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# SYMMETRY BREAKING OPERATORS FOR DUAL PAIRS WITH ONE MEMBER COMPACT

M. MCKEE, A. PASQUALE, AND T. PRZEBINDA

ABSTRACT. We consider a dual pair  $(G, G')$ , in the sense of Howe, with  $G$  compact acting on  $L^2(\mathbb{R}^n)$ , for an appropriate  $n$ , via the Weil representation  $\omega$ . Let  $\tilde{G}$  be the preimage of  $G$  in the metaplectic group. Given a genuine irreducible unitary representation  $\Pi$  of  $\tilde{G}$ , let  $\Pi'$  be the corresponding irreducible unitary representation of  $\tilde{G}'$  in the Howe duality. The orthogonal projection onto  $L^2(\mathbb{R}^n)_\Pi$ , the  $\Pi$ -isotypic component, is the essentially unique symmetry breaking operator in  $\text{Hom}_{\tilde{G}\tilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ . We study this operator by computing its Weyl symbol. Our results allow us to compute the wavefront set of  $\Pi'$  by elementary means.

## CONTENTS

1. <b>Notation and preliminaries.</b>	4
2. <b>The center of the metaplectic group.</b>	5
3. <b>Dual pairs as Lie supergroups.</b>	7
4. <b>Main results.</b>	11
5. <b>The integral over <math>-G^0</math> as an integral over <math>\mathfrak{g}</math>.</b>	16
6. <b>The invariant integral over <math>\mathfrak{g}</math> as an integral over <math>\mathfrak{h}</math>.</b>	17
7. <b>An intertwining distribution in terms of orbital integrals on the symplectic space.</b>	20
8. <b>The special case for the pair <math>O_{2l}, \text{Sp}_{2l'}(\mathbb{R})</math>.</b>	32
9. <b>The special case for the pair <math>O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})</math>, with <math>1 \leq l \leq l'</math>.</b>	35
10. <b>The special case for the pair <math>O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})</math> with <math>l &gt; l'</math>.</b>	37
11. <b>The sketch of a computation of the wave front of <math>\Pi'</math>.</b>	39
Appendix A. <b>The Jacobian of the Cayley transform</b>	40
Appendix B. <b>Proof of formula (86)</b>	41
Appendix C. <b>The special functions <math>P_{a,b}</math> and <math>Q_{a,b}</math></b>	43
Appendix D. <b>The covering <math>\tilde{G} \rightarrow G</math></b>	49
References	52

Let  $W$  be a finite dimensional vector space over  $\mathbb{R}$  equipped with a non-degenerate symplectic form  $\langle \cdot, \cdot \rangle$  and let  $\text{Sp}(W)$  denote the corresponding symplectic group. Write

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$\widetilde{\text{Sp}}(W)$  for the metaplectic group. Let us fix the character  $\chi$  of  $\mathbb{R}$  given by  $\chi(r) = e^{2\pi ir}$ ,  $r \in \mathbb{R}$ . Then the Weil representation of  $\widetilde{\text{Sp}}(W)$  associated to  $\chi$  is denoted by  $(\omega, \mathcal{H}_\omega)$ .

For  $G, G' \subseteq \text{Sp}(W)$  forming a reductive dual pair in the sense of Howe, let  $\widetilde{G}, \widetilde{G}'$  denote their preimages in  $\widetilde{\text{Sp}}(W)$ . Howe's correspondence (or local  $\theta$ -correspondence) for  $\widetilde{G}, \widetilde{G}'$  is a bijection  $\Pi \leftrightarrow \Pi'$  between the irreducible admissible representations of  $\widetilde{G}$  and  $\widetilde{G}'$  which occur as smooth quotients of  $\omega$ , [How89b]. It can be formulated as follows. Assume that  $\text{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty) \neq 0$ . Then  $\text{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty)$  is a  $\widetilde{G}'$ -module under the action via  $\omega$ . Howe proved that it has a unique irreducible quotient, which is an irreducible admissible representation  $(\Pi', \mathcal{H}_{\Pi'})$  of  $\widetilde{G}'$ . Conversely,  $\text{Hom}_{\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty)$  is a  $\widetilde{G}$ -module which has a unique irreducible admissible quotient, infinitesimally equivalent to  $(\Pi, \mathcal{H}_\Pi)$ . Furthermore,  $\Pi \otimes \Pi'$  occurs as a quotient of  $\omega^\infty$  in a unique way, i.e.

$$\dim \text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = 1. \quad (1)$$

In [Kob15], the elements of

$$\text{Hom}_{\widetilde{G}}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty), \quad \text{Hom}_{\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_{\Pi'}^\infty) \quad \text{and} \quad \text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$$

are called symmetry breaking operators. Their construction is part of Stage C of Kobayashi's program for branching problems in the representation theory of real reductive groups.

Since the last space is one dimensional, it deserves a closer look. The explicit construction of the (essentially unique) symmetry breaking operator in  $\text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$  provides an alternative and direct approach to Howe's correspondence. To do this is the aim of the present paper.

Our basic assumption is that  $(G, G')$  is an irreducible dual pair with  $G$  compact. As shown by Howe [How79], up to an isomorphism,  $(G, G')$  is one of the pairs

$$(\text{O}_d, \text{Sp}_{2m}(\mathbb{R})), \quad (\text{U}_d, \text{U}_{p,q}), \quad (\text{Sp}_d, \text{O}_{2m}^*). \quad (2)$$

Then the representations  $\Pi$  and  $\Pi'$  together with their contragredients are arbitrary irreducible unitary highest weight representations. They have been defined by Harish-Chandra in [Har55], were classified in [EHW83] and have been studied in terms of Zuckerman functors in [Wal84], [Ada83] and [Ada87].

The crucial fact for constructing the symmetry breaking operator in  $\text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$  is that, up to a non-zero constant multiple, there is a unique  $\text{GG}'$ -invariant tempered distribution  $f_{\Pi \otimes \Pi'}$  on  $W$  such that

$$\text{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) = \mathbb{C}(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'}), \quad (3)$$

where  $\text{Op}$  and  $\mathcal{K}$  are classical transformations which we shall review in section 1. In [Prz93],  $f_{\Pi \otimes \Pi'}$  is called the intertwining distribution associated to  $\Pi \otimes \Pi'$ . In fact, if we work in a Schrödinger model of  $\omega$ , then  $f_{\Pi \otimes \Pi'}$  happens to be the Weyl symbol, [Hör83], of the operator  $(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$ .

Suppose that the group  $G$  is compact. Let  $\Theta_\Pi$  and  $d_\Pi$  respectively denote the character and the degree of  $\Pi$ . Then the projection on the  $\Pi$ -isotypic component of  $\omega$  is equal to  $d_\Pi/2$  times

$$\int_{\widetilde{G}} \omega(\tilde{g}) \check{\Theta}_\Pi(\tilde{g}) d\tilde{g} = \omega(\check{\Theta}_\Pi), \quad (4)$$

where  $\check{\Theta}_\Pi(\tilde{g}) = \Theta_\Pi(\tilde{g}^{-1})$  and we normalize the Haar measure  $d\tilde{g}$  of  $\tilde{G}$  to have the total mass 2. (This explains the constant multiple  $\frac{1}{2}$  needed for the projection. In this way, the mass of  $G$  is equal to 1.) By Howe's correspondence with  $G$  compact, the projection onto the  $\Pi$ -isotypic component of  $\omega$  is a symmetry breaking operator for  $\Pi \otimes \Pi'$ . The intertwining distribution for  $\Pi \otimes \Pi'$  is therefore determined by the equation

$$(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'}) = \omega(\check{\Theta}_\Pi). \quad (5)$$

There are more cases when  $f_{\Pi \otimes \Pi'}$  may be computed via the formula (5), see [Prz93]. However, if the group  $G$  is compact then the distribution character  $\Theta_{\Pi'}$  may also be recovered from  $f_{\Pi \otimes \Pi'}$  via an explicit formula, see [Prz91]. Thus, in this case, we have a diagram

$$\Theta_\Pi \longrightarrow f_{\Pi \otimes \Pi'} \longrightarrow \Theta_{\Pi'}. \quad (6)$$

In general, the asymptotic properties of  $f_{\Pi \otimes \Pi'}$  relate the associated varieties of the primitive ideals of  $\Pi$  and  $\Pi'$  and, under some more assumptions, the wave front sets of these representations, see [Prz93] and [Prz91].

The usual, often very successful, approach to Howe's correspondence avoids any work with distributions on the symplectic space. Instead, one finds Langlands parameters (see [Moe89], [AB95], [Pau98], [Pau00], [Pau05], [LPTZ03]), character formulas (see [Ada98], [Ren98], [DP96]), or candidates for character formulas (as in [BP14]), or one establishes preservation of unitarity (as in [Li89], [He03], [Prz93] or [ABP<sup>+</sup>07]). However, in the background (explicit or not), there is the orbit correspondence induced by the unnormalized moment maps

$$\mathfrak{g}^* \longleftarrow W \longrightarrow \mathfrak{g}'^*,$$

where  $\mathfrak{g}$  and  $\mathfrak{g}'$  denote the Lie algebras of  $G$  and  $G'$ , respectively, and  $\mathfrak{g}^*$  and  $\mathfrak{g}'^*$  are their duals. This correspondence of orbits has been studied in [DKP97], [DKP05] and [Pan10]. Furthermore, in their recent work, [LM15], H. Y. Loke and J. J. Ma computed the associated variety of the representations for the dual pairs in the stable range in terms of the orbit correspondence. The  $p$ -adic case was studied in detail in [Moe98].

Working with the  $GG'$ -invariant distributions on  $W$  is a more direct approach than relying on the orbit correspondence and provides different insights and results. As a complementary contribution to all work mentioned above, we compute the intertwining distributions  $f_{\Pi \otimes \Pi'}$  explicitly, see section 4. As an application, we obtain the wave front set of  $\Pi'$  by elementary means. The computation will be sketched in section 11, but the detailed proofs will appear in a forthcoming paper, [MPP21]. Another application of the methods presented in this paper leads to the explicit formula for the character of the corresponding irreducible unitary representation  $\Pi'$  of  $\tilde{G}'$ . This can be found in [Mer17, Mer20].

Finally, the explicit formulas for the intertwining distribution provide important information on the nature of the symmetry breaking operators. Namely, they show that none of the symmetry breaking operators of the form  $(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$  is a differential operator. For the present situation, this answers in the negative the question on the existence of differential symmetry breaking operators, addressed in different contexts by several authors (see for instance [KP16a, KP16b] and the references given there). This property is the content of Corollary 7.

## 1. Notation and preliminaries.

Let us first recall the construction of the metaplectic group  $\widetilde{\mathrm{Sp}}(W)$  and the Weil representation  $\omega$ . We are using the approach of [AP14, Section 4], to which we refer the reader for more details.

Let  $\mathfrak{sp}$  denote the Lie algebra of  $\mathrm{Sp}(W)$ . Fix a positive definite compatible complex structure  $J$  on  $W$ , that is an element  $J \in \mathfrak{sp}$  such that  $J^2 = -1$  (minus the identity on  $W$ ) and the symmetric bilinear form  $\langle J\cdot, \cdot \rangle$  is positive definite on  $W$ . For an element  $g \in \mathrm{Sp}(W)$ , let  $J_g = J^{-1}(g - 1)$ . The adjoint of  $J_g$  with respect to the form  $\langle J\cdot, \cdot \rangle$  is  $J_g^* = Jg^{-1}(1 - g)$ . In particular,  $J_g$  and  $J_g^*$  have the same kernel. Hence the image of  $J_g$  is

$$J_g W = (\mathrm{Ker} J_g^*)^\perp = (\mathrm{Ker} J_g)^\perp,$$

where  $\perp$  denotes the orthogonal with respect to  $\langle J\cdot, \cdot \rangle$ . Therefore, the restriction of  $J_g$  to  $J_g W$  defines an invertible element. Thus it makes sense to talk about  $\det(J_g)_{J_g W}^{-1}$ , the reciprocal of the determinant of the restriction of  $J_g$  to  $J_g W$ . With this notation, we have

$$\widetilde{\mathrm{Sp}}(W) = \{\tilde{g} = (g; \xi) \in \mathrm{Sp}(W) \times \mathbb{C}, \quad \xi^2 = i^{\dim(g-1)W} \det(J_g)_{J_g W}^{-1}\}. \quad (7)$$

There exists a specific 2-cocycle  $C : \mathrm{Sp}(W) \times \mathrm{Sp}(W) \rightarrow \mathbb{C}$ , see [AP14, Proposition 4.13], such that  $\widetilde{\mathrm{Sp}}(W)$  is a group with respect to the multiplication

$$(g_1; \xi_1)(g_2; \xi_2) = (g_1 g_2; \xi_1 \xi_2 C(g_1, g_2)) \quad (8)$$

and the homomorphism

$$\widetilde{\mathrm{Sp}}(W) \ni (g; \xi) \rightarrow g \in \mathrm{Sp}(W) \quad (9)$$

does not split.

Let  $\mu_W$  (or simply  $dw$ ) be the Lebesgue measure on  $W$  normalized by the condition that the volume of the unit cube with respect to the form  $\langle J\cdot, \cdot \rangle$  is 1. (Since all positive complex structures are conjugate by elements of  $\mathrm{Sp}$ , this normalization does not depend on the particular choice of  $J$ .) Let  $W = X \oplus Y$  be a complete polarization. Similar normalizations are fixed for the Lebesgue measures on every vector subspace of  $W$ , for instance on  $X$  and on  $Y$ . Furthermore, for every vector space  $V$ , we write  $\mathcal{S}(V)$  for the Schwartz space on  $V$  and  $\mathcal{S}^*(V)$  for the space of tempered distributions on  $V$ .

Each element  $K \in \mathcal{S}^*(X \times X)$  defines an operator  $\mathrm{Op}(K) \in \mathrm{Hom}(\mathcal{S}(X), \mathcal{S}^*(X))$  by

$$\mathrm{Op}(K)v(x) = \int_X K(x, x')v(x') dx'. \quad (10)$$

The map

$$\mathrm{Op} : \mathcal{S}^*(X \times X) \rightarrow \mathrm{Hom}(\mathcal{S}(X), \mathcal{S}^*(X)) \quad (11)$$

is an isomorphism of linear topological spaces. This is known as the Schwartz Kernel Theorem, [Hör83, Theorem 5.2.1].

The Weyl transform is the linear isomorphism  $\mathcal{K} : \mathcal{S}^*(W) \rightarrow \mathcal{S}^*(X \times X)$  defined for  $f \in \mathcal{S}(W)$  by

$$\mathcal{K}(f)(x, x') = \int_Y f(x - x' + y) \chi\left(\frac{1}{2}\langle y, x + x' \rangle\right) dy, \quad (12)$$

(Recall that  $\chi$  is the character of  $\mathbb{R}$  we fixed at the beginning of the introduction.)

For  $g \in \mathrm{Sp}(W)$ , let

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle (g+1)(g-1)^{-1}u, u \rangle\right) \quad (u = (g-1)w, w \in W). \quad (13)$$

Notice that, if  $g-1$  is invertible on  $W$ , then

$$\chi_{c(g)}(u) = \chi\left(\frac{1}{4}\langle c(g)u, u \rangle\right),$$

where  $c(g) = (g+1)(g-1)^{-1}$  is the usual Cayley transform.

Following [AP14, Definition 4.23 and (114)], we define

$$\begin{aligned} t : \mathrm{Sp}(W) \ni g &\longrightarrow \chi_{c(g)}\mu_{(g-1)W} \in \mathcal{S}^*(W), \\ T : \widetilde{\mathrm{Sp}}(W) \ni \tilde{g} = (g; \xi) &\longrightarrow \xi t(g) \in \mathcal{S}^*(W), \end{aligned} \quad (14)$$

where  $\mu_{(g-1)W}$  is the Lebesgue measure on the subspace  $(g-1)W$  normalized as above. Set

$$\omega = \mathrm{Op} \circ \mathcal{K} \circ T. \quad (15)$$

As proved in [AP14, Theorem 4.27],  $\omega$  is a unitary representation of  $\widetilde{\mathrm{Sp}}$  on  $L^2(X)$ . In fact,  $(\omega, L^2(X))$  is the Schrödinger model of Weil representation of  $\widetilde{\mathrm{Sp}}$  attached to the character  $\chi$  and the polarization  $W = X \oplus Y$ . In this realization,  $\mathcal{H}_\omega = L^2(X)$  and  $\mathcal{H}_\omega^\infty = \mathcal{S}(X)$ .

The distribution character of the Weil representation turns out to be the function

$$\Theta : \widetilde{\mathrm{Sp}}(W) \ni (g; \xi) \rightarrow \xi \in \mathbb{C}^\times, \quad (16)$$

[AP14, Proposition 4.27]. Hence for  $\tilde{g} \in \widetilde{\mathrm{Sp}}(W)$  in the preimage of  $g \in \mathrm{Sp}(W)$  under the double covering map (9), we have

$$T(\tilde{g}) = \Theta(\tilde{g})t(g) \quad (\tilde{g} \in \widetilde{\mathrm{Sp}}(W)). \quad (17)$$

Suppose now that  $G, G' \subseteq \mathrm{Sp}(W)$  is a dual pair. Every irreducible admissible representation  $\Pi \otimes \Pi'$  of  $\widetilde{G} \times \widetilde{G}'$  occurring in Howe's correspondence may be realized, up to infinitesimal equivalence, as a subspace of  $\mathcal{H}_\omega^{\infty*} = \mathcal{S}^*(X)$ . Hence

$$\mathrm{Hom}_{\widetilde{G}\widetilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty) \subseteq \mathrm{Hom}(\mathcal{S}(X), \mathcal{S}^*(X)).$$

The existence of the intertwining distribution  $f_{\Pi \otimes \Pi'} \in \mathcal{S}^*(X)$  defined (up to a multiplicative constant) by (3) is thus a consequence of (1), (11) and (12).

Finally, because of (15), the equation (4) leads to the equality

$$f_{\Pi \otimes \Pi'} = T(\check{\Theta}_\Pi). \quad (18)$$

The problem of finding an explicit expression for  $f_{\Pi \otimes \Pi'}$  is hence transformed into the task of computing the right-hand side of (18).

## 2. The center of the metaplectic group.

Let  $Z = \{1, -1\}$  be the center of the symplectic group  $\mathrm{Sp}(W)$ . Then  $(Z, \mathrm{Sp}(W))$  is a dual pair in  $\mathrm{Sp}(W)$  with compact member  $Z$ . Let  $(\widetilde{Z}, \widetilde{\mathrm{Sp}}(W))$  be the corresponding dual pair in the metaplectic group  $\widetilde{\mathrm{Sp}}(W)$ . Then  $\widetilde{Z}$  coincides with the center of  $\widetilde{\mathrm{Sp}}(W)$  and is equal to

$$\widetilde{Z} = \{(1; 1), (1; -1), (-1; \zeta), (-1; -\zeta)\}, \quad (19)$$

where  $\zeta = \left(\frac{i}{2}\right)^{\frac{1}{2} \dim W}$ .

In this section we illustrate how to compute the intertwining distributions for the pair  $(Z, \mathrm{Sp}(W))$ . At the same time, we introduce some facts that will be needed in the rest of the paper.

The formula for the cocycle in (8) is particularly simple over  $Z$ :

$$C(1, \pm 1) = C(-1, 1) = 1 \quad \text{and} \quad C(-1, -1) = 2^{\dim W}.$$

Also,  $C(g, 1) = C(1, g) = 1$  for all  $g \in \mathrm{Sp}(W)$  by [AP14, Proposition 4.13]. Notice that

$$(-1, \pm \zeta)^2 = (1, \zeta^2 C(-1, -1)) = (1, (-1)^{\frac{1}{2} \dim W}),$$

Hence the covering (9) restricted to  $\tilde{Z}$ ,

$$\tilde{Z} \ni \tilde{z} \rightarrow z \in Z \tag{20}$$

splits if and only if  $\frac{1}{2} \dim W$  is even.

By (14) we have

$$t(1) = \delta \quad \text{and} \quad t(-1) = \mu_W,$$

and

$$\begin{aligned} T(1; 1) &= \delta, & T(1; -1) &= -\delta, \\ T(-1; \zeta) &= \zeta \mu_W, & T(-1; -\zeta) &= -\zeta \mu_W. \end{aligned}$$

Moreover, [AP14, Proposition 4.28] shows that for  $v \in L^2(X)$  and  $x \in X$ ,

$$\begin{aligned} \omega(1; 1)v(x) &= v(x), & \omega(1; -1)v(x) &= -v(x), \\ \omega(-1; \zeta)v(x) &= \frac{\zeta}{|\zeta|}v(-x), & \omega(-1; -\zeta)v(x) &= -\frac{\zeta}{|\zeta|}v(-x). \end{aligned}$$

Since  $T(\tilde{z}) = \Theta(\tilde{z})t(z)$  for  $\tilde{z} \in \tilde{Z}$ , it follows that

$$\omega(\tilde{z})v(x) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|}v(zx) \quad (\tilde{z} \in \tilde{Z}). \tag{21}$$

The fraction

$$\chi_+(\tilde{z}) = \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad (\tilde{z} \in \tilde{Z}) \tag{22}$$

defines an irreducible character  $\chi_+$  of the group  $\tilde{Z}$ . Let  $\varepsilon$  be the unique non-trivial irreducible character of the two element group  $Z$ . Then

$$\chi_-(\tilde{z}) = \varepsilon(z) \frac{\Theta(\tilde{z})}{|\Theta(\tilde{z})|} \quad (\tilde{z} \in \tilde{Z}) \tag{23}$$

is also an irreducible character of  $\tilde{Z}$ .

Let  $L^2(X)_+ \subseteq L^2(X)$  denote the subspace of the even functions and let  $L^2(X)_- \subseteq L^2(X)$  denote the subspace of the odd functions. Then, as is well known, [KV78, (6.9)], the restriction  $\omega_\pm$  of  $\omega$  to  $L^2(X)_\pm$  is irreducible. As we have seen above, the center  $\tilde{Z}$  acts on  $L^2(X)_\pm$  via the character  $\chi_\pm$ . Thus  $\chi_\pm$  is the central character of  $\omega_\pm$ .

Hence, in the case of the dual pair  $(Z, \mathrm{Sp}(W))$ , Howe's correspondence looks as follows

$$(\chi_+, \mathbb{C}) \leftrightarrow (\omega_+, L^2(X)_+) \quad \text{and} \quad (\chi_-, \mathbb{C}) \leftrightarrow (\omega_-, L^2(X)_+). \tag{24}$$

The projections

$$L^2(X) \rightarrow L^2(X)_+ \quad \text{and} \quad L^2(X) \rightarrow L^2(X)_-$$

are respectively given by

$$\frac{1}{2}\omega(\check{\chi}_+) = \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_+(\tilde{z})\omega(\tilde{z}) \quad \text{and} \quad \frac{1}{2}\omega(\check{\chi}_-) = \frac{1}{4} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_-(\tilde{z})\omega(\tilde{z}).$$

The corresponding intertwining distributions are

$$\begin{aligned} f_{\chi_+ \otimes \omega_+} &= \frac{1}{2} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_+(\tilde{z})T(\tilde{z}) = \delta + 2^{-\frac{1}{2} \dim W} \mu_W, \\ f_{\chi_- \otimes \omega_-} &= \frac{1}{2} \sum_{\tilde{z} \in \tilde{Z}} \check{\chi}_-(\tilde{z})T(\tilde{z}) = \delta - 2^{-\frac{1}{2} \dim W} \mu_W, \end{aligned} \tag{25}$$

where we normalize the total mass of  $Z$  to be 1, as we did for a general dual pair  $(G, G')$  with  $G$  compact.

The right-hand side of (25) is a sum of two homogenous distributions of different homogeneity degrees. So, asymptotically, they can be isolated. This allows us to recover  $\mu_W$ , and hence  $\tau_{\text{sp}(W)}(W)$ , the wave front of  $\omega_{\pm}$ , out of the intertwining distribution.

### 3. Dual pairs as Lie supergroups.

To present the main results of this paper, we need the realization of dual pairs with one member compact as Lie supergroups. The content of this section is taken from [Prz06] and [MPP15]. We recall the relevant material for making our exposition self-contained.

For a dual pair  $(G, G')$  as in (2), there is a division algebra  $\mathbb{D} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  with an involution over  $\mathbb{R}$ , a finite dimensional left  $\mathbb{D}$ -vector space  $V$  with a positive definite hermitian form  $(\cdot, \cdot)$  and a finite dimensional left  $\mathbb{D}$ -vector space  $V'$  with a positive definite skew-hermitian form  $(\cdot, \cdot)'$  such that  $G$  coincides with the isometry group of  $(\cdot, \cdot)$  and  $G'$  coincides with the isometry group of  $(\cdot, \cdot)'$ .

Let  $V_{\bar{0}} = V$ ,  $d = \dim_{\mathbb{D}} V_{\bar{0}}$ ,  $V_{\bar{1}} = V'$  and  $d' = \dim_{\mathbb{D}} V_{\bar{1}}$ . We assume that both  $V_{\bar{0}}$  and  $V_{\bar{1}}$  are left vector spaces over  $\mathbb{D}$ . Set  $V = V_{\bar{0}} \oplus V_{\bar{1}}$  and define an element  $S \in \text{End}(V)$  by

$$S(v_0 + v_1) = v_0 - v_1 \quad (v_0 \in V_{\bar{0}}, v_1 \in V_{\bar{1}}).$$

Let

$$\begin{aligned} \text{End}(V)_{\bar{0}} &= \{x \in \text{End}(V); Sx = xS\}, \\ \text{End}(V)_{\bar{1}} &= \{x \in \text{End}(V); Sx = -xS\}, \\ \text{GL}(V)_{\bar{0}} &= \text{End}(V)_{\bar{0}} \cap \text{GL}(V). \end{aligned}$$

Denote by  $(\cdot, \cdot)''$  the direct sum of the two forms  $(\cdot, \cdot)$  and  $(\cdot, \cdot)'$ . Let

$$\begin{aligned} \mathfrak{s}_{\bar{0}} &= \{x \in \text{End}(V)_{\bar{0}}; (xu, v)'' = -(u, xv)'', u, v \in V\}, \\ \mathfrak{s}_{\bar{1}} &= \{x \in \text{End}(V)_{\bar{1}}; (xu, v)'' = (u, Sxv)'', u, v \in V\}, \\ \mathfrak{s} &= \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}, \\ S &= \{s \in \text{GL}(V)_{\bar{0}}; (su, sv)'' = (u, v)'', u, v \in V\}, \\ \langle x, y \rangle &= \text{tr}_{\mathbb{D}/\mathbb{R}}(Sxy). \end{aligned} \tag{26}$$

(Here  $\text{tr}_{\mathbb{D}/\mathbb{R}}(x)$  denotes the trace of  $x$  considered as a real endomorphism of  $V$ .) Then  $(S, \mathfrak{s})$  is a real Lie supergroup, i.e. a real Lie group  $S$  together with a real Lie superalgebra  $\mathfrak{s} = \mathfrak{s}_{\bar{0}} \oplus \mathfrak{s}_{\bar{1}}$ , whose even component  $\mathfrak{s}_{\bar{0}}$  is the Lie algebra of  $S$ . (In terms of [DM99, §3.8],



( $S, \mathfrak{s}$ ) is a Harish-Chandra pair.) We shall write  $\mathfrak{s}(\mathbf{V})$  instead of  $\mathfrak{s}$  whenever we want to specify the Lie superalgebra  $\mathfrak{s}$  constructed as above from  $\mathbf{V}$  and  $(\cdot, \cdot)''$ . By restriction, we have the identification

$$\mathfrak{s}_{\bar{1}} = \text{Hom}_{\mathbb{D}}(\mathbf{V}_{\bar{1}}, \mathbf{V}_{\bar{0}}). \quad (28)$$

The group  $S$  acts on  $\mathfrak{s}$  by conjugation and  $(\cdot, \cdot)''$  is a non-degenerate  $S$ -invariant form on the real vector space  $\mathfrak{s}$ , whose restriction to  $\mathfrak{s}_{\bar{0}}$  is symmetric and restriction to  $\mathfrak{s}_{\bar{1}}$  is skew-symmetric. We shall employ the notation

$$s.x = \text{Ad}(s)x = sxs^{-1} \quad (s \in S, x \in \mathfrak{s}), \quad (29)$$

$$x(w) = \text{ad}(x)(w) = xw - wx \quad (x \in \mathfrak{s}_{\bar{0}}, w \in \mathfrak{s}_{\bar{1}}). \quad (30)$$

In terms of the notation introduced at the beginning of this section,

$$\mathfrak{g} = \mathfrak{s}_{\bar{0}}|_{\mathbf{V}_{\bar{0}}}, \quad \mathfrak{g}' = \mathfrak{s}_{\bar{0}}|_{\mathbf{V}_{\bar{1}}}, \quad G = S|_{\mathbf{V}_{\bar{0}}}, \quad G' = S|_{\mathbf{V}_{\bar{1}}}.$$

Define  $W = \text{Hom}_{\mathbb{D}}(\mathbf{V}_{\bar{1}}, \mathbf{V}_{\bar{0}})$ . Then, by restriction, we have the identification

$$W = \mathfrak{s}_{\bar{1}}. \quad (31)$$

Under this identification, the adjoint action of  $G$  on  $\mathfrak{s}_{\bar{1}}$  becomes the action on  $W$  by the left (post)multiplication. Similarly, the adjoint action of  $G'$  on  $\mathfrak{s}_{\bar{1}}$  becomes the action of  $G'$  on  $W$  via the right (pre)multiplication by the inverse. Also, we have the unnormalized moment maps

$$\tau : W \ni w \rightarrow w^2|_{\mathbf{V}_{\bar{0}}} \in \mathfrak{g}, \quad \tau' : W \ni w \rightarrow w^2|_{\mathbf{V}_{\bar{1}}} \in \mathfrak{g}'. \quad (32)$$

An element  $x \in \mathfrak{s}$  is called semisimple (resp., nilpotent) if  $x$  is semisimple (resp., nilpotent) as an endomorphism of  $\mathbf{V}$ . We say that a semisimple element  $x \in \mathfrak{s}_{\bar{1}}$  is regular if it is nonzero and  $\dim(S.x) \geq \dim(S.y)$  for all semisimple  $y \in \mathfrak{s}_{\bar{1}}$ . Let  $x \in \mathfrak{s}_{\bar{1}}$  be fixed. For  $x, y \in \mathfrak{s}_{\bar{1}}$  let  $\{x, y\} = xy + yx \in \mathfrak{s}_{\bar{0}}$  be their anticommutant.

The anticommutant and the double anticommutant of  $x$  in  $\mathfrak{s}_{\bar{1}}$  are

$$\begin{aligned} {}^x\mathfrak{s}_{\bar{1}} &= \{y \in \mathfrak{s}_{\bar{1}} : \{x, y\} = 0\}, \\ {}^{x\mathfrak{s}_{\bar{1}}}\mathfrak{s}_{\bar{1}} &= \bigcap_{y \in {}^x\mathfrak{s}_{\bar{1}}} {}^y\mathfrak{s}_{\bar{1}}, \end{aligned}$$

respectively. A Cartan subspace  $\mathfrak{h}_{\bar{1}}$  of  $\mathfrak{s}_{\bar{1}}$  is defined as the double anticommutant of a regular semisimple element  $x \in \mathfrak{s}_{\bar{1}}$ . We denote by  $\mathfrak{h}_{\bar{1}}^{reg}$  the set of regular elements in  $\mathfrak{h}_{\bar{1}}$ .

Next we describe the Cartan subspaces  $\mathfrak{h}_{\bar{1}} \subseteq \mathfrak{s}_{\bar{1}}$ . We refer to [Prz06, §6] and [MPP15, §4] for the proofs omitted here. Let  $l$  be the rank of  $\mathfrak{g}$ ,  $l'$  the rank of  $\mathfrak{g}'$ , and set

$$l'' = \min\{l, l'\}. \quad (33)$$

Given a Cartan subspace  $\mathfrak{h}_{\bar{1}}$ , there are  $\mathbb{Z}/2\mathbb{Z}$ -graded subspaces  $\mathbf{V}^j \subseteq \mathbf{V}$  such that the restriction of the form  $(\cdot, \cdot)''$  to each  $\mathbf{V}^j$  is non-degenerate,  $\mathbf{V}^j$  is orthogonal to  $\mathbf{V}^k$  for  $j \neq k$  and

$$\mathbf{V} = \mathbf{V}^0 \oplus \mathbf{V}^1 \oplus \mathbf{V}^2 \oplus \cdots \oplus \mathbf{V}^{l''}. \quad (34)$$

The subspace  $\mathbf{V}^0$  coincides with the intersection of the kernels of the elements of  $\mathfrak{h}_{\bar{1}}$  (equivalently,  $\mathbf{V}^0 = \text{Ker}(x)$  if  $\mathfrak{h}_{\bar{1}} = {}^{x\mathfrak{s}_{\bar{1}}}\mathfrak{s}_{\bar{1}}$ ). For  $1 \leq j \leq l''$ , the subspaces  $\mathbf{V}^j = \mathbf{V}_{\bar{0}}^j \oplus \mathbf{V}_{\bar{1}}^j$  are

described as follows. Suppose  $\mathbb{D} = \mathbb{R}$ . Then there is a basis  $v_0, v'_0$  of  $\mathbf{V}_0^j$  and basis  $v_1, v'_1$  of  $\mathbf{V}_1^j$  such that

$$\begin{aligned} (v_0, v_0)'' &= (v'_0, v'_0)'' = 1, & (v_0, v'_0)'' &= 0, \\ (v_1, v_1)'' &= (v'_1, v'_1)'' = 0, & (v_1, v'_1)'' &= 1. \end{aligned} \quad (35)$$

The following formulas define an element  $u_j \in \mathfrak{s}_{\bar{1}}(\mathbf{V}^j)$ ,

$$\begin{aligned} u_j(v_0) &= \frac{1}{\sqrt{2}}(v_1 - v'_1), & u_j(v_1) &= \frac{1}{\sqrt{2}}(v_0 - v'_0), \\ u_j(v'_0) &= \frac{1}{\sqrt{2}}(v_1 + v'_1), & u_j(v'_1) &= \frac{1}{\sqrt{2}}(v_0 + v'_0). \end{aligned}$$

Suppose  $\mathbb{D} = \mathbb{C}$ . Then  $\mathbf{V}_0^j = \mathbb{C}v_0$ ,  $\mathbf{V}_1^j = \mathbb{C}v_1$ , where  $(v_0, v_0)'' = 1$  and  $(v_1, v_1)'' = \delta_j i$ , with  $\delta_j = \pm 1$ . The following formulas define an element  $u_j \in \mathfrak{s}_{\bar{1}}(\mathbf{V}^j)$ ,

$$u_j(v_0) = e^{-i\delta_j \frac{\pi}{4}} v_1, \quad u_j(v_1) = e^{-i\delta_j \frac{\pi}{4}} v_0. \quad (36)$$

Suppose  $\mathbb{D} = \mathbb{H}$ . Then  $\mathbf{V}_0^j = \mathbb{H}v_0$ ,  $\mathbf{V}_1^j = \mathbb{H}v_1$ , where  $(v_0, v_0)'' = 1$  and  $(v_1, v_1)'' = i$ . The following formulas define an element  $u_j \in \mathfrak{s}_{\bar{1}}(\mathbf{V}^j)$ ,

$$u_j(v_0) = e^{-i\frac{\pi}{4}} v_1, \quad u_j(v_1) = e^{-i\frac{\pi}{4}} v_0.$$

In any case, by extending each  $u_j$  by zero outside  $\mathbf{V}^j$ , we have

$$\mathfrak{h}_{\bar{1}} = \sum_{j=1}^{l''} \mathbb{R}u_j. \quad (37)$$

The formula (37) describes a maximal family of mutually non-conjugate Cartan subspaces of  $\mathfrak{s}_{\bar{1}}$ . Notice that there is only one such subspace unless the dual pair  $(G, G')$  is isomorphic to  $(U_l, U_{p,q})$  with  $l'' = l < p+q$ . In the last case there are  $\min(l, p) - \max(l-q, 0) + 1$  such subspaces, assuming  $p \leq q$ . For each  $m$  such that  $\max(l-q, 0) \leq m \leq \min(p, l)$  there is a Cartan subspace  $\mathfrak{h}_{\bar{1},m}$  determined by the condition that  $m$  is the number of the positive  $\delta_j$  in (36). We may assume that  $\delta_1 = \cdots = \delta_m = 1$  and  $\delta_{m+1} = \cdots = \delta_l = -1$ . The choice of the spaces  $\mathbf{V}_0^j$  may be done independently of  $m$ . The spaces  $\mathbf{V}_1^j$  depend on  $m$ .

The Weyl group  $W(S, \mathfrak{h}_{\bar{1}})$  is the quotient of the stabilizer of  $\mathfrak{h}_{\bar{1}}$  in  $S$  by the subgroup  $S^{\mathfrak{h}_{\bar{1}}}$  fixing each element of  $\mathfrak{h}_{\bar{1}}$ . If  $\mathbb{D} \neq \mathbb{C}$ , then  $W(S, \mathfrak{h}_{\bar{1}})$  acts by all sign changes and all permutations of the  $u_j$ 's. If  $\mathbb{D} = \mathbb{C}$ , the Weyl group acts by all sign changes and all permutations of the  $u_j$ 's and all permutations which preserve  $(\delta_1, \dots, \delta_{l''})$ , see [Prz06, (6.3)].

Set  $\delta_j = 1$  for all  $1 \leq j \leq l''$ , if  $\mathbb{D} \neq \mathbb{C}$ , and define

$$J_j = \delta_j \tau(u_j), \quad J'_j = \delta_j \tau'(u_j) \quad (1 \leq j \leq l''). \quad (38)$$

Then  $J_j, J'_j$  are complex structures on  $\mathbf{V}_0^j$  and  $\mathbf{V}_1^j$  respectively. Explicitly,

$$\begin{aligned} J_j(v_0) &= -v'_0, & J_j(v'_0) &= v_0, & J'_j(v_1) &= -v'_1, & J'_j(v'_1) &= v_1, & \text{if } \mathbb{D} = \mathbb{R}, \\ J_j(v_0) &= -iv_0, & J'_j(v_1) &= -iv_1, & & & & & \text{if } \mathbb{D} = \mathbb{C} \text{ or } \mathbb{D} = \mathbb{H}. \end{aligned} \quad (39)$$

(The point of the multiplication by the  $\delta_j$  in (38) is that the complex structures  $J_j, J'_j$  do not depend on the Cartan subspace  $\mathfrak{h}_{\overline{1}}$ .) In particular, if  $w = \sum_{j=1}^{l''} w_j u_j \in \mathfrak{h}_{\overline{1}}$ , then

$$\tau(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J_j \quad \text{and} \quad \tau'(w) = \sum_{j=1}^{l''} w_j^2 \delta_j J'_j. \quad (40)$$

Let  $\mathfrak{h}_{\overline{1}}^2 \subseteq \mathfrak{s}_{\overline{0}}$  be the subspace spanned by all the squares  $w^2$ ,  $w \in \mathfrak{h}_{\overline{1}}$ . Then

$$\mathfrak{h}_{\overline{1}}^2 = \sum_{j=1}^{l''} \mathbb{R}(J_j + J'_j). \quad (41)$$

We shall use the following identification

$$\mathfrak{h}_{\overline{1}}^2|_{\mathfrak{V}_{\overline{0}}} \ni \sum_{j=1}^{l''} y_j J_j = \sum_{j=1}^{l''} y_j J'_j \in \mathfrak{h}_{\overline{1}}^2|_{\mathfrak{V}_{\overline{1}}} \quad (42)$$

and denote both spaces by  $\mathfrak{h}$ . Then  $\mathfrak{h}$  is diagonally embedded in  $\mathfrak{g}$  and in  $\mathfrak{g}'$ , via (41). If  $l'' = l$ , then  $\mathfrak{h}$  is an elliptic Cartan subalgebra of  $\mathfrak{g}$  and if  $l'' = l'$  then  $\mathfrak{h}$  an elliptic Cartan subalgebra of  $\mathfrak{g}'$ . If  $l < l'$  we denote by  $\mathfrak{z}'$  be the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}'$ . Similarly, when  $l' < l$  we denote by  $\mathfrak{z}$  the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$ .

Suppose  $l' < l$ . Then  $\mathfrak{V}_{\overline{1}}^0 = 0$ ,  $\mathfrak{V}_{\overline{0}}^0 \neq 0$  and

$$\mathfrak{V} = \mathfrak{V}_{\overline{0}} = \mathfrak{V}_{\overline{0}}^0 \oplus \mathfrak{V}_{\overline{0}}^1 \oplus \mathfrak{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathfrak{V}_{\overline{0}}^{l''} \quad (43)$$

is a direct sum orthogonal decomposition with respect to the positive definite hermitian form  $(\cdot, \cdot)$ . We extend  $\mathfrak{h} \subseteq \mathfrak{g}$  to a Cartan subalgebra  $\mathfrak{h}(\mathfrak{g}) \subseteq \mathfrak{g}$  as follows. The restriction of  $\mathfrak{h}(\mathfrak{g})$  to  $\mathfrak{V}_{\overline{0}}^1 \oplus \mathfrak{V}_{\overline{0}}^2 \oplus \cdots \oplus \mathfrak{V}_{\overline{0}}^{l''}$  coincides with  $\mathfrak{h}$ . Pick an orthogonal direct sum decomposition

$$\mathfrak{V}_{\overline{0}}^0 = \mathfrak{V}_{\overline{0}}^{0,0} \oplus \mathfrak{V}_{\overline{0}}^{0,l''+1} \oplus \mathfrak{V}_{\overline{0}}^{0,l''+2} \oplus \cdots \oplus \mathfrak{V}_{\overline{0}}^{0,l} \quad (44)$$

where for  $j > l''$ ,  $\dim \mathfrak{V}_{\overline{0}}^{0,j} = 2$  if  $\mathbb{D} = \mathbb{R}$  and  $\dim \mathfrak{V}_{\overline{0}}^{0,j} = 1$  if  $\mathbb{D} \neq \mathbb{R}$ . Also  $\mathfrak{V}_{\overline{0}}^{0,0} = 0$  unless  $G = O_{2l+1}$ , in which case  $\dim \mathfrak{V}_{\overline{0}}^{0,0} = 1$ . In each space  $\mathfrak{V}_{\overline{0}}^{0,j}$ , with  $j > l''$ , we pick an orthonormal basis and define  $J_j$  as in (39). Then

$$\mathfrak{h}(\mathfrak{g}) = \sum_{j=1}^l \mathbb{R} J_j. \quad (45)$$

If  $l \leq l'$ , then we set  $\mathfrak{h}(\mathfrak{g}) = \mathfrak{h}$ .

Let  $J_j^*$ ,  $1 \leq j \leq l$ , be the basis of the space  $\mathfrak{h}(\mathfrak{g})^*$  which is dual to  $J_1, \dots, J_l$ , and set

$$e_j = -iJ_j^*, \quad 1 \leq j \leq l. \quad (46)$$

If  $\mu \in i\mathfrak{h}(\mathfrak{g})^*$ , then  $\mu = \sum_{j=1}^l \mu_j e_j$  with  $\mu_j \in \mathbb{R}$ . We say that  $\mu$  is strictly dominant if  $\mu_1 > \mu_2 > \cdots > \mu_l$ .

#### 4. Main results.

Suppose an irreducible representation  $\Pi$  of  $\widetilde{G}$  occurs in Howe's correspondence. This means that there is a subspace  $\mathcal{H}_\Pi \subseteq L^2(\mathbf{X})$  on which the restriction of  $\omega$  coincides with  $\Pi$ . Since  $\widetilde{Z} \subseteq \widetilde{G} \cap \widetilde{G}'$ , then either  $\mathcal{H}_\Pi \subseteq L^2(\mathbf{X})_+$  or  $\mathcal{H}_\Pi \subseteq L^2(\mathbf{X})_-$ . In the first case the restriction of the central character  $\chi_\Pi$  of  $\Pi$  to  $\widetilde{Z}$  is equal to  $\chi_+$  and in the second case to  $\chi_-$ . Thus for  $\tilde{z} \in \widetilde{Z}$  and  $\tilde{g} \in \widetilde{G}$ ,

$$\begin{aligned} \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_+(\tilde{z})\Theta_\Pi(\tilde{g}) & \text{if } \mathcal{H}_\Pi \subseteq L^2(\mathbf{X})_+, \\ \Theta_\Pi(\tilde{z}\tilde{g}) &= \chi_-(\tilde{z})\Theta_\Pi(\tilde{g}) & \text{if } \mathcal{H}_\Pi \subseteq L^2(\mathbf{X})_-. \end{aligned} \quad (47)$$

We see from equations (17), (22), (23) and (47) that the function

$$\widetilde{G} \ni \tilde{g} \rightarrow T(\tilde{g})\check{\Theta}_\Pi(\tilde{g}) \in \mathcal{S}^*(\mathbf{W})$$

is constant on the fibers of the covering map (9). The following lemma is a restatement of (18). Our main results will be the explicit expressions of the various integrals appearing on the right-hand sides of the equations below.

**Lemma 1.** *Let  $G^0 \subseteq G$  denote the connected identity component. Suppose  $(G, G') = (U_d, U_{p,q})$  or  $(Sp_d, O_{2m}^*)$ . Then  $G = G^0 = -G^0$  and*

$$f_{\Pi \otimes \Pi'} = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg. \quad (48)$$

Formula (48) holds also if  $(G, G') = (O_d, Sp_{2m}(\mathbb{R}))$  with  $d$  even and  $\Theta_\Pi$  supported in  $\widetilde{G}^0$ , because  $G^0 = SO_d = -SO_d = -G^0$ . In the remaining cases

$$f_{\Pi \otimes \Pi'} = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg + \int_{G \setminus (-G^0)} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg. \quad (49)$$

The integrals over  $-G^0$  in (48) and (49) are given in Theorems 2 and 3 below, proved in section 7. The integrals over the other connected component, in the cases where (49) holds, are computed in Theorems 4, 5 and 6, respectively, and proved in sections 8, 9, and 10.

**Remark 1.** Notice that, since the character  $\Theta_\Pi$  is conjugation invariant,

$$\int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \int_G \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi^G) dg$$

where  $\phi^G$  is the projection of  $\phi$  onto the space of the  $G$ -invariants in  $\mathcal{S}(\mathbf{W})$ ,

$$\phi^G(w) = \int_G \phi(g.w) dg \quad (w \in \mathbf{W}) \quad (50)$$

(Recall that we have normalized the Haar measure on  $G$  so that its mass is 1.)

Let

$$\iota = \begin{cases} 1 & \text{if } \mathbb{D} = \mathbb{R} \text{ or } \mathbb{C} \\ \frac{1}{2} & \text{if } \mathbb{D} = \mathbb{H}. \end{cases} \quad (51)$$

and let

$$r = \frac{2 \dim \mathfrak{g}}{\dim \mathbf{V}_\mathbb{R}}, \quad (52)$$

where the subscript  $\mathbb{R}$  indicates that we are viewing  $\mathbf{V}$  as a vector space over  $\mathbb{R}$ . Explicitly,

$$r = \begin{cases} 2l - 1 & \text{if } G = O_{2l}, \\ 2l & \text{if } G = O_{2l+1}, \\ l & \text{if } G = U_l, \\ l + \frac{1}{2} & \text{if } G = \mathrm{Sp}_l. \end{cases} \quad (53)$$

Let

$$\delta = \frac{1}{2l}(d' - r + \iota) \quad \text{and} \quad \beta = \frac{\pi}{l}. \quad (54)$$

Fix an irreducible representation  $\Pi$  of  $\tilde{G}$  that occurs in the restriction of the Weil representation  $\omega$  to  $\tilde{G}$ . Let  $\mu \in i\mathfrak{h}(\mathfrak{g})^*$  be the Harish-Chandra parameter of  $\Pi$  with  $\mu_1 > \mu_2 > \dots$ . Let  $P_{a,b}$  and  $Q_{a,b}$  be the polynomials defined in (C.8) and (C.9). Define

$$\begin{aligned} a_j &= -\mu_j - \delta + 1, & b_j &= \mu_j - \delta + 1 \\ p_j(y_j) &= P_{a_j, b_j}(\beta y_j) e^{-\beta|y_j|}, & q_j(y_j) &= \beta^{-1} Q_{a_j, b_j}(\beta^{-1} y_j) \quad (1 \leq j \leq l''), \end{aligned} \quad (55)$$

where  $\delta$  and  $\beta$  are as in (54).

**Theorem 2.** *Let  $l \leq l'$ . Then there is a non-zero constant  $C$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = C \int_{\mathfrak{h} \cap \tau(W)} \left( \prod_{j=1}^{l'} (p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)) \right) \cdot F_\phi(y) dy, \quad (56)$$

where  $\delta_0$  is the Dirac delta at 0 and  $F_\phi(y)$  is the image of  $\phi$  under the Harish-Chandra regular almost-elliptic orbital integral on  $W$  (see [MPP20, Definition 4] and Lemma 14 below). Furthermore, the  $q_j$ 's are zero (and hence there are no derivatives in formula (56)) if and only if  $l = l'$  for  $\mathbb{D} \neq \mathbb{C}$  and  $l' \in \{l, l+1\}$  for  $\mathbb{D} = \mathbb{C}$ .

**Remark 2.** The integration domain  $\mathfrak{h} \cap \tau(W)$  appearing in Theorem 2 was explicitly determined in [MPP20, Lemma 3.5]. It agrees with  $\mathfrak{h}$  when  $\mathbb{D} \neq \mathbb{C}$ .

In the case  $l > l'$ , up to conjugation, there is a unique Cartan subspace  $\mathfrak{h}_\Gamma$  in  $W$ .

**Theorem 3.** *Let  $l > l'$ . Then there is a constant  $C$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = C \int_{\tau'(\mathfrak{h}_\Gamma^{\mathrm{reg}})} \left( \prod_{j=1}^{l'} p_j(y_j) \right) \cdot F_\phi(y) dy. \quad (57)$$

In particular,  $\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg$  is a locally integrable function whose restriction to  $\mathfrak{h}_\Gamma^{\mathrm{reg}}$  is equal to a non-zero constant multiple of

$$\frac{\sum_{s \in W(G', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta J_j^*(\tau'(w)))}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w))} e^{-\beta \sum_{j=1}^{l'} |\delta_j J_j^*(\tau'(w))|}, \quad (58)$$

where

$$a_{s,j} = (s\mu)_j - \delta + 1, \quad b_{s,j} = -(s\mu)_j - \delta + 1 \quad (s \in W(G', \mathfrak{h}'), 1 \leq j \leq l').$$

The fraction in (58) is the pullback by  $\tau'$  of a  $G'$ -invariant polynomial on  $\mathfrak{g}'$ . The exponential in (58) extends to a  $GG'$ -invariant function. It is a Gaussian on  $W$  if and only if  $G'$  is compact.

Now we consider the integrals over  $G \setminus (-G^0)$  in the cases where (49) holds.

Suppose first that  $(G, G') = (O_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$  and that the character  $\Theta_\Pi$  is not supported on the preimage  $\widetilde{G}^0$  of the connected identity component  $G^0 \subseteq G$ . As is well known from the correspondence of the highest weights, see for example [Prz96, page 99], this may happen if and only if  $l \leq l'$  and, by [Prz89, Appendix C],  $(l, l') \neq (1, 1)$ . So, the pair  $(O_2, \mathrm{Sp}_2(\mathbb{R}))$  is excluded (see also Remark 3 below). In this case, the graded vector space (34) is equal to

$$\mathbf{V} = \mathbf{V}^1 \oplus \mathbf{V}^2 \oplus \dots \oplus \mathbf{V}^l.$$

Recall from (35) that in each  $\mathbf{V}_0^j$  we selected an orthonormal basis  $v_0, v'_0$ . For convenience, we introduce the index  $j$  in the notation and we write  $v_{2j-1} = v_0$  and  $v_{2j} = v'_0$ , for  $1 \leq j \leq l$ . Then  $v_1, v_2, \dots, v_{2l}$  is an orthonormal basis of  $\mathbf{V}_{\overline{0}}$  and

$$J_j v_{2j-1} = v_{2j}, \quad J_j v_{2j} = -v_{2j-1} \quad (1 \leq j \leq l).$$

In terms of the dual basis (46) of  $\mathfrak{h}_{\mathbb{C}}^*$ , the positive roots are

$$e_j \pm e_k \quad (1 \leq j < k \leq l).$$

Define an element  $s \in G$  by

$$sv_1 = v_1, \quad sv_2 = v_2, \quad \dots, \quad sv_{2l-1} = v_{2l-1}, \quad sv_{2l} = -v_{2l}. \quad (59)$$

Then  $G^0 = -G^0$  and  $G = G^0 \cup G^0 s$  is the disjoint union of two connected components. Set

$$\mathbf{V}_{\overline{0},s} = \mathbf{V}_{\overline{0}}^1 \oplus \mathbf{V}_{\overline{0}}^2 \oplus \dots \oplus \mathbf{V}_{\overline{0}}^{l-1} \oplus \mathbb{R}v_{2l}, \quad \mathbf{V}_s = \mathbf{V}_{\overline{0},s} \oplus \mathbf{V}_{\overline{1}}.$$

The dual pair corresponding to  $(\mathbf{V}_{\overline{0},s}, \mathbf{V}_{\overline{1}})$  is  $(G_s, G'_s) = (O_{2l-1}, \mathrm{Sp}_{2l'}(\mathbb{R}))$  acting on the symplectic space  $W_s = \mathrm{Hom}(\mathbf{V}_{\overline{1}}, \mathbf{V}_{\overline{0},s})$ . Let  $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R}J_j$  be the centralizer of  $s$  in  $\mathfrak{h}$ . This is a Cartan subalgebra of  $\mathfrak{g}_s$ , the Lie algebra of  $G_s$ .

Denote by

$$\lambda = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_{l-1} e_{l-1} \quad (60)$$

the highest weight of  $\Pi$ . (Here  $\lambda_l = 0$  because the character  $\Theta_\Pi$  is not supported on  $\widetilde{G}^0$ . See [GW09, p. 277].)

**Theorem 4.** *Let  $(G, G') = (O_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$ . Assume that the character  $\Theta_\Pi$  is not supported on  $\widetilde{G}^0$ . Then  $l \leq l'$  and the pair  $(O_2, \mathrm{Sp}_2(\mathbb{R}))$  is excluded. For all  $\phi \in \mathcal{S}(W)$*

$$\int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi) dg = C(\Pi) \left(\frac{i}{2}\right)^{l'} \int_{G_s^0} \check{\Theta}_{\Pi_s}(\tilde{g}) T_s(\tilde{g})(\phi^G|_{W_s}) dg, \quad (61)$$

where  $\Pi_s$  is the genuine irreducible representation of  $\widetilde{G}_s^0$  which occurs in the restriction of  $\Pi$  to  $\widetilde{G}_s^0$  and has highest weight equal to the restriction of  $\lambda$  to  $\mathfrak{h}_s$ , and  $C(\Pi)$  is a constant equal to  $\pm 1$ , see (142).

**Remark 3.** We consider here the case  $(G, G') = (O_2, \mathrm{Sp}_2(\mathbb{R}))$ . By (D.37) and Proposition D.7, we can identify

$$\widetilde{O}_2 = \{(g, \zeta) \in O_2 \times \mathbb{C}^\times; \zeta^2 = \det g\}$$

and the  $\det^{1/2}$ -covering  $\pi_1 : \widetilde{O}_2 \ni (g, \zeta) \rightarrow g \in O_2$  does not split. Let  $\Pi \in \widetilde{O}_2$  occur in Howe's correspondence and let  $\nu : \widetilde{O}_2 \rightarrow \mathbb{C}^\times$  be the character of  $\widetilde{O}_2$  defined by  $\nu(g, \zeta) = \zeta$ . Then  $\mathrm{supp}(\Theta_\Pi) = \mathrm{supp}(\nu\Theta_\Pi) = \mathrm{supp}(\Theta_{\Pi \otimes \nu})$ . By Remark 6,  $(\Pi \otimes \nu)(g, \zeta) = \zeta \Pi(g, \zeta)$  is

independent of  $\zeta$ . Hence there is  $\Xi \in \widehat{O_2}$  such that  $\Xi(g) = (\Pi \otimes \nu)(g, \zeta)$  for all  $\zeta$ . Therefore  $\pi_1(\text{supp}(\Theta_\Pi)) = \text{supp}(\Theta_\Xi)$ . Since  $\pi_1 : \widetilde{SO_2} \rightarrow SO_2$  splits by (D.41), we conclude that  $\Theta_\Pi$  is supported on  $\widetilde{G^0} = \widetilde{SO_2}$  if and only if  $\Theta_\Xi$  is supported on  $SO_2$ .

For  $n \in \mathbb{Z}$ , let  $\rho_n$  be the character of  $SO_2$  defined by  $\rho_n(e^{i\theta}) = e^{in\theta}$ . Up to equivalence, the irreducible representations of  $O_2$  are of the form  $\Xi_n = \text{Ind}_{SO_2}^{O_2}(\rho_n)$  with  $n > 0$ , together with the trivial representation 1 and det. (Moreover,  $\Xi_n \simeq \Xi_{-n}$  and  $\Xi_0 = 1 \oplus \text{det}$ .) Hence  $\Theta_\Xi$  is not supported on  $\widetilde{SO_2}$  if and only if  $\Xi|_{SO_2} = 1$ . Also,  $\Xi = \text{det}$  is excluded because det does not occur in  $\omega|_{O_2}$  by [Prz96, page 99]. So,

$$\int_{O_2} \Theta_{\text{det}}(g)\omega(g) dg = 0.$$

Suppose then that  $\Xi = 1$ . Since  $(\Theta_1 + \Theta_{\text{det}})|_{O_2 \setminus SO_2} = 0$  and  $\Theta_1|_{SO_2} = \Theta_{\text{det}}|_{SO_2} = 1$ , we conclude that

$$\begin{aligned} \int_{O_2} \Theta_1(g)\omega(g) dg &= \int_{O_2} (\Theta_1 + \Theta_{\text{det}})(g)\omega(g) dg \\ &= \int_{SO_2} (\Theta_1 + \Theta_{\text{det}})(g)\omega(g) dg = 2 \int_{SO_2} \Theta_1(g)\omega(g) dg. \end{aligned}$$

Thus  $f_{1 \otimes \Pi'} = 2 \int_{-G^0} \Theta_1(g)\omega(g) dg$  is computed by Theorem 2.

Now we consider the case  $(G, G') = (O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R}))$  with  $1 \leq l \leq l'$ . Recall from (34) the graded vector space  $\mathbf{V}$ . In the case we consider,  $\dim \mathbf{V}_0^0 = 1$ ,  $\dim \mathbf{V}_1^0 = 2(l' - l)$  and for  $1 \leq j \leq l$ ,  $\dim \mathbf{V}_0^j = \dim \mathbf{V}_1^j = 2$ . Let

$$W_s = \text{Hom}(\mathbf{V}_1, \mathbf{V}_0^1 \oplus \dots \oplus \mathbf{V}_0^l) \quad \text{and} \quad W_s^\perp = \text{Hom}(\mathbf{V}_1, \mathbf{V}_0^0).$$

Then

$$W = W_s \oplus W_s^\perp \tag{62}$$

is a direct sum orthogonal decomposition. Let  $G_s \subseteq G$  be the subgroup acting trivially on the space  $\mathbf{V}_0^0$  and let  $G'_s = G'$ . Then the dual pair corresponding to  $W_s$  is  $(G_s, G'_s) = (O_{2l}, \text{Sp}_{2l'}(\mathbb{R}))$  and dual pair corresponding to  $W_s^\perp$  is  $(O_1, \text{Sp}_{2l'}(\mathbb{R}))$ . The groups  $G$  and  $G_s$  share the same Cartan subgroup  $H = H_s$  and the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}_s$  share the same Cartan subalgebra  $\mathfrak{h} = \mathfrak{h}_s$ .

**Theorem 5.** *Let  $(G, G') = (O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R}))$  with  $1 \leq l \leq l'$ . There is a non-zero constant  $C$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\begin{aligned} \int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1 - g)T_s(\tilde{g})(\phi^G|_{W_s}) dg \\ &= C \int_{\mathfrak{h}} \left( \prod_{j=1}^l (p_j(y_j) + q_j(-\partial_{y_j})\delta_0(y_j)) \right) \cdot \prod_{j=1}^l \partial(J_j) \cdot F_{\phi^G|_{W_s}}(y) dy, \end{aligned} \tag{63}$$

where  $T_s$  is the operator  $T$ , see (14), corresponding to the symplectic space  $W_s$ .

**Remark 4.** The pair  $(O_1, \text{Sp}_{2l'}(\mathbb{R}))$  was studied in detail in section 2.

Now we consider the case  $(G, G') = (O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R}))$  with  $l > l'$ . Recall the graded vector space  $\mathbf{V}$ , (34) and (43),

$$\mathbf{V} = \mathbf{V}^0 \oplus \mathbf{V}^1 \oplus \cdots \oplus \mathbf{V}^{l'},$$

where

$$\begin{aligned} \mathbf{V}_0^0 &= \mathbf{V}_0^{0,0} \oplus (\mathbf{V}_0^{0,0})^\perp, \\ \mathbf{V}_1^0 &= 0, \end{aligned}$$

with  $\dim \mathbf{V}_0^{0,0} = 1$  and  $\dim (\mathbf{V}_0^{0,0})^\perp = 2(l - l')$ . Let

$$W_s = \text{Hom}(\mathbf{V}_1^1 \oplus \cdots \oplus \mathbf{V}_1^{l'}, (\mathbf{V}_0^{0,0})^\perp \oplus \mathbf{V}_0^1 \oplus \cdots \oplus \mathbf{V}_0^{l'}), \quad W_s^\perp = \text{Hom}(\mathbf{V}_1^1, \mathbf{V}_0^{0,0}).$$

Then

$$W = W_s \oplus W_s^\perp \tag{64}$$

is a direct sum orthogonal decomposition. Let  $G_s \subseteq G$  be the subgroup acting trivially on the space  $\mathbf{V}_0^{0,0}$  and let  $G'_s = G'$ . The dual pair corresponding to  $W_s$  is  $(G_s, G'_s) = (O_{2l}, \text{Sp}_{2l'}(\mathbb{R}))$  and dual pair corresponding to  $W_s^\perp$  is  $(O_1, \text{Sp}_{2l'}(\mathbb{R}))$ .

**Theorem 6.** *Let  $(G, G') = (O_{2l+1}, \text{Sp}_{2l'}(\mathbb{R}))$  with  $l > l'$ . There is a constant  $C$  which depends only of the dual pair  $(G, G')$  such that for all  $\phi \in \mathcal{S}(W)$*

$$\begin{aligned} \int_{G^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi) dg &= \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1 - g) T_s(\tilde{g})(\phi^G|_{W_s}) dg \\ &= C \int_{\tau'(\mathfrak{h}_1^{\text{reg}})} \prod_{j=1}^{l'} r_j(y_j) \cdot F_{\phi^G|_{W_s}}(y) dy, \end{aligned} \tag{65}$$

where  $\beta$  is as in (113) and  $r_j(y_j) = R_{a_j, b_j}(\beta y_j) e^{-\beta |y_j|}$ , see Proposition C.6.

The proofs of the theorems stated above are included in the following sections.

**Remark 5.** The term  $\det(1 - g)$  admits a representation theoretical interpretation. Indeed, let  $\sigma$  be the spin representation of  $G_s^0$ . Then, by [Lit06, Ch. XI, III., p. 254]

$$\Theta_{\sigma \otimes \sigma^c}(g) = |\Theta_\sigma(g)|^2 = \det(1 + g) \quad (g \in G_s^0). \tag{66}$$

So  $\det(1 - g) = \Theta_{\sigma \otimes \sigma^c}(-g)$ .

We conclude this section with a corollary on the non-differential nature of the symmetry breaking operators in  $\text{Hom}_{\tilde{G}\tilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$ .

**Corollary 7.** *Let  $(G, G')$  be a real reductive dual pair with one member compact. Then the essentially unique non-zero symmetry breaking operator in*

$$\text{Hom}_{\tilde{G}\tilde{G}'}(\mathcal{H}_\omega^\infty, \mathcal{H}_\Pi^\infty \otimes \mathcal{H}_{\Pi'}^\infty)$$

*is not a differential operator.*

*Proof.* We are going to show that  $(\text{Op} \circ \mathcal{K})(f_{\Pi \otimes \Pi'})$  is not a differential operator.

Let  $f \in \mathcal{S}^*(W)$  and recall the definition of  $\mathcal{K}(f)$  in (12). According to [Hör83, Theorems 5.2.1 (the Schwartz kernel theorem) and 5.2.3], the continuous linear map  $\text{Op} \circ \mathcal{K}(f)$  is a distribution-valued differential operator if and only if  $\mathcal{K}(f) \in \mathcal{S}^*(X \times X)$  is supported by the diagonal  $\Delta = \{(x, x); x \in X\}$ . This implies that  $f$  is supported in  $Y$ . Indeed, given



$\varphi \in \mathcal{S}(\mathbf{X} \times \mathbf{X})$ , let  $\psi \in \mathcal{S}(\mathbf{X} \times \mathbf{X})$  be defined by  $\varphi(x, x') = \psi(x - x', x + x')$  for all  $x, x' \in \mathbf{X}$ . Furthermore, define

$$\psi(a, \widehat{y}) = \int_{\mathbf{X}} \chi\left(\frac{1}{2}\langle a, b \rangle\right) \psi(a, b) db.$$

Then

$$\text{supp } \varphi \cap \Delta = \emptyset \quad \text{if and only if} \quad \text{supp } \psi(\cdot, \widehat{\cdot}) \cap (\{0\} \times \mathbf{Y}) = \emptyset.$$

Since  $\mathcal{K}(f)(\varphi) = f(\psi(\cdot, \widehat{\cdot}))$ , as claimed.

Notice that this cannot happen in our case. Indeed, the support of  $f_{\Pi \otimes \Pi'}$  is  $\text{GG}'$ -invariant. Since the pair is of type I,  $\text{GG}'$  acts irreducibly on  $W$ . Therefore the inclusion  $\text{supp } f_{\Pi \otimes \Pi'} \subseteq \mathbf{Y}$  would imply  $\text{supp } f_{\Pi \otimes \Pi'} = \{0\}$ . But we see from Lemma 1 and the theorems in this section that this is not the case.  $\square$

### 5. The integral over $-G^0$ as an integral over $\mathfrak{g}$ .

Let  $\mathfrak{sp}(W)$  be the Lie algebra of  $\text{Sp}(W)$ . Set

$$\mathfrak{sp}(W)^c = \{x \in \mathfrak{sp}(W); x - 1 \text{ is invertible}\}, \quad (67)$$

$$\text{Sp}(W)^c = \{g \in \text{Sp}(W); g - 1 \text{ is invertible}\}. \quad (68)$$

The Cayley transform  $c : \mathfrak{sp}(W)^c \rightarrow \text{Sp}(W)^c$  is the bijective rational map defined by  $c(x) = (x + 1)(x - 1)^{-1}$ . Its inverse  $c^{-1} : \text{Sp}(W)^c \rightarrow \mathfrak{sp}(W)^c$  is given by the same formula,  $c^{-1}(g) = (g + 1)(g - 1)^{-1}$ .

Since all eigenvalues of  $x \in \mathfrak{g} \subseteq \text{End}(W)$  are purely imaginary,  $x - 1$  is invertible. Therefore  $\mathfrak{g} \subseteq \mathfrak{sp}(W)^c$ . Moreover,  $c(\mathfrak{g}) \subseteq G$ . Since the map  $c$  is continuous, the range  $c(\mathfrak{g})$  is connected. Also,  $-1 = c(0)$  is in  $c(\mathfrak{g})$ . Furthermore, for  $x \in \mathfrak{g}$ ,

$$c(x) - 1 = (x + 1)(x - 1)^{-1} - (x - 1)(x - 1)^{-1} = 2(x - 1)^{-1}$$

is invertible. Hence  $c(\mathfrak{g}) \subseteq G \cap \text{Sp}(W)^c$ . This is an equality since  $c(c(y)) = y$  and  $c(G) \subseteq \mathfrak{g}$ . Thus

$$c(\mathfrak{g}) = \{g \in G; \det(g - 1) \neq 0\}.$$

This is an open dense subset of  $G$  if  $G = \text{U}_d$  or  $G = \text{Sp}_d$ . If  $G = \text{O}_{2l}$  then  $c(\mathfrak{g})$  is dense in  $\text{SO}_{2l}$ . If  $G = \text{O}_{2l+1}$  then  $c(\mathfrak{g})$  is equal to the connected component of  $-1$ , which coincides with  $-\text{SO}_{2l+1}$ . Therefore,

$$\int_{-G^0} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg = \int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg, \quad (69)$$

if  $G \neq \text{O}_{2l+1}$  and

$$\begin{aligned} \int_{G^0} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg &= \int_{-c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg, \\ \int_{-G^0} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg &= \int_{c(\mathfrak{g})} T(\tilde{g}) \check{\Theta}_{\Pi}(\tilde{g}) dg, \end{aligned} \quad (70)$$

if  $G = \text{O}_{2l+1}$ . Let

$$\tilde{c} : \mathfrak{g} \rightarrow \widetilde{G} \quad (71)$$

be a real analytic lift of  $c$ . Set  $\tilde{c}_-(x) = \tilde{c}(x)\tilde{c}(0)^{-1}$ . Then  $\tilde{c}_-(0)$  is the identity of the group  $\widetilde{\text{Sp}}(W)$ .

By (14), for  $x \in \mathfrak{g}$  we have an explicit formula,

$$t(c(x)) = \chi_x \mu_W,$$

where

$$\chi_x(w) = \chi\left(\frac{1}{4}\langle x(w), w \rangle\right) \quad (w \in W). \quad (72)$$

Hence

$$T(\tilde{c}(x)) = \Theta(\tilde{c}(x)) \chi_x \mu_W. \quad (73)$$

Therefore, for a suitable normalization of the Lebesgue measure on  $\mathfrak{g}$ ,

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_x \mu_W dx, \quad (74)$$

where  $j_{\mathfrak{g}}(x)$  is the Jacobian of the map  $c : \mathfrak{g} \rightarrow c(\mathfrak{g})$  (see Appendix A for its explicit expression). Also, since  $\tilde{c}(0)$  is in the center of the metaplectic group,

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg = \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \chi_x \mu_W dx. \quad (75)$$

In the rest of this paper we shall write  $dw = d\mu_W(w)$ , when convenient.

## 6. The invariant integral over $\mathfrak{g}$ as an integral over $\mathfrak{h}$ .

Let  $H \subseteq G$  be a Cartan subgroup. Fix a system of positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ . For any positive root  $\alpha$  let  $\mathfrak{g}_{\mathbb{C},\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$  be the corresponding  $\text{ad}(\mathfrak{h}_{\mathbb{C}})$ -eigenspace and let  $X_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}$  be a non-zero vector. Then there is a character (group homomorphism)  $\xi_\alpha : H \rightarrow \mathbb{C}^\times$  such that

$$\text{Ad}(h)X_\alpha = \xi_\alpha(h)X_\alpha \quad (h \in H).$$

The derivative of  $\xi_\alpha$  at the identity coincides with  $\alpha$ . Let  $\rho \in \mathfrak{h}_{\mathbb{C}}^*$  denote one half times the sum of all the positive roots. Then in all cases except when  $G = O_{2l+1}$  or  $G = U_l$  with  $l$  even, there is a character  $\xi_\rho : H \rightarrow \mathbb{C}^\times$  whose derivative at the identity is equal to  $\rho$ , see [GW09, (2.21) and p. 145]. When  $G = O_{2l+1}$  or  $G = U_l$  with  $l$  even, the character  $\xi_\rho$  exists as a map defined on a non-trivial double cover

$$\widehat{H} \ni \widehat{h} \rightarrow h \in H. \quad (76)$$

of  $H$ . In particular the Weyl denominator

$$\Delta(h) = \xi_\rho(h) \prod_{\alpha > 0} (1 - \xi_{-\alpha}(h)) \quad (77)$$

is defined for  $h \in H$  or  $h \in \widehat{H}$  according to the cases described above. The sign representation  $\text{sgn}_{\mathfrak{g}/\mathfrak{h}}$  of the Weyl group  $W(G, \mathfrak{h})$  is defined by

$$\Delta(sh) = \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(s)\Delta(h) \quad (s \in W(G, \mathfrak{h})), \quad (78)$$

where either  $h \in H$  or  $h \in \widehat{H}$ . Notice that  $W(G, \mathfrak{h}) = W(\widetilde{G}, \mathfrak{h})$  because  $\widetilde{G}$  is a central extension of  $G$ .

We now determine the preimage  $\widetilde{\mathbb{H}}$  of  $\mathbb{H}$  in  $\widetilde{\mathbb{G}}$  and, where needed, a double cover  $\widehat{\mathbb{H}}$  of  $\widetilde{\mathbb{H}}$  on which all functions  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  can be defined. At the same time, we define a lift to  $\widetilde{\mathbb{H}}$ , or to  $\widehat{\mathbb{H}}$ , of the modified Cayley transform defined on  $\mathfrak{h}$  by

$$c_-(x) = (1+x)(1-x)^{-1} = -c(x).$$

Suppose first that  $\mathbb{G} = \mathbb{O}_{2l+1}$ . Then  $\mathbb{H} = \mathbb{H}^0 \cdot \mathbb{Z}$  is the direct product of the connected identity component  $\mathbb{H}^0$  of  $\mathbb{H}$  and the center  $\mathbb{Z}$  of  $\mathrm{Sp}(\mathbb{W})$ . According to Proposition D.7,  $\widetilde{\mathbb{G}} \rightarrow \mathbb{G}$  is isomorphic to the  $\det^{1/2}$ -covering (D.43). Hence  $\widetilde{\mathbb{H}^0} \rightarrow \mathbb{H}^0$  is isomorphic to the  $\det^{1/2}$ -covering  $\sqrt{\mathbb{H}^0} = \{(h, \zeta) \in \mathbb{H}^0 \times \mathbb{C}^\times : \zeta^2 = 1\} = \{(h, \pm 1) : h \in \mathbb{H}^0\}$ , whose identity component is isomorphic to  $\mathbb{H}^0$ . This connected component will be again denoted by  $\mathbb{H}^0$ . In this notation,  $\widetilde{\mathbb{H}} = \mathbb{H}^0 \cdot \widetilde{\mathbb{Z}}$ .

We have a chain of double covering homomorphisms

$$\begin{aligned} \widehat{\mathbb{H}^0} \times \widetilde{\mathbb{Z}} &\rightarrow \mathbb{H}^0 \times \widetilde{\mathbb{Z}} \rightarrow \mathbb{H}^0 \times \mathbb{Z} \rightarrow \mathbb{H}^0 \\ (\widehat{h}, \widetilde{z}) &\longrightarrow (h, \widetilde{z}) \longrightarrow (h, z) \longrightarrow h, \end{aligned} \tag{79}$$

where  $\widehat{\mathbb{H}^0}$  is the preimage of  $\mathbb{H}^0$  in  $\widehat{\mathbb{H}}$ . We set  $\widehat{\mathbb{H}} = \widehat{\mathbb{H}^0} \times \widetilde{\mathbb{Z}}$ . We lift  $\Delta$  and  $\xi_\mu$  to functions on  $\widehat{\mathbb{H}}$  constant on the fibers of composition of all the maps (79) and the character  $\check{\Theta}_\Pi$  to a function on  $\widehat{\mathbb{H}^0} \times \widetilde{\mathbb{Z}}$  constant on the fibers of the leftmost map in (79).

The subset  $c_-(\mathfrak{h}) \subseteq \mathbb{H}^0$  consists of all  $h \in \mathbb{H}^0$  such that  $h+1$  is invertible. In particular, it is dense in  $\mathbb{H}^0$ . We choose a real analytic section

$$\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\mathbb{H}^0}$$

to the covering map (76) and set

$$\widehat{c}_- : \mathfrak{h} \ni x \rightarrow (\sigma(c_-(x)), 1) \in \widehat{\mathbb{H}} = \widehat{\mathbb{H}^0} \times \widetilde{\mathbb{Z}}. \tag{80}$$

Suppose now that  $\mathbb{G} = \mathbb{U}_l$  with  $l$  even. If  $\mathbb{G}' = \mathbb{U}_{p,q}$  with  $p+q$  odd, then the covering  $\widetilde{\mathbb{H}} \rightarrow \mathbb{H}$  does not split (see the proof of Proposition D.7). Hence  $\widehat{\mathbb{H}} = \widetilde{\mathbb{H}}$ . Therefore  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  are defined on  $\widehat{\mathbb{H}}$ . We have the Cayley transform  $c_- : \mathfrak{h} \rightarrow \mathbb{H}$ , an analytic section  $\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\mathbb{H}}$  and the map

$$\widehat{c}_- : \mathfrak{h} \ni x \rightarrow \sigma(c_-(x)) \in \widehat{\mathbb{H}}. \tag{81}$$

If  $\mathbb{G}' = \mathbb{U}_{p,q}$  with  $p+q$  even, then  $\widetilde{\mathbb{H}} = \mathbb{H} \times \{1, \tilde{1}\}$  splits by Proposition D.7, and we have maps

$$\begin{aligned} \widehat{\mathbb{H}} &\rightarrow \mathbb{H} \longrightarrow \widetilde{\mathbb{H}} \longrightarrow \mathbb{H}, \\ \widehat{h} &\rightarrow h \rightarrow (h, 1) \rightarrow h. \end{aligned} \tag{82}$$

Again  $\Delta$ ,  $\xi_\mu$  and  $\check{\Theta}_\Pi$  are defined on  $\widehat{\mathbb{H}}$  and (81) defines the lift of the Cayley transform we shall use. In this case, we set then  $\widehat{\mathbb{H}} = \widehat{\mathbb{H}}$ .

For the remaining dual pairs,  $\widehat{\mathbb{H}} = \mathbb{H}$  and we lift  $\Delta$  and  $\xi_\mu$  to functions on  $\widetilde{\mathbb{H}}$  constant on the fibers of the covering map  $\widetilde{\mathbb{H}} \rightarrow \mathbb{H}$  and write  $\widehat{c}_-$  for  $\widetilde{c}_-$ , which was defined under the equation (71).

Let  $\Pi$  be an irreducible representation of  $\widetilde{G}$ , and let  $\mu \in i\mathfrak{h}^*$  represent the infinitesimal character of  $\Pi$ . When  $\mu$  is dominant, then we will refer to it as the Harish-Chandra parameter of  $\Pi$ . This is consistent with the usual terminology; see e.g. [Kna86, Theorem 9.20]. Then the corresponding character  $\xi_\mu$  is defined as  $\xi_\mu = \xi_\rho \xi_{\mu-\rho}$ , where  $\xi_{\mu-\rho}$  is one of the extremal H-weights of  $\Pi$ . In these terms, Weyl's character formula looks as follows,

$$\Theta_\Pi(h)\Delta(h) = \sum_{s \in W(G, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{s\mu}(h), \quad (83)$$

where  $h \in \widetilde{H}$  or  $h \in \widehat{H}$ , according to the cases above.

**Lemma 8.** *Let*

$$\kappa(x) = i^{-m} \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \quad (x \in \mathfrak{h}),$$

where  $m$  is the number of positive roots. Then, for a suitable normalization of the Lebesgue measure on  $\mathfrak{h}$  and any  $\phi \in \mathcal{S}(W)$ ,

$$\begin{aligned} \int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \frac{\check{\chi}_\Pi(\tilde{c}(0))}{|W(G, \mathfrak{h})|} \int_{\mathfrak{h}} (\check{\Theta}_\Pi\Delta)(\widehat{c}_-(x)) \cdot \kappa(x) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx \\ &= \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \cdot \kappa(x) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx, \end{aligned}$$

where  $\phi^G$  is as in (50) and each consecutive integral is absolutely convergent.

*Proof.* Applied to a test function  $\phi \in \mathcal{S}(W)$ , the integral (70) over  $c(\mathfrak{g})$  is absolutely convergent because both, the character and the function  $T(\tilde{g})(\phi)$  are continuous and bounded (see for example [Prz93, Proposition 1.13]) and the group  $G$  is compact. Hence, each consecutive integral in the formula (75) applied to  $\phi$ ,

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \check{\chi}_\Pi(\tilde{c}(0)) \int_{\mathfrak{g}} \check{\Theta}_\Pi(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_W \chi_x(w) \phi(w) dw dx, \quad (84)$$

is absolutely convergent. Since

$$\chi_{g.x}(w) = \chi_x(g^{-1}.w)$$

and the Lebesgue measure  $dw$  is  $G$ -invariant,

$$\int_G \int_W \chi_{g.x}(w) \phi(w) dw dg = \int_W \chi_x(w) \phi^G(w) dw.$$

Observe also that  $\widetilde{\operatorname{Ad}}(\tilde{g}) = \operatorname{Ad}(g)$  and the characters  $\check{\Theta}_\Pi$  and  $\Theta$  are  $\widetilde{G}$ -invariant. Therefore the Weyl integration formula on  $\mathfrak{g}$  shows that (84) is equal to  $\frac{\check{\chi}_\Pi(\tilde{c}(0))}{|W(G, \mathfrak{h})|}$  times

$$\begin{aligned} &\int_{\mathfrak{h}} |\pi_{\mathfrak{g}/\mathfrak{h}}(x)|^2 \check{\Theta}_\Pi(\tilde{c}_-(x)) \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \int_W \chi_x(w) \phi^G(w) dw dx \\ &= \int_{\mathfrak{h}} \check{\Theta}_\Pi(\widehat{c}_-(x)) \Delta(\widehat{c}_-(x)) \left( i^{-m} \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\Delta(\widehat{c}_-(x))} \Theta(\tilde{c}(x)) j_{\mathfrak{g}}(x) \right) \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx. \end{aligned}$$

(Here, we suppose that the Haar measure on  $H$  is normalized to have total mass 1.) This verifies the first equality and the absolute convergence. Since,

$$\check{\Theta}_\Pi(\widehat{c}_-(x))\Delta(\widehat{c}_-(x)) = \sum_{s \in W(\mathfrak{G}, \mathfrak{h})} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}}(s) \xi_{-s\mu}(\widehat{c}_-(x)),$$

the second equality follows via a straightforward computation.  $\square$

Since any element  $x \in \mathfrak{g}$ , viewed as an endomorphism of  $V$  over  $\mathbb{R}$ , has imaginary eigenvalues which come in complex conjugate pairs, we have  $\det(1 - x)_{V_{\mathbb{R}}} \geq 1$ . Define

$$\operatorname{ch}(x) = \det(1 - x)_{V_{\mathbb{R}}}^{1/2} \quad (x \in \mathfrak{g}). \quad (85)$$

Recall the symbols  $r$  and  $\iota$  from (52) and (51).

**Lemma 9.** *There is a constant  $C$  which depends only on the dual pair  $(G, G')$  such that*

$$\kappa(x) = C \operatorname{ch}^{d' - r - \iota}(x) \quad (x \in \mathfrak{h}).$$

*Proof.* Recall [Prz93, Lemma 5.7] that  $\pi_{\mathfrak{g}/\mathfrak{h}}(x)$  is a constant multiple of  $\Delta(\widehat{c}_-(x)) \operatorname{ch}^{r - \iota}(x)$ ,

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) = C \Delta(\widehat{c}_-(x)) \operatorname{ch}^{r - \iota}(x). \quad (86)$$

(For the reader's convenience, this is verified in Appendix B.) As is well known, [AP14, Definition 4.16],

$$\Theta(\tilde{c}(x))^2 = i^{\dim W} \det(2^{-1}(x - 1))_W \quad (x \in \mathfrak{sp}(W), \det(x - 1) \neq 0). \quad (87)$$

Hence there is a choice of  $\tilde{c}$  so that

$$\Theta(\tilde{c}(x)) = \left(\frac{i}{2}\right)^{\frac{1}{2} \dim W} \det(1 - x)_{\mathbb{W}}^{\frac{1}{2}} \quad (x \in \mathfrak{g}). \quad (88)$$

Furthermore, since the symplectic space may be realized as  $W = \operatorname{Hom}_{\mathbb{D}}(V', V)$ , see (31), we obtain that

$$\det(1 - x)_W = \det(1 - x)_{V_{\mathbb{R}}}^{d'} \quad (x \in \mathfrak{g}). \quad (89)$$

Also, as checked in [Prz91, (3.11)], the Jacobian of  $\tilde{c}_- : \mathfrak{g} \rightarrow G$  is a constant multiple of  $\operatorname{ch}^{-2r}(x)$ . (Again, for reader's convenience a –slightly different– proof is included in Appendix A.) Hence the claim follows.  $\square$

**Corollary 10.** *For any  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = C \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d' - r - \iota}(x) \cdot \pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw dx,$$

where  $C$  is a constant,  $\phi^G$  in as in (50), and each consecutive integral is absolutely convergent.

### 7. An intertwining distribution in terms of orbital integrals on the symplectic space.

We keep the notation introduced in section 3. Let

$$W(G, \mathfrak{h}(\mathfrak{g})) = \begin{cases} \Sigma_l & \text{if } \mathbb{D} = \mathbb{C}, \\ \Sigma_l \times \{\pm 1\}^l & \text{otherwise.} \end{cases} \quad (90)$$

Denote the elements of  $\Sigma_l$  by  $\sigma$  and the elements of  $\{\pm 1\}^l$  by  $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_l)$ , so that an arbitrary element of the group (90) is of the form  $t = \epsilon\sigma$ , with  $\epsilon = (1, 1, \dots, 1)$ , if  $\mathbb{D} = \mathbb{C}$ . This group acts on  $\mathfrak{h}(\mathfrak{g})$ , see (45), as follows: for  $t = \epsilon\sigma$ ,

$$t \left( \sum_{j=1}^l y_j J_j \right) = \sum_{j=1}^l \epsilon_j y_{\sigma^{-1}(j)} J_j. \quad (91)$$

As the indicated by the notation,  $W(G, \mathfrak{h}(\mathfrak{g}))$  coincides with the Weyl group, equal to the quotient of the normalizer of  $\mathfrak{h}(\mathfrak{g})$  in  $G$  by the centralizer of  $\mathfrak{h}(\mathfrak{g})$  in  $G$ .

The action of  $W(G, \mathfrak{h}(\mathfrak{g}))$  on  $\mathfrak{h}(\mathfrak{g})$  extends by duality to  $i\mathfrak{h}(\mathfrak{g})^*$ . More precisely, let  $e_j$  be as in (46). If  $\mu \in \mathfrak{h}(\mathfrak{g})^*$ , then  $\mu = \sum_{j=1}^l \mu_j e_j$  with all  $\mu_j \in \mathbb{R}$ . If  $t = \epsilon\sigma \in W(G, \mathfrak{h}(\mathfrak{g}))$ , then

$$t \left( \sum_{j=1}^l \mu_j e_j \right) = \sum_{j=1}^l \epsilon_{\sigma^{-1}(j)} \mu_{\sigma^{-1}(j)} e_j. \quad (92)$$

Recall the notation of Lemma 8 and the symbol  $\delta$  from (54).

**Lemma 11.** *The following formulas hold for any  $y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}(\mathfrak{g})$ ,*

$$\xi_{-\mu}(\widehat{c}_-(ty)) = \xi_{-t^{-1}\mu}(\widehat{c}_-(y)) \quad (t \in W(G, \mathfrak{h})) \quad (93)$$

and

$$\xi_{-\mu}(\widehat{c}_-(y)) \operatorname{ch}^{d-r-\iota}(y) = \prod_{j=1}^l (1 + iy_j)^{-\mu_j + \delta - 1} (1 - iy_j)^{\mu_j + \delta - 1}, \quad (94)$$

where all the exponents are integers:

$$\pm \mu_j + \delta \in \mathbb{Z} \quad (1 \leq j \leq l). \quad (95)$$

In particular, (94) is a rational function in the variables  $y_1, y_2, \dots, y_l$ .

*Proof.* Since

$$\xi_{-\mu}(\widehat{c}_-(y)) = \prod_{j=1}^l \left( \frac{1 + iy_j}{1 - iy_j} \right)^{-\mu_j} = \prod_{j=1}^l (1 + iy_j)^{-\mu_j} (1 - iy_j)^{\mu_j},$$

(93) and (94) follow from the definition of the action of  $W(G, \mathfrak{h})$ , the definition of  $\operatorname{ch}$  in (85), and the following easy-to-check formula:

$$\operatorname{ch}(y) = \prod_{j=1}^l (1 + y_j^2)^{\frac{1}{2i}} = \prod_{j=1}^l (1 + iy_j)^{\frac{1}{2i}} (1 - iy_j)^{\frac{1}{2i}}. \quad (96)$$

Let  $\lambda = \sum_{j=1}^l \lambda_j e_j$  be the highest weight of the representation  $\Pi$  and let  $\rho = \sum_{j=1}^l \rho_j e_j$  be one half times the sum of the positive roots of  $\mathfrak{h}(\mathfrak{g})$  in  $\mathfrak{g}_{\mathbb{C}}$ . If  $\mu$  is the Harish-Chandra parameter of  $\Pi$ , then  $\lambda + \rho = \mu = \sum_{j=1}^l \mu_j e_j$ . Hence, the statement (95) is equivalent to

$$\lambda_j + \rho_j + \frac{1}{2l}(d' - r + \iota) \in \mathbb{Z}. \quad (97)$$

Indeed, if  $G = O_d$ , then with the standard choice of the positive root system,  $\rho_j = \frac{d}{2} - j$ . Also,  $\lambda_j \in \mathbb{Z}$ ,  $\iota = 1$ ,  $r = d - 1$ . Hence, (97) follows. Similarly, if  $G = U_d$ , then  $\rho_j = \frac{d+1}{2} - j$ ,  $\lambda_j + \frac{d'}{2} \in \mathbb{Z}$ ,  $\iota = 1$ ,  $r = d$ , which implies (97). If  $G = Sp_d$ , then  $\rho_j = d + 1 - j$ ,  $\lambda_j \in \mathbb{Z}$ ,  $\iota = \frac{1}{2}$ ,  $r = d + \frac{1}{2}$ , and (97) follows.  $\square$

Our next goal is to understand the integral

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_W \chi_x(w) \phi^G(w) dw,$$

occurring in the formula for  $\int_{-G^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g}) dg$  in Lemma 8 and Corollary 10, in terms of orbital integrals on the symplectic space  $W$ . The results depend on whether  $l \leq l'$  or  $l > l'$  and will be given in Lemmas 14 and 15. We first need two other lemmas.

**Lemma 12.** *Fix an element  $z \in \mathfrak{h}(\mathfrak{g})$ . Let  $\mathfrak{z} \subseteq \mathfrak{g}$  and  $Z \subseteq G$  denote the centralizer of  $z$ . (Then  $Z$  is a real reductive group with Lie algebra  $\mathfrak{z}$ .) Denote by  $\mathfrak{c}$  the center of  $\mathfrak{z}$  and by  $\pi_{\mathfrak{g}/\mathfrak{z}}$  the product of the positive roots for  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}(\mathfrak{g})_{\mathbb{C}})$  which do not vanish on  $z$ . Let  $B(\cdot, \cdot)$  be any non-degenerate symmetric  $G$ -invariant real bilinear form on  $\mathfrak{g}$ . Then there is a constant  $C_{\mathfrak{z}}$  such that for any  $x \in \mathfrak{h}(\mathfrak{g})$  and  $x' \in \mathfrak{c}$ ,*

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \pi_{\mathfrak{g}/\mathfrak{z}}(x') & \int_G e^{iB(g.x, x')} dg \\ & = C_{\mathfrak{z}} \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}x) e^{iB(x, t(x'))}. \end{aligned} \quad (98)$$

(Here  $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})} = 1$  if  $\mathfrak{z} = \mathfrak{h}$ . Recall also the notation  $g.x = gxg^{-1}$ .)

*Proof.* The proof is a straightforward modification of the argument proving Harish-Chandra's formula for the Fourier transform of a regular semisimple orbit, [Har57, Theorem 2, page 104]. A more general, and by now classical, result is [DV90, Proposition 34, p. 49].  $\square$

The symplectic form  $\langle \cdot, \cdot \rangle$  on  $W$  according to the Lie superalgebra structure introduced in (27) is

$$\langle w', w \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}w'w) \quad (w', w \in W). \quad (99)$$

Hence

$$\langle xw, w \rangle = \text{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) \quad (x \in \mathfrak{g} \oplus \mathfrak{g}', w \in W). \quad (100)$$

Set

$$B(x, y) = \pi \text{tr}_{\mathbb{D}/\mathbb{R}}(xy) \quad (x, y \in \mathfrak{g}). \quad (101)$$

**Lemma 13.** *Recall the Gaussian  $\chi_x$  from (72). Then*

$$\chi_x(w) = e^{iB(x, \tau(w))} \quad (x \in \mathfrak{g}, w \in W). \quad (102)$$

*Proof.* Notice that

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_0}) - \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_1}),$$

where

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_0}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(x|_{\mathfrak{V}_1}w|_{\mathfrak{V}_1}w|_{\mathfrak{V}_0}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w|_{\mathfrak{V}_0}x|_{\mathfrak{V}_1}w|_{\mathfrak{V}_1}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathfrak{V}_1})$$

and similarly

$$\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xw^2|_{\mathfrak{V}_1}) = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(wxw|_{\mathfrak{V}_0}).$$

Hence

$$\langle xw, w \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) = -\mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}wxw) = -\langle wx, w \rangle.$$

Therefore

$$\langle x(w), w \rangle = 2 \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(\mathbf{S}xw^2) \quad (x \in \mathfrak{g} \oplus \mathfrak{g}', w \in \mathbf{W}). \quad (103)$$

Then (100) and (32) show that

$$\frac{\pi}{2} \langle x(w), w \rangle = B(x, \tau(w)) \quad (x \in \mathfrak{g}, w \in \mathbf{W}),$$

which completes the proof.  $\square$

The Harish-Chandra regular almost semisimple orbital integral  $F(y)$ ,  $y \in \mathfrak{h}$ , was defined in [MPP20, Definition 3 and Theorems 5 and 7]. In particular, [MPP20, Theorem 7] implies that, in the statements below, all the integrals over  $\mathfrak{h}$  involving  $F(y)$  are absolutely convergent. Recall the notation  $F_\phi(y)$  for  $F(y)(\phi)$ .

**Lemma 14.** *Suppose  $l \leq l'$ . Then, with the notation of Lemma 8,*

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \int_{\mathbf{W}} \chi_x(w) \phi^G(w) dw = C \int_{\mathfrak{h} \cap \tau(\mathbf{W})} e^{iB(x,y)} F_\phi(y) dy.$$

where  $C$  is a non-zero constant which depends only of the dual pair  $(G, G')$ .

*Proof.* Here we use the notation introduced in [MPP20, Definition 4]. The Weyl integration formula on  $\mathbf{W}$  (see [MPP15, Theorem 21] and [MPP20, (16), (23)]) shows that

$$\int_{\mathbf{W}} \chi_x(w) \phi^G(w) dw = \sum_{\mathfrak{h}_\Gamma} \int_{\tau(\mathfrak{h}_\Gamma^\pm)} \pi_{\mathfrak{g}/\mathfrak{h}}(\tau(w)) \pi_{\mathfrak{g}'/\mathfrak{g}'}(\tau(w)) C(\mathfrak{h}_\Gamma) \mu_{\mathcal{O}(w)}(\chi_x \phi^G) d\tau(w), \quad (104)$$

where  $C(\mathfrak{h}_\Gamma)$  is a constant determined in [MPP20, Lemma 2.1]. Let us consider first the case of a semisimple orbital integral

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{\mathbf{S}/\mathbf{S}^{\mathfrak{h}_\Gamma}} (\chi_x \phi^G)(s.w) d(\mathbf{S}^{\mathfrak{h}_\Gamma}),$$

where  $\mathbf{S}^{\mathfrak{h}_\Gamma}$  is the centralizer of  $\mathfrak{h}_\Gamma$  in  $\mathbf{S}$ . Recall the identification  $y = \tau(w) = \tau'(w)$  and let us write  $s = gg'$ , where  $g \in G$  and  $g' \in G'$ . Then

$$\chi_x(s.w) = e^{i\frac{\pi}{2} \langle x(s.w), s.w \rangle} = e^{iB(x, \tau(s.w))} = e^{iB(x, g, \tau(w))} = e^{iB(x, g, y)} \quad (105)$$

and

$$\phi^G(s.w) = \phi^G(g'.w). \quad (106)$$



Since  $l \leq l'$ , we have  $S^{\mathfrak{h}_\Gamma} = H \times Z'$ , where  $H$  is the Cartan subgroup of  $G$  corresponding to  $\mathfrak{h}$  and  $Z' \subseteq G'$  is the centralizer of  $\mathfrak{h}$ . Hence  $S/S^{\mathfrak{h}_\Gamma} = G/H \cdot G'/Z'$  and there is a positive constant  $C_1$  such that

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_1 \int_G e^{iB(x,g,y)} dg \int_{G'/Z'} \phi^G(g'.w) d(g'Z').$$

However we know from Harish-Chandra (Lemma 12) that

$$\pi_{\mathfrak{g}/\mathfrak{h}}(x) \left( \int_G e^{iB(x,g,y)} dg \right) \pi_{\mathfrak{g}/\mathfrak{h}}(y) = C_2 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) e^{iB(x,t,y)}.$$

Hence, using (104) and [MPP20, Definition 3.2 and Lemma 3.4], we obtain for some suitable positive constants  $C_k$ ,

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}}(x) & \int_W \chi_x(w) \phi^G(w) dw & (107) \\ &= C_3 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \sum_{\mathfrak{h}_\Gamma} \int_{\tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} C(\mathfrak{h}_\Gamma) \pi_{\mathfrak{g}'/Z'}(y) \int_{G'/Z'} \phi^G(g'.w) d(g'Z') dy \\ &= C_4 \sum_{t \in W(G,\mathfrak{h})} \text{sgn}_{\mathfrak{g}/\mathfrak{h}}(t) \int_{\bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} F_{\phi^G}(y) dy \\ &= C_4 \sum_{t \in W(G,\mathfrak{h})} \int_{\bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+)} e^{iB(x,t,y)} F_{\phi^G}(t,y) dy \\ &= C_4 \int_{W(G,\mathfrak{h})(\bigcup_{\mathfrak{h}_\Gamma} \tau(\mathfrak{h}_\Gamma^+))} e^{iB(x,y)} F_{\phi^G}(y) dy \\ &= C_4 \int_{\mathfrak{h} \cap \tau(W)} e^{iB(x,y)} F_{\phi^G}(y) dy. \end{aligned}$$

Since  $F_{\phi^G} = \text{vol}(G)F_\phi = F_\phi$ , the formula follows.

Then  $G = \text{O}_{2l+1}$ ,  $G' = \text{Sp}_{2l'}(\mathbb{R})$  and

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{S/S^{\mathfrak{h}_\Gamma + w_0}} (\chi_x \phi^G)(s.(w + w_0)) d(sS^{\mathfrak{h}_\Gamma + w_0}).$$

Since the Cartan subspace  $\mathfrak{h}_\Gamma$  preserves the decomposition (34),  $(w + w_0)^2 = w^2 + w_0^2$ . Hence,  $(s.(w + w_0))^2 = s.(w^2 + w_0^2)$ . The element  $x \in \mathfrak{h}$  acts by zero on  $\mathfrak{g}'$ . Therefore  $x(s.(w + w_0))^2 = x(s.(w + w_0))^2|_{\mathfrak{V}_0}$ . Since  $S(\mathfrak{V}^0) = \text{O}_1 \times \text{Sp}_{2(l'-l)}(\mathbb{R})$  we see that  $w_0^2|_{\mathfrak{V}_0} = 0$ . Thus  $xs.w_0^2|_{\mathfrak{V}_0} = 0$ . Therefore, by (26),

$$\langle x(s.(w + w_0)), s.(w + w_0) \rangle = \text{tr}(x(s.(w + w_0))^2) = \text{tr}(xs.w^2|_{\mathfrak{V}_0}) = \text{tr}(xg.\tau(w)),$$

because  $s = gg'$ . Hence,

$$\chi_x(s.(w + w_0)) = e^{i\frac{\pi}{2} \langle x(s.(w+w_0)), s.(w+w_0) \rangle} = e^{iB(x,g.\tau(w))} = e^{iB(x,g.y)}$$

and

$$\phi^G(s.(w + w_0)) = \phi^G(g'.(w + w_0)).$$

Therefore, with  $n = \tau'(w_0)$ ,

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_1 \int_G e^{iB(x,g,y)} dg \int_{G'/Z'^n} \phi^G(g'.w) d(g'Z'^n).$$

Thus, the computation (107) holds again, and we are done.  $\square$

**Lemma 15.** *Suppose  $l > l'$ . Let  $\mathfrak{z} \subseteq \mathfrak{g}$  and  $Z \subseteq G$  be the centralizers of  $\tau(\mathfrak{h}_\Gamma)$ . Then for  $\phi \in \mathcal{S}(W)$*

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw \\ = C \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} e^{iB(x,t,y)} F_\phi(y) dy, \end{aligned}$$

where  $C$  is a non-zero constant which depends only on the dual pair  $(G, G')$ .

*Proof.* By the Weyl integration formula with the roles of  $G$  and  $G'$  reversed, see [MPP20, (23)],

$$\int_W \chi_x(w) \phi^G(w) dw = C_1 \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \pi_{\mathfrak{g}/\mathfrak{z}}(\tau'(w)) \pi_{\mathfrak{g}'/\mathfrak{h}'(\tau'(w))} \mu_{\mathcal{O}(w)}(\chi_x \phi^G) d\tau'(w),$$

where

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = \int_{S/S^{\mathfrak{h}_\Gamma}} (\chi_x \phi^G)(s.w) d(sS^{\mathfrak{h}_\Gamma}).$$

Recall the identification  $y = \tau(w) = \tau'(w)$  and let us write  $s = gg'$ , where  $g \in G$  and  $g' \in G'$ . Then, as in (105) and (106),

$$\chi_x(s.w) = e^{iB(x,g,y)} \quad \text{and} \quad \phi^G(s.w) = \phi^G(g'.w).$$

Since  $l > l'$ , according the notation introduced in (42),  $\mathfrak{h}$  is a Cartan subalgebra of  $G'$ , with corresponding Cartan subgroup  $H'$ . Then  $S^{\mathfrak{h}_\Gamma} = Z \times H'$ . Hence  $S/S^{\mathfrak{h}_\Gamma} = G/Z \cdot G'/H'$  and there is a constant  $C_2$  such that

$$\mu_{\mathcal{O}(w)}(\chi_x \phi^G) = C_2 \int_G e^{iB(x,g,y)} dg \int_{G'/H'} \phi^G(g'.w) d(g'H').$$

By (98) in Lemma 12 and [MPP20, (34)], we obtain for some constants  $C_k$

$$\begin{aligned} \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw & \tag{108} \\ = C_3 \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} e^{iB(x,t,y)} \pi_{\mathfrak{g}'/\mathfrak{h}}(y) \\ & \times \int_{G'/H'} \phi^G(g'.w) d(g'H') dy \\ = C_4 \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} e^{iB(x,t,y)} F_{\phi^G}(y) dy. \end{aligned}$$

Since  $F_{\phi^G} = \text{vol}(G)F_\phi = F_\phi$ , the formula follows.  $\square$

**Lemma 16.** *Suppose  $l \leq l'$ . Then there is a seminorm  $q$  on  $\mathcal{S}(W)$  such that*

$$\left| \int_{\mathfrak{h} \cap \tau(W)} F_\phi(y) e^{iB(x,y)} dy \right| \leq q(\phi) \operatorname{ch}(x)^{-d'+r-1} \quad (x \in \mathfrak{h}, \phi \in \mathcal{S}(W)).$$

*Proof.* The boundedness of the function  $T(\tilde{g})(\phi)$ ,  $\tilde{g} \in \tilde{G}$ , means that there is a seminorm  $q(\phi)$  on  $\mathcal{S}(\mathfrak{g})$  such that

$$\left| \Theta(\tilde{c}(x)) \int_W \chi_x(w) \phi(w) dw \right| \leq q(\phi) \quad (x \in \mathfrak{g}). \quad (109)$$

Equivalently, replacing  $q(\phi)$  with a constant multiple of  $q(\phi)$ , and using (85), (88) and (89), we see that

$$\left| \int_W \chi_x(w) \phi(w) dw \right| \leq q(\phi) \operatorname{ch}^{-d'}(x) \quad (x \in \mathfrak{g}). \quad (110)$$

Since  $l \leq l'$ , Lemma 14 together with (110) prove that (again up to a multiplicative constant that can be absorbed by  $q(\phi)$ ),

$$\left| \int_{\mathfrak{h} \cap \tau(W)} F_\phi(y) e^{iB(x,y)} dy \right| \leq q(\phi) |\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \operatorname{ch}(x)^{-d'}.$$

One may choose the system of the positive roots of  $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$  so that for every  $y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}$

$$\pi_{\mathfrak{g}/\mathfrak{h}}(y) = \begin{cases} \prod_{1 \leq j < k \leq l} i(-y_j + y_k) & \text{if } \mathbb{D} = \mathbb{C}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^l 2iy_j & \text{if } \mathbb{D} = \mathbb{H}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l}, \\ \prod_{1 \leq j < k \leq l} (-y_j^2 + y_k^2) \cdot \prod_{j=1}^l iy_j & \text{if } \mathbb{D} = \mathbb{R} \text{ and } \mathfrak{g} = \mathfrak{so}_{2l+1}. \end{cases} \quad (111)$$

Recall the constants  $r$  and  $\iota$  from (53) and (51). Then

$$\max\{\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}}; 1 \leq j \leq l\} = \frac{1}{\iota}(r-1), \quad (112)$$

where  $\deg_{y_j} \pi_{\mathfrak{g}/\mathfrak{h}}$  denotes the degree of  $\pi_{\mathfrak{g}/\mathfrak{h}}(y)$  with respect to the variable  $y_j$ .

Also, (112) and (96) imply that

$$|\pi_{\mathfrak{g}/\mathfrak{h}}(x)| \leq C_5 \operatorname{ch}^{r-1}(x) \leq C_5 \operatorname{ch}^{r-\iota}(x) \quad (x \in \mathfrak{h}),$$

where  $C_5$  is a constant. Thus, the claim follows.  $\square$

Lemmas 14 and 15 allows us to restate Corollary 10 in terms of orbital integrals on the symplectic space  $W$ .

**Corollary 17.** *Suppose  $l \leq l'$ . Then for any  $\phi \in \mathcal{S}(W)$*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g})(\phi) dg = C \int_{\mathfrak{h}} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\mathfrak{h} \cap \tau(W)} e^{iB(x,y)} F_\phi(y) dy dx,$$

where  $C$  is a constant that depends only on the dual pair  $(G, G')$  and each consecutive integral is absolutely convergent.

*Proof.* The equality is immediate from Corollary 10 and Lemma 14. The absolute convergence of the outer integral over  $\mathfrak{h}$  follows from Lemma 16.  $\square$

**Corollary 18.** *Suppose  $l > l'$ . Then for any  $\phi \in \mathcal{S}(W)$ ,*

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg \\ = C \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-s\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx,$$

where  $C$  is a constant that depends only on the dual pair  $(G, G')$  and each consecutive integral is absolutely convergent.

*Proof.* The formula is immediate from Corollary 10, Lemma 15 and formula (93):

$$\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg(\phi) \\ = C_1 \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \left( \pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x) \int_W \chi_x(w) \phi^G(w) dw \right) dx \\ = C_2 \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ \times \left( \sum_{tW(Z, \mathfrak{h}(\mathfrak{g})) \in W(G, \mathfrak{h}(\mathfrak{g}))/W(Z, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,t,y)} F_\phi(y) dy \right) dx \\ = \frac{C_2}{|W(Z, \mathfrak{h}(\mathfrak{g}))|} \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ \times \left( \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,t,y)} F_\phi(y) dy \right) dx \\ = C_3 \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-\mu}(\widehat{c}_-(t \cdot x)) \text{ch}^{d'-r-\iota}(t \cdot x) \\ \times \left( \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(t \cdot x, t \cdot y)} F_\phi(y) dy \right) dx \\ = C_3 \sum_{t \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(t) \int_{\mathfrak{h}(\mathfrak{g})} \xi_{-t^{-1}\mu}(\widehat{c}_-(x)) \text{ch}^{d'-r-\iota}(x) \\ \times \left( \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy \right) dx.$$

Let  $G''$  be the isometry group of the restriction of the form  $(\cdot, \cdot)$  to  $V_0^{0,0}$  and let  $\mathfrak{h}'' = \sum_{j=l'+1}^l \mathbb{R}J_j$ . Then, as in (112), we check that

$$\max\{\deg_{x_j} \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}; 1 \leq j \leq l\} = \max\{\deg_{x_j} \pi_{\mathfrak{z}''/\mathfrak{h}''}; l' + 1 \leq j \leq l\} = \frac{1}{l}(r'' - 1),$$

where  $r'' = \frac{2 \dim \mathfrak{g}_{\mathbb{R}}'}{\dim \mathbb{V}_{0,0}^{\mathbb{R}}}$  is defined as in (52). Since  $r - r'' = d'$ , we see that

$$\mathrm{ch}^{d'-r-1}(x) |\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x)| \leq \mathrm{const} \mathrm{ch}^{d'-r-1+r''-1}(x) = \mathrm{const} \mathrm{ch}^{-2}(x).$$

Furthermore,  $F_\phi$  is absolutely integrable. Therefore, the absolute convergence of the last integral over  $\mathfrak{h}$  follows from the fact that  $\mathrm{ch}^{-2\iota}$  is absolutely integrable.  $\square$

To prove Theorem 2 (and Theorem 3), we still need the following explicit formula for the form  $B(x, y)$ . Let  $\beta = \frac{2\pi}{\iota}$ , where  $\iota$  is as in (51). Then

$$B(x, y) = -\beta \sum_{j=1}^l x_j y_j \quad \left( x = \sum_{j=1}^l x_j J_j, y = \sum_{j=1}^l y_j J_j \in \mathfrak{h}(\mathfrak{g}) \right). \quad (113)$$

Indeed, the definition of the form  $B$ , (101), shows that

$$\begin{aligned} B(x, y) &= \pi \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(xy) = \pi \sum_{j,k} \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(J_j J_k) x_j y_k \\ &= \pi \sum_j \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(-1_{\mathbb{V}_0^j}) x_j y_j = -\frac{2\pi}{\iota} \sum_j x_j y_j. \end{aligned} \quad (114)$$

*Proof of Theorem 2.* By [MPP20, Theorem 7], the function  $F_\phi$  has the required number of continuous derivatives for the formula (56) to make sense. The operators appearing in the integrand of (56) act on different variables and therefore commute. Also, the constants  $a_j, b_j$  are integers by (95). Hence, equation (56) follows from Corollary 17, Lemma 11 and Proposition C.4.

For the last statement about the derivatives, notice that the degree of the polynomial  $q_j$ , or equivalently  $Q_{a_j, b_j}$ , is  $-a_j - b_j = 2\delta - 2$ . Hence, there are no derivatives in the formula (56) if and only if  $2\delta - 2 \leq 0$ . However,

$$2\delta - 2 = \frac{1}{\iota}(d' - r - \iota)$$

(where  $\iota = 1/2$  if  $\mathbb{D} = \mathbb{H}$  and 1 otherwise) and

$$d' - r - \iota = \begin{cases} 2l' - 2l & \text{if } (G, G') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}), \\ 2l' - 2l - 1 & \text{if } (G, G') = (\mathrm{O}_{2l+1}, \mathrm{Sp}_{2l'}), \\ l' - l - 1 & \text{if } (G, G') = (\mathrm{U}_l, \mathrm{U}_{p,q}), p + q = l', \\ l' - l & \text{if } (G, G') = (\mathrm{Sp}_l, \mathrm{O}_{2l'}^*). \end{cases}$$

Thus, since we assume  $l \leq l'$ , there are no derivatives in the formula (56) if and only if

$$d' - r - \iota \leq 0,$$

which means that  $l = l'$  if  $\mathbb{D} \neq \mathbb{C}$  and  $l' \in \{l, l+1\}$  if  $\mathbb{D} = \mathbb{C}$ . This completes the proof.  $\square$

Suppose now  $l > l'$ . Let  $\mathfrak{h}'' = \sum_{j=l'+1}^l \mathbb{R}J_j$ , so that

$$\mathfrak{h}(\mathfrak{g}) = \mathfrak{h} \oplus \mathfrak{h}''. \quad (115)$$

Then the centralizer of  $\tau(\mathfrak{h}_{\overline{1}})$  coincides with the centralizer of  $\mathfrak{h}$  in  $\mathfrak{g}$  and is equal to  $\mathfrak{z} = \mathfrak{h} \oplus \mathfrak{g}''$ , where  $\mathfrak{g}''$  is the Lie algebra of the group  $G''$  of the isometries of the restriction

of the form  $(\cdot, \cdot)$  to  $\mathbf{V}_0^0$ . Furthermore, the derived Lie algebras of  $\mathfrak{z}$  and  $\mathfrak{g}''$  coincide (i.e.  $[\mathfrak{z}, \mathfrak{z}] = [\mathfrak{g}'', \mathfrak{g}'']$ ) and  $\mathfrak{h}''$  is a Cartan subalgebra of  $\mathfrak{g}''$ .

**Lemma 19.** *Suppose  $l > l'$ . In terms of Corollary 18 and the decomposition (115)*

$$\begin{aligned} & \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \\ &= \left( \xi_{-s\mu}(\widehat{c}_-(x')) \operatorname{ch}^{d'-r-\iota}(x') \right) \left( \xi_{-s\mu}(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') \right), \end{aligned} \quad (116)$$

where  $x = x' + x'' \in \mathfrak{h}(\mathfrak{g})$ , with  $x' \in \mathfrak{h}'$  and  $x'' \in \mathfrak{h}''$ . Moreover,

$$\begin{aligned} & \int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r-\iota}(x'') \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') dx'' \\ &= C \sum_{s'' \in W(\mathbf{G}'', \mathfrak{h}'')} \operatorname{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''), \end{aligned} \quad (117)$$

where  $C$  is a constant,  $\rho''$  is one half times the sum of positive roots for  $(\mathfrak{g}'', \mathfrak{h}'')$  and  $\mathbb{I}_{\{0\}}$  is the indicator function of zero.

*Proof.* Formula (116) is obvious, because  $\pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x' + x'') = \pi_{\mathfrak{g}''/\mathfrak{h}''}(x'')$ . We shall verify (117). Let  $r''$  denotes the number defined in (52) for the Lie algebra  $\mathfrak{g}''$ . A straightforward computation verifies the following table:

$\mathfrak{g}$	$r$	$r''$	$d' - r + r''$
$\mathfrak{u}_d$	$d$	$d - d'$	0
$\mathfrak{o}_d$	$d - 1$	$d - d' - 1$	0
$\mathfrak{sp}_d$	$d + \frac{1}{2}$	$d - d' + \frac{1}{2}$	0

By (86) applied to  $\mathbf{G}'' \supseteq \mathbf{H}''$  and  $\mathfrak{g}'' \supseteq \mathfrak{h}''$ ,

$$\pi_{\mathfrak{g}''/\mathfrak{h}''}(x'') = C_1'' \Delta''(\widehat{c}_-(x'')) \operatorname{ch}^{r''-\iota}(x'') \quad (x = x' + x'', x' \in \mathfrak{h}', x'' \in \mathfrak{h}''),$$

where  $\Delta''$  is the Weyl denominator for  $\mathbf{G}''$ ,

$$\Delta'' = \sum_{s'' \in W(\mathbf{G}'', \mathfrak{h}'')} \operatorname{sgn}(s'') \xi_{s''\rho''}. \quad (118)$$

Hence, by (96), the integral (117) is a constant multiple of

$$\int_{\mathfrak{h}''} \xi_{-s\mu}(\widehat{c}_-(x'')) \Delta''(\widehat{c}_-(x'')) \operatorname{ch}^{d'-r+r''}(x'') \cdot \operatorname{ch}^{-2\iota}(x'') dx'' = 2^{\dim \mathfrak{h}''} \int_{\widehat{c}_-(\mathfrak{h}'')} \xi_{-s\mu}(h) \Delta''(h) dh, \quad (119)$$

where  $\widehat{c}_-(\mathfrak{h}'') \subseteq \widehat{\mathbf{H}}''^0$ .

Notice that the function

$$\widehat{\mathbf{H}}''^0 \ni h \rightarrow \xi_{-s\mu}(h) \Delta''(h) \in \mathbb{C} \quad (120)$$

is constant on the fibers of the covering map

$$\widehat{\mathbf{H}}''^0 \rightarrow \mathbf{H}''^0. \quad (121)$$

Indeed, the covering (121) is non-trivial only in two cases, namely  $\mathbf{G}'' = \mathbf{O}_{2l''+1}$  and  $\mathbf{G}'' = \mathbf{U}_{l''}$  with  $l''$  even; see (76). In these cases, (118) shows that this claim is true

provided that the weight  $-s\mu + s''\rho''$  is integral for the Cartan subgroup  $H''$  (i.e. it is equal to the derivative of a character of  $H''$ ).

Suppose  $G'' = O_{2l''+1}$ . Then  $G = O_{2l+1}$ ,  $\lambda_j \in \mathbb{Z}$  and  $\rho_j \in \mathbb{Z} + \frac{1}{2}$ . Hence,  $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$ . Since,  $\rho_j'' \in \mathbb{Z} + \frac{1}{2}$ , we see that  $(-s\mu)_j + \rho_j'' \in \mathbb{Z}$ .

Suppose now that  $G'' = U_{l''}$  with  $l''$  even. Then  $G = U_l$  and  $(-s\mu)_j \in \mathbb{Z} + \frac{1}{2}$ . In fact, if  $l'$  is even, i.e.  $l = l' + l''$  is even, then  $\lambda_j \in \mathbb{Z}$  and  $\rho_j \in \mathbb{Z} + \frac{1}{2}$ . If  $l'$  is odd, i.e.  $l = l' + l''$  is odd, then  $\lambda_j \in \mathbb{Z} + \frac{1}{2}$  and  $\rho_j \in \mathbb{Z}$ . Since  $\rho_j'' \in \mathbb{Z} + \frac{1}{2}$ , in both cases, we conclude that  $(-s\mu)_j + \rho_j'' \in \mathbb{Z}$ .

Therefore, (119) is a constant multiple of

$$\begin{aligned} & \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}(s'') \int_{H''^0} \xi_{-s\mu}(h) \xi_{s''\rho''}(h) dh \\ &= \begin{cases} \operatorname{vol}(H''^0) \operatorname{sgn}(s'') & \text{if } (s\mu)|_{\mathfrak{h}''} = s''\rho'', \\ 0 & \text{if } (s\mu)|_{\mathfrak{h}''} \notin W(G'', \mathfrak{h}'')\rho'', \end{cases} \\ &= \operatorname{vol}(H''^0) \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho''). \end{aligned} \quad (122)$$

□

**Corollary 20.** *Suppose  $l > l'$ . Then for any  $\phi \in \mathcal{S}(W)$*

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g}) dg(\phi) \\ &= C \sum_{s \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d-r-l}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx \\ &= C' \int_{\mathfrak{h}'} \xi_{-\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d-r-l}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx \end{aligned} \quad (123)$$

where  $C$  is a non-zero constant,  $C' = C|W(G', \mathfrak{h}')|$  and each consecutive integral is absolutely convergent. The expression (123) is zero unless one can choose the Harish-Chandra parameter  $\mu$  of  $\Pi$  so that

$$\mu|_{\mathfrak{h}''} = \rho''. \quad (124)$$

*Proof.* The second equality in (123) follows from the fact that  $F_\phi(sy) = \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) F_\phi(y)$ ,  $s \in W(G', \mathfrak{h}')$ .

Observe that  $B(x' + x'', y) = B(x', y)$  for  $x' \in \mathfrak{h}'$ ,  $x'' \in \mathfrak{h}''$  and  $y \in \tau'(\mathfrak{h}_1^{reg}) \subseteq \mathfrak{h}'$ . We see therefore from Corollary 18 and Lemma 19 that

$$\begin{aligned} & \int_{-G^0} \check{\Theta}_\Pi(\tilde{g}) T(\tilde{g}) dg(\phi) \\ &= C \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}/(\mathfrak{h})/\mathfrak{h}}(s) \operatorname{sgn}(s'') \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') \times \\ & \quad \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d-r-l}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx. \end{aligned} \quad (125)$$

Notice that for  $x \in \mathfrak{h}'$  and  $s'' \in W(G'', \mathfrak{h}'')$ , we have  $s''x = x$ . Thus  $\xi_{-s\mu}(\widehat{c}_-(x)) = \xi_{-s''s\mu}(\widehat{c}_-(x))$  by (93). Notice also that  $W(G'', \mathfrak{h}'') \subseteq W(G, \mathfrak{h})$  and  $\operatorname{sgn}(s'') = \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s'')$ .

Moreover,  $\mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + s''\rho'') = \mathbb{I}_{\{0\}}(-(s''^{-1}s\mu)|_{\mathfrak{h}''} + \rho'')$ . Hence, replacing  $s$  by  $s''s$  in (125), we see that this expression is equal to

$$\begin{aligned} C \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g}))} \sum_{s'' \in W(G'', \mathfrak{h}'')} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \mathbb{I}_{\{0\}}(-(s\mu)|_{\mathfrak{h}''} + \rho'') \times \\ \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx \\ = C |W(G'', \mathfrak{h}'')| \sum_{s \in W(G, \mathfrak{h}(\mathfrak{g})), (s\mu)|_{\mathfrak{h}''} = \rho''} \operatorname{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(s) \times \\ \int_{\mathfrak{h}'} \xi_{-s\mu}(\widehat{c}_-(x)) \operatorname{ch}^{d'-r-\iota}(x) \int_{\tau'(\mathfrak{h}_1^{reg})} e^{iB(x,y)} F_\phi(y) dy dx. \end{aligned} \quad (126)$$

Clearly (126) is zero if there is no  $s$  such that  $(s\mu)|_{\mathfrak{h}''} = \rho''$ . Since  $\mu$  is determined only up to the conjugation by the Weyl group, we may thus assume that  $\mu|_{\mathfrak{h}''} = \rho''$  and (124) follows. Under this assumption the summation in (126) is over the  $s$  such that  $(s\mu)|_{\mathfrak{h}''} = \rho''$ . For such  $s$  we have

$$(s\mu)|_{\mathfrak{h}'} + \rho'' = s\mu = s(\mu|_{\mathfrak{h}'} + \rho''). \quad (127)$$

Since  $\mu$  is regular, (127) shows that  $s \in W(G', \mathfrak{h}')$ . Hence, (123) follows. The absolute convergence of the integrals was checked in the proof of Corollary 18.  $\square$

*Proof of Theorem 3.* The formula (57) follows from (123), Lemma 11 and from Proposition C.5 because, under our assumption,  $a_{s,j} + b_{s,j} = -2\delta + 2 = -\frac{1}{\iota}(d' - r) + 1 \geq 1$ . The Weyl integration formula on  $W$  (see [MPP15, Theorem 21] together with (57), (40) and (55) implies that  $\int_{-G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g}) dg$  is a locally integrable function whose restriction to  $\mathfrak{h}_1^{reg}$  is equal to a non-zero constant multiple of

$$\frac{1}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(\tau'(w))} \sum_{s \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \prod_{j=1}^{\iota'} P_{a_{s,j}, b_{s,j}}(\beta J_j'^*(\tau'(w))) e^{-\beta |\delta_j J_j'^*(\tau'(w))|} \quad (w \in \mathfrak{h}_1^{reg}).$$

Notice that for  $w = \sum_{j=1}^{\iota'} w_j u_j \in \mathfrak{h}_1$ ,

$$\sum_{j=1}^{\iota'} |\delta_j J_j'^*(\tau'(w))| = \sum_{j=1}^{\iota'} w_j^2 = \sum_{j=1}^{\iota'} \delta_j J_j'^*(\tau'(w)) = \left( \sum_{j=1}^{\iota'} \delta_j J_j'^* \right) \circ \tau'(w).$$

This is a quadratic polynomial on  $\mathfrak{h}_1$ , invariant under the Weyl group  $W(S, \mathfrak{h}_1)$ . Such a polynomial has no  $GG'$ -invariant extension to  $W$ , unless  $G'$  is compact.

Indeed, suppose  $P$  is a real-valued  $GG'$ -invariant polynomial on  $W$  such that

$$P(w) = \left( \sum_{j=1}^{\iota'} \delta_j J_j'^* \right) \circ \tau'(w) \quad (w \in \mathfrak{h}_1).$$

Then  $P$  extends uniquely to a complex-valued  $G_{\mathbb{C}}G'_{\mathbb{C}}$ -invariant polynomial on the complexification  $W_{\mathbb{C}}$  of  $W$ . Hence, by the Classical Invariant Theory, [How89a, Theorems 1A



and 1B] there is a  $G'_\mathbb{C}$ -invariant polynomial  $Q$  on  $\mathfrak{g}'_\mathbb{C}$  such that  $P = Q \circ \tau'$ . Hence,

$$Q(\tau'(w)) = P(w) = \left( \sum_{j=1}^{l'} \delta_j J'_j{}^* \right) \circ \tau'(w) \quad (w \in \mathfrak{h}_{\overline{1}}).$$

Since  $\tau'(\mathfrak{h}_{\overline{1}})$  spans  $\mathfrak{h}'$ , we see that the restriction of  $Q$  to  $\mathfrak{h}'$

$$Q|_{\mathfrak{h}'} = \sum_{j=1}^{l'} \delta_j J'_j{}^* \in \mathfrak{h}'_\mathbb{C}.$$

Since  $Q$  is  $G'_\mathbb{C}$ -invariant, the restriction  $Q|_{\mathfrak{h}'}$  has to be invariant under the corresponding Weyl group. There are no linear invariants if  $G' = \mathrm{Sp}_{2l'}(\mathbb{R})$  or  $\mathrm{O}_{2l'}^*$ . Therefore  $G' = \mathrm{U}_{p,q}$ ,  $p+q = l'$ . But in this case the invariance means that all the  $\delta_j$  are equal. Hence  $G' = \mathrm{U}_{l'}$  is compact. In the case  $G' = \mathrm{U}_{l'}$ , the sum of squares coincides with  $\langle J(w), w \rangle$  for a positive complex structure  $J$  on  $W$  which commutes with  $G$  and  $G'$  and therefore

$$e^{-\beta \sum_{j=1}^{l'} |\delta_j J'_j{}^*(\tau'(w))|} \quad (128)$$

extends to a Gaussian on  $W$ . If  $G'$  is not compact that (128) extends to a  $GG'$ -invariant function on  $W$ , which is bounded but is not a Gaussian.

The polynomial

$$\sum_{s \in W(G', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta y_j) \quad (y \in \mathfrak{h}')$$

is  $W(G', \mathfrak{h}')$ -skew-invariant. Hence it is divisible by  $\pi_{\mathfrak{g}'/\mathfrak{h}'}(y)$ . Therefore the fraction

$$\frac{\sum_{s \in W(G', \mathfrak{h}')} \mathrm{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(s) \prod_{j=1}^{l'} P_{a_{s,j}, b_{s,j}}(\beta y_j)}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(y)} \quad (y \in \mathfrak{h}')$$

is a  $W(G', \mathfrak{h}')$ -invariant polynomial. By Chevalley's restriction theorem, [Wal88, 3.1.3] it extends to a  $G'$ -invariant polynomial on  $\mathfrak{g}'$ . Thus the pullback by  $\tau'$  is a  $GG'$ -invariant polynomial on  $W$ . The proof is now complete.  $\square$

## 8. The special case for the pair $\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R})$ .

Here we consider the case  $(G, G') = (\mathrm{O}_{2l}, \mathrm{Sp}_{2l'}(\mathbb{R}))$  and suppose that the character  $\Theta_\Pi$  is not supported on the preimage of the connected identity component  $\widetilde{G}^0$ . As is well known from the correspondence of the highest weights, see for example [Prz96, page 99], this may happen if and only if  $l \leq l'$  and, by [Prz89, Appendix C],  $(l, l') \neq (1, 1)$ . Recall the element  $s \in G$ , (59), with centralizer in  $\mathfrak{h}$  equal to  $\mathfrak{h}_s = \sum_{j=1}^{l-1} \mathbb{R} J_j$ .

**Lemma 21.** *Let  $H_s \subseteq H$  be the centralizer of  $s$ . Then every element of the connected component  $G^0 s$  is  $G$ -conjugate to an element of  $H_s s$ .*

*Proof.* Fix an element  $g \in G$ . As shown in [Cur84, page 114],  $g$  preserves a subspace of  $V$  of dimension 1 or 2. Hence  $V$  decomposes into a direct sum of  $g$ -irreducible subspaces of dimension 1 or 2, and the claim follows.  $\square$

Let

$$\Delta_s(h) = \prod_{1 \leq j < k \leq l-1} (1 - \xi_{e_j - e_k}(h))(1 - \xi_{e_j + e_k}(h)) \cdot \prod_{j=1}^{l-1} (1 - \xi_{2e_j}(h)) \quad (h \in H_s) \quad (129)$$

and let  $W(G, \mathfrak{h})_s \subseteq W(G, \mathfrak{h})$  be the centralizer of  $s$ . The elements of  $W(G, \mathfrak{h})_s$  act by sign changes and permutations of the  $J_1, \dots, J_{l-1}$ , but fix  $J_l$ . The following two lemmas follow respectively from [Wen01, Theorems 2.5 and 2.6].

**Lemma 22.** *For any continuous  $G$ -invariant function  $f : G^0 s \rightarrow \mathbb{C}$ ,*

$$\int_{G^0 s} f(g) dg = \frac{1}{|W(G, \mathfrak{h})_s|} \int_{H_s} f(hs) |\Delta_s(h)|^2 dh.$$

Recall the spaces

$$V_{\bar{0},s} = V_0^1 \oplus V_0^2 \oplus \dots \oplus V_0^{l-1} \oplus \mathbb{R}v_{2l}, \quad V_s = V_{\bar{0},s} \oplus V_{\bar{1}}.$$

The corresponding dual pair is  $(G_s, G'_s) = (O_{2l-1}, \text{Sp}_{2l'}(\mathbb{R}))$  acting on the symplectic space  $W_s = \text{Hom}(V_{\bar{1}}, V_{\bar{0},s})$ . Furthermore we have  $\Theta_s, t_s, T_s$ , as defined as in (16) and (14).

Since the group  $H_s$  acts as the identity on  $\mathbb{R}v_{2l-1} \oplus \mathbb{R}v_{2l}$  we may identify it with a Cartan subgroup of  $G_s$ . Notice that the coverings

$$\widetilde{G^0 s} \rightarrow G^0 s, \quad \widetilde{G^0} \rightarrow G^0$$

split (see Appendix D). Hence we may choose continuous sections

$$H_s s \ni hs \rightarrow \widetilde{hs} \in \widetilde{H_s s} \quad \text{and} \quad H_s \ni h \rightarrow \widetilde{h} \in \widetilde{H_s}. \quad (130)$$

Let  $\rho_s$  denote the one half times the sum of the roots that occur in (129). Explicitly

$$\rho_s = (l-1)e_1 + (l-2)e_2 + \dots + e_{l-1}. \quad (131)$$

**Lemma 23.** *Pick a continuous section*

$$H_s \ni h \rightarrow \widetilde{hs} \in \widetilde{H_s s}. \quad (132)$$

Then

$$\Theta_{\Pi}(\widetilde{hs}) = \frac{\Theta_{\Pi}(\tilde{s})}{\Theta_{\Pi_s}(1)} \Theta_{\Pi_s}(h) \quad (h \in H_s), \quad (133)$$

where

$$\Theta_{\Pi_s}(h) = \frac{\sum_{w \in W(G, \mathfrak{h})_s} \text{sgn}(w) \xi_{w \cdot (\lambda + \rho_s)}(h)}{\Delta_s(h)} \quad (h \in H_s), \quad (134)$$

with  $\text{sgn} = \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}_s}$ , is the character of the irreducible representation  $\Pi_s$  of  $G_s^0$  with highest weight  $\lambda$ , see (60), and

$$\frac{\Theta_{\Pi}(\tilde{s})}{\Theta_{\Pi_s}(1)} = \pm 1. \quad (135)$$

To compute  $\Theta(\widetilde{hs})$ , we need the following lemma. Recall from [AP14, Definition 4.16, (35) and (101)] that

$$\Theta^2(g) = i^{\dim(g-1)W} \det(J^{-1}(g-1))_{J^{-1}(g-1)} \quad (g \in \text{Sp}(W)). \quad (136)$$

Recall also the map  $t$  from (14).

**Lemma 24.** *For  $h \in H_s$  we have*

$$\Theta^2(hs) = \left(\frac{i}{2}\right)^{2l'} \Theta_s^2(h) \quad (137)$$

and

$$t(hs) = t_s(h) \otimes \delta_0, \quad (138)$$

where  $\delta_0$  is the Dirac delta on  $W_s^\perp$ .

*Proof.* By definition

$$W = W_s \oplus W_s^\perp \quad \text{where} \quad W_s^\perp = \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l-1}).$$

Since

$$h|_{W_s^\perp} = 1|_{W_s^\perp} \quad \text{and} \quad s|_{W_s^\perp} = 1|_{W_s^\perp}$$

we have  $(hs - 1)W = (hs - 1)W_s$ . Furthermore

$$W_s = \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1}) \oplus \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l})$$

and

$$\begin{aligned} (hs - 1)|_{\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1})} &= (h - 1)|_{\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1})}, \\ (hs - 1)|_{\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l})} &= -2I_{\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l})}. \end{aligned}$$

Therefore

$$\det(hs - 1)_{(hs-1)W} = \det(h - 1)_{\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1})} \cdot (-2)^{2l'}.$$

Moreover,

$$(h - 1)W_s = (h - 1) \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1}).$$

Hence

$$\det(hs - 1)_{(hs-1)W} = \det(h - 1)_{(h-1)W_s} \cdot 2^{2l'}$$

and

$$\dim(hs - 1)W = \dim(h - 1)W_s + 2l'.$$

We may fix the complex structure  $J$  that acts as an element of  $G'$ , see the proof of Lemma 5.4 in [DKP05]. Then  $J^{-1}(hs - 1) = (hs - 1)J^{-1}$  so that  $J^{-1}(hs - 1)W = (hs - 1)W$ . Hence,

$$\det(J^{-1}(hs - 1))_{J^{-1}(hs-1)W} = \det(hs - 1)_{(hs-1)W}$$

and similarly for  $W_s$ . By (136), this verifies (137).

Let  $w_s \in W_s$  and let  $w_s^\perp \in W_s^\perp$ . Then

$$\begin{aligned} \langle c(hs)(hs - 1)(w_s + w_s^\perp), (hs - 1)(w_s + w_s^\perp) \rangle &= \langle c(hs)(hs - 1)w_s, (hs - 1)w_s \rangle \\ &= \langle (hs + 1)w_s, (hs - 1)w_s \rangle. \end{aligned} \quad (139)$$

Let us write  $w_s = w_s^{l-1} + w_s^l$ , where

$$w_s^{l-1} \in \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1}), \quad w_s^l \in \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l}).$$

Then

$$(hs + 1)w_s = (h + 1)w_s^{l-1} \quad \text{and} \quad (hs - 1)w_s = (h - 1)w_s^{l-1} - 2w_s^l.$$

Hence (139) is equal to

$$\begin{aligned} \langle (h+1)w_s^{l-1}, (h-1)w_s^{l-1} - 2w_s^l \rangle &= \langle (h+1)w_s^{l-1}, (h-1)w_s^{l-1} \rangle \\ &= \langle c(h)(h-1)w_s^{l-1}, (h-1)w_s^{l-1} \rangle = \langle c(h)(h-1)w_s, (h-1)w_s \rangle. \end{aligned}$$

Since

$$\begin{aligned} (hs-1)W &= (hs-1)W_s \\ &= (h-1)\text{Hom}(\mathbf{V}_{\bar{1}}, \mathbf{V}_0^1 \oplus \mathbf{V}_0^2 \oplus \cdots \oplus \mathbf{V}_0^{l-1}) \oplus \text{Hom}(\mathbf{V}_{\bar{1}}, \mathbb{R}v_{2l}) = (h-1)W_s, \end{aligned}$$

equation (138) follows.  $\square$

*Proof of Theorem 4.* Let us choose  $\tilde{h} = h$  in (130) and define the section  $\mathbb{H}_s s \rightarrow \widetilde{\mathbb{H}_s s}$  so that

$$\Theta(\widetilde{hs}) = \Theta_s(\tilde{h}) \left( \frac{i}{2} \right)^{l'}. \quad (140)$$

Then

$$T(\widetilde{hs}) = \left( \frac{i}{2} \right)^{l'} T_s(\tilde{h}) \otimes \delta_0. \quad (141)$$

In these terms, Lemmas 22 and 24 show that

$$\begin{aligned} &\int_{\widetilde{\mathbb{G}}_s^0} \check{\Theta}_{\Pi}(\tilde{g}) T(\tilde{g})(\phi) dg \\ &= \frac{1}{|W(\mathbb{G}, \mathfrak{h})_s|} \int_{\mathbb{H}_s} \check{\Theta}_{\Pi}(\widetilde{hs}) \Delta_s(h) \overline{\Delta_s(h)} T(\widetilde{hs})(\phi^{\mathbb{G}}) dh \\ &= \left( \frac{i}{2} \right)^{l'} \frac{1}{|W(\mathbb{G}, \mathfrak{h})_s|} \int_{\mathbb{H}_s} \frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_s}(1)} \check{\Theta}_{\Pi_s}(h) \Delta_s(h) \overline{\Delta_s(h)} T_s(\tilde{h})(\phi^{\mathbb{G}}|_{W_s}) dh \\ &= \left( \frac{i}{2} \right)^{l'} \frac{\check{\Theta}_{\Pi}(\tilde{s})}{\check{\Theta}_{\Pi_s}(1)} \int_{\mathbb{G}_s^0} \check{\Theta}_{\Pi_s}(\tilde{g}) T_s(\tilde{g})(\phi^{\mathbb{G}}|_{W_s}) dg. \end{aligned} \quad (142)$$

Now we lift  $\Pi_s$  to be a genuine irreducible representation of  $\widetilde{\mathbb{G}}_s^0$  so that the product under the integral does not depend on the lift  $\tilde{h}$ .  $\square$

### 9. The special case for the pair $\text{O}_{2l+1}, \text{Sp}_{2l'}(\mathbb{R})$ , with $1 \leq l \leq l'$ .

Recall the decomposition (62). Let us denote the objects corresponding to  $W_s$  by the subscript  $s$ :  $\Theta_s, t_s$  and  $T_s$ . The groups  $\mathbb{G}$  and  $\mathbb{G}_s$  share the same Cartan subgroup  $\mathbb{H} = \mathbb{H}_s$  and the Lie algebras  $\mathfrak{g}$  and  $\mathfrak{g}_s$  share the same Cartan subalgebra  $\mathfrak{h} = \mathfrak{h}_s$ . Since any element  $h \in \mathbb{H}$  acts trivially on  $W_s^{\perp}$ , we see that  $(h-1)W = (h-1)W_s$ . Hence,

$$\det(h-1)_{(h-1)W} = \det(h-1)_{(h-1)W_s}$$

and, just as in section 8, we check that

$$\Theta(\tilde{h}) = \Theta_s(\tilde{h}), \quad t(\tilde{h}) = t_s(\tilde{h}) \otimes \delta_0, \quad (143)$$

where  $\delta_0$  is the Dirac delta on  $W_s^{\perp}$ .

*Proof of Theorem 5.* It is easy to check that

$$|\Delta(h)|^2 = |\Delta_s(h)|^2 \det(1-h) \quad (h \in \mathbb{H}), \quad (144)$$

where

$$\Delta_s(h) = \prod_{1 \leq j \leq k \leq l} (1 - \xi_{e_j - e_k}(h))(1 - \xi_{e_j + e_k}(h)).$$

Hence for  $\phi \in \mathcal{S}(W)$ ,

$$\begin{aligned} \int_{\widetilde{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \frac{1}{|W(G, \mathfrak{h})|} \int_{\mathbb{H}} \check{\Theta}_\Pi(\tilde{h})\Delta(h)\overline{\Delta(h)}T(\tilde{h})(\phi^G) dh \\ &= \frac{1}{|W(G_s, \mathfrak{h}_s)|} \int_{\mathbb{H}_s} \check{\Theta}_\Pi(\tilde{h}) \det(1-h)\Delta_s(h)\overline{\Delta_s(h)}\Theta_s(\tilde{h})t_s(h)(\phi^G|_{W_s}) dh \\ &= \int_{\widetilde{G}^0} \check{\Theta}_\Pi(\tilde{g}) \det(1-g)T_s(\tilde{g})(\phi|_{W_s}) dg. \end{aligned} \quad (145)$$

On the other hand,

$$\begin{aligned} \int_{\widetilde{G}^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg &= \frac{1}{|W(G_s, \mathfrak{h}_s)|} \int_{\mathbb{H}_s} \check{\Theta}_\Pi(\tilde{h})\Delta(h)\overline{\Delta(h)}\Theta_s(\tilde{h})t_s(h)(\phi^G|_{W_s}) dh \\ &= \frac{1}{|W(G_s, \mathfrak{h}_s)|} \int_{\mathbb{H}_s} \left( \check{\Theta}_\Pi(\tilde{h})\Delta(h) \right) \left( \frac{\overline{\Delta(h)}}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(c(h))} \Theta_s(\tilde{h}) \right) \\ &\quad \times \pi_{\mathfrak{g}_s/\mathfrak{h}_s}(c(h)) \int_{W_s} \chi_{c(h)}(w) (\phi^G|_{W_s})(w) dw dh. \end{aligned} \quad (146)$$

Since  $\overline{\Delta(h)}$  is a constant multiple of  $\Delta(h)$ , Appendices A and B show that there is a constant  $C_1$  such that

$$\begin{aligned} \frac{\overline{\Delta(h)}}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(c(h))} \cdot \Theta_s(\tilde{h}) \cdot dh &= C_1 \frac{\pi_{\mathfrak{g}/\mathfrak{h}}(x)}{\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x)} \text{ch}^{-r+1}(x) \cdot \text{ch}^{2l'}(x) \cdot \text{ch}^{-2}(x) dx \\ &= C_1 \prod_{j=1}^l e_j(x) \cdot \text{ch}^{2l'-r-1}(x) dx \quad (h = c(x)). \end{aligned}$$

Lemma 14 shows that there is a constant  $C_2$  such that

$$\pi_{\mathfrak{g}_s/\mathfrak{h}_s}(x) \int_{W_s} \chi_x(w) (\phi^G|_{W_s})(w) dw = C_2 \int_{\mathfrak{h}_s} e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy.$$

This shows that (145) is equal to a constant multiple of

$$\int_{\mathfrak{h}_s} \xi_{-\mu}(c(x)) \text{ch}^{2l'-r-1}(x) \prod_{j=1}^l e_j(x) \int_{\mathfrak{h}_s} e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx.$$

Notice that

$$(2l' - r - 1) + 1 = 2l' - 2l = 2l' - (2l - 1) - 1 = 2l' - r_s - 1.$$

Hence [MPP20, Theorem 3.5] with  $d' = 2l'$  implies that the function  $F_{\phi^G|_{W_s}}(y)$ , which is an orbital integral for the pair  $(G_s, G') = (O_{2l}, \text{Sp}_{2l'}(\mathbb{R}))$ , is regular enough so that we may rewrite the last expression as a constant multiple of

$$\int_{\mathfrak{h}_s} \xi_{-\mu}(c(x)) \text{ch}^{2l'-r-1}(x) \int_{\mathfrak{h}_s} e^{iB(x,y)} \prod_{j=1}^l \partial_{y_j} F_{\phi^G|_{W_s}}(y) dy dx.$$

We can then proceed as in the proof of Theorem 2 and complete the argument.  $\square$

### 10. The special case for the pair $O_{2l+1}, \mathrm{Sp}_{2l'}(\mathbb{R})$ with $l > l'$ .

Recall the decomposition (64). Let us denote the objects corresponding to  $W_s$  by the subscript  $s$ :  $\Theta_s$ ,  $t_s$  and  $T_s$ . Since any element  $h \in H(\mathfrak{g})$ , see (45), acts trivially on  $W_s^\perp$ , we see that

$$(h - 1)W = (h - 1)W_s.$$

Hence

$$\det(h - 1)_{(h-1)W} = \det(h - 1)_{(h-1)W_s}.$$

Hence, as in section 8, we check that

$$\Theta(\tilde{h}) = \Theta_s(\tilde{h}), \quad t(h) = t_s(h) \otimes \delta_0, \quad (147)$$

where  $\delta_0$  is the Dirac delta on  $W_s^\perp$ .

*Proof of Theorem 6.* As in section 9, we check that for  $\phi \in \mathcal{S}(W)$ ,

$$\int_{G^0} \check{\Theta}_\Pi(\tilde{g})T(\tilde{g})(\phi) dg = \int_{G_s^0} \check{\Theta}_\Pi(\tilde{g}) \det(1 - g)T_s(\tilde{g})(\phi^G|_{W_s}) dg, \quad (148)$$

which verifies the first equality in (65). On the other hand, (147) implies that the left hand side of (148) is equal to

$$\frac{1}{|W(G, \mathfrak{h}(\mathfrak{g}))|} \int_{H(\mathfrak{g})} \left( \check{\Theta}_\Pi(\tilde{h})\Delta(h) \right) \left( \frac{\overline{\Delta(h)}}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(c(h))} \Theta_s(\tilde{h}) \right) \pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(c(h)) \int_{W_s} \chi_{c(h)}(w) (\phi^G|_{W_s})(w) dw dh. \quad (149)$$

Lemma 15 shows that there is a constant  $C$  such that

$$\begin{aligned} & \pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x) \int_{W_s} \chi_x(w) \phi^G|_{W_s}(w) dw \\ &= C \int_{\tau'(\mathfrak{h}_1^{reg})} \sum_{tW(\mathfrak{Z}_s, \mathfrak{h}(\mathfrak{g})) \in W(G_s, \mathfrak{h}(\mathfrak{g}))/W(\mathfrak{Z}_s, \mathfrak{h}(\mathfrak{g}))} \mathrm{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1} \cdot x) e^{iB(x, t \cdot y)} F_{\phi^G|_{W_s}}(y) dy, \end{aligned}$$

where  $\mathfrak{z}_s \subseteq \mathfrak{g}_s$  is the centralizer of  $\mathfrak{h} = \mathfrak{h}'$ .

As in section 9, and with  $d' = 2l'$ ,

$$\begin{aligned} & \frac{\overline{\Delta(h)}}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(c(h))} \cdot \Theta_s(\tilde{h}) \cdot dh \\ &= C \frac{\pi_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(x)}{\pi_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(x)} \mathrm{ch}^{-r+1}(x) \cdot \mathrm{ch}^{d'}(x) \cdot \mathrm{ch}^{-2}(x) dx = C \prod_{j=1}^l e_j(x) \cdot \mathrm{ch}^{d'-r-1}(x) dx. \end{aligned}$$

This shows that (149) is equal to a constant multiple of

$$\begin{aligned} & \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \xi_{-\mu}(c(u^{-1}.x)) \cdot \prod_{j=1}^l e_j(x) \cdot \text{ch}^{d'-r-1}(x) \\ & \quad \times \sum_{tW(Z_s, \mathfrak{h}(\mathfrak{g})) \in W(G_s, \mathfrak{h}(\mathfrak{g}))/W(Z_s, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(t^{-1}.x) e^{iB(x,t,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Now we notice that

$$\text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \prod_{j=1}^l e_j(x) = \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(u) \prod_{j=1}^l e_j(u^{-1}.x), \quad (150)$$

change the variable of integration  $x$  to  $t.x$  and use the equation  $\text{ch}(t.x) = \text{ch}(x)$ , to see that (149) is a constant multiple of

$$\begin{aligned} & \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(tu) \xi_{-\mu}(c(u^{-1}.x)) \cdot \prod_{j=1}^l e_j(u^{-1}.x) \cdot \text{ch}^{d'-r-1}(x) \\ & \quad \times \sum_{t \in W(G_s, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Since  $\text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(tu) \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(t) = \text{sgn}_{\mathfrak{g}_s/\mathfrak{h}(\mathfrak{g})}(u)$ , (150) implies that (149) is a constant multiple of

$$\begin{aligned} & \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \xi_{-\mu}(c(u^{-1}.x)) \cdot \prod_{j=1}^l e_j(x) \cdot \text{ch}^{d'-r-1}(x) \\ & \quad \times \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \quad (151) \end{aligned}$$

Now we use the direct sum decomposition (115) of  $\mathfrak{h}(\mathfrak{g})$  and the equality

$$\prod_{j=l'+1}^l e_j(x) \cdot \pi_{\mathfrak{z}_s/\mathfrak{h}(\mathfrak{g})}(x) = \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x)$$

to rewrite (151) as

$$\begin{aligned} & \int_{\mathfrak{h}(\mathfrak{g})} \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \xi_{-u,\mu}(c(x)) \cdot \text{ch}^{d'-r-1}(x) \cdot \pi_{\mathfrak{z}/\mathfrak{h}(\mathfrak{g})}(x) \\ & \quad \prod_{j=1}^{l'} e_j(x) \cdot e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \end{aligned}$$

Lemma 19 implies that this last expression is a constant multiple of

$$\int_{\mathfrak{h}'} \int_{\tau'(\mathfrak{h}_\Gamma^{reg})} \sum_{u \in W(G, \mathfrak{h}(\mathfrak{g}))} \text{sgn}_{\mathfrak{g}/\mathfrak{h}(\mathfrak{g})}(u) \xi_{-u,\mu}(c(x)) \cdot \text{ch}^{d'-r-1}(x) \prod_{j=1}^{l'} e_j(x) \cdot e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx.$$

From the proof of Corollary 20 we see that the above expression is a constant multiple of

$$\int_{\mathfrak{h}'} \int_{\tau'(\mathfrak{h}_1^{reg})} \sum_{u \in W(G', \mathfrak{h}')} \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(u) \xi_{-u, \mu}(c(x)) \cdot \operatorname{ch}^{d'-r-1}(x) \prod_{j=1}^{l'} e_j(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \quad (152)$$

Now we transport  $u \in W(G', \mathfrak{h}')$  from  $\xi_{-u, \mu}(c(x))$  to  $F_{\phi^G|_{W_s}}(y)$ , use the fact that  $F_{\phi^G|_{W_s}}(u \cdot y) = \operatorname{sgn}_{\mathfrak{g}'/\mathfrak{h}'}(u) F_{\phi^G|_{W_s}}(y)$  to see that (152) is a constant multiple of

$$\int_{\mathfrak{h}'} \int_{\tau'(\mathfrak{h}_1^{reg})} \xi_{-\mu}(c(x)) \operatorname{ch}^{d'-r-1}(x) \prod_{j=1}^{l'} e_j(x) e^{iB(x,y)} F_{\phi^G|_{W_s}}(y) dy dx. \quad (153)$$

Finally we rewrite (153) as

$$\begin{aligned} & \int_{\mathfrak{h}'} \int_{\mathfrak{h}'} \xi_{-u, \mu}(c(x)) \operatorname{ch}^{d'-r-1}(x) \prod_{j=1}^{l'} e_j(x) e^{iB(x,y)} \mathbb{I}_{\tau'(\mathfrak{h}_1^{reg})}(y) F_{\phi^G|_{W_s}}(y) dy dx \\ &= C \int_{\mathfrak{h}'} \int_{\mathfrak{h}'} \xi_{-u, \mu}(c(x)) \operatorname{ch}^{d'-r-1}(x) e^{iB(x,y)} \prod_{j=1}^{l'} \partial(J_j) (\mathbb{I}_{\tau'(\mathfrak{h}_1^{reg})} F_{\phi^G|_{W_s}}) (y) dy dx. \end{aligned} \quad (154)$$

This completes the argument.  $\square$

## 11. The sketch of a computation of the wave front of $\Pi'$ .

**Corollary 25.** *For any representation  $\Pi \otimes \Pi'$  which occurs in the restriction of the Weil representation to the dual pair  $(\widetilde{G}, \widetilde{G}')$ ,*

$$WF(\Pi') = \tau'(\tau^{-1}(0)).$$

Here  $WF(\Pi')$  stands for the wave front of the character  $\Theta_{\Pi'}$  at the identity and  $0 = WF(\Pi)$  since  $\Pi$  is finite dimensional.

The complete proof is rather lengthy but unlike the one provided in [Prz91, Theorem 6.11], it is independent of [Vog78]. We sketch the main steps below. The details are going to appear in [MPP21].

The variety  $\tau^{-1}(0) \subseteq W$  is the closure of a single  $GG'$ -orbit  $\mathcal{O}$ ; see e.g. [Prz91, Lemma 2.16]. There is a positive  $GG'$ -invariant measure  $\mu_{\mathcal{O}}$  on this orbit which defines a homogeneous distribution. We denote its degree by  $\deg \mu_{\mathcal{O}}$ .

Recall that if  $V$  is a  $n$ -dimensional real vector space,  $t > 0$  and  $M_t v = tv$  for  $v \in V$ , then the pullback of  $u \in \mathcal{S}^*(V)$  by  $M_t$  is  $M_t^* u \in \mathcal{S}^*(V)$ , defined by

$$(M_t^* u)(\phi) = t^{-n} u(\phi \circ M_{t^{-1}}) \quad (\phi \in \mathcal{S}(V)).$$

In particular, for  $V = W$

$$M_t^* \mu_{\mathcal{O}} = t^{\deg \mu_{\mathcal{O}}} \mu_{\mathcal{O}}.$$

Define  $\tau'_* : \mathcal{S}^*(W) \rightarrow \mathcal{S}^*(\mathfrak{g}')$  by  $\tau'_*(u)(\psi) = u(\psi \circ \tau')$ . Then, for  $t > 0$ ,

$$t^{2 \dim \mathfrak{g}'} M_{t^2}^* \circ \tau'_* = t^{\dim W} \tau'_* \circ M_t^*. \quad (155)$$



A rather lengthy but straightforward computation based on Theorems 2, 3 and 4, shows that

$$t^{\deg \mu_{\mathcal{O}}} M_{t^{-1}}^* f_{\Pi \otimes \Pi'} \xrightarrow[t \rightarrow 0]{} C \mu_{\mathcal{O}}, \quad (156)$$

as tempered distributions on  $W$ , where  $C$  is a non-zero constant.

Let  $\mathcal{F}$  indicate a Fourier transform on  $\mathcal{S}^*(\mathfrak{g}')$ . Then, for  $t > 0$ ,

$$M_t^* \circ \mathcal{F} = t^{-\dim \mathfrak{g}'} \mathcal{F} \circ M_{t^{-1}}. \quad (157)$$

Hence, in the topology of  $\mathcal{S}^*(\mathfrak{g}')$ ,

$$t^{2 \deg \mu_{\mathcal{O}'}} M_{t^2}^* \mathcal{F} \tau'_*(f_{\Pi \otimes \Pi'}) \xrightarrow[t \rightarrow 0^+]{} C \mathcal{F} \mu_{\mathcal{O}'} \quad (158)$$

where  $C \neq 0$  and  $\mathcal{O}' = \tau'(\mathcal{O})$ .

There is an easy to verify inclusion  $WF(\Pi') \subseteq \overline{\mathcal{O}'}$ , [Prz91, (6.14)] and a formula for the character  $\Theta_{\Pi'}$  in terms of  $\mathcal{F}(\tau'_*(f_{\Pi \otimes \Pi'}))$ ,

$$\frac{1}{\sigma} \cdot \tilde{c}_-^* \Theta_{\Pi'} = \tau'_*(\widehat{f_{\Pi \otimes \Pi'}}), \quad (159)$$

where  $\sigma$  is a smooth function, [Prz91, Theorem 6.7]. By combining this with the following elementary lemma, one completes the argument.

**Lemma 26.** *Suppose  $f, u \in \mathcal{S}^*(\mathbb{R}^n)$  and  $u$  is homogeneous of degree  $d \in \mathbb{C}$ . Suppose*

$$t^d M_{t^{-1}}^* f(\psi) \xrightarrow[t \rightarrow 0^+]{} u(\psi) \quad (\psi \in \mathcal{S}(\mathbb{R}^n)). \quad (160)$$

Then

$$WF_0(\hat{f}) \supseteq \text{supp } u, \quad (161)$$

where the subscript 0 indicates the wave front set at zero and

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(y) e^{2\pi i x \cdot y} dy.$$

## APPENDIX A. The Jacobian of the Cayley transform

Here we determine the Jacobian of the Cayley transform  $c_- : \mathfrak{g} \rightarrow \mathbb{G}$ . A straightforward computation shows that for a fixed  $x \in \mathfrak{g}$ ,

$$c_-(x+y)c_-(x)^{-1} - 1 = (1-x-y)^{-1} 2y(1+x)^{-1} \quad (y \in \mathfrak{g}).$$

Hence the derivative (tangent map) is given by

$$c'_-(x)y = (1-x)^{-1} 2y(1-x)^{-1} \quad (y \in \mathfrak{g}). \quad (\text{A.1})$$

Recall that  $\mathbb{G}$  is the isometry group of a hermitian form  $(\cdot, \cdot)$  on  $\mathbb{V}$ . Hence we have the adjoint

$$\text{End}_{\mathbb{D}}(\mathbb{V}) \ni g \rightarrow g^* \in \text{End}_{\mathbb{D}}(\mathbb{V})$$

defined by

$$(gu, v) = (u, g^*v) \quad (u, v \in \mathbb{V}).$$

Let us view the Lie algebra  $\mathfrak{g}$  as a real vector space and consider the map

$$\gamma : \text{GL}_{\mathbb{D}}(\mathbb{V}) \rightarrow \text{GL}(\mathfrak{g}), \quad \gamma(g)(y) = gyg^*.$$

Then  $\det \circ \gamma : \mathrm{GL}_{\mathbb{D}}(\mathbf{V}) \rightarrow \mathbb{R}^{\times}$  is a group homomorphism. Hence there is a number  $s \in \mathbb{R}$  such that

$$\det(\gamma(g)) = (\det(g)_{\mathbf{V}_{\mathbb{R}}})^s \quad (g \in \mathrm{GL}_{\mathbb{D}}(\mathbf{V})).$$

On the other hand, for a fixed number  $a \in \mathbb{R}^{\times}$ ,

$$\det(\gamma(aI_{\mathbf{V}})) = a^{2 \dim \mathfrak{g}} \quad \text{and} \quad \det(aI_{\mathbf{V}})_{\mathbf{V}_{\mathbb{R}}} = a^{\dim \mathbf{V}_{\mathbb{R}}}.$$

Hence,

$$\det(\gamma(g)) = (\det(g)_{\mathbf{V}_{\mathbb{R}}})^{\frac{2 \dim \mathfrak{g}}{\dim \mathbf{V}_{\mathbb{R}}}} \quad (g \in \mathrm{GL}_{\mathbb{D}}(\mathbf{V})).$$

If  $x \in \mathfrak{g}$ , then  $1 \pm x \in \mathrm{GL}_{\mathbb{D}}(\mathbf{V})$  and

$$(1 \pm x)^* = 1 \mp x \quad \text{and} \quad ((1 \pm x)^{-1})^* = (1 \mp x)^{-1}.$$

Hence

$$c'_-(x)y = 2(1-x)^{-1}y(1+x)^{-1}c_-(x)y = 2c_-(x)\gamma(c_-(x)^{-1})\gamma((1-x)^{-1})y \quad (y \in \mathfrak{g}).$$

Notice that that  $|\det(c_-(x))| = 1$  because  $c(\mathfrak{g}) \subseteq \mathrm{G}$ . Therefore

$$|\det(c'_-(x))| = 2^{\dim \mathfrak{g}} \det(1-x)_{\mathbf{V}_{\mathbb{R}}}^{-\frac{2 \dim \mathfrak{g}}{\dim \mathbf{V}_{\mathbb{R}}}} = 2^{\dim \mathfrak{g}} \mathrm{ch}(x)^{-2r} \quad (x \in \mathfrak{g}), \quad (\text{A.2})$$

where  $\mathrm{ch}$  and  $r$  are as in (85) and (52), respectively.

#### APPENDIX B. Proof of formula (86)

Here we verify the formula (86) for the orthogonal matrix group

$$\mathrm{G} = \mathrm{O}_{2l+1} = \{g \in \mathrm{GL}_{2l+1}(\mathbb{R}); gg^t = I\}.$$

The other cases are similar. Fix the diagonal Cartan subgroup

$$\mathrm{H} = \{\mathrm{diag}(u_1, u_2, \dots, u_l, \pm 1); u_j \in \mathrm{SO}_2(\mathbb{R}), 1 \leq j \leq l\}.$$

Then the connected identity component is

$$\mathrm{H}^0 = \{\mathrm{diag}(u_1, u_2, \dots, u_l, 1); u_j \in \mathrm{SO}_2(\mathbb{R}); 1 \leq j \leq l\}.$$

The group  $\widehat{\mathrm{H}}^0$  may be realized as

$$\widehat{\mathrm{H}}^0 = \{\mathrm{diag}(u_1, u_2, \dots, u_l, 1; \xi); u_j, \xi \in \mathrm{SO}_2(\mathbb{R}), 1 \leq j \leq l; \xi^2 = u_1 u_2 \cdots u_l\}$$

and the covering map (76) is

$$\mathrm{diag}(u_1, u_2, \dots, u_l, 1; \xi) \longrightarrow \mathrm{diag}(u_1, u_2, \dots, u_l, 1).$$

We identify

$$a + ib = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad (a, b \in \mathbb{R}).$$

Then

$$\mathrm{SO}_2(\mathbb{R}) = \{u \in \mathbb{C}; |u| = 1\}.$$

In these terms, the usual choice of the positive roots  $e_j \pm e_k$ , with  $1 \leq j < k \leq l$ , and  $e_j$ , with  $1 \leq j \leq l$ , gives

$$\begin{aligned}\xi_{e_j - e_k}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j u_k^{-1}, \\ \xi_{e_j + e_k}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j u_k, \\ \xi_{e_j}(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_j, \\ \xi_\rho(\text{diag}(u_1, u_2, \dots, u_l, 1; \xi)) &= u_1^{l-1} u_2^{l-2} \cdots u_{l-1} \xi.\end{aligned}$$

We shall use the following branch of the square root

$$\sqrt{r e^{i\theta}} = \sqrt{r} e^{i\frac{\theta}{2}} \quad (r < 0, -\pi < \theta < \pi).$$

It is easy to check that for two complex numbers  $\zeta_1, \zeta_2$  in the open right half-plane,

$$\sqrt{\zeta_1 \zeta_2} = \sqrt{\zeta_1} \sqrt{\zeta_2}. \quad (\text{B.3})$$

Furthermore, if  $z \in \mathbb{C} \setminus ]-\infty, 0]$ , then

$$z = \sqrt{z} \sqrt{z} \quad \text{and} \quad \sqrt{\frac{1}{z}} = \frac{1}{\sqrt{z}}. \quad (\text{B.4})$$

If  $x = \text{diag}(ix_1, ix_2, \dots, ix_l, 0) \in \mathfrak{h}$ , then  $c_-(x) = \text{diag}(u_1, u_2, \dots, u_l, 1)$  has coordinates

$$u_j = \frac{1 + z_j}{1 - z_j}, \quad z_j = ix_j, \quad 1 \leq j \leq l,$$

with  $|u_j| = 1$  and  $u_j \neq -1$  for all  $j$ . Hence we may choose the section  $\sigma : c_-(\mathfrak{h}) \rightarrow \widehat{\mathbb{H}}^0$  of the covering map as follows:

$$\sigma(\text{diag}(u_1, u_2, \dots, u_l, 1)) = \text{diag}(u_1, u_2, \dots, u_l, 1; \sqrt{u_1} \sqrt{u_2} \cdots \sqrt{u_l}).$$

Recall that  $\widehat{c}_-(x) = (\sigma(c_-(x)), 1)$ , (80). A straightforward computation using (B.4) shows that

$$\Delta(\widehat{c}_-(x)) = \prod_{1 \leq j < k \leq l} \left( \sqrt{u_j} \frac{1}{\sqrt{u_k}} - \frac{1}{\sqrt{u_j}} \sqrt{u_k} \right) \left( \sqrt{u_j} \sqrt{u_k} - \frac{1}{\sqrt{u_j}} \frac{1}{\sqrt{u_k}} \right) \cdot \prod_{j=1}^l \left( \sqrt{u_j} - \frac{1}{\sqrt{u_j}} \right).$$

Hence (B.3) and (B.4) applied to the  $1 \pm z_j$ 's show that

$$\begin{aligned}\sqrt{u_j} \frac{1}{\sqrt{u_k}} - \frac{1}{\sqrt{u_j}} \sqrt{u_k} &= \frac{2(z_j - z_k)}{\sqrt{1 - z_j} \sqrt{1 + z_j} \sqrt{1 - z_k} \sqrt{1 + z_k}}, \\ \sqrt{u_j} \sqrt{u_k} - \frac{1}{\sqrt{u_j}} \frac{1}{\sqrt{u_k}} &= \frac{2(z_j + z_k)}{\sqrt{1 - z_j} \sqrt{1 + z_j} \sqrt{1 - z_k} \sqrt{1 + z_k}}, \\ \sqrt{u_j} - \frac{1}{\sqrt{u_j}} &= \frac{2z_j}{\sqrt{1 - z_j} \sqrt{1 + z_j}}.\end{aligned}$$

Therefore

$$\Delta(\widehat{c}_-(x)) = \prod_{1 \leq j < k \leq l} \frac{4(z_j - z_k)(z_j + z_k)}{(1 - z_j)(1 + z_j)(1 - z_k)(1 + z_k)} \cdot \prod_{j=1}^l \frac{2z_j}{\sqrt{1 - z_j} \sqrt{1 + z_j}}.$$

Since

$$\prod_{1 \leq j < k \leq l} (1 - z_j)(1 + z_j)(1 - z_k)(1 + z_k) = \left( \prod_{j=1}^l (1 - z_j^2) \right)^{l-1} = \left( \prod_{j=1}^l (1 + x_j^2) \right)^{l-1},$$

the formula (86) follows.

### APPENDIX C. The special functions $P_{a,b}$ and $Q_{a,b}$

For two integers  $a$  and  $b$  define the following functions in the real variable  $\xi$ ,

$$P_{a,b,2}(\xi) = \begin{cases} \sum_{k=0}^{b-1} \frac{a(a+1)\dots(a+k-1)}{k!(b-1-k)!} 2^{-a-k} \xi^{b-1-k} & \text{if } b \geq 1 \\ 0 & \text{if } b \leq 0, \end{cases} \quad (\text{C.5})$$

$$P_{a,b,-2}(\xi) = \begin{cases} (-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1)\dots(b+k-1)}{k!(a-1-k)!} (-2)^{-b-k} \xi^{a-1-k} & \text{if } a \geq 1 \\ 0 & \text{if } a \leq 0, \end{cases} \quad (\text{C.6})$$

where  $a(a+1)\dots(a+k-1) = 1$  if  $k = 0$ . Notice that

$$P_{a,b,-2}(\xi) = P_{b,a,2}(-\xi) \quad (\xi \in \mathbb{R}, a, b \in \mathbb{Z}). \quad (\text{C.7})$$

Set

$$\begin{aligned} P_{a,b}(\xi) &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{a,b,-2}(\xi)\mathbb{I}_{\mathbb{R}^-}(\xi)) \\ &= 2\pi(P_{a,b,2}(\xi)\mathbb{I}_{\mathbb{R}^+}(\xi) + P_{b,a,2}(-\xi)\mathbb{I}_{\mathbb{R}^+}(-\xi)), \end{aligned} \quad (\text{C.8})$$

where  $\mathbb{I}_S$  denotes the indicator function of the set  $S$ . Also, let

$$Q_{a,b}(iy) = 2\pi \begin{cases} 0 & \text{if } a + b \geq 1, \\ \sum_{k=b}^{-a} \frac{a(a+1)\dots(a+k-1)}{k!} 2^{-a-k} (1 - iy)^{k-b} & \text{if } -a > b - 1 \geq 0, \\ \sum_{k=a}^{-b} \frac{b(b+1)\dots(b+k-1)}{k!} 2^{-b-k} (1 + iy)^{k-a} & \text{if } -b > a - 1 \geq 0, \\ (1 + iy)^{-a} (1 - iy)^{-b} & \text{if } a \leq 0 \text{ and } b \leq 0. \end{cases} \quad (\text{C.9})$$

**Proposition C.1.** *For any  $a, b \in \mathbb{Z}$ , the formula*

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \phi(y) dy \quad (\phi \in \mathcal{S}(\mathbb{R})) \quad (\text{C.10})$$

*defines a tempered distribution on  $\mathbb{R}$ . The restriction of the Fourier transform of this distribution to  $\mathbb{R} \setminus \{0\}$  is a function given by*

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} e^{-iy\xi} dy = P_{a,b}(\xi) e^{-|\xi|}. \quad (\text{C.11})$$

*The right hand side of (C.11) is an absolutely integrable function on the real line and thus defines a tempered distribution on  $\mathbb{R}$ . Furthermore,*

$$(1 + iy)^{-a} (1 - iy)^{-b} = \frac{1}{2\pi} \int_{\mathbb{R}} P_{a,b}(\xi) e^{-|\xi|} e^{iy\xi} dy + \frac{1}{2\pi} Q_{a,b}(iy) \quad (\text{C.12})$$

*and hence,*

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} e^{-iy\xi} dy = P_{a,b}(\xi) e^{-|\xi|} + Q_{a,b}\left(-\frac{d}{d\xi}\right) \delta_0(\xi). \quad (\text{C.13})$$

*Proof.* Since,  $|1 \pm iy| = \sqrt{1 + y^2}$ , (C.10) is clear. The integral (C.11) is equal to

$$\begin{aligned} & \frac{1}{i} \int_{i\mathbb{R}} (1+z)^{-a}(1-z)^{-b} e^{-z\xi} dz \\ &= 2\pi(-\mathbb{I}_{\mathbb{R}^+}(\xi) \operatorname{res}_{z=1}(1+z)^{-a}(1-z)^{-b} e^{-z\xi} + \mathbb{I}_{\mathbb{R}^-}(\xi) \operatorname{res}_{z=-1}(1+z)^{-a}(1-z)^{-b} e^{-z\xi}). \end{aligned} \quad (\text{C.14})$$

The computation of the two residues is straightforward and (C.11) follows.

Since

$$\int_0^\infty e^{-\xi} e^{i\xi y} d\xi = (1 - iy)^{-1},$$

we have

$$\int_0^\infty \xi^m e^{-\xi} e^{i\xi y} d\xi = \left( \frac{d}{d(iy)} \right)^m (1 - iy)^{-1} = m!(1 - iy)^{-m-1} \quad (m = 0, 1, 2, \dots). \quad (\text{C.15})$$

Thus, if  $b \geq 1$ , then

$$\begin{aligned} & \int_0^\infty P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y} d\xi = \sum_{k=0}^{b-1} \frac{a(a+1)\dots(a+k-1)}{k!} 2^{-a-k} (1 - iy)^{-b+k} \\ &= (1 - iy)^{-b} 2^{-a} \sum_{k=0}^{b-1} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left( -\frac{1}{2}(1 - iy) \right)^k. \end{aligned}$$

Also, if  $a \leq 0$ , then

$$\begin{aligned} & 2^a (1 + iy)^{-a} = \left( 1 - \frac{1}{2}(1 - iy) \right)^{-a} = \sum_{k=0}^{-a} \binom{-a}{k} \left( -\frac{1}{2}(1 - iy) \right)^k \\ &= \sum_{k=0}^{-a} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left( -\frac{1}{2}(1 - iy) \right)^k. \end{aligned}$$

Hence,

$$\begin{aligned} & \int_0^\infty P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y} d\xi - (1 + iy)^{-a} (1 - iy)^{-b} \\ &= (1 - iy)^{-b} 2^{-a} \left( \sum_{k=0}^{b-1} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left( -\frac{1}{2}(1 - iy) \right)^k \right. \\ & \quad \left. - \sum_{k=0}^{-a} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left( -\frac{1}{2}(1 - iy) \right)^k \right). \end{aligned} \quad (\text{C.16})$$

Recall that  $P_{a,b,-2} = 0$  if  $a \leq 0$ . Hence, (C.11) shows that (C.16) is the inverse Fourier transform of a distribution supported at  $\{0\}$  - a polynomial.

Suppose  $-a < b - 1$ . Then (C.16) is equal to

$$2^{-a} \sum_{k=-a+1}^{b-1} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left( -\frac{1}{2} \right)^k (1 - iy)^{k-b},$$

which tends to zero if  $y$  goes to infinity. The only polynomial with this property is the zero polynomial. Thus in this case (C.16) is zero. If  $-a = b - 1$ , then (C.16) is obviously zero.

Suppose  $-a > b - 1$ . Then (C.16) is equal to

$$-2^{-a} \sum_{k=b}^{-a} \frac{(-a)(-a-1)\dots(-a-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1-iy)^{k-b}. \quad (\text{C.17})$$

As in (C.15) we have

$$\int_{-\infty}^0 \xi^m e^{\xi} e^{i\xi y} d\xi = \left(\frac{d}{d(iy)}\right)^m (1+iy)^{-1} = (-1)^m m! (1+iy)^{-m-1} \quad (m = 0, 1, 2, \dots).$$

Suppose  $a \geq 1$ . Then

$$\begin{aligned} \int_{-\infty}^0 P_{a,b,-2}(\xi) e^{\xi} e^{i\xi y} d\xi &= (-1)^{a+b-1} \sum_{k=0}^{a-1} \frac{b(b+1)\dots(b+k-1)}{k!} (-2)^{-b-k} (-1)^{a-1+k} (1+iy)^{-a+k} \\ &= (1+iy)^{-a} 2^{-b} \sum_{k=0}^{a-1} \frac{(-b)(-b-1)\dots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k. \end{aligned}$$

Also, if  $b \leq 0$ , then

$$2^b (1-iy)^{-b} = \sum_{k=0}^{-b} \frac{(-b)(-b-1)\dots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k.$$

Hence,

$$\begin{aligned} \int_{-\infty}^0 P_{a,b,-2}(\xi) e^{\xi} e^{i\xi y} d\xi - (1+iy)^{-a} (1-iy)^{-b} & \quad (\text{C.18}) \\ &= (1+iy)^{-a} 2^{-b} \left( \sum_{k=0}^{a-1} \frac{(-b)(-b-1)\dots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k \right. \\ & \quad \left. - \sum_{k=0}^{-b} \frac{(-b)(-b-1)\dots(-b-k+1)}{k!} \left(-\frac{1}{2}(1+iy)\right)^k \right). \end{aligned}$$

As before, we show that (C.18) is zero if  $-b \leq a - 1$ . If  $-b > a - 1$ , then (C.18) is equal to

$$-2^{-b} \sum_{k=a}^{-b} \frac{(-b)(-b-1)\dots(-b-k+1)}{k!} \left(-\frac{1}{2}\right)^k (1+iy)^{k-a}.$$

If  $a \geq 1$  and  $b \geq 1$ , then our computations show that

$$\int_0^{\infty} P_{a,b,2}(\xi) e^{-\xi} e^{i\xi y} d\xi + \int_{-\infty}^0 P_{a,b,-2}(\xi) e^{\xi} e^{i\xi y} d\xi - (1+iy)^{-a} (1-iy)^{-b} \quad (\text{C.19})$$

is a polynomial which tends to zero if  $y$  goes to infinity. Thus (C.19) is equal zero. This completes the proof of (C.12). The statement (C.13) is a direct consequence of (C.12).  $\square$

The test functions which occur in Proposition C.1 don't need to be in the Schwartz space. In fact the test functions we shall use in our applications are not necessarily smooth. Therefore we'll need a more precise version of the formula (C.13). This requires a definition and two well known lemmas.

Following Harish-Chandra denote by  $\mathcal{S}(\mathbb{R}^\times)$  the space of the smooth complex valued functions defined on  $\mathbb{R}^\times$  whose all derivatives are rapidly decreasing at infinity and have limits at zero from both sides. For  $\psi \in \mathcal{S}(\mathbb{R}^\times)$  let

$$\psi(0+) = \lim_{x \rightarrow 0+} \psi(x), \quad \psi(0-) = \lim_{x \rightarrow 0-} \psi(x), \quad \langle \psi \rangle_0 = \psi(0+) - \psi(0-).$$

In particular the condition  $\langle \psi \rangle_0 = 0$  means that  $\psi$  extends to a continuous function on  $\mathbb{R}$ .

**Lemma C.2.** *Let  $c = 0, 1, 2, \dots$  and let  $\psi \in \mathcal{S}(\mathbb{R}^\times)$ . Suppose*

$$\langle \psi \rangle_0 = \dots = \langle \psi^{(c-1)} \rangle_0 = 0. \quad (\text{C.20})$$

(The condition (C.20) is empty if  $c = 0$ .) Then

$$\left| \int_{\mathbb{R}^\times} e^{-iy\xi} \psi(\xi) d\xi \right| \leq \min\{1, |y|^{-c-1}\} (|\langle \psi^{(c)} \rangle_0| + \|\psi^{(c+1)}\|_1 + \|\psi\|_1) \quad (\text{C.21})$$

*Proof.* Integration by parts shows that for  $z \in \mathbb{C}^\times$

$$\begin{aligned} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi &= z^{-1} \psi(0+) + \dots + z^{-c-1} \psi^{(c)}(0+) + z^{-c-1} \int_{\mathbb{R}^+} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi, \\ \int_{\mathbb{R}^-} e^{-z\xi} \psi(\xi) d\xi &= -z^{-1} \psi(0-) - \dots - z^{-c-1} \psi^{(c)}(0-) + z^{-c-1} \int_{\mathbb{R}^-} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_{\mathbb{R}^\times} e^{-z\xi} \psi(\xi) d\xi \\ &= z^{-1} \langle \psi \rangle_0 + \dots + z^{-c} \langle \psi^{(c-1)} \rangle_0 + z^{-c-1} \langle \psi^{(c)} \rangle_0 + z^{-c-1} \int_{\mathbb{R}^\times} e^{-z\xi} \psi^{(c+1)}(\xi) d\xi \end{aligned}$$

and (C.21) follows.  $\square$

**Lemma C.3.** *Under the assumptions of Lemma C.2, with  $1 \leq c$ ,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \psi^{(k)}(0) \quad (0 \leq k \leq c-1),$$

where each consecutive integral is absolutely convergent.

*Proof.* Since

$$\int_{\mathbb{R}} |y|^{c-1} \min\{1, |y|^{-c-1}\} dy < \infty,$$

the absolute convergence follows from Lemma C.2. Since the Fourier transform of  $\psi$  is absolutely integrable and since  $\psi$  is continuous at zero, Fourier inversion formula [Hör83, (7.1.4)] shows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^\times} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \psi(0). \quad (\text{C.22})$$

Also, for  $0 < k$ ,

$$\begin{aligned}
 & \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi = \int_{\mathbb{R}^\times} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi \\
 &= \int_{\mathbb{R}^+} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi + \int_{\mathbb{R}^-} (-\partial_\xi) ((iy)^{k-1} e^{-iy\xi}) \psi(\xi) d\xi \\
 &= (iy)^{k-1} \psi(0+) + \int_{\mathbb{R}^+} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi \\
 &\quad - (iy)^{k-1} \psi(0-) + \int_{\mathbb{R}^-} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi \\
 &= (iy)^{k-1} \langle \psi \rangle_0 + \int_{\mathbb{R}^\times} (iy)^{k-1} e^{-iy\xi} \psi'(\xi) d\xi.
 \end{aligned}$$

Hence, by induction on  $k$  and by our assumption

$$\begin{aligned}
 \int_{\mathbb{R}^\times} (iy)^k e^{-iy\xi} \psi(\xi) d\xi &= (iy)^{k-1} \langle \psi \rangle_0 + (iy)^{k-2} \langle \psi' \rangle_0 + \cdots + \langle \psi^{(k-1)} \rangle_0 \\
 &\quad + \int_{\mathbb{R}^\times} e^{-iy\xi} \psi^{(k)}(\xi) d\xi \\
 &= \int_{\mathbb{R}^\times} e^{-iy\xi} \psi^{(k)}(\xi) d\xi.
 \end{aligned}$$

Therefore our lemma follows from (C.22).  $\square$

The following proposition is an immediate consequence of Lemmas C.2, C.3, and the formula (C.12).

**Proposition C.4.** *Fix two integers  $a, b \in \mathbb{Z}$  and a function  $\psi \in \mathcal{S}(\mathbb{R}^\times)$ . Let  $c = -a - b$ . If  $c \geq 0$  assume that*

$$\langle \psi \rangle_0 = \cdots = \langle \psi^{(c)} \rangle_0 = 0. \tag{C.23}$$

Then

$$\begin{aligned}
 & \int_{\mathbb{R}} \int_{\mathbb{R}^\times} (1+iy)^{-a} (1-iy)^{-b} e^{-iy\xi} \psi(\xi) d\xi dy \\
 &= \int_{\mathbb{R}^\times} P_{a,b}(\xi) e^{-|\xi|} \psi(\xi) d\xi + Q_{a,b}(\partial_\xi) \psi(\xi)|_{\xi=0} \\
 &= \int_{\mathbb{R}} (P_{a,b}(\xi) e^{-|\xi|} + Q_{a,b}(-\partial_\xi) \delta_0(\xi)) \psi(\xi) d\xi, .
 \end{aligned} \tag{C.24}$$

where  $\delta_0$  denotes the Dirac delta at 0.

(Recall that  $Q_{a,b} = 0$  if  $c < 0$  and  $Q_{a,b}$  is a polynomial of degree  $c$  if  $c \geq 0$ .)

Let  $\mathcal{S}(\mathbb{R}^+)$  be the space of the smooth complex valued functions whose all derivatives are rapidly decreasing at infinity and have limits at zero. Then  $\mathcal{S}(\mathbb{R}^+)$  may be viewed as the subspace of the functions in  $\mathcal{S}(\mathbb{R}^\times)$  which are zero on  $\mathbb{R}^-$ . Similarly we define  $\mathcal{S}(\mathbb{R}^-)$ . The following proposition is a direct consequence of Proposition C.4. We sketch an independent proof below.



**Proposition C.5.** *There is a seminorm  $p$  on the space  $\mathcal{S}(\mathbb{R}^+)$  such that*

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right| \leq \min\{1, |z|^{-1}\} p(\psi) \quad (\psi \in \mathcal{S}(\mathbb{R}^+), \operatorname{Re} z \geq 0), \quad (\text{C.25})$$

and similarly for  $\mathcal{S}(\mathbb{R}^-)$ .

Fix integers  $a, b \in \mathbb{Z}$  with  $a + b \geq 1$ . Then for any function  $\psi \in \mathcal{S}(\mathbb{R}^+)$ ,

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^+} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^+} P_{a,b,2}(\xi) e^{-\xi} \psi(\xi) d\xi, \quad (\text{C.26})$$

and any function  $\psi \in \mathcal{S}(\mathbb{R}^-)$ ,

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^-} e^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^-} P_{a,b,-2}(\xi) e^{\xi} \psi(\xi) d\xi, \quad (\text{C.27})$$

where each consecutive integral is absolutely convergent.

*Proof.* Clearly

$$\left| \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right| \leq \int_{\mathbb{R}^+} e^{-\operatorname{Re} z\xi} |\psi(\xi)| d\xi \leq \|\psi\|_1.$$

Integration by parts shows that for  $z \neq 0$ ,

$$\int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi = z^{-1} \psi(0) + z^{-1} \int_{\mathbb{R}^+} e^{-z\xi} \psi'(\xi) d\xi.$$

Hence (C.25) follows with  $p(\psi) = |\psi(0)| + \|\psi\|_1 + \|\psi'\|_1$ .

Let  $a, b \in \mathbb{Z}$  be such that  $a + b \geq 1$ . Then the function

$$(1 + z)^{-a} (1 - z)^{-b} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi$$

is continuous on  $\operatorname{Re} z \geq 0$  and meromorphic on  $\operatorname{Re} z > 0$  and (C.25) shows that it is dominated by  $|z|^{-2}$ . Therefore Cauchy's Theorem implies that the left hand side of (C.26) is equal to

$$-2\pi \operatorname{res}_{z=1} \left( (1 + z)^{-a} (1 - z)^{-b} \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right).$$

The computation of this residue is straightforward. This verifies (C.26). The proof of (C.27) is entirely analogous.  $\square$

**Proposition C.6.** *Fix integers  $a, b \in \mathbb{Z}$  with  $a + b \geq 2$ . Set  $R_{a,b,2}(\xi) = P_{a,b,2}(\xi) - P_{a,b-1,2}(\xi)$ . Then for any function  $\psi \in \mathcal{S}(\mathbb{R}^+)$ ,*

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^+} iye^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^+} R_{a,b,2}(\xi) e^{-\xi} \psi(\xi) d\xi, \quad (\text{C.28})$$

and any function  $\psi \in \mathcal{S}(\mathbb{R}^-)$ ,

$$\int_{\mathbb{R}} (1 + iy)^{-a} (1 - iy)^{-b} \int_{\mathbb{R}^-} iye^{-iy\xi} \psi(\xi) d\xi dy = 2\pi \int_{\mathbb{R}^-} R_{a,b,-2}(\xi) e^{\xi} \psi(\xi) d\xi, \quad (\text{C.29})$$

where each consecutive integral is absolutely convergent.

*Proof.* The function

$$(1+z)^{-a}(1-z)^{-b}z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi$$

is continuous on  $\operatorname{Re} z \geq 0$  and meromorphic on  $\operatorname{Re} z > 0$  and since  $a+b \geq 2$ , (C.25) shows that it is dominated by  $|z|^{-2}$ . Therefore Cauchy's Theorem implies that the left hand side of (C.26) is equal to

$$-2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-b}z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right). \quad (\text{C.30})$$

Since

$$\begin{aligned} (1+z)^{-a}(1-z)^{-b}z &= (1+z)^{-a}(1-z)^{-b}(z-1) + (1+z)^{-a}(1-z)^{-b} \\ &= -(1+z)^{-a}(1-z)^{-(b-1)} + (1+z)^{-a}(1-z)^{-b}, \end{aligned}$$

(C.30) equals

$$\begin{aligned} 2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-(b-1)}z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right) \\ - 2\pi \operatorname{res}_{z=1} \left( (1+z)^{-a}(1-z)^{-b}z \int_{\mathbb{R}^+} e^{-z\xi} \psi(\xi) d\xi \right). \quad (\text{C.31}) \end{aligned}$$

These residues were computed in the proof of Proposition C.5. This verifies (C.28). The proof of (C.27) is entirely analogous.  $\square$

#### APPENDIX D. The covering $\tilde{G} \rightarrow G$

In this appendix we recall some results about the splitting of the restrictions  $\tilde{L} \rightarrow L$  of the metaplectic covering

$$1 \rightarrow \{\pm 1\} \rightarrow \tilde{\operatorname{Sp}}(W) \rightarrow \operatorname{Sp}(W) \rightarrow 1 \quad (\text{D.32})$$

to a subgroup  $L$  of the compact member  $G$  of a dual pair  $(G, G')$  as in (2). This is well known, but we could not find a reference sketching the proofs of the results we are using in this paper. We are therefore providing a short and complete argument.

If  $K$  is a maximal compact subgroup of  $\operatorname{Sp}(W)$ , then  $\tilde{K}$  is a maximal compact subgroup of  $\tilde{\operatorname{Sp}}(W)$ . The group  $\tilde{\operatorname{Sp}}(W)$  is connected, noncompact, semisimple and with finite center  $\tilde{Z}$ . (Since  $\tilde{\operatorname{Sp}}(W)$  is a double cover of  $\operatorname{Sp}(W)$ , only the connectedness needs to be commented. It follows from the fact that the covering (D.32) does not split; see e.g. [AP14, Proposition 4.20] or the original proof [Wei64, p. 199]). The maximal compact subgroup  $\tilde{K}$  is therefore connected; see e.g. [Hel78, Chapter VI, Theorem 1.1]. Hence the covering

$$\tilde{K} \rightarrow K \quad (\text{D.33})$$

does not split.

As is well known,  $K$  is isomorphic to a compact unitary group. In fact, if  $W = \mathbb{R}^{2n}$  and

$$J_{2n} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad (\text{D.34})$$

then

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix}; a, b \in \mathrm{GL}_n(\mathbb{R}), ab^t = b^t a, aa^t + bb^t = I_n \right\} \quad (\text{D.35})$$

is a maximal compact subgroup of  $\mathrm{Sp}_{2n}(\mathbb{R})$  and

$$\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}} \ni \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightarrow a + ib \in \mathrm{U}_n \quad (\text{D.36})$$

is a Lie group isomorphism. Any two maximal compact subgroups of  $\mathrm{Sp}(W)$  are conjugate by an inner automorphism. Let  $\mathrm{K} \rightarrow \mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$  be the corresponding isomorphism. Composition with (D.36) fixes then an isomorphism  $\phi : \mathrm{K} \rightarrow \mathrm{U}_n$ . Set

$$\tilde{\mathrm{K}}^\phi = \{(u, \zeta) \in \mathrm{K} \times \mathbb{C}^\times; \det(\phi(u)) = \zeta^2\} \quad (\text{D.37})$$

Recall the bijection between equivalence classes of  $n$ -fold path-connected coverings and the conjugacy classes of index- $n$  subgroups of the fundamental group (see e.g. [Hat01, Theorem 1.38]). Then, up to an isomorphism of coverings,  $\mathrm{U}_n$  has only one connected double cover. Hence (D.33) is isomorphic to

$$\tilde{\mathrm{K}}^\phi \ni (u, \zeta) \rightarrow u \in \mathrm{K}. \quad (\text{D.38})$$

Let  $\mathrm{L} \subseteq \mathrm{K}$  be any subgroup and

$$\tilde{\mathrm{L}} \rightarrow \mathrm{L} \quad (\text{D.39})$$

the restriction of the covering (D.33) to  $\mathrm{L}$ . Let  $\tilde{\mathrm{L}}^\phi$  be the preimage of  $\mathrm{L}$  in  $\tilde{\mathrm{K}}^\phi$ . Then (D.39) splits if and only if

$$\tilde{\mathrm{L}}^\phi \rightarrow \mathrm{L} \quad (\text{D.40})$$

splits, i.e. there is a group homomorphism  $\mathrm{L} \ni g \rightarrow \zeta(g) \in \mathrm{U}_1 \subset \mathbb{C}^\times$  such that  $\zeta(g)^2 = \det(\phi(g))$  for all  $g \in \mathrm{L}$ . For instance, if  $\mathrm{L}$  is a connected subgroup of  $\mathrm{K}$  such that

$$\mathrm{L} \subseteq \{u \in \mathrm{K}; \det(\phi(u)) = 1\}, \quad (\text{D.41})$$

then (D.39) splits.

To fix  $\phi$ , let  $(V, (\cdot, \cdot))$  and  $(V', (\cdot, \cdot)')$  be the defining spaces of  $G$  and  $G'$ , respectively, with  $\dim_{\mathbb{D}} V = d$  and  $\dim_{\mathbb{D}} V' = d'$ . Realize  $W$  as  $V \otimes_{\mathbb{D}} V'$ , considered as a real symplectic space, with symplectic form  $\langle \cdot, \cdot \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}((\cdot, \cdot) \otimes (\cdot, \cdot)')$ , where  $\mathrm{tr}_{\mathbb{D}/\mathbb{R}}$  denotes the reduced trace; see [How79, §5] and [Wei73][p. 169]. Then the group  $G$  is viewed as a subgroup of  $\mathrm{Sp}(W)$  via the identification  $G \ni g \rightarrow g \otimes 1 \in \mathrm{Sp}(W)$ .<sup>1</sup> Similarly,  $G'$  is viewed as a subgroup of  $\mathrm{Sp}(W)$  via the identification  $G' \ni g' \rightarrow 1 \otimes g' \in \mathrm{Sp}(W)$ . Recall that  $n$ -by- $n$ -matrices over  $\mathbb{C}$  can be identified with  $2n$ -by- $2n$  matrices over  $\mathbb{R}$  under the isomorphism

$$\alpha : M \rightarrow \begin{pmatrix} \mathrm{Re} M & -\mathrm{Im} M \\ \mathrm{Im} M & \mathrm{Re} M \end{pmatrix}.$$

Moreover,  $n$ -by- $n$ -matrices over  $\mathbb{H}$  can be identified with  $2n$ -by- $2n$  matrices over  $\mathbb{C}$  under the isomorphism

$$\beta : M \rightarrow \begin{pmatrix} z_1(M) & -\overline{z_2(M)} \\ z_2(M) & z_1(M) \end{pmatrix}.$$

Here, for  $v \in \mathbb{H}$ , we write  $v = z_1(v) + jz_2(v)$  with  $z_1(v), z_2(v) \in \mathbb{C}$ , and we similarly define  $z_1(M)$  and  $z_2(M)$  if  $M$  is a matrix over  $\mathbb{H}$ .

<sup>1</sup>Following the notation at the beginning of Section 3, one should identify  $g$  and  $(g^{-1})^t \otimes 1$ .

Since  $G$  is compact, there is a compatible positive complex structure  $J$  on  $W$  such that the maximal compact subgroup  $K = \mathrm{Sp}(W)^J$  of  $\mathrm{Sp}(W)$  contains  $G$ . Moreover, since  $G$  commutes with  $J$ , there is  $J' \in G'$  such that  $J = 1 \otimes J'$ . Set  $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}$ . Then, the explicit expressions of  $J'$  with respect to the standard basis of  $V \simeq \mathbb{D}^d$  and of  $J$  with respect to the standard basis of  $W \simeq \mathbb{R}^{2n}$  are given as follows:

$(G, G')$	$J'$	$n$	$J$
$(O_d, \mathrm{Sp}_{2m}(\mathbb{R}))$	$J_{2m}$	$md$	$J_{2md}$
$(U_d, U_{p,q})$	$-iI_{p,q}$	$d(p+q)$	$\begin{pmatrix} 0 & I_{dp,dq} \\ -I_{dp,dq} & 0 \end{pmatrix}$
$(\mathrm{Sp}_d, O_{2m}^*)$	$-jI_m$	$2md$	$\begin{pmatrix} J_{2pm} & 0 \\ 0 & J_{2pm} \end{pmatrix}$

Notice that in the  $(U_d, U_{p,q})$ -case we have  $SJS^{-1} = J_{2d(p+q)}$  for  $S = \begin{pmatrix} I_{d(p+q)} & 0 \\ 0 & I_{dp,dq} \end{pmatrix}$ ;

in the  $(\mathrm{Sp}_d, O_{2m}^*)$ -case,  $TJT^{-1} = J_{4pm}$  for  $T = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$ . Hence, in all cases we

can embed  $G$  in (D.35) from the identification  $g \rightarrow g \otimes 1 \in \mathrm{Sp}(W)^J$  followed by the isomorphism of  $\mathrm{Sp}(W)^J$  and  $\mathrm{Sp}_{2n}(\mathbb{R})^{J_{2n}}$  corresponding to the conjugations by  $S$  or  $T$ , and then apply (D.36). We obtain:

$$\det(\phi(g)) = \begin{cases} \det(g)_V^m & \text{if } (G, G') = (O_d, \mathrm{Sp}_{2m}(\mathbb{R})) \\ \det(g)_V^{p-q} & \text{if } (G, G') = (U_d, U_{p,q}) \\ 1 & \text{if } (G, G') = (\mathrm{Sp}_d, O_{2m}^*) \end{cases}, \quad (\text{D.42})$$

where  $\det(g)_V$  denotes the determinant of  $g$  as an element of  $G \subseteq \mathrm{GL}_{\mathbb{D}}(V)$ . (The determinant of an  $n$ -by- $n$  matrix over  $\mathbb{H}$  can be reduced to a determinant of a  $2n$ -by- $2n$  matrix over  $\mathbb{C}$  via the isomorphism  $\beta$ . For elements of  $\mathrm{Sp}(d)$ , this notion of determinant coincides with other possible notions of quaternionic determinants; see [Asl96] for additional information.)

**Proposition D.7.** *The covering  $\tilde{G} \rightarrow G$  splits if and only if  $\det(\phi(g))$  is a square. This happens for all pairs  $(G, G')$  different from  $(O_d, \mathrm{Sp}_{2m}(\mathbb{R}))$  with  $m$  odd and  $(U_d, U_{p,q})$  with  $p+q$  odd. In these two non-splitting cases, the covering  $\tilde{G} \rightarrow G$  is isomorphic to the  $\det^{1/2}$ -covering*

$$\sqrt{G} \ni (g, \zeta) \rightarrow g \in G \quad (\text{D.43})$$

where

$$\sqrt{G} = \{(g, \zeta) \in G \times \mathbb{C}^\times; \zeta^2 = \det(g)_V\}. \quad (\text{D.44})$$

*Proof.* By (D.42) there is a group homomorphism  $G \ni g \rightarrow \zeta(g) \in U_1 \subseteq \mathbb{C}^\times$  so that  $\zeta(g)^2 = \det(\phi(g))$  for all pairs  $(G, G')$  except at most the two cases listed in the statement of the Proposition.

Suppose that  $G' = \mathrm{Sp}_{2m}(\mathbb{R})$ , and let  $\zeta : O_d \rightarrow U_1$  be a continuous group homomorphism so that  $\zeta(g)^2 = \det(g)_V^m = (\pm 1)^m$ . Then  $\zeta(O_d) \subseteq \{\pm 1, \pm i\}$  and it is a subgroup with at most two elements. So  $\zeta(O_d) \subseteq \{\pm 1\}$ . On the other hand, if  $g \in O_d \setminus \mathrm{SO}_d$ , then  $\det(g)_G = -1$ . Thus  $\zeta(g)^2 \neq \det(g)_V^m$  if  $m$  is odd.

Suppose now that  $G' = U_{p,q}$ , and let  $\zeta : U_d \rightarrow U_1$  be a continuous group homomorphism so that  $\zeta(g)^2 = \det(g)_V^{p-q}$ . Restriction to  $U_1 \equiv \{\mathrm{diag}(h, 1, \dots, 1); h \in U_1\} \subseteq U_d$  yields a continuous group homomorphism  $h \in U_1 \rightarrow \zeta(h) \in U_1$ . Thus, there is  $k \in \mathbb{Z}$  so that  $\zeta(h) = h^k$  for all  $h \in U_1$ . So  $h^{2k} = \zeta(h)^2 = \det(\mathrm{diag}(h, 1, \dots, 1))^{p-q}$  implies that  $p + q$  must be even.

For the last statement, consider for  $k \in \mathbb{Z}$  the covering  $M_k = \{(g, \zeta) \in G \times \mathbb{C}^\times; \zeta^2 = \det(g)_V^{2k+1}\}$  of  $G$ . Then  $(g, \zeta) \rightarrow (g, \zeta^{\frac{1}{2k+1}})$  is a covering isomorphism between  $M_k$  and  $M_0$ .  $\square$

**Remark 6.** Keep the notation of (D.37) and let  $\alpha : \tilde{K}^\phi \rightarrow \tilde{K}$  be the isomorphism lifting  $\phi^{-1} : U_n \rightarrow K$ . Then, by [Fol89, Proposition 4.39] or [Prz89, (1.4.17)], the map

$$(u, \zeta) \rightarrow \zeta^{-1} \omega(\alpha(u, \zeta))$$

is independent of  $\zeta$ .

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