\overline{1}})$  has empty interior.

We now prove that if  $j_0 \neq p$  the same holds for the support of  $\prod_{j=1}^{l'} P_{a_{s, j}, b_{s, j}}$  for every  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  such that  $(s\mu)|_{\mathfrak{h}''} = \rho''$ .

**Lemma 26.** *Let  $\mathbb{D} = \mathbb{C}$ . Suppose that  $\mu$  and  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  satisfy (153) for  $j_0 \in \{0, 1, \dots, l'\}$ . If  $j_0 \neq p$ , then the intersection of the support of  $\prod_{j=1}^{l'} P_{a_{s, j}, b_{s, j}}$  with  $\tau'(\mathfrak{h}_{\overline{1}})$  has empty interior.*

*Proof.* Since

$$s_{0, j_0}(J_{j_0+j}) = J_{l'+j}, \quad s(J_{j_0+j}) = J_{l'+j} \quad (1 \leq j \leq l-l'),$$

the composition  $s^{-1}s_{0, j_0}$  fixes the elements of  $\{J_{j_0+1}, \dots, J_{j_0+l-l'}\}$  and permutes those of  $\{J_1, \dots, J_{j_0}\} \cup \{J_{j_0+l-l'+1}, \dots, J_l\}$ . Then  $s^{-1} = (s^{-1}s_{0, j_0})s_{0, j_0}^{-1}$  maps  $\{J_{l'+1}, \dots, J_l\}$  onto  $\{J_{j_0+1}, \dots, J_{j_0+l-l'}\}$  and hence  $\{J_1, \dots, J_{l'}\}$  bijectively onto  $\{J_1, \dots, J_{j_0}\} \cup \{J_{j_0+l-l'+1}, \dots, J_l\}$ . Therefore  $\{(s\mu)_j = \mu_{s^{-1}(j)}; 1 \leq j \leq l'\}$  is a permutation of  $\{\mu_j; 1 \leq j \leq j_0\} \cup \{\mu_j; j_0 + l -$

$l' + 1 \leq j \leq l\}$ . By (164), there are  $j_0$  negative  $a_j$  and  $l' - j_0$  negative  $b_j$  for  $1 \leq j \leq l'$ . The same is then true for the  $a_{s,j}$  and the  $b_{s,j}$ . The support of  $\prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}$  is therefore a product (in some order) of  $j_0$  copies of  $\mathbb{R}^+$  and  $l' - j_0$  copies of  $\mathbb{R}^-$ . Its intersection with  $\tau'(\mathfrak{h}_{\bar{1}})$  has therefore empty interior if  $j_0 \neq p$ .  $\square$

When the intersection of the support of  $\prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}$  and  $\tau'(\mathfrak{h}_{\bar{1}})$  has empty interior, the integral on the right-hand side of (154) that corresponds to  $s$  vanishes. Lemma 26 shows that every such integral is zero when  $j_0 \neq p$ . This yields the following corollary. Recall that we are supposing that  $q \geq p$ .

**Corollary 27.** *Suppose that  $\Pi$  is a genuine representation of  $\tilde{U}_l$  with Harish-Chandra  $\mu$  satisfying (153) for  $j_0 \in \{0, 1, \dots, l'\}$ . If  $j_0 \neq p$  then*

$$f_{\Pi \otimes \Pi'} = \int_{U_l} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg = 0.$$

Thus, if  $\Pi$  is a genuine representation of  $\tilde{U}_l$  occurs in Howe's correspondence, then its highest weight must be of the form  $\lambda = \sum_{j=1}^l \left(\frac{q-p}{2} + \nu_j\right) e_j$  where

$$\nu_1 \geq \nu_2 \geq \dots \geq \nu_p \geq \nu_{p+1} = \dots = \nu_{l-q} = 0 \geq \nu_{l-q+1} \geq \dots \geq \nu_l.$$

In the proof of Theorem 3 we will see that the condition on the highest weight of  $\Pi$  is also sufficient for the nonvanishing of the intertwining distributions.

Because of Corollary 27, we can restrict ourselves to the case  $j_0 = p$  when  $G = U_l$ . In this case, to simplify notation, we will write  $s_0$  instead of  $s_{0,p}$ . Hence

$$s_0 = 1 \quad (\text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{H}) \quad \text{and} \quad s_0 = s_{0,p} \quad (\text{if } \mathbb{D} = \mathbb{C}). \quad (165)$$

Observe that this notation allows us to write

$$\prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta y_j) = (2\pi)^{l'} \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta y_j) \mathbb{I}_{\delta_j \mathbb{R}^+}(y_j), \quad (166)$$

which unifies (161) and (162).

Suppose that  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  satisfies (150) and  $j_0 = p$  if  $\mathbb{D} = \mathbb{C}$ . Then

$$ss_0^{-1}|_{\mathfrak{h}''} = 1 \quad \text{and} \quad ss_0^{-1}(\mathfrak{h}) = \mathfrak{h}. \quad (167)$$

The condition  $ss_0^{-1}(\mathfrak{h}) = \mathfrak{h}$  and the identification (42), allow us to consider  $ss_0^{-1}$  as isomorphisms of  $\mathfrak{h}'$ . In the followin lemma we prove that such a  $s$  contributes to the right-hand side of (154) if and only if  $ss_0^{-1} \in W(G', \mathfrak{h}')$ . Moreover, in this case, the contribution from  $s$  agree with that of  $s_0$ .

**Lemma 28.** *Let  $l > l'$  and let  $\mu$  and  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  satisfy (150) with  $j_0 = p$  if  $\mathbb{D} = \mathbb{C}$ . The integral*

$$\int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-l'}(x) \int_{\tau'(\mathfrak{h}_{\bar{1}}^{reg})} e^{iB(x,y)} F_{\phi}(y) dy dx \quad (168)$$

is zero:

- (a) if  $ss_0^{-1}|_{\mathfrak{h}}$  acts by some sign changes, when  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ ,
- (b) if  $ss_0^{-1}|_{\mathfrak{h}}$  does not stabilize  $\{J_1, \dots, J_p\}$  (and  $\{J_{p+1}, \dots, J_{l'}\}$ ), when  $\mathbb{D} = \mathbb{C}$ .

Equivalently, by identifying  $\mathfrak{h}$  and  $\mathfrak{h}'$  via (42), the integral (168) is zero unless  $ss_0^{-1} \in W(G', \mathfrak{h}')$ . Moreover, (154) becomes: for any  $\phi \in \mathcal{S}(W)$

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg(\phi) = C \check{\chi}_\Pi(\tilde{c}(0)) \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \left( \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta y_j) \right) e^{-\beta \sum_{j=1}^{l'} |y_j|} F_\phi(y) dy, \quad (169)$$

where  $C$  is a non-zero constant which depends on the dual pair  $(G, G')$ .

*Proof.* Let  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$ . Suppose that  $ss_0^{-1}(J_j) = -J_j$  for some  $j \in \{1, \dots, l'\}$ . Then  $(s\mu)_j = -(s_0\mu)_j$ , which interchanges the  $j$ -th indices  $a$  and  $b$  of  $s\mu$  and  $s_0\mu$ . Thus  $P_{a_{s,j}, b_{s,j}}$  is supported in  $\mathbb{R}^-$ , and the support of (158) has a lower dimensional intersection with  $\tau'(\mathfrak{h}_\Gamma)$ .

The case  $\mathbb{D} = \mathbb{C}$  is similar: if  $ss_0^{-1}(J_i) = J_j$  where  $1 \leq i \leq p < j \leq l'$ , then  $(s\mu)_j = (s_0\mu)_i$ , which interchanges the  $i$ -th and  $j$ -th indices  $a$  and  $b$  of  $s\mu$  and  $s_0\mu$ . The support of (158) has therefore a lower dimensional intersection with  $\tau'(\mathfrak{h}_\Gamma)$ .

By the above and by identifying  $\mathfrak{h}$  and  $\mathfrak{h}'$  via (42), we can restrict the sum on the right-hand side of (154) to the set of  $s \in W(G, \mathfrak{h}(\mathfrak{g}))$  such that  $ss_0^{-1}|_{\mathfrak{h}} \in W(\mathfrak{g}', \mathfrak{h}')$  and  $ss_0^{-1}|_{\mathfrak{h}'} = 1$ . Therefore, the sum can be parametrized by  $W(\mathfrak{g}', \mathfrak{h}')$ . By (158) and since  $\text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(ss_0^{-1}) = \text{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(ss_0^{-1})$ , we obtain that  $\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg(\phi)$  is  $\check{\chi}_\Pi(\tilde{c}(0))$  times a constant multiple of

$$\begin{aligned} & \sum_{s' \in W(G', \mathfrak{h}')} \text{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \int_{\mathfrak{h}'} \xi_{-s's_0\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-l}(x) \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} e^{iB(x,y)} F_\phi(y) dy dx \\ &= \sum_{s' \in W(G', \mathfrak{h}')} \text{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \left( \prod_{j=1}^{l'} P_{a_{s's_0,j}, b_{s's_0,j}}(\beta y_j) \right) e^{-\beta \sum_{j=1}^{l'} |y_j|} F_\phi(y) dy. \end{aligned}$$

Observe that

$$\prod_{j=1}^{l'} P_{a_{s's_0,j}, b_{s's_0,j}}(\beta y_j) = \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta (s'^{-1}y)_j)$$

because  $s' \in W(\mathfrak{g}', \mathfrak{h}')$  permutes the indices  $1 \leq j \leq l'$ . Recall also that  $F_\phi(y)$  transforms as the sign representation with respect to the action of  $W(G', \mathfrak{h}')$ . Formula (169) therefore follows. The new non-zero constant  $C$  is the one appearing in (154) times  $|W(\mathfrak{g}', \mathfrak{h}')|$  times  $\text{sign}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s_0)$ , which is equal to 1 if  $\mathbb{D} = \mathbb{R}$  or  $\mathbb{H}$  and  $(-1)^{q(l-l')}$  if  $\mathbb{D} = \mathbb{C}$ .  $\square$

*Proof of Theorem 3.* The conditions on the highest weight  $\lambda$  rephrase the condition (150) for the Harish-Chandra parameter  $\mu$  of  $\Pi$  determined in Corollary 23 and its proof. The case of  $\mathbb{D} = \mathbb{C}$  was further considered in Corollary 27. Hence  $\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = 0$  unless  $\lambda$  satisfies them.

Formula (62) holds because, by Lemma 25, it is a restatement of (169).

By (166), the function  $\prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}}(\beta y_j)$  has support equal to  $\tau'(\mathfrak{h}_\Gamma)$  and we can rewrite the right-hand side of (169) as a constant multiple of

$$\int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \left( \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta y_j) \right) e^{-\beta \sum_{j=1}^{l'} |y_j|} F_\phi(y) dy, \quad (170)$$

By the  $W(G', \mathfrak{h}')$ -invariance of  $F_\phi$ , we can replace the term

$$\left( \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta y_j) \right) e^{-\beta \sum_{j=1}^{l'} |y_j|}$$

in the integral (170) by its  $W(G', \mathfrak{h}')$ -skew-invariant component

$$\frac{1}{|W(G', \mathfrak{h}')|} \sum_{s' \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j) e^{-\beta \sum_{j=1}^{l'} |y_j|}. \quad (171)$$

Here we have used that  $\sum_{j=1}^{l'} |(s'y)_j| = \sum_{j=1}^{l'} |y_j|$ . Notice that

$$\prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j) = \prod_{j=1}^{l'} P_{a_{s'_{-1} s_0,j}, b_{s'_{-1} s_0,j}, 2\delta_j}(\beta y_j)$$

because  $W(G', \mathfrak{h}')$  only permutes the  $y$ -coordinates for which the  $\delta_j$ 's have equal sign. Moreover, (171) is non-zero because  $P_{a_{s'_{-1} s_0,j}, b_{s'_{-1} s_0,j}, 2\delta_j}(\beta y_j)$  is not  $W(G', \mathfrak{h}')$ -invariant when  $W(G', \mathfrak{h}') \neq 1$ . Indeed, the condition  $\mu_1 > \mu_2 > \dots > \mu_{l'}$  implies  $b_1 > b_2 > \dots > b_{l'}$  and  $a_1 < a_2 < \dots < a_{l'}$ . If  $W(G', \mathfrak{h}') \neq 1$ , then there are at least two indices  $j \neq j'$  such that  $\delta_j = \delta_{j'}$  and the corresponding factors in (171) have different degrees. (If  $b \geq 1$  then the degree of  $P_{a,b,2}$  is  $b-1$  and if  $a \geq 1$  then that of  $P_{a,b,-2}$  is  $a-1$ .) Recall from [MPP20, Definition 3.2] that

$$F_\phi(y) = C_{\mathfrak{h}_\Gamma} \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \int_{S/S^{\mathfrak{h}_\Gamma}} \phi(s \cdot y) d(sS^{\mathfrak{h}_\Gamma}) \quad (\phi \in \mathcal{S}(W), y \in \tau(\mathfrak{h}_\Gamma^{\text{reg}})), \quad (172)$$

where  $S^{\mathfrak{h}_\Gamma}$  is the centralizer of  $\mathfrak{h}_\Gamma$  in  $S$  and  $C_{\mathfrak{h}_\Gamma}$  is an explicit constant. By (172), [MPP20, Lemma 2.1] and the definition of  $C_{\mathfrak{h}_\Gamma}$ ,

$$\begin{aligned} & C_{\mathfrak{h}_\Gamma} \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} \frac{\sum_{s' \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{g}/\mathfrak{z}}(y)} e^{-\beta \sum_{j=1}^{l'} |y_j|} \\ & \quad \times \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \pi_{\mathfrak{g}/\mathfrak{z}}(y) \int_{S/S^{\mathfrak{h}_\Gamma}} \phi(s \cdot y) d(sS^{\mathfrak{h}_\Gamma}) dy \\ & = i^{\dim \mathfrak{g}/\mathfrak{h}} \int_{\tau'(\mathfrak{h}_\Gamma^{\text{reg}})} \frac{\sum_{s' \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{g}/\mathfrak{z}}(y)} e^{-\beta \sum_{j=1}^{l'} |y_j|} \\ & \quad \times |\pi_{\mathfrak{g}_0/\mathfrak{h}_\Gamma^2}(w^2)| \int_{S/S^{\mathfrak{h}_\Gamma}} \phi(s \cdot y) d(sS^{\mathfrak{h}_\Gamma}) dy \quad (w \in \mathfrak{h}_\Gamma^{\text{reg}}, y = \tau'(w)). \end{aligned} \quad (173)$$

We now prove that

$$\frac{\sum_{s' \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0,j}, b_{s_0,j}, 2\delta_j}(\beta(s'\tau'(w))_j)}{\pi_{\mathfrak{g}/\mathfrak{z}}(\tau'(w))} e^{-\beta \sum_{j=1}^{l'} |\tau'(w)_j|} \quad (174)$$

is the restriction to  $\mathfrak{h}_\Gamma^{\text{reg}}$  of a locally integrable function, say  $\Phi$ , on  $W$ .

We can suppose to have chosen the system of the positive roots of  $\mathfrak{h}'$  in  $\mathfrak{g}'_{\mathbb{C}}$  so that their product is given by the formula

$$\pi_{\mathfrak{g}'/\mathfrak{h}'}\left(\sum_{j=1}^{l'} y_j J'_j\right) = \begin{cases} \prod_{1 \leq j < k \leq l'} i(-y_j + y_k) & \text{if } \mathbb{D} = \mathbb{C}, \\ \prod_{1 \leq j < k \leq l'} (-y_j^2 + y_k^2) & \text{if } \mathbb{D} = \mathbb{H}, \\ \prod_{1 \leq j < k \leq l'} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l'} 2iy_j & \text{if } \mathbb{D} = \mathbb{R}. \end{cases} \quad (175)$$

Moreover, we may have chosen the positive roots of  $\mathfrak{h}$  in  $\mathfrak{g}_{\mathbb{C}}/\mathfrak{z}_{\mathbb{C}}$  so that their product is equal to

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{z}}\left(\sum_{j=1}^{l'} y_j J_j\right) & \quad (176) \\ & = \begin{cases} \prod_{1 \leq j < k \leq l'} i(-y_j + y_k) \cdot \prod_{j=1}^{l'} (-iy_j)^{d-d'} & \text{if } \mathbb{D} = \mathbb{C}, \\ \prod_{1 \leq j < k \leq l'} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l'} 2iy_j \cdot \prod_{j=1}^{l'} (-y_j^2)^{d-d'} & \text{if } \mathbb{D} = \mathbb{H}, \\ \prod_{1 \leq j < k \leq l'} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l'} (iy_j)^{d-d'} & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l}, \\ \prod_{1 \leq j < k \leq l'} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^{l'} iy_j \cdot \prod_{j=1}^{l'} (iy_j)^{d-d'} & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1}. \end{cases} \end{aligned}$$

It follows that there is a non-zero constant  $C_3$ , depending of  $(G, G')$ , such that

$$\pi_{\mathfrak{g}/\mathfrak{z}}(y) = C_3 \pi_{\mathfrak{g}'/\mathfrak{h}'}(y) \det(y)_{V'}^{\gamma}, \quad (y = \tau(w) = \tau'(w), w \in \mathfrak{h}_{\overline{\mathbb{T}}}), \quad (177)$$

where

$$\gamma = \begin{cases} l - l' & \text{if } \mathbb{D} = \mathbb{C} \\ l - l' + \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1} \text{ or } \mathbb{D} = \mathbb{H} \\ l - l' - \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l} \end{cases} \quad (178)$$

and  $\det(g')_{V'}$  denotes the determinant of  $g'$  as an element of  $G' \subseteq \mathrm{GL}_{\mathbb{D}}(V')$ . (See the remark after (D.44) in Appendix D for the case  $\mathbb{D} = \mathbb{H}$ .)

The polynomial in (171) is  $W(G', \mathfrak{h}')$ -skew-invariant. Hence it is divisible by  $\pi_{\mathfrak{g}'/\mathfrak{h}'}(y)$ . Therefore the fraction

$$\frac{\sum_{s' \in W(G', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s') \prod_{j=1}^{l'} P_{a_{s_0, j}, b_{s_0, j}, 2\delta_j}(\beta(s'y)_j)}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(y)} \quad (y \in \mathfrak{h}')$$

is a  $W(G', \mathfrak{h}')$ -invariant polynomial. By Chevalley's restriction theorem, see e.g. [Wal88, 3.1.3], it extends to a  $G'$ -invariant polynomial on  $\mathfrak{g}'$ . Thus its pullback by  $\tau'$  is a  $GG'$ -invariant polynomial on  $W$ .

We now consider the term  $e^{-\beta \sum_{j=1}^{l'} |y_j|}$ . Notice that for  $w = \sum_{j=1}^{l'} w_j u_j \in \mathfrak{h}_{\overline{\mathbb{T}}}$ ,

$$\sum_{j=1}^{l'} |y_j| = \sum_{j=1}^{l'} |J'_j{}^*(\tau'(w))| = \sum_{j=1}^{l'} w_j^2 = \sum_{j=1}^{l'} \delta_j J'_j{}^*(\tau'(w)) = \left( \sum_{j=1}^{l'} \delta_j J'_j{}^* \right) \circ \tau'(w).$$

This is a quadratic polynomial on  $\mathfrak{h}_{\overline{\mathbb{T}}}$ , invariant under the Weyl group  $W(S, \mathfrak{h}_{\overline{\mathbb{T}}})$ . Such a polynomial has no  $GG'$ -invariant extension to  $W$ , unless  $G'$  is compact.

Indeed, suppose  $P$  is a real-valued  $GG'$ -invariant polynomial on  $W$  such that

$$P(w) = \left( \sum_{j=1}^{l'} \delta_j J'_j{}^* \right) \circ \tau'(w) \quad (w \in \mathfrak{h}_{\overline{\mathbb{T}}}).$$



Then  $P$  extends uniquely to a complex-valued  $G_{\mathbb{C}}G'_{\mathbb{C}}$ -invariant polynomial on the complexification  $W_{\mathbb{C}}$  of  $W$ . Hence, by the Classical Invariant Theory, [How89a, Theorems 1A and 1B] there is a  $G'_{\mathbb{C}}$ -invariant polynomial  $Q$  on  $\mathfrak{g}'_{\mathbb{C}}$  such that  $P = Q \circ \tau'$ . Hence,

$$Q(\tau'(w)) = P(w) = \left( \sum_{j=1}^{l'} \delta_j J'_j{}^* \right) \circ \tau'(w) \quad (w \in \mathfrak{h}_{\overline{1}}).$$

Since  $\tau'(\mathfrak{h}_{\overline{1}})$  spans  $\mathfrak{h}'$ , we see that the restriction of  $Q$  to  $\mathfrak{h}'$  is

$$Q|_{\mathfrak{h}'} = \sum_{j=1}^{l'} \delta_j J'_j{}^* \in \mathfrak{h}'_{\mathbb{C}}.$$

Since  $Q$  is  $G'_{\mathbb{C}}$ -invariant, the restriction  $Q|_{\mathfrak{h}'}$  has to be invariant under the corresponding Weyl group. There are no linear invariants if  $G' = \mathrm{Sp}_{2l'}(\mathbb{R})$  or  $\mathrm{O}_{2l'}^*$ . Therefore  $G' = \mathrm{U}_{p,q}$ ,  $p+q = l'$ . But in this case the invariance means that all the  $\delta_j$  are equal. Hence  $G' = \mathrm{U}_{l'}$  is compact. In the case  $G' = \mathrm{U}_{l'}$ , the sum of squares coincides with  $\langle J(w), w \rangle$  for a positive complex structure  $J$  on  $W$  which commutes with  $G$  and  $G'$  and therefore

$$e^{-\beta \sum_{j=1}^{l'} |\delta_j J'_j{}^*(\tau'(w))|} \quad (179)$$

extends to a Gaussian on  $W$ . If  $G'$  is not compact than (179) extends to a  $GG'$ -invariant function on  $W$ , which is bounded but is not a Gaussian.

Finally, the function  $\det(y)_{\mathbb{V}'}^{\gamma}$ , appearing in (177) is the restriction to  $\tau'(\mathfrak{h}_{\overline{1}}^{\mathrm{reg}})$  of the  $G'$ -invariant polynomial  $x \rightarrow \det(x)_{\mathbb{V}'}^{\gamma}$ , on  $\mathfrak{g}'$ . Its pullback by  $\tau'$  is a  $GG'$ -invariant polynomial on  $W$ .

The above constructions define the  $GG'$ -invariant function  $\Phi$  on  $W$  restricting to (174) on  $\mathfrak{h}_{\overline{1}}^{\mathrm{reg}}$ . It remains to prove that  $\Phi$  is locally integrable. For this, it is enough to prove that the integral  $\int_W \Phi(w)\phi(w) dw$  is finite for every  $\phi \in C_c^\infty(W)$ . Weyl's integration formula on  $W$  with  $l > l'$  (see [MPP15, Theorem 21]) together with the  $GG'$ -invariance of  $\Phi$ , yields

$$\int_W \Phi(w)\phi(w) dw = \int_{\tau'(\mathfrak{h}_{\overline{1}}^{\mathrm{reg}})} \Phi(y) |\pi_{\mathfrak{s}_{\overline{0}}/\mathfrak{h}_{\overline{1}}}^2(w^2)| \int_{S/S^{\mathfrak{h}_{\overline{1}}}} \phi(s.y) d(sS^{\mathfrak{h}_{\overline{1}}}) dy \quad (y = w^2),$$

which coincides up to a non-zero constant with the finite integral  $\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g}) dg(\phi)$  we started with.

It is still left to prove that if the highest weight  $\lambda$  of  $\Pi$  satisfies the conditions (a) or (b), then the integral (61) is nonzero. Let

$$\tau'^* : \mathcal{S}(\mathfrak{g}') \in \psi \rightarrow \psi \circ \tau' \in \mathcal{S}(W)^G$$

be the pullback by  $\tau'$ . According to [MPP20, (25)], there is a surjective map  $\tau'_* : \mathcal{S}(W)^G \rightarrow \mathcal{S}(\mathfrak{g}')$  such that

$$\tau'^* \circ \tau'_*(\phi) = \phi \quad (\phi \in \mathcal{S}(W)^G).$$

By [MPP20, Theorem 3.4 and (39)], since  $l > l'$ ,

$$F_\phi(y) = C'_{\mathfrak{h}_{\overline{1}}} \pi_{\mathfrak{g}/\mathfrak{h}'}(y) \int_{G'/H'} \tau'_*(\phi)(g'.y) d(g'H'),$$

where  $H' \subseteq G'$  is the Cartan subgroup corresponding to  $\mathfrak{h}'$  and  $C'_{\mathfrak{h}'_{\mathbb{T}}}$  is a suitable non-zero constant. Observe that (171) is a constant multiple of a not-identically zero  $W(G', \mathfrak{h}')$ -invariant real-valued continuous function, say  $f$ . Hence (173) can be written as

$$\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g})T(\tilde{g}) dg(\phi) = C' \int_{\tau'(\mathfrak{h}'_{\mathbb{T}}{}^{reg})} f(y) |\pi_{\mathfrak{s}'_0/\mathfrak{h}'_{\mathbb{T}}}{}^2(w^2)| \int_{G'/H'} \tau'_*(\phi)(g'.y) d(g'H') dy$$

$$(w \in \mathfrak{h}'_{\mathbb{T}}{}^{reg}, y = \tau'(w)), \quad (180)$$

where  $C'$  is a non-zero constant. Recall from [MPP20, (17)] that  $\pi_{\mathfrak{s}'_0/\mathfrak{h}'_{\mathbb{T}}}{}^2(w^2) \neq 0$  for all  $w \in \mathfrak{h}'_{\mathbb{T}}{}^{reg}$ . Choose a  $W(G', \mathfrak{h}')$ -invariant open set  $U_0 \subseteq \tau'(\mathfrak{h}'_{\mathbb{T}}{}^{reg})$  on which  $f$  has constant sign, say positive. Let  $\psi \in C_c^\infty(\mathfrak{g}')$  be positive and with support contained in  $U_0$ . Since  $\tau'_*$  is surjective, there is  $\phi \in \mathcal{S}(W)^G$  such that  $\tau'_*(\phi) = \psi$ . For all  $y \in \tau'(\mathfrak{h}'_{\mathbb{T}}{}^{reg})$ , we have  $(G'.y) \cap \tau'(\mathfrak{h}'_{\mathbb{T}}{}^{reg}) = W(G', \mathfrak{h}')y$ . Hence, with this choice of  $\phi$ , the orbital integral in (180) vanishes for  $y \notin W(G', \mathfrak{h}')U_0 = U_0$ . The right-hand side of (180) is therefore equal to  $C'$  times a positive constant.

The proof is now complete.  $\square$

### 9. The special case for the pair $(O_{2l}, Sp_{2l'}(\mathbb{R}))$ .

Here we consider the case  $(G, G') = (O_{2l}, Sp_{2l'}(\mathbb{R}))$  and suppose that the character  $\Theta_{\Pi}$  is not supported on the preimage of the connected identity component  $\widetilde{G}^0$ . Since the case  $l > l'$  was considered in Theorem 5 and the dual pair  $(O_2, Sp_2(\mathbb{R}))$  was treated in Remark 5, we will suppose in the following that  $l \leq l'$  and  $(l, l') \neq (1, 1)$ . Recall the element  $s \in G$ , (65), with centralizer in  $\mathfrak{h}$  equal to  $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R}J_j$ , and the spaces

$$V_{\bar{0},s} = V_{\bar{0}}^1 \oplus V_{\bar{0}}^2 \oplus \cdots \oplus V_{\bar{0}}^{l-1} \oplus \mathbb{R}v_{2l}, \quad V_s = V_{\bar{0},s} \oplus V_{\mathbb{T}}.$$

The corresponding dual pair is  $(G_s, G'_s) = (O_{2l-1}, Sp_{2l'}(\mathbb{R}))$  acting on the symplectic space  $W_s = \text{Hom}(V_{\mathbb{T}}, V_{\bar{0},s})$ .

**Lemma 29.** *Let  $H_s \subseteq H$  be the identity connected component of the centralizer of  $s$ . Then every element of the connected component  $G^0s$  is  $G$ -conjugate to an element of  $H_s s$ .*

*Proof.* Fix an element  $g \in G$ . As shown in [Cur84, page 114],  $g$  preserves a subspace of  $V$  of dimension 1 or 2. Hence  $V$  decomposes into a direct sum of  $g$ -irreducible subspaces of dimension 1 or 2, and the claim follows.  $\square$

Let us consider the  $\rho$ -function and the Weyl denominator for the root system of type  $C_{l-1}$  which is dual to that of  $(\mathfrak{g}_s, \mathfrak{h}_s)$ :

$$\rho_s^D = (l-1)e_1 + (l-2)e_2 + \cdots + e_{l-1} \quad (181)$$

and

$$\Delta_s^D(h) = \xi_{\rho_s^D}(h) \prod_{1 \leq j < k \leq l-1} (1 - \xi_{e_k - e_j}(h))(1 - \xi_{-e_j - e_k}(h)) \cdot \prod_{j=1}^{l-1} (1 - \xi_{-2e_j}(h)) \quad (h \in H_s). \quad (182)$$

Observe that the Weyl group of the root system of type  $C_{l-1}$  coincides with  $W(G_s^0, \mathfrak{h}_s)$ . Its elements act by sign changes and permutations of the  $J_1, \dots, J_{l-1}$ . The following two lemmas follow respectively from [Wen01, Theorems 2.5 and 2.6].

**Lemma 30.** *For any continuous  $G$ -invariant function  $f : G^0 s \rightarrow \mathbb{C}$ ,*

$$\int_{G^0 s} f(g) dg = \frac{1}{|W(G_s^0, \mathfrak{h}_s)|} \int_{H_s} f(hs) |\Delta_s^D(h)|^2 dh.$$

Notice that the coverings

$$\widetilde{G^0 s} \rightarrow G^0 s, \quad \widetilde{G^0} \rightarrow G^0$$

split (see Appendix D). Hence we may choose continuous sections

$$H_s s \ni hs \rightarrow \widetilde{hs} \in \widetilde{H_s s} \quad \text{and} \quad H_s \ni h \rightarrow \widetilde{h} \in \widetilde{H_s}. \quad (183)$$

**Lemma 31.** *Pick a continuous section*

$$H_s \ni h \rightarrow \widetilde{hs} \in \widetilde{H_s s}.$$

Then

$$\Theta_{\Pi}(\widetilde{hs}) = \frac{\Theta_{\Pi}(\widetilde{s})}{\Theta_{\Pi_s}(1)} \Theta_{\Pi_s}(h) \quad (h \in H_s), \quad (184)$$

where

$$\Theta_{\Pi_s}(h) = \frac{\sum_{w \in W(G_s^0, \mathfrak{h}_s)} \text{sgn}(w) \xi_{w \cdot (\lambda + \rho_s^D)}(h)}{\Delta_s^D(h)} \quad (h \in H_s), \quad (185)$$

$\lambda$  is the highest weight of  $\Pi$ , see (66), and

$$\frac{\Theta_{\Pi}(\widetilde{s})}{\Theta_{\Pi_s}(1)} = \pm 1. \quad (186)$$

Let

$$\rho_s = \rho_s^D - \frac{1}{2} \sum_{j=1}^{l-1} e_j \quad (187)$$

be the  $\rho$ -function of  $(\mathfrak{so}_{2l-1}, \mathfrak{h}_s)$ . Then  $\xi_{\rho_s}$  is well defined on the double covering

$$\widehat{H_s} \ni \widehat{h} \rightarrow h \in H_s$$

introduced in (100). Set

$$\Delta_s(\widehat{h}) = \xi_{\rho_s}(\widehat{h}) \prod_{1 \leq j < k \leq l-1} (1 - \xi_{e_k - e_j}(h)) (1 - \xi_{-e_j - e_k}(h)) \cdot \prod_{j=1}^{l-1} (1 - \xi_{-e_j}(h)) \quad (h \in H_s). \quad (188)$$

**Lemma 32.** *Let  $\lambda$  be the highest weight of  $\Pi$ . Then*

$$\lambda_s = \lambda + \frac{1}{2} \sum_{j=1}^{l-1} e_j \quad (189)$$

is the highest weight of an irreducible representation  $\Pi_{\lambda_s}$  of  $\text{Spin}_{2l-1}$ . Let  $\Theta_{\Pi_{\lambda_s}}$  denote its character. Then

$$\Theta_{\Pi_s}(h) \Delta_s^D(h) \overline{\Delta_s^D(h)} = \Theta_{\Pi_{\lambda_s}}(\widehat{h}) \Theta_{\sigma}(\widehat{h}) \Delta_s(\widehat{h}) \overline{\Delta_s(\widehat{h})} \quad (h \in H_s), \quad (190)$$

where  $\Theta_{\sigma}$  is the character of the spin representation  $\sigma$  of  $\text{Spin}_{2l-1}$ .

*Proof.* By (182) and (188),

$$\Delta_s^{\mathbb{D}}(h) = \Theta_\sigma(\widehat{h}) \Delta_s(\widehat{h}) \quad (\widehat{h} \in \widehat{H}_s), \quad (191)$$

where, by [Lit06, Ch. XI, III., p. 254],

$$\Theta_\sigma(\widehat{h}) = \prod_{j=1}^{l-1} \xi_{e_j/2}(\widehat{h})(1 + \xi_{-e_j}(\widehat{h}))$$

is the character of the spin representation  $\sigma$  of  $\text{Spin}_{2l-1}$ . Notice that this character is real valued. Since  $\lambda + \rho_s^{\mathbb{D}} = \lambda_s + \rho_s$ , formula (190) follows immediately from (185) and (191).  $\square$

**Remark 10.** The same computations leading to (190) show that

$$\Theta_{\Pi_s}(h) = \frac{\Theta_{\Pi_{\lambda_s}}(\widehat{h})}{\Theta_\sigma(\widehat{h})} \quad (h \in H_s).$$

This equality shows that the right-hand side is a continuous  $W(G_s^0, \mathfrak{h}_s)$ -invariant function on  $\widehat{H}_s$ . It is therefore the restriction to  $\widehat{H}_s$  of a central function  $\Phi$  on  $\text{Spin}_{2l-1}$ . By (185),  $\Theta_{\Pi_s}$  has highest weight  $\lambda$ . Hence  $\Phi$  is a finite linear combination of irreducible characters of  $\text{Spin}_{2l-1}$ . Thus, despite the fact that the denominator of (185) is the Weyl denominator for a root system of type  $C_{l-1}$ , the function  $\Theta_{\Pi_s}$  lifts to a finite linear combination of irreducible characters of  $\text{Spin}_{2l-1}$ .

To compute  $\Theta(\widetilde{hs})$ , we need the following lemma. Recall from [AP14, Definition 4.16, (35) and (101)] that

$$\Theta^2(g) = i^{\dim(g-1)W} \det(J^{-1}(g-1))_{J^{-1}(g-1)W}^{-1} \quad (g \in \text{Sp}(W)). \quad (192)$$

Recall also the map  $t$  from (14), and  $\Theta_s, t_s, T_s$ , as defined as in (16) and (14). Finally, let  $\kappa$  denote the map from  $H_s \cup H_{s^*}$  into the Cartan subgroup of  $G_s$  obtained by restriction to  $V_{\overline{0},s}$ .

**Lemma 33.** *For  $h \in H_s$  we have*

$$\Theta^2(hs) = \Theta_s^2(\kappa(hs)) \quad (193)$$

and

$$t(hs) = t_s(\kappa(hs)) \otimes \delta_0, \quad (194)$$

where  $\delta_0$  is the Dirac delta on  $W_s^\perp$ .

*Proof.* By definition

$$W = W_s \oplus W_s^\perp \quad \text{where} \quad W_s^\perp = \text{Hom}(V_{\overline{1}}, \mathbb{R}v_{2l-1}).$$

Since

$$h|_{W_s^\perp} = 1|_{W_s^\perp} \quad \text{and} \quad s|_{W_s^\perp} = 1|_{W_s^\perp}$$

we have  $(hs - 1)W = (hs - 1)W_s = (\kappa(hs) - 1)W_s$ . Furthermore

$$W_s = \text{Hom}(V_{\overline{1}}, V_{\overline{0}}^1 \oplus V_{\overline{0}}^2 \oplus \cdots \oplus V_{\overline{0}}^{l-1}) \oplus \text{Hom}(V_{\overline{1}}, \mathbb{R}v_{2l}),$$

and

$$\begin{aligned} (hs - 1)|_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{V}_0^1 \oplus \mathbb{V}_0^2 \oplus \cdots \oplus \mathbb{V}_0^{l-1})} &= (h - 1)|_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{V}_0^1 \oplus \mathbb{V}_0^2 \oplus \cdots \oplus \mathbb{V}_0^{l-1})} \\ &= (\kappa(hs) - 1)|_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{V}_0^1 \oplus \mathbb{V}_0^2 \oplus \cdots \oplus \mathbb{V}_0^{l-1})} \end{aligned}$$

whereas

$$(hs - 1)|_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{R}v_{2l})} = -2 \mathbf{1}_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{R}v_{2l})} = (\kappa(hs) - 1)|_{\text{Hom}(\mathbb{V}_{\bar{1}}, \mathbb{R}v_{2l})}.$$

We may fix the complex structure  $J$  that acts as an element of  $G'$ , see the proof of Lemma 5.4 in [DKP05]. Then  $J^{-1}(hs - 1) = (hs - 1)J^{-1}$  so that  $J^{-1}(hs - 1)W = (hs - 1)W = (\kappa(hs) - 1)W_s = J^{-1}(\kappa(hs) - 1)W_s$ . Hence,

$$\det(J^{-1}(hs - 1))_{J^{-1}(hs-1)W} = \det(\kappa(hs) - 1)_{J^{-1}(\kappa(hs)-1)W_s}.$$

By (192), this verifies (193).

Let  $w_s \in W_s$  and  $w_s^\perp \in W_s^\perp$ . Then, writing  $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{W_s} + \langle \cdot, \cdot \rangle_{W_s^\perp}$ , we have

$$\begin{aligned} \langle c(hs)(hs - 1)(w_s + w_s^\perp), (hs - 1)(w_s + w_s^\perp) \rangle &= \langle c(hs)(hs - 1)w_s, (hs - 1)w_s \rangle \\ &= \langle (hs + 1)w_s, (hs - 1)w_s \rangle \\ &= \langle c(\kappa(hs))(\kappa(hs) - 1)w_s, (\kappa(hs) - 1)w_s \rangle_{W_s}, \end{aligned}$$

which proves equation (194).  $\square$

*Proof of Theorem 4.* Let us choose  $\tilde{h} = h$  in (183) and define the section  $H_{s,s} \rightarrow \widetilde{H}_{s,s}$  so that

$$\Theta(\widetilde{hs}) = \Theta_s(\widetilde{\kappa(hs)}). \quad (195)$$

Then

$$T(\widetilde{hs}) = T_s(\widetilde{\kappa(hs)}) \otimes \delta_0. \quad (196)$$

In these terms, Lemmas 30, 32 and 33 show that

$$\begin{aligned} &\int_{\widetilde{G_s^0}} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg \\ &= \frac{1}{|W(G_s^0, \mathfrak{h}_s)|} \int_{H_s} \check{\Theta}_\Pi(\widetilde{hs})\Delta_s^D(h)\overline{\Delta_s^D(h)}T(\widetilde{hs})(\phi^G) dh \\ &= \frac{\check{\Theta}_\Pi(\tilde{s})}{\check{\Theta}_{\Pi_s}(1)} \frac{1}{|W(G_s^0, \mathfrak{h}_s)|} \int_{H_s} \check{\Theta}_{\Pi_s}(h)\Delta_s^D(h)\overline{\Delta_s^D(h)}T_s(\kappa(\widetilde{hs}))(\phi^G|_{W_s}) dh \\ &= \frac{\check{\Theta}_\Pi(\tilde{s})}{\check{\Theta}_{\Pi_s}(1)} \frac{1}{|W(G_s^0, \mathfrak{h}_s)|} \int_{H_s} \check{\Theta}_{\Pi_{\lambda_s}}(\hat{h})\Theta_\sigma(\hat{h})\Delta_s(\hat{h})\overline{\Delta_s(\hat{h})}T_s(\kappa(\widetilde{hs}))(\phi^G|_{W_s}) dh \\ &= \frac{\check{\Theta}_\Pi(\tilde{s})}{\check{\Theta}_{\Pi_s}(1)} \int_{-G_s^0} \check{\Theta}_{\Pi_{\lambda_s} \otimes \sigma}(\tilde{g})T_s(\tilde{g})(\phi^G|_{W_s}) dg. \end{aligned} \quad (197)$$

$\square$

10. **The special case for the pair  $O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})$  with  $1 \leq l \leq l'$ .**

Recall the decomposition (69). Let us denote the objects corresponding to  $W_s$  by the subscript  $s$ :  $\Theta_s, t_s$  and  $T_s$ . If  $H$  is a Cartan subgroup of  $G$ , then  $H^0 = H_s$  is a Cartan subgroup of  $G_s$ , and the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}_s$  share the same Cartan subalgebra  $\mathfrak{h} = \mathfrak{h}_s$ . Since any element  $h \in H$  acts trivially on  $W_s^\perp$ , we see that  $(h-1)W = (h-1)W_s$ . Hence,

$$\det(h-1)_{(h-1)W} = \det(h-1)_{(h-1)W_s}$$

and, just as in section 9, we check that

$$\Theta(\tilde{h}) = \Theta_s(\tilde{h}), \quad t(h) = t_s(h) \otimes \delta_0 \quad (h \in H^0), \quad (198)$$

where  $\delta_0$  is the Dirac delta on  $W_s^\perp$ .

*Proof of Theorem 6.* Let  $\widehat{H} \ni \widehat{h} \rightarrow h \in H$  be the double covering of  $H$  on which the functions  $\xi_\rho$  and  $\Delta$  are well-defined; see (100). It is easy to check that

$$|\Delta(\widehat{h})|^2 = |\Delta_s(h)|^2 \det(1-h) \quad (h \in H), \quad (199)$$

where

$$\Delta_s(h) = \xi_{-\rho_s}(h) \prod_{1 \leq j < k \leq l} (1 - \xi_{-e_j + e_k}(h))(1 - \xi_{-e_j - e_k}(h)).$$

Hence, for  $\phi \in \mathcal{S}(W)$ ,

$$\begin{aligned} \int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \frac{1}{|W(G^0, \mathfrak{h})|} \int_{H^0} \check{\Theta}_\Pi(\tilde{h})\Delta(\widehat{h})\overline{\Delta(\widehat{h})}T(\tilde{h})(\phi^G) dh \\ &= \frac{1}{|W(G_s, \mathfrak{h}_s)|} \int_{H_s} \check{\Theta}_\Pi(\tilde{h}) \det(1-h) \Delta_s(h) \overline{\Delta_s(h)} \Theta_s(\tilde{h}) t_s(h) (\phi^G|_{W_s}) dh \\ &= \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1-g) T_s(\tilde{g}) (\phi^G|_{W_s}) dg. \end{aligned}$$

This proves (71). On the other hand, by (198) and since  $c_-(\mathfrak{h}_s)$  is dense in  $H_s = H^0$ ,

$$\begin{aligned} \int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \frac{1}{|W(G_s, \mathfrak{h}_s)|} \int_{H_s} \check{\Theta}_\Pi(\tilde{h})\Delta(\widehat{h})\overline{\Delta(\widehat{h})}\Theta_s(\tilde{h})t_s(h)(\phi^G|_{W_s}) dh \\ &= \frac{4^l}{|W(G_s, \mathfrak{h}_s)|} \int_{\mathfrak{h}_s} \check{\Theta}_\Pi(\widehat{c}_-(x))\overline{\Delta(\widehat{c}_-(x))}\Delta(\widehat{c}_-(x))\Theta_s(\tilde{h})\frac{1}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)} \\ &\quad \times \pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x) \int_{W_s} \chi_x(w) (\phi^G|_{W_s})(w) dw \cdot \text{ch}^{-2}(x) dx, \quad (200) \end{aligned}$$

where the jacobian of the map  $c_- : \mathfrak{h}_s \rightarrow H_s$  is computed using Appendix A for  $G = \text{SO}_2$ . We now proceed as in Lemma 11:

$$\check{\Theta}_\Pi(\widehat{c}_-(x))\overline{\Delta(\widehat{c}_-(x))} = (-1)^m \Theta_\Pi(\widehat{c}_-(x)^{-1})\Delta(\widehat{c}_-(x)^{-1}) = (-1)^m \sum_{s \in W(G, \mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{-s\mu}(\widehat{c}_-(x)),$$

Appendix B shows that there is a constant  $C_1$  such that

$$\begin{aligned} \Delta(\widehat{c}_-(x))\Theta_s(\tilde{h})\frac{1}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)}\text{ch}^{-2}(x) &= C_1\frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)}\text{ch}^{-r+1}(x)\text{ch}^{2l'}(x)\text{ch}^{-2}(x) \\ &= C_1\prod_{j=1}^l e_j(x)\text{ch}^{2l'-r-1}(x). \end{aligned}$$

By Lemma 17, there is a constant  $C_2$  such that

$$\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)\int_{\mathfrak{W}_s}\chi_x(w)(\phi^G|_{\mathfrak{W}_s})(w)dw = C_2\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^G|_{\mathfrak{W}_s}}(y)dy. \quad (201)$$

This, together with Lemma 14, implies that (200) is equal to a constant multiple of

$$\begin{aligned} \int_{\mathfrak{h}_s}\xi_{-\mu}(\widehat{c}_-(x))\text{ch}^{2l'-r-1}(x)\prod_{j=1}^l e_j(x)\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^G|_{\mathfrak{W}_s}}(y)dydx \\ = \int_{\mathfrak{h}_s}\prod_{j=1}^l(1+ix_j)^{\mu_j+\delta-1}(1-ix_j)^{-\mu_j+\delta-1}x_j\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^G|_{\mathfrak{W}_s}}(y)dydx \end{aligned} \quad (202)$$

where  $\delta = \frac{1}{2}(2l' - r - 1)$ , see (54).

Writing  $x_j = -i(1 + ix_j) + i$ , we see that the right-hand side of (202) is  $i$  times

$$\begin{aligned} \int_{\mathfrak{h}_s}\prod_{j=1}^l(1+ix_j)^{-a_j}(1-ix_j)^{-b_j}\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^G|_{\mathfrak{W}_s}}(y)dydx \\ - \int_{\mathfrak{h}_s}\prod_{j=1}^l(1+ix_j)^{-(a_j-1)}(1-ix_j)^{-b_j}\int_{\mathfrak{h}_s}e^{iB(x,y)}F_{\phi^G|_{\mathfrak{W}_s}}(y)dydx, \end{aligned} \quad (203)$$

where  $a_j$  and  $b_j$  are as in (55).

Notice that

$$(2l' - r - 1) + 1 = 2l' - 2l = 2l' - (2l - 1) - 1 = 2l' - r_s - 1.$$

By [MPP20, Theorem 3.6] with  $d' = 2l'$ , for every  $1 \leq j \leq l$  the function  $F_{\phi^G|_{\mathfrak{W}_s}}(y)$  admits continuous partial derivatives in  $y_j$  of order  $\leq 2l' - r_s - 1$ . (Observe that this means, in particular, that  $F_{\phi^G|_{\mathfrak{W}_s}}$  extends continuously on  $\mathfrak{h}_s$ , but needs not be differentiable when  $l = l'$ .)

Setting

$$c_j = -a_j - b_j = 2\delta - 2 = 2(l' - l) - 1 \quad \text{and} \quad c'_j = -(a_j - 1) - b_j = 2(l' - l),$$

we see that  $F_{\phi^G|_{\mathfrak{W}_s}}$  has sufficiently many continuous derivatives so that Proposition C.5 applies. This proves (72).

The properties of the distribution (73) are an immediate consequence of the definitions of the functions  $p_j^b$  and  $q_j^b$  in terms of the functions  $P_{a,b}$  and  $Q_{a,b}$  from (C.8) and (C.9).  $\square$

11. **The special case for the pair  $O_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R})$  with  $l > l'$ .**

Recall the decompositions (74) and (143). As before, we denote the objects corresponding to  $W_s$  by the subscript  $s$ :  $\Theta_s$ ,  $t_s$  and  $T_s$ . In particular,  $\mathfrak{h}_s = \mathfrak{h}(\mathfrak{g})$ , see (45), and  $H_s = H(\mathfrak{g})^0$ . Since any element  $h \in H(\mathfrak{g})$  acts trivially on  $W_s^\perp$ , we see that

$$(h - 1)W = (h - 1)W_s.$$

Hence

$$\det(h - 1)_{(h-1)W} = \det(h - 1)_{(h-1)W_s}.$$

Hence, as in section 9, we check that

$$\Theta(\tilde{h}) = \Theta_s(\tilde{h}), \quad t(h) = t_s(h) \otimes \delta_0, \quad (204)$$

where  $\delta_0$  is the Dirac delta on  $W_s^\perp$ .

*Proof of Theorem 7.* As in section 10, we check that for  $\phi \in \mathcal{S}(W)$ ,

$$\int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1 - g)T_s(\tilde{g})(\phi^G|_{W_s}) dg, \quad (205)$$

which verifies the first equality in (76). On the other hand, similar computations as those done in section 10 and (204) imply that the left-hand side of (205) is a constant multiple of

$$\begin{aligned} & \frac{1}{|W(G^0, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} (\Theta_\Pi(\tilde{c}_-(x)^{-1})\Delta(\tilde{c}_-(x)^{-1})) \left( \frac{\Delta(\tilde{c}_-(x))}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \Theta_s(\tilde{c}_-(x)) \right) \\ & \quad \times \pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x) \int_{W_s} \chi_x(w) (\phi^G|_{W_s})(w) dw \mathrm{ch}^{-2}(x) dx. \end{aligned} \quad (206)$$

Lemma 18 shows that there is a constant  $C_1$  such that

$$\begin{aligned} & \pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x) \int_{W_s} \chi_x(w) \phi^G|_{W_s}(w) dw \\ & = C_1 \int_{\tau'(\mathfrak{h}_1^{\mathrm{reg}})} \sum_{tW(Z_s, \mathfrak{h}(\mathfrak{g})) \in W(G_s, \mathfrak{h}(\mathfrak{g}))/W(Z_s, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) e^{iB(x, t \cdot y)} F_{\phi^G|_{W_s}}(y) dy, \end{aligned}$$

where  $\mathfrak{z}_s \subseteq \mathfrak{g}_s$  is the centralizer of  $\mathfrak{h} = \mathfrak{h}'$ .

As in section 10, with  $d' = 2l'$  and for a suitable constant  $C_2$ ,

$$\begin{aligned} \frac{\Delta(\tilde{c}_-(x))}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \Theta_s(\tilde{c}_-(x)) \mathrm{ch}^{-2}(x) & = C_2 \frac{\pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x)}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \mathrm{ch}^{-r+1}(x) \mathrm{ch}^{d'}(x) \mathrm{ch}^{-2}(x) \\ & = C_2 \prod_{j=1}^l e_j(x) \cdot \mathrm{ch}^{d'-r-1}(x). \end{aligned}$$



This, together with (14), shows that (206) is equal to a constant multiple of

$$\begin{aligned} & \sum_{u \in W(\mathfrak{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_1^{reg})} \xi_{-\mu}(\widehat{c}_-(u^{-1}x)) \cdot \prod_{j=1}^l e_j(x) \cdot \operatorname{ch}^{d'-r-1}(x) \\ & \times \sum_{tW(\mathfrak{Z}_s, \mathfrak{h}(\mathfrak{g})) \in W(\mathfrak{G}_s, \mathfrak{h}(\mathfrak{g}))/W(\mathfrak{Z}_s, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Now notice that

$$\operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \prod_{j=1}^l e_j(x) = \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(u) \prod_{j=1}^l e_j(u^{-1}.x). \quad (207)$$

Changing the variable of integration  $x$  to  $t.x$  and using that  $\operatorname{ch}(t.x) = \operatorname{ch}(x)$ , we see that (206) is a constant multiple of

$$\begin{aligned} & \sum_{u \in W(\mathfrak{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(tu) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_1^{reg})} \xi_{-\mu}(\widehat{c}_-(u^{-1}x)) \cdot \prod_{j=1}^l e_j(u^{-1}x) \cdot \operatorname{ch}^{d'-r-1}(x) \\ & \times \sum_{t \in W(\mathfrak{G}_s, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Since  $\operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(tu) \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) = \operatorname{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(u)$ , (207) implies that (206) is a constant multiple of

$$\begin{aligned} & \sum_{u \in W(\mathfrak{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_1^{reg})} \xi_{-\mu}(\widehat{c}_-(u^{-1}x)) \cdot \prod_{j=1}^l e_j(x) \cdot \operatorname{ch}^{d'-r-1}(x) \\ & \times \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \quad (208) \end{aligned}$$

Now we use the direct sum decomposition (143) of  $\mathfrak{h}(\mathfrak{g})$ , the equality

$$\prod_{j=l'+1}^l e_j(x) \cdot \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) = \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x),$$

and (119) to rewrite (208) as

$$\begin{aligned} & \sum_{u \in W(\mathfrak{G}, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-u\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-1}(x) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \cdot \prod_{j=1}^{l'} e_j(x) \\ & \times \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Because of the additional term  $\prod_{j=1}^{l'} e_j(x)$ , we cannot repeat the argument used in the proof of Corollary 21 to prove the absolute convergence of this integral. Lemma 22

implies that this last expression is a constant multiple of

$$\begin{aligned} & \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}'} \int_{\tau'(\mathfrak{h}_1^{reg})} \xi_{-u\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-1}(x) \\ & \times \left( \sum_{u'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}''/\mathfrak{h}''}(u'') \mathbb{I}_{\{0\}}(-(u\mu)|_{\mathfrak{h}''} + u''\rho'') \right) \prod_{j=1}^{l'} e_j(x) \cdot e^{iB(x,y)} F_{\phi_G|_{W_s}}(y) dy dx. \end{aligned} \quad (209)$$

From Corollary 23 we see that (209) is zero unless there is  $u \in W(G, \mathfrak{h}(\mathfrak{g}))$  satisfying (151). As in Theorem 3, this imposes the condition (a) on the highest weight  $\lambda$  of  $\Pi$ .

Suppose that (151) holds for some  $u \in W(G, \mathfrak{h}(\mathfrak{g}))$ . Then, as in Corollary 23, (209) is equal to

$$\sum_{\substack{u \in W(G, \mathfrak{h}(\mathfrak{g})) \\ (u\mu)_{\mathfrak{h}''} = \rho''}} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\mathfrak{h}'} \xi_{-u\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-1}(x) \prod_{j=1}^{l'} e_j(x) \cdot \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_{\phi_G|_{W_s}}(y) dy dx. \quad (210)$$

Lemma 24 must be modified, using the same method used in the proof of Theorem 7, to take the additional term  $\prod_{j=1}^{l'} e_j(x)$  into account. For  $u \in W(G, \mathfrak{h}(\mathfrak{g}))$  and  $1 \leq j \leq l$ , recall the parameters  $a_{u,j}$  and  $b_{u,j}$  introduced in (64). Notice that  $Q_{a_{u,j}, b_{u,j}} = Q_{a_{u,j-1}, b_{u,j}} = 0$  for  $1 \leq j \leq l$  because  $a_{u,j} + b_{u,j} = 2(l-j) + 1 \geq 2$  for  $l > j$ . Hence, by Proposition C.5, formula (210) becomes  $i$  times

$$\begin{aligned} & \sum_{\substack{u \in W(G, \mathfrak{h}(\mathfrak{g})) \\ (u\mu)_{\mathfrak{h}''} = \rho''}} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \int_{\tau'(\mathfrak{h}_1^{reg})} \prod_{j=1}^{l'} \left( P_{a_{u,j}, b_{u,j}}(2\pi y_j) - P_{a_{u,j-1}, b_{u,j}}(2\pi y_j) \right) \\ & \times e^{-2\pi \sum_{j=1}^{l'} |y_j|} F_{\phi_G|_{W_s}}(y) dy. \end{aligned}$$

The final formula (76), as well as the non-vanishing of the distribution  $\int_{G_0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg$  when the highest weight  $\lambda$  satisfies condition (a), now follows using the steps leading to Theorem 3.  $\square$

## 12. A sketch of a computation of the wave front of $\Pi'$ .

**Corollary 34.** *For any representation  $\Pi \otimes \Pi'$  which occurs in the restriction of the Weil representation to the dual pair  $(\tilde{G}, \tilde{G}')$ ,*

$$WF(\Pi') = \tau'(\tau^{-1}(0)).$$

Here  $WF(\Pi')$  stands for the wave front of the character  $\Theta_{\Pi'}$  at the identity and  $0 = WF(\Pi)$  since  $\Pi$  is finite dimensional.

The complete proof is rather lengthy but unlike the one provided in [Prz91, Theorem 6.11], it is independent of [Vog78]. We sketch the main steps below. The details are going to appear in [MPP21].

The variety  $\tau^{-1}(0) \subseteq W$  is the closure of a single  $GG'$ -orbit  $\mathcal{O}$ ; see e.g. [Prz91, Lemma 2.16]. There is a positive  $GG'$ -invariant measure  $\mu_{\mathcal{O}}$  on this orbit which defines a homogeneous distribution. We denote its degree by  $\deg \mu_{\mathcal{O}}$ .

Recall that if  $V$  is a  $n$ -dimensional real vector space,  $t > 0$  and  $M_t v = tv$  for  $v \in V$ , then the pullback of  $u \in \mathcal{S}'(V)$  by  $M_t$  is  $M_t^* u \in \mathcal{S}'(V)$ , defined by

$$(M_t^* u)(\phi) = t^{-n} u(\phi \circ M_{t^{-1}}) \quad (\phi \in \mathcal{S}(V)).$$

In particular, for  $V = W$

$$M_t^* \mu_{\mathcal{O}} = t^{\deg \mu_{\mathcal{O}}} \mu_{\mathcal{O}}.$$

Define  $\tau'_* : \mathcal{S}'(W) \rightarrow \mathcal{S}'(\mathfrak{g}')$  by  $\tau'_*(u)(\psi) = u(\psi \circ \tau')$ . Then, for  $t > 0$ ,

$$t^{2 \dim \mathfrak{g}'} M_{t^2}^* \circ \tau'_* = t^{\dim W} \tau'_* \circ M_t^*. \quad (211)$$

A rather lengthy but straightforward computation based on Theorems 2, 3 and 4, shows that

$$t^{\deg \mu_{\mathcal{O}}} M_{t^{-1}}^* f_{\Pi \otimes \Pi'} \xrightarrow[t \rightarrow 0]{} C \mu_{\mathcal{O}}, \quad (212)$$

as tempered distributions on  $W$ , where  $C$  is a non-zero constant.

Let  $\mathcal{F}$  indicate a Fourier transform on  $\mathcal{S}'(\mathfrak{g}')$ . Then, for  $t > 0$ ,

$$M_t^* \circ \mathcal{F} = t^{-\dim \mathfrak{g}'} \mathcal{F} \circ M_{t^{-1}}. \quad (213)$$

Hence, in the topology of  $\mathcal{S}'(\mathfrak{g}')$ ,

$$t^{2 \deg \mu_{\mathcal{O}'}} M_{t^2}^* \mathcal{F} \tau'_*(f_{\Pi \otimes \Pi'}) \xrightarrow[t \rightarrow 0^+]{} C \mathcal{F} \mu_{\mathcal{O}'} \quad (214)$$

where  $C \neq 0$  and  $\mathcal{O}' = \tau'(\mathcal{O})$ .

There is an easy to verify inclusion  $WF(\Pi') \subseteq \overline{\mathcal{O}'}$ , [Prz91, (6.14)] and a formula for the character  $\Theta_{\Pi'}$  in terms of  $\mathcal{F}(\tau'_*(f_{\Pi \otimes \Pi'}))$ ,

$$\frac{1}{\sigma} \cdot \tilde{c}_-^* \Theta_{\Pi'} = \tau'_*(\widehat{f_{\Pi \otimes \Pi'}}), \quad (215)$$

where  $\sigma$  is a smooth function, [Prz91, Theorem 6.7]. By combining this with the following elementary lemma, one completes the argument.

**Lemma 35.** *Suppose  $f, u \in \mathcal{S}'(\mathbb{R}^n)$  and  $u$  is homogeneous of degree  $d \in \mathbb{C}$ . Suppose*

$$t^d M_{t^{-1}}^* f(\psi) \xrightarrow[t \rightarrow 0^+]{} u(\psi) \quad (\psi \in \mathcal{S}(\mathbb{R}^n)). \quad (216)$$

Then

$$WF_0(\hat{f}) \supseteq \text{supp } u, \quad (217)$$

where the subscript 0 indicates the wave front set at zero and

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x \cdot y} dy.$$

### APPENDIX A. The Jacobian of the Cayley transform

Here we determine the Jacobian of the Cayley transform  $c_- : \mathfrak{g} \rightarrow G$ . A straightforward computation shows that for a fixed  $x \in \mathfrak{g}$ ,

$$c_-(x+y)c_-(x)^{-1} - 1 = (1-x-y)^{-1}2y(1+x)^{-1} \quad (y \in \mathfrak{g}).$$

Hence the derivative (tangent map) is given by

$$c'_-(x)y = (1-x)^{-1}2y(1-x)^{-1} \quad (y \in \mathfrak{g}). \quad (\text{A.1})$$

Recall that  $G$  is the isometry group of a hermitian form  $(\cdot, \cdot)$  on  $V$ . Hence we have the adjoint

$$\text{End}_{\mathbb{D}}(V) \ni g \rightarrow g^* \in \text{End}_{\mathbb{D}}(V)$$

defined by

$$(gu, v) = (u, g^*v) \quad (u, v \in V).$$

Let us view the Lie algebra  $\mathfrak{g}$  as a real vector space and consider the map

$$\gamma : \text{GL}_{\mathbb{D}}(V) \rightarrow \text{GL}(\mathfrak{g}), \quad \gamma(g)(y) = gyg^*.$$

Then  $\det \circ \gamma : \text{GL}_{\mathbb{D}}(V) \rightarrow \mathbb{R}^\times$  is a group homomorphism. Hence there is a number  $s \in \mathbb{R}$  such that

$$\det(\gamma(g)) = (\det(g)_{V_{\mathbb{R}}})^s \quad (g \in \text{GL}_{\mathbb{D}}(V)),$$

where the subscript  $\mathbb{R}$  indicates that we are viewing  $V$  as a vector space over  $\mathbb{R}$ . On the other hand, for a fixed number  $a \in \mathbb{R}^\times$ ,

$$\det(\gamma(aI_V)) = a^{2 \dim \mathfrak{g}} \quad \text{and} \quad \det(aI_V)_{V_{\mathbb{R}}} = a^{\dim V_{\mathbb{R}}}.$$

Hence,

$$\det(\gamma(g)) = (\det(g)_{V_{\mathbb{R}}})^{\frac{2 \dim \mathfrak{g}}{\dim V_{\mathbb{R}}}} \quad (g \in \text{GL}_{\mathbb{D}}(V)).$$

If  $x \in \mathfrak{g}$ , then  $1 \pm x \in \text{GL}_{\mathbb{D}}(V)$  and

$$(1 \pm x)^* = 1 \mp x \quad \text{and} \quad ((1 \pm x)^{-1})^* = (1 \mp x)^{-1}.$$

Hence

$$c'_-(x)y = 2(1-x)^{-1}y(1+x)^{-1}c_-(x)y = 2(\gamma((1-x)^{-1})y)c_-(x) \quad (y \in \mathfrak{g}).$$

Notice that  $|\det(c_-(x))| = 1$  because  $c(\mathfrak{g}) \subseteq G$ . Therefore

$$|\det(c'_-(x))| = 2^{\dim \mathfrak{g}} \det(1-x)_{V_{\mathbb{R}}}^{-\frac{2 \dim \mathfrak{g}}{\dim V_{\mathbb{R}}}} = 2^{\dim \mathfrak{g}} \text{ch}(x)^{-2r} \quad (x \in \mathfrak{g}), \quad (\text{A.2})$$

where  $\text{ch}$  and  $r$  are as in (111) and (52), respectively.

APPENDIX B. **Proof of formula (112)**

Here we verify the formula (112) for the orthogonal matrix group

$$G = O_{2l+1} = \{g \in GL_{2l+1}(\mathbb{R}); gg^t = I\}.$$

The other cases are similar.

We identify

$$a + ib = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (a, b \in \mathbb{R}).$$

Then

$$SO_2(\mathbb{R}) = \{u \in \mathbb{C}; |u| = 1\}.$$

Fix the diagonal Cartan subgroup

$$H = \{\text{diag}(u_1, u_2, \dots, u_l, \pm 1); u_j \in SO_2(\mathbb{R}), 1 \leq j \leq l\}.$$

Then the connected identity component of  $H$  is

$$H^0 = \{\text{diag}(u_1, u_2, \dots, u_l, 1); u_j \in SO_2(\mathbb{R}); 1 \leq j \leq l\}.$$

The group  $\widehat{H}^0$  may be realized as

$$\widehat{H}^0 = \{\text{diag}(u_1, u_2, \dots, u_l, 1; \xi); u_j, \xi \in SO_2(\mathbb{R}), 1 \leq j \leq l; \xi^2 = u_1 u_2 \cdots u_l\}$$

and the covering map (100) is

$$\text{diag}(u_1, u_2, \dots, u_l, 1; \xi) \longrightarrow \text{diag}(u_1, u_2, \dots, u_l, 1).$$

In these terms, the usual choice of the positive roots  $e_j \pm e_k$ , with  $1 \leq j < k \leq l$ , and  $e_j$ , with  $1 \leq j \leq l$ , gives

$$\begin{aligned} \xi_{-e_j+e_k}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j^{-1} u_k, \\ \xi_{-e_j-e_k}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j^{-1} u_k^{-1}, \\ \xi_{-e_j}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j^{-1}, \\ \xi_\rho(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_1^{l-1} u_2^{l-2} \cdots u_{l-1} \xi. \end{aligned}$$

We shall use the following branch of the square root

$$\sqrt{re^{i\theta}} = \sqrt{r}e^{i\frac{\theta}{2}} \quad (r > 0, -\pi < \theta < \pi).$$

It is easy to check that for two complex numbers  $\zeta_1, \zeta_2$  in the open right half-plane,

$$\sqrt{\zeta_1 \zeta_2} = \sqrt{\zeta_1} \sqrt{\zeta_2}. \quad (\text{B.3})$$

Furthermore, if  $z \in \mathbb{C} \setminus ]-\infty, 0]$ , then

$$z = \sqrt{z} \sqrt{z} \quad \text{and} \quad \sqrt{\frac{1}{z}} = \frac{1}{\sqrt{z}}. \quad (\text{B.4})$$

If  $x = \text{diag}(ix_1, ix_2, \dots, ix_l, 0) \in \mathfrak{h}$ , then  $c_-(x) = \text{diag}(u_1, u_2, \dots, u_l, 1)$  has coordinates

$$u_j = \frac{1 + z_j}{1 - z_j}, \quad z_j = ix_j, \quad 1 \leq j \leq l,$$

with  $|u_j| = 1$  and  $u_j \neq -1$  for all  $j$ . Hence we may choose the section  $\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\mathbb{H}}^0$  of the covering map as follows:

$$\sigma(\text{diag}(u_1, u_2, \dots, u_l, 1)) = \text{diag}(u_1, u_2, \dots, u_l, 1; \sqrt{u_1}\sqrt{u_2}\cdots\sqrt{u_l}).$$

Recall that  $\widehat{c}_-(x) = (\sigma(c_-(x)), 1)$ , (104). A straightforward computation using (B.4) shows that

$$\Delta(\widehat{c}_-(x)) = \prod_{1 \leq j < k \leq l} \left( \frac{1}{\sqrt{u_j}} \sqrt{u_k} - \sqrt{u_j} \frac{1}{\sqrt{u_k}} \right) \left( \frac{1}{\sqrt{u_j}} \frac{1}{\sqrt{u_k}} - \sqrt{u_j} \sqrt{u_k} \right) \cdot \prod_{j=1}^l \left( \sqrt{u_j} - \frac{1}{\sqrt{u_j}} \right).$$

Hence (B.3) and (B.4) applied to the  $1 \pm z_j$ 's show that

$$\begin{aligned} \frac{1}{\sqrt{u_j}} \sqrt{u_k} - \sqrt{u_j} \frac{1}{\sqrt{u_k}} &= \frac{2(-z_j + z_k)}{\sqrt{1 - z_k} \sqrt{1 + z_k} \sqrt{1 - z_j} \sqrt{1 + z_j}}, \\ \frac{1}{\sqrt{u_j}} \frac{1}{\sqrt{u_k}} - \sqrt{u_j} \sqrt{u_k} &= \frac{2(-z_j - z_k)}{\sqrt{1 - z_k} \sqrt{1 + z_k} \sqrt{1 - z_j} \sqrt{1 + z_j}}, \\ \sqrt{u_j} - \frac{1}{\sqrt{u_j}} &= \frac{2z_j}{\sqrt{1 - z_j} \sqrt{1 + z_j}}. \end{aligned}$$

Therefore

$$\Delta(\widehat{c}_-(x)) = \prod_{1 \leq j < k \leq l} \frac{4(z_j - z_k)(z_j + z_k)}{(1 - z_j)(1 + z_j)(1 - z_k)(1 + z_k)} \cdot \prod_{j=1}^l \frac{2z_j}{\sqrt{1 - z_j} \sqrt{1 + z_j}}.$$

Since

$$\prod_{1 \leq j < k \leq l} (1 - z_j)(1 + z_j)(1 - z_k)(1 + z_k) = \left( \prod_{j=1}^l (1 - z_j^2) \right)^{l-1} = \left( \prod_{j=1}^l (1 + x_j^2) \right)^{l-1},$$

the formula (112) follows.

### APPENDIX C. The special functions $P_{a,b}$ and $Q_{a,b}$

For two integers  $a$  and  $b$  define the following functions in the real variable  $\xi$ ,

$$P_{a,b,2}(\xi) = \begin{cases} \sum_{k=0}^{b-1} \frac{a(a+1)\cdots(a+k-1)}{k!(b-1-k)!} 2^{-a-k} \xi^{b-1-k} & \text{if } b \geq 1 \\ 0 & \text{if } b \leq 0, \end{cases} \quad (\text{C.5})$$

$$P_{a,b,-2}(\xi) = \begin{cases} (-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1)\cdots(b+k-1)}{k!(a-1-k)!} (-2)^{-b-k} \xi^{a-1-k} & \text{if } a \geq 1 \\ 0 & \text{if } a \leq 0, \end{cases} \quad (\text{C.6})$$

where  $a(a+1)\cdots(a+k-1) = 1$  if  $k = 0$ . Notice that

$$P_{a,b,-2}(\xi) = P_{b,a,2}(-\xi) \quad (\xi \in \mathbb{R}, a, b \in \mathbb{Z}). \quad (\text{C.7})$$

Set

$$\begin{aligned} P_{a,b}(\xi) &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{a,b,-2}(\xi)\mathbb{I}_{\mathbb{R}^-}(\xi)) \\ &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{b,a,2}(-\xi)\mathbb{I}_{\mathbb{R}^+}(-\xi)), \end{aligned} \quad (\text{C.8})$$

where  $\mathbb{I}_S$  denotes the indicator function of the set  $S$ . Also, let

$$Q_{a,b}(iy) = 2\pi \begin{cases} 0 & \text{if } a + b \geq 1, \\ \sum_{k=b}^{-a} \frac{a(a+1)\cdots(a+k-1)}{k!} 2^{-a-k} (1-iy)^{k-b} & \text{if } -a > b-1 \geq 0, \\ \sum_{k=a}^{-b} \frac{b(b+1)\cdots(b+k-1)}{k!} 2^{-b-k} (1+iy)^{k-a} & \text{if } -b > a-1 \geq 0, \\ (1+iy)^{-a} (1-iy)^{-b} & \text{if } a \leq 0 \text{ and } b \leq 0. \end{cases} \quad (\text{C.9})$$

Observe also that

$$P_{b,a}(\xi) = P_{a,b}(-\xi) \quad \text{and} \quad Q_{b,a}(iy) = Q_{a,b}(-iy). \quad (\text{C.10})$$

The following elementary fact will be crucial at several points.

**Lemma C.1.** *Suppose that  $a + b \leq 1$ . Then at most one between  $P_{a,b,2}$  and  $P_{a,b,-2}$  can be non-zero. Hence  $P_{a,b}$  is either 0 or a the restriction of a polynomial to a half line.*

**Remark 11.** Let  $\Gamma$  denote the gamma function. If  $k$  is a nonnegative integer, then

$$a(a+1)\cdots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)},$$

which is often shortened by the Pochhammer symbol  $(a)_k$ . Another useful formula is

$$a(a+1)\cdots(a+k-1) = (-1)^k (-a)(-a-1)\cdots(-a-k+1) = (-1)^k \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)}.$$

In this notation, for an integer  $b \geq 1$  and  $h = 0, 1, \dots, b-1$ ,

$$(b-1-h)! = \frac{(b-1)!}{(-b+1)_h} \quad \text{and} \quad \Gamma(-a-b+2+h) = \Gamma(-a-b+2) (-a-b+2)_h.$$

Hence

$$\begin{aligned} P_{a,b,2}(\xi) &= \sum_{k=0}^{b-1} (-1)^k \frac{\Gamma(-a+1)}{\Gamma(-a+1-k)} \frac{1}{k!(b-1-k)!} 2^{-a-k} \xi^{b-1-k} \\ &= \Gamma(-a+1) \sum_{h=0}^{b-1} (-1)^{b-1-h} \frac{1}{\Gamma(-a-b+2+h)} \frac{1}{(b-1-h)!h!} 2^{-a-b+1+h} \xi^h \\ &= (-1)^{b-1} 2^{-a-b+1} \frac{\Gamma(-a+1)}{\Gamma(-a-b+2)(b-1)!} \sum_{h=0}^{b-1} \frac{(-b+1)_h}{(-a-b+2)_h h!} (2\xi)^h \\ &= (-1)^{b-1} 2^{-a-b+1} \frac{\Gamma(-a+1)}{\Gamma(-a-b+2)(b-1)!} {}_1F_1(-b+1; -a-b+2; 2\xi) \\ &= (-1)^{b-1} 2^{-a-b+1} L_{b-1}^{-a-b+1}(2\xi), \end{aligned}$$

where  ${}_1F_1$  is the confluent hypergeometric function and  $L_n^\alpha(x)$  is a Laguerre polynomial. See [Erd53, 6.9(36), §10.12].

**Proposition C.2.** *For any  $a, b \in \mathbb{Z}$ , the formula*

$$\int_{\mathbb{R}} (1+iy)^{-a} (1-iy)^{-b} \phi(y) dy \quad (\phi \in \mathcal{S}(\mathbb{R})) \quad (\text{C.11})$$

defines a tempered distribution on  $\mathbb{R}$ . The restriction of the Fourier transform of this distribution to  $\mathbb{R} \setminus \{0\}$  is a function given by

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi} dy = P_{a,b}(\xi)e^{-|\xi|}. \quad (\text{C.12})$$

The right-hand side of (C.12) is an absolutely integrable function on the real line and thus defines a tempered distribution on  $\mathbb{R}$ . Furthermore,

$$(1+iy)^{-a}(1-iy)^{-b} = \frac{1}{2\pi} \int_{\mathbb{R}} P_{a,b}(\xi)e^{-|\xi|}e^{iy\xi} dy + \frac{1}{2\pi} Q_{a,b}(iy) \quad (\text{C.13})$$

and hence,

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b}e^{-iy\xi} dy = P_{a,b}(\xi)e^{-|\xi|} + Q_{a,b}\left(-\frac{d}{d\xi}\right)\delta_0(\xi). \quad (\text{C.14})$$

*Proof.* Since,  $|1 \pm iy| = \sqrt{1+y^2}$ , (C.11) is clear. The integral (C.12) is equal to

$$\begin{aligned} & \frac{1}{i} \int_{i\mathbb{R}} (1+z)^{-a}(1-z)^{-b}e^{-z\xi} dz \\ &= 2\pi(-\mathbb{I}_{\mathbb{R}^+}(\xi) \operatorname{res}_{z=1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi} + \mathbb{I}_{\mathbb{R}^-}(\xi) \operatorname{res}_{z=-1}(1+z)^{-a}(1-z)^{-b}e^{-z\xi}). \end{aligned} \quad (\text{C.15})$$

The computation of the two residues is straightforward and (C.12) follows.

Since

$$\int_0^\infty e^{-\xi}e^{i\xi y} d\xi = (1-iy)^{-1},$$

we have

$$\int_0^\infty \xi^m e^{-\xi}e^{i\xi y} d\xi = \left(\frac{d}{d(iy)}\right)^m (1-iy)^{-1} = m!(1-iy)^{-m-1} \quad (m = 0, 1, 2, \dots). \quad (\text{C.16})$$

Thus, if  $b \geq 1$ , then

$$\begin{aligned} & \int_0^\infty P_{a,b,2}(\xi)e^{-\xi}e^{i\xi y} d\xi \\ &= \sum_{k=0}^{b-1} \frac{a(a+1)\cdots(a+k-1)}{k!} 2^{-a-k}(1-iy)^{-b+k} \\ &= (1-iy)^{-b} 2^{-a} \sum_{k=0}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k. \end{aligned}$$

Also, if  $a \leq 0$ , then

$$\begin{aligned} 2^a(1+iy)^{-a} &= \left(1 - \frac{1}{2}(1-iy)\right)^{-a} = \sum_{k=0}^{-a} \binom{-a}{k} \left(-\frac{1}{2}(1-iy)\right)^k \\ &= \sum_{k=0}^{-a} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k. \end{aligned}$$



Hence,

$$\begin{aligned} & \int_0^\infty P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y} d\xi - (1+iy)^{-a} (1-iy)^{-b} \\ &= (1-iy)^{-b} 2^{-a} \left( \sum_{k=0}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k \right. \\ & \quad \left. - \sum_{k=0}^{-a} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}(1-iy)\right)^k \right). \end{aligned} \quad (\text{C.17})$$

Recall that  $P_{a,b,-2} = 0$  if  $a \leq 0$ . Hence, (C.12) shows that (C.17) is the inverse Fourier transform of a distribution supported at  $\{0\}$ , hence a polynomial.

Suppose  $-a < b - 1$ . Then (C.17) is equal to

$$2^{-a} \sum_{k=-a+1}^{b-1} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1-iy)^{k-b},$$

which is zero because  $(-a)(-a-1)\cdots(-a-k+1) = 0$  for  $k \geq -a + 1$ . If  $-a = b - 1$ , then (C.17) is obviously zero.

Suppose  $-a > b - 1$ . Then (C.17) is equal to

$$-2^{-a} \sum_{k=b}^{-a} \frac{(-a)(-a-1)\cdots(-a-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1-iy)^{k-b}. \quad (\text{C.18})$$

As in (C.16) we have

$$\int_{-\infty}^0 \xi^m e^\xi e^{i\xi y} d\xi = \left(\frac{d}{d(iy)}\right)^m (1+iy)^{-1} = (-1)^m m! (1+iy)^{-m-1} \quad (m = 0, 1, 2, \dots).$$

Suppose  $a \geq 1$ . Then

$$\begin{aligned} & \int_{-\infty}^0 P_{a,b,-2}(\xi) e^\xi e^{i\xi y} d\xi \\ &= (-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1)\cdots(b+k-1)}{k!} (-2)^{-b-k} (-1)^{a-1+k} (1+iy)^{-a+k} \\ &= (1+iy)^{-a} 2^{-b} \sum_{k=0}^{a-1} \frac{(-b)(-b-1)\cdots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k. \end{aligned}$$

Also, if  $b \leq 0$ , then

$$2^b (1-iy)^{-b} = \sum_{k=0}^{-b} \frac{(-b)(-b-1)\cdots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k.$$

Hence,

$$\begin{aligned} & \int_{-\infty}^0 P_{a,b,-2}(\xi) e^\xi e^{i\xi y} d\xi - (1+iy)^{-a} (1-iy)^{-b} \\ &= (1+iy)^{-a} 2^{-b} \left( \sum_{k=0}^{a-1} \frac{(-b)(-b-1)\cdots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k \right. \\ & \quad \left. - \sum_{k=0}^{-b} \frac{(-b)(-b-1)\cdots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k \right). \end{aligned} \quad (\text{C.19})$$

As before, we show that (C.19) is zero if  $-b \leq a-1$ . If  $-b > a-1$ , then (C.19) is equal to

$$-2^{-b} \sum_{k=a}^{-b} \frac{(-b)(-b-1)\cdots(-b-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1+iy)^{k-a}.$$

If  $a \geq 1$  and  $b \geq 1$ , then our computations show that

$$\int_0^\infty P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y} d\xi + \int_{-\infty}^0 P_{a,b,-2}(\xi) e^\xi e^{i\xi y} d\xi - (1+iy)^{-a} (1-iy)^{-b} \quad (\text{C.20})$$

is a polynomial which tends to zero if  $y$  goes to infinity. Thus (C.20) is equal zero. This completes the proof of (C.13). The statement (C.14) is a direct consequence of (C.13).  $\square$

The test functions which occur in Proposition C.2 need not be in the Schwartz space. In fact the test functions we shall use in our applications are not necessarily smooth. Therefore we shall need a more precise version of the formula (C.14). This requires a definition and two well-known lemmas.

Following Harish-Chandra denote by  $\mathcal{S}(\mathbb{R}^\times)$  the space of the smooth complex valued functions defined on  $\mathbb{R}^\times$  whose all derivatives are rapidly decreasing at infinity and have limits at zero from both sides. For  $\psi \in \mathcal{S}(\mathbb{R}^\times)$  let

$$\psi(0+) = \lim_{x \rightarrow 0+} \psi(x), \quad \psi(0-) = \lim_{x \rightarrow 0-} \psi(x), \quad \langle \psi \rangle_0 = \psi(0+) - \psi(0-).$$

In particular the condition  $\langle \psi \rangle_0 = 0$  means that  $\psi$  extends to a continuous function on  $\mathbb{R}$ .

**Lemma C.3.** *Let  $c = 0, 1, 2, \dots$  and let  $\psi \in \mathcal{S}(\mathbb{R}^\times)$ . Suppose*

$$\langle \psi \rangle_0 = \dots = \langle \psi^{(c-1)} \rangle_0 = 0. \quad (\text{C.21})$$

(The condition (C.21) is empty if  $c = 0$ .) Then

$$\left| \int_{\mathbb{R}^\times} e^{-iy\xi} \psi(\xi) d\xi \right| \leq \min\{1, |y|^{-c-1}\} (|\langle \psi^{(c)} \rangle_0| + \|\psi^{(c+1)}\|_1 + \|\psi\|_1) \quad (\text{C.22})$$

*Proof.* Integration by parts shows that for  $z \in \mathbb{C}^\times$

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi &= z^{-1} \psi(0+) + \dots + z^{-c-1} \psi^{(c)}(0+) + z^{-c-1} \int_{\mathbb{R}^+} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi, \\ \int_{\mathbb{R}^-} e^{-z\xi} \psi(\xi) d\xi &= -z^{-1} \psi(0-) - \dots - z^{-c-1} \psi^{(c)}(0-) + z^{-c-1} \int_{\mathbb{R}^-} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{\mathbb{R}^\times} e^{-z\xi} \psi(\xi) d\xi \\ &= z^{-1} \langle \psi \rangle_0 + \cdots + z^{-c} \langle \psi^{(c-1)} \rangle_0 + z^{-c-1} \langle \psi^{(c)} \rangle_0 + z^{-c-1} \int_{\mathbb{R}^\times} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi \end{aligned}$$

and (C.22) follows.  $\square$

**Lemma C.4.** *Under the assumptions of Lemma C.3, with  $1 \leq c$ ,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \psi^{(k)}(0) \quad (0 \leq k \leq c-1),$$

where each consecutive integral is absolutely convergent.

*Proof.* Since

$$\int_{\mathbb{R}} |y|^{c-1} \min\{1, |y|^{-c-1}\} dy < \infty,$$

the absolute convergence follows from Lemma C.3. Since the Fourier transform of  $\psi$  is absolutely integrable and since  $\psi$  is continuous at zero, Fourier inversion formula [Hör83, (7.1.4)] shows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^\times} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \psi(0). \quad (\text{C.23})$$

Also, for  $0 < k$ ,

$$\begin{aligned} & \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi = \int_{\mathbb{R}^\times} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi \\ &= \int_{\mathbb{R}^+} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi + \int_{\mathbb{R}^-} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi \\ &= (iy)^{k-1} \psi(0+) + \int_{\mathbb{R}^+} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi \\ &\quad - (iy)^{k-1} \psi(0-) + \int_{\mathbb{R}^-} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi \\ &= (iy)^{k-1} \langle \psi \rangle_0 + \int_{\mathbb{R}^\times} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi. \end{aligned}$$

Hence, by induction on  $k$  and by our assumption

$$\begin{aligned} \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi &= (iy)^{k-1} \langle \psi \rangle_0 + (iy)^{k-2} \langle \psi' \rangle_0 + \cdots + \langle \psi^{(k-1)} \rangle_0 \\ &\quad + \int_{\mathbb{R}^\times} e^{-iy\xi} \psi^{(k)}(\xi) d\xi \\ &= \int_{\mathbb{R}^\times} e^{-iy\xi} \psi^{(k)}(\xi) d\xi. \end{aligned}$$

Therefore our lemma follows from (C.23).  $\square$

The following proposition is an immediate consequence of Lemmas C.3, C.4, and the formula (C.13).

**Proposition C.5.** Fix two integers  $a, b \in \mathbb{Z}$  and a function  $\psi \in \mathcal{S}(\mathbb{R}^\times)$ . Let  $c = -a - b$ . If  $c \geq 0$  assume that

$$\langle \psi \rangle_0 = \cdots = \langle \psi^{(c)} \rangle_0 = 0. \quad (\text{C.24})$$

Then

$$\begin{aligned} & \int_{\mathbb{R}} \int_{\mathbb{R}^\times} (1 + iy)^{-a} (1 - iy)^{-b} e^{-iy\xi} \psi(\xi) d\xi dy \\ &= \int_{\mathbb{R}^\times} P_{a,b}(\xi) e^{-|\xi|} \psi(\xi) d\xi + Q_{a,b}(\partial_\xi) \psi(\xi) |_{\xi=0} \\ &= \int_{\mathbb{R}} (P_{a,b}(\xi) e^{-|\xi|} + Q_{a,b}(-\partial_\xi) \delta_0(\xi)) \psi(\xi) d\xi, \end{aligned} \quad (\text{C.25})$$

where  $\delta_0$  denotes the Dirac delta at 0.

(Recall that  $Q_{a,b} = 0$  if  $c < 0$  and  $Q_{a,b}$  is a polynomial of degree  $c$  if  $c \geq 0$ .)

Let  $\mathcal{S}(\mathbb{R}^+)$  be the space of the smooth complex valued functions whose all derivatives are rapidly decreasing at infinity and have limits at zero. Then  $\mathcal{S}(\mathbb{R}^+)$  may be viewed as the subspace of the functions in  $\mathcal{S}(\mathbb{R}^\times)$  which are zero on  $\mathbb{R}^-$ . Similarly we define  $\mathcal{S}(\mathbb{R}^-)$ . The following propositions are direct consequences of Proposition C.5. We sketch independent proofs below.

**Proposition C.6.** There is a seminorm  $p$  on the space  $\mathcal{S}(\mathbb{R}^+)$  such that

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right| \leq \min\{1, |z|^{-1}\} p(\psi) \quad (\psi \in \mathcal{S}(\mathbb{R}^+), \operatorname{Re} z \geq 0), \quad (\text{C.26})$$

and similarly for  $\mathcal{S}(\mathbb{R}^-)$ .

Fix integers  $a, b \in \mathbb{Z}$  with  $a + b \geq 1$ . Then for any function  $\psi \in \mathcal{S}(\mathbb{R}^+)$ ,

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^+} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^+} P_{a,b,2}(\xi) e^{-\xi} \psi(\xi) d\xi, \quad (\text{C.27})$$

and any function  $\psi \in \mathcal{S}(\mathbb{R}^-)$ ,

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^-} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^-} P_{a,b,-2}(\xi) e^{\xi} \psi(\xi) d\xi, \quad (\text{C.28})$$

where each consecutive integral is absolutely convergent.

*Proof.* Clearly

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right| \leq \int_{\mathbb{R}^+} e^{-\operatorname{Re} z\xi} |\psi(\xi)| d\xi \leq \|\psi\|_1.$$

Integration by parts shows that for  $z \neq 0$ ,

$$\int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi = z^{-1} \psi(0) + z^{-1} \int_{\mathbb{R}^+} e^{-z\xi} \psi'(\xi) d\xi.$$

Hence (C.26) follows with  $p(\psi) = |\psi(0)| + \|\psi\|_1 + \|\psi'\|_1$ .

Let  $a, b \in \mathbb{Z}$  be such that  $a + b \geq 1$ . Then the function

$$(1 + z)^{-a} (1 - z)^{-b} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi$$

is continuous on  $\operatorname{Re} z \geq 0$  and meromorphic on  $\operatorname{Re} z > 0$  and (C.26) shows that it is dominated by  $|z|^{-2}$ . Therefore Cauchy's Theorem implies that the left-hand side of (C.27) is equal to

$$-2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-b} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right).$$

The computation of this residue is straightforward. This verifies (C.27). The proof of (C.28) is entirely analogous.  $\square$

Let  $a$  and  $b$  be integers. For  $\xi, y \in \mathbb{R}$  define

$$P_{a,b,\pm 2}^b(\xi) = P_{a,b,\pm 2}(\xi) - P_{a-1,b,\pm 2}(\xi), \quad (\text{C.29})$$

where, as before,  $\mathbb{I}_S$  denotes the indicator function of the set  $S$ .

**Proposition C.7.** *Fix integers  $a, b \in \mathbb{Z}$  with  $a+b \geq 2$ . Then for any function  $\psi \in \mathcal{S}(\mathbb{R}^+)$ ,*

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^+} iye^{-iy\xi} \psi(\xi) d\xi dy = -2\pi \int_{\mathbb{R}^+} P_{a,b,2}^b(\xi) e^{-\xi} \psi(\xi) d\xi, \quad (\text{C.30})$$

and any function  $\psi \in \mathcal{S}(\mathbb{R}^-)$ ,

$$\int_{\mathbb{R}} (1+iy)^{-a}(1-iy)^{-b} \int_{\mathbb{R}^-} iye^{-iy\xi} \psi(\xi) d\xi dy = -2\pi \int_{\mathbb{R}^-} P_{a,b,-2}^b(\xi) e^{\xi} \psi(\xi) d\xi, \quad (\text{C.31})$$

where each consecutive integral is absolutely convergent.

*Proof.* The function

$$(1+z)^{-a}(1-z)^{-b} z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi$$

is continuous on  $\operatorname{Re} z \geq 0$  and meromorphic on  $\operatorname{Re} z > 0$  and since  $a+b \geq 2$ , (C.26) shows that it is dominated by  $|z|^{-2}$ . Therefore Cauchy's Theorem implies that the left-hand side of (C.27) is equal to

$$-2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-b} z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right). \quad (\text{C.32})$$

Since

$$(1+z)^{-a}(1-z)^{-b} z = (1+z)^{-(a-1)}(1-z)^{-b} - (1+z)^{-a}(1-z)^{-b}$$

(C.32) equals

$$\begin{aligned} & 2\pi \operatorname{res}_{z=1} \left( (1+z)^{-(a-1)}(1-z)^{-b} z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right) \\ & - 2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-b} z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right). \end{aligned} \quad (\text{C.33})$$

These residues were computed in the proof of Proposition C.6. This verifies (C.30). The proof of (C.31) is entirely analogous.  $\square$

**Remark 12.** By a change of variables in the integrals of Proposition C.7, one sees that  $P_{b,a,2}^b(\xi) = -P_{a,b,-2}^b(-\xi)$ . One can also deduce this property from (C.10), by observing that  $P_{a,b,\pm 2}^b = \frac{1}{2}(P_{a,b-1,\pm 2} - P_{a-1,b,\pm 2})$ .

APPENDIX D. **The covering**  $\widetilde{G} \rightarrow G$ 

In this appendix we recall some results about the splitting of the restrictions  $\widetilde{L} \rightarrow L$  of the metaplectic covering

$$1 \rightarrow \{\pm 1\} \rightarrow \widetilde{\mathrm{Sp}}(W) \rightarrow \mathrm{Sp}(W) \rightarrow 1 \quad (\text{D.34})$$

to a subgroup  $L$  of the compact member  $G$  of a dual pair  $(G, G')$  as in (2). This is well known, but we could not find a reference sketching the proofs of the results we are using in this paper. We are therefore providing a short and complete argument.

If  $K$  is a maximal compact subgroup of  $\mathrm{Sp}(W)$ , then  $\widetilde{K}$  is a maximal compact subgroup of  $\widetilde{\mathrm{Sp}}(W)$ . The group  $\widetilde{\mathrm{Sp}}(W)$  is connected, noncompact, semisimple and with finite center  $\widetilde{Z}$ . (Since  $\widetilde{\mathrm{Sp}}(W)$  is a double cover of  $\mathrm{Sp}(W)$ , only the connectedness needs to be commented. It follows from the fact that the covering (D.34) does not split; see e.g. [AP14, Proposition 4.20] or the original proof [Wei64, p. 199]). The maximal compact subgroup  $\widetilde{K}$  is therefore connected; see e.g. [Hel78, Chapter VI, Theorem 1.1]. Hence the covering

$$\widetilde{K} \rightarrow K \quad (\text{D.35})$$

does not split.

As is well known,  $K$  is isomorphic to a compact unitary group. In fact, if  $W = \mathbb{R}^{2n}$  and

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad (\text{D.36})$$

then

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b \in \mathrm{GL}_n(\mathbb{R}), ab^t = ba^t, aa^t + bb^t = I_n \right\} \quad (\text{D.37})$$

is a maximal compact subgroup of  $\mathrm{Sp}_{2n}(\mathbb{R})$  and

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} \ni \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightarrow a + ib \in U_n \quad (\text{D.38})$$

is a Lie group isomorphism. Any two maximal compact subgroups of  $\mathrm{Sp}(W)$  are conjugate by an inner automorphism. Let  $K \rightarrow \mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$  be the corresponding isomorphism. Composition with (D.38) fixes then an isomorphism  $\phi : K \rightarrow U_n$ . Set

$$\widetilde{K}^\phi = \{(u, \zeta) \in K \times \mathbb{C}^\times; \det(\phi(u)) = \zeta^2\} \quad (\text{D.39})$$

Recall the bijection between equivalence classes of  $n$ -fold path-connected coverings and the conjugacy classes of index- $n$  subgroups of the fundamental group (see e.g. [Hat01, Theorem 1.38]). Then, up to an isomorphism of coverings,  $U_n$  has only one connected double cover. Hence (D.35) is isomorphic to

$$\widetilde{K}^\phi \ni (u, \zeta) \rightarrow u \in K. \quad (\text{D.40})$$

Let  $L \subseteq K$  be any subgroup and

$$\widetilde{L} \rightarrow L \quad (\text{D.41})$$

the restriction of the covering (D.35) to  $L$ . Let  $\widetilde{L}^\phi$  be the preimage of  $L$  in  $\widetilde{K}^\phi$ . Then (D.41) splits if and only if

$$\widetilde{L}^\phi \rightarrow L \quad (\text{D.42})$$

splits, i.e. there is a group homomorphism  $L \ni g \rightarrow \zeta(g) \in U_1 \subset \mathbb{C}^\times$  such that  $\zeta(g)^2 = \det(\phi(g))$  for all  $g \in L$ . For instance, if  $L$  is a connected subgroup of  $K$  such that

$$L \subseteq \{u \in K; \det(\phi(u)) = 1\}, \quad (\text{D.43})$$

then (D.41) splits.

To fix  $\phi$ , let  $(V, (\cdot, \cdot))$  and  $(V', (\cdot, \cdot)')$  be the defining spaces of  $G$  and  $G'$ , respectively, with  $\dim_{\mathbb{D}} V = d$  and  $\dim_{\mathbb{D}} V' = d'$ . Realize  $W$  as  $V \otimes_{\mathbb{D}} V'$ , considered as a real symplectic space, with symplectic form  $\langle \cdot, \cdot \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}((\cdot, \cdot) \otimes (\cdot, \cdot)')$ , where  $\text{tr}_{\mathbb{D}/\mathbb{R}}$  denotes the reduced trace; see [How79, §5] and [Wei73, p. 169]. Then the group  $G$  is viewed as a subgroup of  $\text{Sp}(W)$  via the identification  $G \ni g \rightarrow g \otimes 1 \in \text{Sp}(W)$ .<sup>1</sup> Similarly,  $G'$  is viewed as a subgroup of  $\text{Sp}(W)$  via the identification  $G' \ni g' \rightarrow 1 \otimes g' \in \text{Sp}(W)$ . Recall that  $n$ -by- $n$ -matrices over  $\mathbb{C}$  can be identified with  $2n$ -by- $2n$  matrices over  $\mathbb{R}$  under the isomorphism

$$\alpha : M \rightarrow \begin{pmatrix} \text{Re } M & -\text{Im } M \\ \text{Im } M & \text{Re } M \end{pmatrix}.$$

Moreover,  $n$ -by- $n$ -matrices over  $\mathbb{H}$  can be identified with  $2n$ -by- $2n$  matrices over  $\mathbb{C}$  under the isomorphism

$$\beta : M \rightarrow \begin{pmatrix} z_1(M) & -\overline{z_2(M)} \\ z_2(M) & \overline{z_1(M)} \end{pmatrix}.$$

Here, for  $v \in \mathbb{H}$ , we write  $v = z_1(v) + jz_2(v)$  with  $z_1(v), z_2(v) \in \mathbb{C}$ , and we similarly define  $z_1(M)$  and  $z_2(M)$  if  $M$  is a matrix over  $\mathbb{H}$ .

Since  $G$  is compact, there is a compatible positive complex structure  $J$  on  $W$  such that the maximal compact subgroup  $K = \text{Sp}(W)^J$  of  $\text{Sp}(W)$  contains  $G$ . Moreover, since  $G$  commutes with  $J$ , there is  $J' \in G'$  such that  $J = 1 \otimes J'$ . Set  $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ . Then, the explicit expressions of  $J'$  with respect to the standard basis of  $V \simeq \mathbb{D}^d$  and of  $J$  with respect to the standard basis of  $W \simeq \mathbb{R}^{2n}$  are given as follows:

$(G, G')$	$J'$	$n$	$J$
$(O_d, \text{Sp}_{2m}(\mathbb{R}))$	$J_{2m}$	$md$	$J_{2md}$
$(U_d, U_{p,q})$	$-iI_{p,q}$	$d(p+q)$	$\begin{pmatrix} 0 & I_{dp,dq} \\ -I_{dp,dq} & 0 \end{pmatrix}$
$(\text{Sp}_d, O_{2m}^*)$	$-jI_m$	$2md$	$\begin{pmatrix} J_{2pm} & 0 \\ 0 & J_{2pm} \end{pmatrix}$

Notice that in the  $(U_d, U_{p,q})$ -case we have  $SJS^{-1} = J_{2d(p+q)}$  for  $S = \begin{pmatrix} I_{d(p+q)} & 0 \\ 0 & I_{dp,dq} \end{pmatrix}$ ;

in the  $(\text{Sp}_d, O_{2m}^*)$ -case,  $TJT^{-1} = J_{4pm}$  for  $T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ . Hence, in all cases we

can embed  $G$  in (D.37) from the identification  $g \rightarrow g \otimes 1 \in \text{Sp}(W)^J$  followed by the

<sup>1</sup>Following the notation at the beginning of Section 3, one should identify  $g$  and  $(g^{-1})^t \otimes 1$ .

isomorphism of  $\mathrm{Sp}(W)^J$  and  $\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$  corresponding to the conjugations by  $S$  or  $T$ , and then apply (D.38). We obtain:

$$\det(\phi(g)) = \begin{cases} \det(g)_{\mathbb{V}}^m & \text{if } (G, G') = (O_d, \mathrm{Sp}_{2m}(\mathbb{R})) \\ \det(g)_{\mathbb{V}}^{p-q} & \text{if } (G, G') = (U_d, U_{p,q}) \\ 1 & \text{if } (G, G') = (\mathrm{Sp}_d, O_{2m}^*) \end{cases}, \quad (\text{D.44})$$

where  $\det(g)_{\mathbb{V}}$  denotes the determinant of  $g$  as an element of  $G \subseteq \mathrm{GL}_{\mathbb{D}}(\mathbb{V})$ . (The determinant of an  $n$ -by- $n$  matrix over  $\mathbb{H}$  can be reduced to a determinant of a  $2n$ -by- $2n$  matrix over  $\mathbb{C}$  via the isomorphism  $\beta$ . For elements of  $\mathrm{Sp}(d)$ , this notion of determinant coincides with other possible notions of quaternionic determinants; see [Asl96] for additional information.)

**Proposition D.8.** *The covering  $\tilde{G} \rightarrow G$  splits if and only if  $\det(\phi(g))$  is a square. This happens for all pairs  $(G, G')$  different from  $(O_d, \mathrm{Sp}_{2m}(\mathbb{R}))$  with  $m$  odd and  $(U_d, U_{p,q})$  with  $p + q$  odd. In these two non-splitting cases, the covering  $\tilde{G} \rightarrow G$  is isomorphic to the  $\det^{1/2}$ -covering*

$$\sqrt{G} \ni (g, \zeta) \rightarrow g \in G \quad (\text{D.45})$$

where

$$\sqrt{G} = \{(g, \zeta) \in G \times \mathbb{C}^\times; \zeta^2 = \det(g)_{\mathbb{V}}\}. \quad (\text{D.46})$$

*Proof.* By (D.44) there is a group homomorphism  $G \ni g \rightarrow \zeta(g) \in U_1 \subseteq \mathbb{C}^\times$  so that  $\zeta(g)^2 = \det(\phi(g))$  for all pairs  $(G, G')$  except at most the two cases listed in the statement of the Proposition.

Suppose that  $G' = \mathrm{Sp}_{2m}(\mathbb{R})$ , and let  $\zeta : O_d \rightarrow U_1$  be a continuous group homomorphism so that  $\zeta(g)^2 = \det(g)_{\mathbb{V}}^m = (\pm 1)^m$ . Then  $\zeta(O_d) \subseteq \{\pm 1, \pm i\}$  and it is a subgroup with at most two elements. So  $\zeta(O_d) \subseteq \{\pm 1\}$ . On the other hand, if  $g \in O_d \setminus \mathrm{SO}_d$ , then  $\det(g)_G = -1$ . Thus  $\zeta(g)^2 \neq \det(g)_{\mathbb{V}}^m$  if  $m$  is odd.

Suppose now that  $G' = U_{p,q}$ , and let  $\zeta : U_d \rightarrow U_1$  be a continuous group homomorphism so that  $\zeta(g)^2 = \det(g)_{\mathbb{V}}^{p-q}$ . Restriction to  $U_1 \equiv \{\mathrm{diag}(h, 1, \dots, 1); h \in U_1\} \subseteq U_d$  yields a continuous group homomorphism  $h \in U_1 \rightarrow \zeta(h) \in U_1$ . Thus, there is  $k \in \mathbb{Z}$  so that  $\zeta(h) = h^k$  for all  $h \in U_1$ . So  $h^{2k} = \zeta(h)^2 = \det(\mathrm{diag}(h, 1, \dots, 1))^{p-q}$  implies that  $p + q$  must be even.

For the last statement, consider for  $k \in \mathbb{Z}$  the covering  $M_k = \{(g, \zeta) \in G \times \mathbb{C}^\times; \zeta^2 = \det(g)_{\mathbb{V}}^{2k+1}\}$  of  $G$ . Then  $(g, \zeta) \rightarrow (g, \zeta^{\frac{1}{2k+1}})$  is a covering isomorphism between  $M_k$  and  $M_0$ .  $\square$

**Remark 13.** Keep the notation of (D.39) and let  $\alpha : \tilde{K}^\phi \rightarrow \tilde{K}$  be the isomorphism lifting  $\phi^{-1} : U_n \rightarrow K$ . Then, by [Fol89, Proposition 4.39] or [Prz89, (1.4.17)], the map

$$(u, \zeta) \rightarrow \zeta^{-1} \omega(\alpha(u, \zeta))$$

is independent of  $\zeta$ .

## APPENDIX E. Highest weights of irreducible genuine representations of $\tilde{G}$

In this appendix we collect some root/weight data for the irreducible genuine representations of  $\tilde{G}$ , where  $G$  is a compact member of a reductive dual pair  $(G, G')$ . Let  $\mathfrak{h}$



be a fixed Cartan subalgebra of the Lie algebra  $\mathfrak{g}$  of  $G$ . We denote by  $\Delta^+$  a choice of positive roots for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  and by  $\rho$  the one-half of their sum. The genuine irreducible representations of  $\tilde{G}$  are parametrized by their highest weights  $\lambda = \sum_{j=1}^l \lambda_j e_j$  listed below.

$$\underline{(G, G') = (U_l, U_{p,q}), \quad l \geq 1, \quad q \geq p \geq 0, \quad p + q \geq 1:}$$

If  $l = 1$ , then  $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ . If  $l \geq 2$ , then:

$$\Delta^+ = \{e_j - e_k; \quad 1 \leq j < k \leq l\} \text{ (type } A_{l-1}), \quad \rho = \sum_{j=1}^l \left(\frac{l+1}{2} - j\right) e_j,$$

$$\lambda_j = \frac{q-p}{2} + \nu_j, \quad \nu_j \in \mathbb{Z}, \quad \nu_1 \geq \nu_2 \geq \cdots \geq \nu_l.$$

$$\underline{(G, G') = (O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})), \quad l \geq 0, \quad l' \geq 1:}$$

If  $l = 0$ , then  $\mathfrak{g} = 0$ . If  $l \geq 1$ , then:

$$\Delta^+ = \{e_j \pm e_k; \quad 1 \leq j < k \leq l\} \cup \{e_j; \quad 1 \leq j \leq l\} \text{ (type } B_l), \quad \rho = \sum_{j=1}^l \left(l + \frac{1}{2} - j\right) e_j,$$

$$\lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0.$$

$$\underline{(G, G') = (\text{Sp}_l, O_{2l'}^*), \quad l \geq 1, \quad l' \geq 2:}$$

$$\Delta^+ = \{e_j \pm e_k; \quad 1 \leq j < k \leq l\} \cup \{2e_j; \quad 1 \leq j \leq l\} \text{ (type } C_l), \quad \rho = \sum_{j=1}^l (l + 1 - j) e_j,$$

$$\lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l \geq 0.$$

$$\underline{(G, G') = (O_{2l}, \text{Sp}_{2l'}(\mathbb{R})), \quad l \geq 1, \quad l' \geq 1:}$$

If  $l = 1$ , then  $\mathfrak{h}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ . If  $l \geq 2$ , then:

$$\Delta^+ = \{e_j \pm e_k; \quad 1 \leq j < k \leq l\} \text{ (type } D_l), \quad \rho = \sum_{j=1}^l (l - j) e_j,$$

$$\lambda_j \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq |\lambda_l|.$$

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