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# ISS LYAPUNOV STRICTIFICATION VIA OBSERVER DESIGN AND INTEGRAL ACTION CONTROL FOR A KORTEWEG-DE VRIES EQUATION

ISMAILA BALOGOUN\*, SWANN MARX\* , AND DANIELE ASTOLFI †

**Abstract.** The article deals with the output regulation of a nonlinear Korteweg-de-Vries (KdV) equation subject to a distributed disturbance. The control input and the regulated output are located at the boundary. To achieve this objective, we follow a Lyapunov approach. To this end inspired by a strictification methodology recently introduced in the finite-dimensional context, we construct an ISS-Lyapunov functional for the KdV equation thanks to the use of an observer which is designed following the backstepping approach. Then, thanks to this Lyapunov functional, we apply the forwarding approach in order to solve the desired output regulation problem.

**Key words.** Input-to-state stability, integral controller, Korteweg-de Vries equation, backstepping, observer, regulation, forwarding.

**1. Introduction.** This paper deals with the output regulation of a Korteweg-de Vries (KdV) equation. The KdV equation is a mathematical model of waves on shallow water surfaces (see e.g., [8] for a survey). Such an equation has been studied in [52, 10, 13] in the controllability context, in [9, 14, 37, 55, 39] in terms of stabilization, and in [54, 12] where some asymptotic analysis of the equilibrium point coinciding with the origin are given. We may also mention [40, 41] where *input-to state stability* (ISS, in short) properties are obtained via feedback stabilization in presence of a saturated damping (we refer to [51, 26, 27] or the recent survey [43] for the characterization of ISS Lyapunov functional in the infinite-dimensional context). Roughly speaking, the output regulation problem consists in designing a feedback-law such that the output converges asymptotically towards a desired reference and such that disturbances are rejected, possibly in spite of some “small” model uncertainties. Following the celebrated *internal-model principle*, a solution to such a problem exists when references and disturbance (denoted generically as exosignals) are generated by a known autonomous dynamical system (denoted as exosystem), and a copy of such a system is embedded in the controller dynamics, see, e.g. [22, 45]. A well known example is the use of integral action for tracking and rejecting constant references and disturbances.

Output regulation is an old topic in the finite-dimensional context, but many results remain to be found in the context of nonlinear systems (see e.g., [2, 24] for recent results in this field), and many further research lines have to be followed when dealing with time-varying references. See, for instance, [1] where a finite-dimensional system is regulated by adding a transport equation for the case of periodic exosignals. For infinite-dimensional systems, even if one can mention some old results such as [18], the topic is still very active. A generalization of internal-model principle has been proposed in [45], but the use of integral action to achieve output regulation in the presence of constant references/perturbations for infinite dimensional systems has been initiated early in [48]. Since then, several methods to design an integral action have been developed for linear dynamics following, for instance, a spectral approach in [49, 58, 45], by using operator and semi-group methods in [31, 59], based on frequency domain methods with Laplace transform in [4, 15] or by relying on Lyapunov techniques in [29], [21, 57]. We may also mention [19, 20] which propose to regulate an output towards time-varying references that are generated by a known linear dynamical system or [30] which extends the sliding mode methodology for hyperbolic systems to reject time-varying disturbances. In the context of nonlinear PDEs, we recall also the works [44, 25, 60]

Among all these techniques, in this article, we are particularly interested in Lyapunov techniques. Indeed, such a methodology has been proved to be efficient to deal with nonlinear systems. Among these techniques, we aim at using the forwarding methodology that has been first introduced for finite-dimensional systems in cascade form [42, 2] and then extended to some hyperbolic systems [56] in the regulation context, and to abstract systems [35] in the stabilization context. In [56], it is shown that a *strict* Lyapunov functional<sup>1</sup> is needed for open-loop stable systems that we aim at regulating. In other words, before adding an integral action, we should be able to show that a strict Lyapunov functional for the open-loop dynamics does exist (or can be obtained after employing a preliminary stabilizing state-feedback, see, e.g. [2] in the finite dimensional

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<sup>1</sup>Strict Lyapunov functionals are Lyapunov functionals whose time-derivative is bounded by a negative function depending on the full-state [33, §2.1].

context). Such Lyapunov functionals are known for hyperbolic systems [3], but it is not the case for the KdV equation. In addition to the existence of this Lyapunov functional, some ISS properties are needed to apply the forwarding method.

In the perspective of addressing the output regulation problem for the KdV equation, we first establish some new results that may have their own interest. In particular, we study the following (nonlinear) KdV equation

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ w_x(t, L) = d_2(t), \\ w(0, x) = w_0(x), \end{cases} \quad (1.1)$$

where  $(t, x) \in \mathbb{R}_+ \times [0, L]$ ,  $L > 0$ ,  $d_1$  and  $d_2$  denote external inputs that might be seen, for instance, as disturbances, and its associated linearized dynamics around the origin described by

$$\begin{cases} w_t + w_x + w_{xxx} = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ w_x(t, L) = d_2(t), \\ w(0, x) = w_0(x). \end{cases} \quad (1.2)$$

where  $(t, x) \in \mathbb{R}_+ \times [0, L]$ . We show that the KdV equations (1.1) and (1.2) satisfy an ISS property with respect to the disturbances  $d_1, d_2$  by explicitly constructing a strict Lyapunov functional. Note that there is no systematic method to build strict Lyapunov functionals either for nonlinear ordinary differential equations or (linear or nonlinear) partial differential equations. However, in many situations, a *weak* Lyapunov functional, i.e., a Lyapunov functional whose time derivative is nonpositive, exists. Often, it also coincides with the energy of the system. It is however difficult to deduce any quantitative robustness properties from a weak Lyapunov functional, and in particular, ISS properties cannot be generically obtained from such functions. For this reason, in the finite-dimensional context, a lot of attention has been put in the *strictification* of weak Lyapunov functions, namely the conception of systematic procedures to modify a weak Lyapunov function in order to make it strict. See, for instance, [33, 50]. To the best of our knowledge, in the infinite-dimensional context, such an approach has been applied only to certain classes of hyperbolic systems [51].

The first contribution of this paper, that might be seen thus of independent interest with respect to the context of output regulation, is the construction of an ISS-Lyapunov functional for our KdV equation via a strictification procedure. The methodology we propose is inspired on [50] and is based on the design of an observer, which is also a new result in the KdV equation context and therefore consists in the second main contribution of this article. Let us illustrate it. Consider system (1.1) with no inputs, namely  $d_1 = d_2 = 0$ . A formal computation shows that the time derivative of the energy  $E$  defined as

$$E(w) := \int_0^L w(t, x)^2 dx \quad (1.3)$$

yields along solutions

$$\dot{E}(w) := \frac{d}{dt} \int_0^L w(t, x)^2 dx = -|w_x(t, 0)|^2. \quad (1.4)$$

These computations are sufficient to establish that the origin is Lyapunov stable but not to conclude stronger properties (such as asymptotic stability or an ISS property if we re-introduce the effect of the disturbances in the computation of the derivative of the energy along the trajectories of (1.1)). In other words, the energy  $E$  is a weak-Lyapunov functional. Since  $w_x(t, 0)$  is an exactly observable output as soon as  $L \notin \mathcal{N}$  with

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N} \right\},$$

then, following [50], our strategy consists in designing an observer with the output  $w_x(t, 0)$ . Such an observer is obtained by using the *backstepping* approach (see, e.g., [32]) and the Fredholm operator (see, e.g., [14] or [23]). The proposed observer differs from the works in [36, 37, 55] in the same context of KdV equations because

a different measured output is considered. Finally, by combining the Lyapunov functional derived from the observer analysis and the energy  $E$ , we obtain a strict Lyapunov functional, that will be used to establish the desired ISS properties for systems (1.1) and (1.2) with respect to the inputs  $d_1$  and  $d_2$ .

Finally, the third contribution of this article consists in addressing the output regulation problem for constant perturbations and references. We suppose that a control is acting at the boundary  $w_x(t, L)$  of the KdV equations (1.1) and (1.2) and that we want to regulate the output  $w_x(t, 0)$  to a desired reference  $r$ . As a consequence, we extend the plant with an integral action processing the error  $w_x(t, 0) - r$  and we show how to design an output-feedback law. The gain of the controller is obtained via the forwarding technique which is employed to construct a strict Lyapunov functional built upon the ISS Lyapunov functional obtained in the first part of this article. Global stability properties are established for the linear model (1.2) while only local ones are proved for the nonlinear one (1.1). Note that, in both cases, we prove pointwise convergence of the tracking error i.e  $\lim_{t \rightarrow \infty} |w_x(t, 0) - r| = 0$ . Also, note that the results of [56, Theorems 1, 2] cannot be used of the shelf for the linear model (1.2) because in our article we consider a control input acting at the boundary (to be more precise, with an unbounded operator). Nevertheless, since we know an ISS-Lyapunov function we can apply the proposed methodology, similarly to what has been done for the hyperbolic equations in [56, Theorem 3]. In the context of output regulation of nonlinear PDEs as in (1.1), there exist very few results. Let us mention for instance [60] which studies quasilinear hyperbolic systems. We recall also [44, 25] in which the local problem is solved for regular linear operators perturbed by nonlinearities satisfying a Lipschitz condition. Note, however, that these results do not directly apply to the KdV nonlinear model because the nonlinearity  $ww_x$  is not Lipschitz in the right space. In this article, we are able to solve the local regulation problem for (1.1) thanks to the strict Lyapunov functional that we established.

This paper is organized as follows. In Section 2, we formulate the problem and state the results about the construction of the ISS Lyapunov functional. In Section 3, an observer is designed using a Fredholm operator. Section 4 contains the proofs of the ISS results of the KdV equations under consideration. Section 5 states and proves some regulation results for the KdV equation. Finally, Section 6 collects concluding remarks and discuss some remaining open problems.

**Notation:** Set  $\mathbb{R}_+ = [0, \infty)$ . The term  $w_t$  stands for the partial derivative of the function  $w$  with respect to  $t$ . The term  $w_x$  (resp.  $w_{xx}, w_{xxx}$ ) stands for the first (resp. second and third) order partial derivative of the function  $w$  with respect to  $x$ . When a function  $V$  (resp.  $M$ ) depends only the time variable  $t$  (resp. the space variable  $x$ ), we use the notation  $\dot{V}(t) = \frac{d}{dt}V(t)$  (resp.  $M'(x) := \frac{d}{dx}M(x)$ ). The functional space  $L^2(0, L)$  denotes the set of (Lebesgue) measurable functions  $f$  such that  $\int_0^L |f(x)|^2 dx < +\infty$ . The associated norm is  $\|f\|_{L^2}^2 := \int_0^L |f(x)|^2 dx$ . We define the functional space  $C^2([0, T])$  as the class of continuous functions on  $[0, T]$ , which have continuous derivatives of order two on  $[0, T]$ , the functional spaces  $H^k(0, L)$ . For any  $p \in [1, \infty]$ , we use the standard notation  $W^{1,p}(0, L)$  for the Sobolev space defined as  $W^{1,p}(0, L) := \{u \in L^p(0, L) : \dot{u} \in L^p(0, L)\}$ .

**2. Construction of an ISS Lyapunov functional.** The objective of this section is to study the ISS properties of the KdV models (1.1) and (1.2) and to establish the existence of a strict ISS-Lyapunov functional. The proof of the main result is postponed to Section 4. Furthermore, as mentioned in the introduction, the proposed ISS-Lyapunov functional will be used in the sequel in order to design an output feedback integral action controller, see Sections 5.1 and 5.2. Note that we will not provide further discussions on the well-posedness of (1.2) and (1.1), since it is not the main topic of this paper. Interested readers may refer to [52, 16, 5] for more information on this issue. We just emphasize on the fact that, when looking at regular solutions, we will consider initial conditions in the space

$$H_L^3(0, L) := \{w \in H^3(0, L) : w(0) = w(L) = 0, w'(L) = d_2(0)\}, \quad (2.1)$$

with  $d_2$  being the perturbation entering at the boundary condition in (1.1) or (1.2). In this case, for any  $T > 0$ , solutions  $w$  to (1.1) or (1.2) belong to the functional space  $C(0, T; H^3(0, L)) \cap C^1(0, T; L^2(0, L))$  and satisfy, for all  $t \in [0, T]$ , the additional compatibility conditions  $w(t, 0) = w(t, L) = 0, w_x(t, L) = d_2(t)$ .

Next, we state the following definition of input-to-state stability for systems (1.1) and (1.2).

**DEFINITION 2.1.** *System (1.1) (resp. (1.2)) is said to be (exponentially) input-to-state stable (ISS), if there exist positive constants  $c_0, c_1, c_2, \mu$ , such that any solution  $w \in C^0(\mathbb{R}_+; L^2(0, L)) \cap L^2(\mathbb{R}_+; H^1(0, L))$  to*

(1.1) (respectively to (1.2)) satisfies for all  $t \geq 0$

$$\|w(t, \cdot)\|_{L^2} \leq c_0 e^{-\mu t} \|w_0\|_{L^2} + c_1 \int_0^t e^{-\mu(t-s)} \|d_1(s, \cdot)\|_{L^2} ds + c_2 \int_0^t e^{-\mu(t-s)} |d_2(s)| ds, \quad (2.2)$$

for any initial condition  $w_0 \in L^2(0, L)$ ,  $d_1 \in L^2([0, t]; L^2(0, L))$  and  $d_2 \in L^2(0, t)$ . Furthermore, if there exists  $\delta > 0$  such that (2.2) holds only with  $w_0, d_1, d_2$  satisfying

$$\|w_0\|_{L^2} + \lim_{t \rightarrow \infty} \int_0^t e^{-\mu(t-s)} (\|d_1(s, \cdot)\|_{L^2} + |d_2(s)|) ds \leq 3\delta$$

then the system (1.1) (resp. (1.2)) is said to be locally (exponentially) input-to-state stable (LISS).

In the literature, the definition (2.2) is related to the notion of the “Fading Memory Input-to-State Stability”, see e.g [28, Chapter 7], due to the presence of weighting exponential functions used in the norms characterizing the gain of the signals  $d_1$  and  $d_2$ . Thus, with some abuse of language, we call it Input-to-State Stability in this paper. Also, it is important to underline that such a definition allows to consider a large class of disturbances  $d_1, d_2$ , which includes, among others, constant and periodic signals.

In general, proving the ISS property defined above needs the knowledge of the trajectories of the system, which is not an easy task. Therefore, in practice, ISS Lyapunov functionals are used to prove the desired ISS properties. To this end, we recall the result in [43, Theorem 3], showing that the existence of an ISS Lyapunov functional is sufficient to establish the ISS properties of Definition 2.1.

Before stating the definition of such Lyapunov functionals, we recall now which type of derivatives we are going to use in this article. Indeed, for any Lyapunov functional  $V$  for solutions to (1.2) or (1.1), one has the following equality:

$$\dot{V}(w) = \frac{d}{dt} V(w) = D_V(w) w_t, \quad (2.3)$$

where  $D_V(w)$  denotes the Fréchet derivative (see for instance [17, Definition A.5.33] for the definition). The proof of this equality follows the same path than the one given in [17, Lemma 11.2.5]. For instance, this means that the time derivative along solutions to (1.2) of  $E(w) = \|w\|_{L^2}^2$  can be computed as

$$\dot{E}(w) = 2 \int_0^L (-w_x - w_{xxx} + d_1) w dx, \quad (2.4)$$

and, for time derivative along solutions to (1.1):

$$\dot{E}(w) = 2 \int_0^L (-w_x - w_{xxx} - w w_x + d_1) w dx. \quad (2.5)$$

showing that the time does not play any role when using the Fréchet derivative. This is why the time will disappear when differentiating Lyapunov functionals in the rest of the paper.

We are now in position to state the following definition of ISS Lyapunov functional.

**DEFINITION 2.2.** A function  $V : L^2(0, L) \rightarrow \mathbb{R}$  is said to be an exponentially ISS Lyapunov functional for the system (1.1) (resp. (1.2)), if there exist positive constants  $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma_1, \sigma_2$  such that:

(i) For all  $w \in L^2(0, L)$ ,

$$\underline{\alpha} \|w\|_{L^2}^2 \leq V(w) \leq \bar{\alpha} \|w\|_{L^2}^2. \quad (2.6)$$

(ii) The time derivative of  $V$  along the trajectories of (1.1) (resp. (1.2)) satisfies

$$\dot{V}(w) \leq -\alpha \|w\|_{L^2}^2 + \sigma_1 \|d_1\|_{L^2}^2 + \sigma_2 |d_2|^2. \quad (2.7)$$

for any  $w \in L^2(0, L)$ ,  $d_1 \in L^2(0, L)$  and  $d_2 \in \mathbb{R}$ . If there exists  $\delta > 0$  such that (ii) holds only if  $\|w\|_{L^2} + \|d_1\|_{L^2} + |d_2| \leq 3\delta$  then  $V$  is said to be a locally exponentially ISS Lyapunov functional for the system (1.1).

As explained in the introduction, for any  $L \notin \mathcal{N}$ , the energy function in  $L^2$ -norm defined in (1.3) is a weak Lyapunov functional for the system (1.1) (resp. (1.2)) in view of (1.4). Indeed, on the right hand side of the inequality, we have a function which depends only on a part of the state  $w(t, x)$ , i.e.,  $w_x(t, 0)$ . From (1.4), one can deduce that the origin of the system of (1.1) with  $d_1 = d_2 = 0$  is Lyapunov stable. In order to show also the exponential stability properties of the origin, one can follow [52, Proposition 3.3], by using the fact that  $w_x(t, 0)$  is exactly observable as soon as  $L \notin \mathcal{N}$ : indeed, using the related observability inequality, and integrating (1.4) between 0 and  $T$ , exponential stability can be established as illustrated in [8, §4.1.]. However, nothing can be easily said in the presence of disturbances. As a consequence, in order to show the desired ISS properties of the system (1.1) (resp. (1.2)), we follow a different approach here: we aim at constructing a strict ISS Lyapunov functional, which is a new result, to the best of our knowledge. Using the observability of the output  $w_x(t, 0)$ , we can follow the methodology described in [50] and that can be decomposed as follows. First, we design an observer for the output  $w_x(t, 0)$ . Then, we consider the sum of the Lyapunov functional coming from the latter observer design and the natural energy, and we prove that this sum boils down to be a strict Lyapunov functional. Finally, thanks to this strict Lyapunov functional, we deduce ISS properties for systems (1.1) and (1.2). These properties are written more precisely in the following theorem, which is our first main result.

**THEOREM 2.3.** *Suppose that  $L \notin \mathcal{N}$ . Then, there exists a functional  $W : L^2(0, L) \rightarrow \mathbb{R}_+$  such that, the function  $V(w) := W(w) + E(w)$  with  $E$  being the energy in  $L^2$ -norm defined in (1.3), is*

- (a) *an exponentially ISS Lyapunov functional for the system (1.2);*
- (b) *a locally exponentially ISS Lyapunov functional for the system (1.1).*

*Moreover, the functional  $W$  is given by  $W(w) := \|\Pi(w)\|_{L^2}^2$  with  $\Pi$  being a continuous linear operator from  $L^2(0, L)$  to  $L^2(0, L)$  with a continuous inverse.*

The proof of Theorem 2.3 is postponed to Section 4. In particular, in the next section, we will first show how to design an ISS observer for the linearized system (1.2) by means of the output  $w_x(t, 0)$ . The proposed design is based on the backstepping method, see, e.g., [32] and on the Fredholm transformation, see, e.g., [23, 14]. Then, in Section 4, we will use the ISS-Lyapunov functional associated to such an observer to build the functional  $W$  claimed in the statement of Theorem 2.3.

The following result will be also useful when dealing with the regulation problem. It is an ISS result for a perturbed version of (1.1) with non-constant (small) coefficients. Its proof is omitted for compactness since it follows the same path used in the proof of Theorem 2.3, item (b).

**COROLLARY 2.4.** *Suppose  $L \notin \mathcal{N}$ . There exists positive real numbers  $\bar{a}, \bar{b}$  such that, for any  $a \in C([0, L])$ ,  $b \in C^1([0, L])$  satisfying  $\|a\|_\infty \leq \bar{a}$  and  $\|b\|_{W^{1,\infty}} \leq \bar{b}$ , the Lyapunov function  $V$  established in Theorem 2.3 is a locally exponentially ISS Lyapunov functional for the following system*

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = a(x)w + b(x)w_x, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L]. \end{cases} \quad (2.8)$$

*Moreover, it is an exponential ISS Lyapunov functional for the linearized dynamics of (2.8), i.e. in absence of the term  $ww_x$ .*

**3. Observer design for a Linear KdV equation.** In this section, we design an observer for the linear KdV equation (1.2) with  $y(t) = w_x(t, 0)$  defined as the output function. In particular, we consider the following system

$$\begin{cases} w_t + w_x + w_{xxx} = d_1, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (3.1)$$

and we design an observer with a distributed correction term of the form

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + p(x)[y(t) - \hat{w}_x(t, 0)] = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \hat{w}(t, 0) = \hat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}_x(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}(0, x) = \hat{w}_0(x), & x \in [0, L], \end{cases} \quad (3.2)$$

where  $p$  is an output injection gain to be designed. Note that the well-posedness of system (3.2) can be proved by following the same approach as in [36]. We define now the estimation error coordinates as follows

$$\hat{w} \mapsto \tilde{w} := w - \hat{w}$$

mapping system (3.2) into

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} - p(x)\tilde{w}_x(t, 0) = d_1, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \tilde{w}_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), & x \in [0, L]. \end{cases} \quad (3.3)$$

The objective of this section is to show that the gain  $p$  can be selected so that to guarantee the system (3.3) to be ISS with respect to the disturbances  $d_1, d_2$ . This, in turns, guarantees the convergence of the solutions of the observer (3.2) towards the trajectories of the observed plant (3.1) in the unperturbed case ( $d_1 = 0, d_2 = 0$ ), and desirable bounded-input bounded-output properties otherwise. This is established in the next theorem claiming the existence of an ISS-Lyapunov functionals for the system (3.3) under an appropriate choice of the function  $p$ .

**THEOREM 3.1.** *Suppose that  $L \notin \mathcal{N}$ . For any  $\lambda > 0$ , there exist a non-zero function  $p \in L^2(0, L)$ , a Lyapunov functional  $U : L^2(0, L) \rightarrow \mathbb{R}$  and positive constants  $\underline{c}, \bar{c}, \varrho_1, \varrho_2$  satisfying the following properties.*

(i) *For all  $\tilde{w} \in L^2(0, L)$*

$$\underline{c}\|\tilde{w}\|_{L^2}^2 \leq U(\tilde{w}) \leq \bar{c}\|\tilde{w}\|_{L^2}^2. \quad (3.4)$$

(ii) *The time derivative of  $U$  along the trajectories of (3.3) satisfies, for all  $w \in L^2(0, L)$ ,  $d_1 \in L^2(0, L)$  and  $d_2 \in \mathbb{R}$ ,*

$$\dot{U}(\tilde{w}) \leq -\lambda U(\tilde{w}) + \varrho_1\|d_1\|_{L^2}^2 + \varrho_2|d_2|^2. \quad (3.5)$$

Moreover, the functional  $U$  is given by  $U(w) := \|\bar{\Pi}^{-1}(w)\|_{L^2}^2$ , with  $\bar{\Pi}$  being a continuous linear operator from  $L^2(0, L)$  to  $L^2(0, L)$  with continuous inverse.

**Proof:** The proof of Theorem 3.1 is divided into two parts. The first step consists in proving the existence of  $p \in L^2(0, L)$  such that the origin of (3.3), in the unperturbed case  $d_1 = 0, d_2 = 0$ , is exponentially stable. The second step is to show the existence of a Lyapunov functional  $U$  satisfying the inequalities (3.4) and (3.5).

Let us start the proof of the first step. Inspired by [14, equation (1.8)], consider the change of coordinates

$$\tilde{w} \mapsto \gamma := \bar{\Pi}^{-1}\tilde{w} \quad (3.6)$$

where the function  $\bar{\Pi}$  is defined thanks to the following Fredholm integral transformation

$$\tilde{w}(x) = \bar{\Pi}(\gamma)(x) := \gamma(x) - \int_0^L P(x, z)\gamma(z)dz, \quad (3.7)$$

for all  $x \in [0, L]$ , where  $\tilde{w}$  satisfies (3.3) with  $d_1 = 0$  and  $d_2 = 0$ ,  $P$  is a function to be defined and  $\gamma$  is the solution to the following system

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \gamma(t, 0) = \gamma(t, L) = \gamma_x(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L], \end{cases} \quad (3.8)$$

with  $\lambda > 0$ . Note that using an integration by parts and the boundary conditions of (3.8), one immediately obtains

$$\frac{d}{dt} \int_0^L |\gamma(t, x)|^2 dx \leq -2\lambda \int_0^L |\gamma(t, x)|^2 dx$$

from which it is straightforward to deduce the exponential stability in the  $L^2$ -norm of  $\gamma$ . As a consequence, the main idea of the proof consists in selecting the function  $p$  such that (3.7) holds. To do so, we need to find the kernel  $P$  such that  $\tilde{w}(t, x) = \bar{\Pi}(\gamma)(t, x)$  satisfies (3.3) when  $d_1 = 0$  and  $d_2 = 0$ . Furthermore, we have also to ensure that the corresponding transformation is invertible and continuous. To this end, we first formally differentiate with respect to the time and with respect to the space the change of coordinates (3.7). We obtain the following identities

$$\tilde{w}_t(t, x) = \gamma_t(t, x) + \int_0^L P(x, z) \left( \lambda \gamma(t, z) + \gamma_z(t, z) + \gamma_{zzz}(t, z) \right) dz, \quad (3.9)$$

$$\tilde{w}_x(t, x) = \gamma_x(t, x) - \int_0^L P_x(x, z) \gamma(t, z) dz, \quad (3.10)$$

$$\tilde{w}_{xxx}(t, x) = \gamma_{xxx}(t, x) - \int_0^L P_{xxx}(x, z) \gamma(t, z) dz, \quad (3.11)$$

in which (3.9) has been obtained by using the  $\gamma$ -dynamics in (3.8). After some integrations by parts, (3.9) gives

$$\begin{aligned} \tilde{w}_t(t, x) &= \gamma_t(t, x) - P(x, 0) \gamma(t, 0) + P(x, L) \gamma(t, L) + P(x, L) \gamma_{xx}(t, L) - P(x, 0) \gamma_{xx}(t, 0) + P_z(x, 0) \gamma_x(t, 0) \\ &\quad - \int_0^L \left( -\lambda P(x, z) + P_z(x, z) + P_{zzz}(x, z) \right) \gamma(t, z) dz - P_z(x, L) \gamma_x(t, L) + P_{zz}(x, L) \gamma(t, L) \\ &\quad - P_{zz}(x, 0) \gamma(t, 0). \end{aligned} \quad (3.12)$$

Then, by adding on both sides the terms  $\tilde{w}_x$ ,  $\tilde{w}_{xxx}$  and  $-p(x)\tilde{w}_x(t, 0)$  and using (3.3), (3.8) and the previous identities (3.10), (3.11), we further obtain

$$\begin{aligned} \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) - p(x)\tilde{w}_x(t, 0) &= \\ &= \gamma_t(t, x) + \gamma_x(t, x) + \gamma_{xxx}(t, x) + \lambda \gamma(t, x) - \int_0^L \left( -\lambda P + P_z + P_{zzz} + P_{xxx} + P_x \right) \gamma(t, z) dz \\ &\quad - \lambda \gamma(t, x) + P(x, L) \gamma_{xx}(t, L) + P_z(x, 0) \gamma_x(t, 0) - P(x, 0) \gamma_{xx}(t, 0) - p(x) \left[ \gamma_x(t, 0) - \int_0^L P_x(0, z) \gamma(t, z) dz \right] \end{aligned}$$

where some arguments are omitted for compactness when clear from the context. Then, using the identity

$$-\lambda \gamma(t, x) = \int_0^L \lambda \delta(x - z) \gamma(t, z) dz,$$

where  $\delta(x - z)$  denotes the Dirac measure on the diagonal of the square  $[0, L] \times [0, L]$ , the previous equation gives

$$\begin{aligned} \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) - p(x)\tilde{w}_x(t, 0) &= \\ &= \gamma_t(t, x) + \gamma_x(t, x) + \gamma_{xxx}(t, x) + \lambda \gamma(t, x) - \int_0^L \left( -\lambda P + P_z + P_{zzz} + P_x + P_{xxx} - \lambda \delta(x - z) \right) \gamma(t, z) dz \\ &\quad - P(x, 0) \gamma_{xx}(t, 0) + P(x, L) \gamma_{xx}(t, L) + p(x) \int_0^L P_x(0, z) \gamma(t, z) dz - \gamma_x(t, 0) [p(x) - P_z(x, 0)]. \end{aligned} \quad (3.13)$$

From equation (3.13), we finally obtain the following conditions for the functions  $P$  and  $p$ .

- (a) The identity  $-\lambda P + P_z + P_{zzz} + P_x + P_{xxx} = \lambda \delta(x - z)$  is satisfied for all  $(x, z) \in [0, L] \times [0, L]$ .
- (b) The boundary conditions  $P(x, 0) = P(x, L) = P_x(0, z) = 0$  are satisfied for all  $(x, z) \in [0, L] \times [0, L]$ .



(c) An appropriate choice of  $p$  is given by  $p(x) := P_z(x, 0)$ , for all  $x \in [0, L]$ .

Moreover, note also that the following.

(d) By setting  $x = 0$  and  $x = L$  in (3.7), we need:  $P(0, z) = P(L, z) = 0$  for all  $z \in [0, L]$ .

(e) By setting  $x = L$  in (3.10), we need:  $P_x(L, z) = 0$  for all  $z \in [0, L]$ .

Therefore, collecting the conditions (a)-(e), we impose the function  $P$  to satisfy the following PDE

$$\begin{cases} -\lambda P + P_z + P_{zzz} + P_x + P_{xxx} = \lambda \delta(x - z), \\ P(x, 0) = P(x, L) = 0, \\ P(L, z) = P(0, z) = 0, \\ P_x(L, z) = P_x(0, z) = 0, \end{cases} \quad (3.14)$$

where  $(x, z) \in [0, L] \times [0, L]$  and  $\delta(x - z)$  denotes the Dirac measure on the diagonal of the square  $[0, L] \times [0, L]$ . Now, in order to show the existence of a solution to (3.14), let us make the following change of variable:

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} := \begin{pmatrix} L - z \\ L - x \end{pmatrix},$$

and define  $G(\bar{x}, \bar{z}) := -P(x, z)$ . From (3.14) it is obtained

$$\begin{cases} \lambda G + G_{\bar{z}} + G_{\bar{z}\bar{z}\bar{z}} + G_{\bar{x}} + G_{\bar{x}\bar{x}\bar{x}} = \lambda \delta(\bar{x} - \bar{z}), \\ G(\bar{x}, 0) = G(\bar{x}, L) = 0, \\ G(L, \bar{z}) = G(0, \bar{x}) = 0, \\ G_{\bar{z}}(\bar{x}, 0) = G_{\bar{z}}(\bar{x}, L) = 0, \end{cases} \quad (3.15)$$

with  $(\bar{x}, \bar{z})$  belonging to  $[0, L] \times [0, L]$ . Note that in [14, Lemma 2.1], it has been proved that, for any  $L \notin \mathcal{N}$ , the system (3.15) admits a unique solution  $G \in H_0^1((0, L) \times (0, L))$ . Therefore, we can conclude that the kernel  $P$  exists. Then according to [14, Lemma 3.1], the transformation  $\bar{\Pi}$  is invertible and continuous on  $L^2(0, L)$  and its inverse is also continuous. As a consequence, we have shown that, for an appropriate choice of the function  $p \in L^2(0, L)$ , the system (3.3) is transformed into the system (3.8) via a linear change of coordinates which is invertible with a continuous inverse. Since the origin of system (3.8) is exponentially stable, we conclude that so is the origin of (3.3) in the non-perturbed case (i.e.,  $d_1 = 0$ ,  $d_2 = 0$ ). Note also that  $p$  is non-zero. Indeed, if  $p = 0$  then, in view of the condition (c), we would have  $P_z(x, 0) = 0$ . Therefore, the system (3.14) would have seven boundary conditions. But then, because of the degree of the first equation of (3.14), the system (3.14) would have no solution. This concludes the first part of the proof.

We want now to prove the existence of a Lyapunov functional which satisfies the inequalities (3.4) and (3.5) in presence of  $d_1, d_2$ . To this end, we choose the following candidate Lyapunov function  $U : L^2(0, L) \rightarrow \mathbb{R}$

$$U(w) := \|\bar{\Pi}^{-1}(w)\|_{L^2}^2 \quad (3.16)$$

Since  $\bar{\Pi}^{-1}$  exists, then  $U$  is well defined in  $L^2(0, L)$ . Moreover, according to the continuity of  $\bar{\Pi}^{-1}$  and  $\bar{\Pi}$  in  $L^2(0, L)$ , there exist two positive constants  $\underline{c}$  and  $\bar{c}$  satisfying inequality (3.4) for all  $w \in L^2(0, L)$ . Note that the function  $w \in L^2(0, L) \mapsto U(w) \in \mathbb{R}_+$  is equivalent to the standard norm on the space  $L^2(0, L)$  according to (3.4). It only remains to prove that  $U$  satisfies the inequality (3.5). To this end, we show inequality (3.5) for  $\tilde{w}_0 \in H_L^3(0, L)$ ,  $d_2 \in C^2([0, T])$  and  $d_1 \in C^1([0, T], L^2(0, L))$ . The result follows for all  $\tilde{w}_0 \in L^2(0, L)$ ,  $d_1 \in L^1([0, T]; L^2(0, L))$  and  $d_2 \in L^2(0, T)$ , by a standard density argument similar to the one used in [34, Lemma 1]. Now, consider again the transformation defined in (3.6), (3.7). Similar computations can be used to show that its inverse transformation is defined by

$$\gamma(x) := \bar{\Pi}^{-1}(\tilde{w})(x) = \tilde{w} + \int_0^L Q(x, z) \tilde{w}(z) dz, \quad (3.17)$$

where  $Q \in H_0^1((0, L) \times (0, L))$  is now the solution of the following system

$$\begin{cases} \lambda Q + Q_z + Q_{zzz} + Q_x + Q_{xxx} = \lambda \delta(x - z), \\ Q(x, 0) = Q(x, L) = 0, \\ Q(L, z) = Q(0, z) = 0, \\ Q_x(L, z) = 0, \end{cases} \quad (3.18)$$

and satisfies  $p(x) + \int_0^L p(z)Q(x, z)dz = Q_z(x, 0)$  for all  $x \in [0, L]$ . Now, consider the solution  $\tilde{w}$  of system (3.3) with  $d_1, d_2$  possibly different from zero. Then, applying the change of coordinates  $\gamma = \bar{\Pi}^{-1}(\tilde{w})$  defined in (3.17), (3.18), we obtain

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = \bar{\Pi}^{-1}(d_1) + Q_z(x, L)d_2, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (3.19)$$

The derivative of  $U$  along the trajectory of (3.3), or equivalently on the trajectory of (3.19), yields

$$\begin{aligned} \dot{U}(w) &= -2 \int_0^L \gamma \left( \gamma_{xxx} + \gamma_x + \lambda\gamma - \bar{\Pi}^{-1}(d_1) - Q_z(x, L)d_2 \right) dx \\ &= -2\lambda \int_0^L |\gamma|^2 dx + 2 \int_0^L \gamma_x \gamma_{xx} dx + 2 \int_0^L \bar{\Pi}^{-1}(d_1) \gamma dx + 2d_2 \int_0^L Q_z(x, L) \gamma dx \\ &\leq -2\lambda \|\gamma\|_{L^2}^2 + 2 \left| \int_0^L \bar{\Pi}^{-1}(d_1) \gamma dx \right| + d_2^2 - \gamma_x(0)^2 + 2 \left| d_2 \int_0^L Q_z(x, L) \gamma dx \right|, \end{aligned} \quad (3.20)$$

where, in the second equation, we have used an integration by parts to compute

$$2 \int_0^L \gamma_x \gamma_{xx} dx = \left[ \gamma_x^2(x) \right]_0^L = d_2^2 - \gamma_x(0)^2.$$

Using first Cauchy-Schwarz's inequality and then Young's inequality  $2ab \leq \nu a^2 + \frac{1}{\nu} b^2$ , for any  $\nu > 0$ , from (3.20) we finally obtain

$$\begin{aligned} \dot{U}(w) &\leq -2\lambda \|\gamma\|_{L^2}^2 + 2\|\gamma\|_{L^2} \|\bar{\Pi}^{-1}(d_1)\|_{L^2} + 2|d_2| \|\gamma\|_{L^2} \|Q_z(\cdot, L)\|_{L^2} + |d_2|^2 \\ &\leq -\lambda \|\gamma\|_{L^2}^2 + \frac{2}{\lambda} \|\bar{\Pi}^{-1}(d_1)\|_{L^2}^2 + \left( 1 + \frac{2}{\lambda} \|Q_z(\cdot, L)\|_{L^2}^2 \right) |d_2|^2. \end{aligned}$$

Using the inequality (3.4) on the term depending on  $d_1$ , we finally obtain

$$\dot{U}(w) \leq -\lambda \|\gamma\|_{L^2}^2 + \frac{2\bar{c}}{\lambda} \|d_1\|_{L^2}^2 + \left( 1 + \frac{2}{\lambda} \|Q_z(\cdot, L)\|_{L^2}^2 \right) |d_2|^2 \quad (3.21)$$

showing the inequality (3.5) with  $\varrho_1 = \frac{2\bar{c}}{\lambda}$ ,  $\varrho_2 = 1 + \frac{2}{\lambda} \|Q_z(\cdot, L)\|_{L^2}^2$ . This completes the proof.  $\square$

From the existence of the ISS Lyapunov functional established in Theorem 3.1, one can immediately deduce the following property for the observer (3.2).

**COROLLARY 3.2.** *For any  $\lambda > 0$ , there exists a function  $p \in L^2(0, L)$  such that the observer (3.2) is an ISS exponential convergent observer for system (3.1) with convergence rate  $\lambda$ , namely, there exist some  $c_0, c_1, c_2 > 0$  such that the following inequality holds*

$$\|\hat{w}(t, \cdot) - w(t, \cdot)\|_{L^2} \leq c_0 e^{-\lambda t} \|\hat{w}_0 - w_0\|_{L^2} + c_1 \int_0^t e^{-\lambda(t-s)} \|d_1(s, \cdot)\|_{L^2} ds + c_2 \int_0^t e^{-\lambda(t-s)} |d_2(s)| ds, \quad (3.22)$$

for any initial conditions  $w_0, \hat{w}_0 \in L^2(0, L)$ , any  $d_1 \in L^2([0, t]; L^2(0, L))$ , any  $d_2 \in L^2(0, t)$  and for all  $t \geq 0$ .

**Proof:** The proof can be directly inherited from Theorem 3.1 by applying Grönwall's lemma to inequality (3.5).  $\square$

As a conclusion of this section, we remark that in view of the exponential stability properties of the observer (3.2), one can also design a local observer for the nonlinear KdV model (1.1). In particular, selecting

the gain  $p$  as in Corollary 3.2 it is possible to show that the following system

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + \hat{w}\hat{w}_x + p(x)[y(t) - \hat{w}_x(t, 0)] = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \hat{w}(t, 0) = \hat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}_x(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}(0, x) = \hat{w}_0(x), & x \in [0, L], \end{cases}$$

is a locally exponentially ISS observer for system (1.1), namely inequality (3.22) holds for all  $w_0, \hat{w}_0, d_1, d_2$  satisfying

$$\|\hat{w}_0\|_{L^2} + \|w_0\|_{L^2} + \lim_{t \rightarrow \infty} \int_0^t e^{-\mu(t-s)} \left( \|d_1(s, \cdot)\|_{L^2} + |d_2(s)| \right) ds \leq \delta$$

for some  $\delta$  small enough. The proof is omitted for space reasons and can be derived by combining the arguments of the proof of Theorem 3.1 with the robustness result established in Corollary 2.4.

**4. Proof of Theorem 2.3.** Let  $T > 0$ . We prove the statement of the Theorem 2.3 for  $w_0 \in H_L^3(0, L)$ ,  $d_2 \in C_0^2([0, T])$  and  $d_1 \in C^1([0, T], L^2(0, L))$ , where we recall that  $H_L^3(0, L)$  is defined in (2.1). Since  $H_L^3(0, L)$ ,  $C^2([0, T])$  and  $C^1([0, T], L^2(0, L))$  are dense in  $L^2(0, L)$ ,  $L^2(0, T)$  and  $L^1([0, T]; L^2(0, L))$ , respectively, the result follows for all  $w_0 \in L^2(0, L)$ ,  $d_1 \in L^1([0, T]; L^2(0, L))$  and  $d_2 \in L^2(0, T)$ , by a standard density argument similar to the one provided in [34, Lemma 1].

**Proof of item (a) of Theorem 2.3.** The derivative of the Energy (1.3) gives along solutions of the linear KdV model (1.2) a negative term in  $w_x(t, 0)$ . Moreover, Theorem 3.1 shows that using such a term in the  $w$ -dynamics, we are able to obtain an ISS-Lyapunov functional  $U$ . As a consequence, the main idea of this proof consists in adding and subtracting the term  $w_x(t, 0)$ , multiplied by a coefficient  $p(x)$ , in the  $w$  dynamics: one term is used to obtain the negativity in the  $L^2$  norm of the full space as in (3.5), while the other is treated as a distributed disturbance  $d_1$  and compensated by the negativity of the Energy.

With the previous points in mind, fix  $\lambda = 1$  and consider the functions  $p$  and  $U$  given by Theorem 3.1. Set  $\bar{p} := \|p\|_{L^2}^2$ . Note that  $\bar{p} \neq 0$  because  $p$  is a non-zero function. We define the operator  $\Pi$  and the function  $W$  as follows

$$W(w) := \frac{1}{2\bar{p}\varrho_1} U(w) = \|\Pi(w)\|_{L^2}^2, \quad \Pi(w) := \frac{1}{\sqrt{2\bar{p}\varrho_1}} \bar{\Pi}^{-1}(w), \quad (4.1)$$

for all  $w \in L^2(0, L)$ , where the operator  $\bar{\Pi}$  and the parameter  $\varrho_1$  are given by Theorem 3.1. We show that the statement of the theorem holds and in particular that the inequalities (2.6), (2.7) are satisfied. First, in view of (3.4), we obtain

$$\frac{\underline{c}}{2\bar{p}\varrho_1} \|w\|_{L^2} \leq W(w) \leq \frac{\bar{c}}{2\bar{p}\varrho_1} \|w\|_{L^2}$$

As a consequence, by recalling that  $E(w) = \|w\|_{L^2}^2$ , the inequality (2.6) is satisfied for the function  $V = E + W$  with  $\underline{\alpha} := 1 + \frac{\underline{c}}{2\bar{p}\varrho_1}$  and  $\bar{\alpha} := 1 + \frac{\bar{c}}{2\bar{p}\varrho_1}$ .

Then, in order to show the inequality (2.7) we compute the derivative of the functional  $V$  along the trajectories of the system (1.2). We first analyze the time derivative of the energy  $E$ . Using (1.4) and adding the effect of the perturbations  $d_1, d_2$ , we obtain

$$\begin{aligned} \dot{E}(w) &= -|w_x(0)|^2 + 2 \int_0^L w(x) d_1(x) dx + |d_2|^2 \\ &\leq -|w_x(0)|^2 + \frac{\underline{c}}{4\bar{p}\varrho_1} \|w\|_{L^2}^2 + \frac{4\bar{p}\varrho_1}{\underline{c}} \|d_1\|_{L^2}^2 + |d_2|^2 \end{aligned} \quad (4.2)$$

where the second inequality has been obtained by using the Cauchy-Schwarz and Young inequalities, and with the parameters  $\underline{c}, \varrho_1$  given by Theorem 3.1. Next, we compute the derivative of  $W$  along the trajectories of system (1.2). To this end, we first add and subtract the term  $p(x)w_x(t, 0)$  to the dynamics, obtaining

$$\begin{cases} w_t + w_x + w_{xxx} - p(x)w_x(t, 0) = -p(x)w_x(t, 0) + d_1(t, x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L]. \end{cases} \quad (4.3)$$

Applying the ISS-Lyapunov inequality (3.5) along solutions to (4.3), the derivative of  $U$  yields

$$\begin{aligned}\dot{U}(w) &\leq -U(w) + \varrho_1 \|d_1 - pw_x(0)\|_{L^2}^2 + \varrho_2 |d_2|^2 \\ &\leq -\underline{c} \|w\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|pw_x(0)\|_{L^2}^2 + \varrho_2 |d_2|^2,\end{aligned}\quad (4.4)$$

where in the second inequality we used again the inequality (3.4). Finally, we can compute the derivative of the function  $V = E + W$ , with  $W$  defined in (4.1), by combining (4.4) and (4.2) and using the identity

$$-|w_x(0)|^2 + \frac{1}{\bar{p}} \|pw_x(0)\|_{L^2}^2 = 0. \quad (4.5)$$

Simple computations (omitted for space reason) give the inequality (2.7) with the choice  $\alpha := \frac{\underline{c}}{4\bar{p}\varrho_1}$ ,  $\sigma_1 = \frac{4\bar{p}\varrho_1}{\underline{c}} + \frac{1}{\bar{p}}$ ,  $\sigma_2 := 1 + \frac{\varrho_2}{2\bar{p}\varrho_1}$ . This concludes the proof of item (a) of Theorem 2.3.

**Proof of the item (b) of Theorem 2.5.** Consider again the function  $V = E + W$  with defined in (4.1). The derivative of the energy (1.3) along the trajectories of the nonlinear system (1.1) is computed as in (4.2) because the contribution of the nonlinear  $ww_x$  is zero. Next, we compute the time derivative of  $W$ . However, due to the presence of the nonlinear term  $ww_x$  we cannot apply off-the-shelf the inequality (3.5) by including such a term in the disturbance  $d_1$ : it would not be bounded with the right norm. As a consequence, unfortunately, we need to revisit and adapt some steps of the proof of Theorem 3.1 and in particular we need to compute the change of coordinates defined in (3.6), (3.17). Recalling that we selected  $\lambda = 1$ , the  $\gamma$ -dynamics reads

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \gamma = -\bar{\Pi}^{-1}(p)w_x(t, 0) + \bar{\Pi}^{-1}(d_1) - \bar{\Pi}^{-1}(ww_x) + Q_z(x, L)d_2, & (t, x) \in \Omega \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+ \\ \gamma_x(t, L) = d_2(t), & t \in \mathbb{R}_+ \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (4.6)$$

where  $Q$  is defined in (3.17). With respect to system (3.19) we have two extra terms to analyse, that are the terms  $\bar{\Pi}^{-1}(p)w_x(0)$  and  $\bar{\Pi}^{-1}(ww_x)$ . As a consequence, we consider again the Lyapunov functional  $U(w) := \|\gamma\|_{L^2}^2$  as in (3.16), and we follow similar computations to those developed from (3.20) to (3.21). Also, as in the proof of item (a), we consider as a full disturbance the term  $d_1 - pw_x(0)$ , see inequality (4.4). In particular, the derivative of  $U$  along the trajectories of system (4.6) satisfies, for all  $w \in L^2(0, L)$

$$\dot{U}(w) \leq -\|\gamma\|_{L^2}^2 + \varrho_1 \|d_1 - pw_x(0)\|_{L^2}^2 + \varrho_2 |d_2|^2 + 2 \left| \int_0^L f(ww_x) \gamma dx \right|$$

where the function  $f$  is defined as  $f(ww_x)(x) := \bar{\Pi}^{-1}(ww_x)(x) = w(x)w_x(x) + \int_0^L Q(x, z)w(z)w_x(z)dz$ . By using the same argument as in [14, Proof of Theorem 1.2, page 1111-1113], we can show the existence of positive constant  $\bar{f}$  that depends only on the function  $Q$ , such that

$$2 \left| \int_0^L f(ww_x)(x) \gamma(x) dx \right| \leq \bar{f} \|\gamma\|_{L^2}^3 \quad \forall w \in L^2(0, L).$$

As a consequence, combining the previous inequalities and following the same computations in (4.4), we obtain, for all  $w \in L^2(0, L)$

$$\dot{U}(w) \leq -(1 - \bar{f} \|\gamma\|_{L^2}) \|\gamma\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|pw_x(0)\|_{L^2}^2 + \varrho_2 |d_2|^2.$$

Therefore, using the inequality (3.4), we obtain

$$\dot{U}(w) \leq -\frac{\underline{c}}{2} \|w\|_{L^2}^2 + 2\varrho_1 \|d_1\|_{L^2}^2 + 2\varrho_1 \|pw_x(0)\|_{L^2}^2 + \varrho_2 |d_2|^2$$

for all  $w$  satisfying  $\|w\|_{L^2} \leq \bar{\delta}$ , with  $\bar{\delta} = (2\sqrt{\underline{c}\bar{f}})^{-1}$ . Using the definition of the function  $W$  in (4.1) and following the same steps of item (a), we obtain the inequality inequality (2.7) with the choice  $\alpha := \frac{\underline{c}}{8\bar{p}\varrho_1}$ ,  $\sigma_1 = \frac{4\bar{p}\varrho_1}{\underline{c}} + \frac{1}{\bar{p}}$ ,  $\sigma_2 := 1 + \frac{\varrho_2}{2\bar{p}\varrho_1}$ , and  $\delta = \frac{1}{3}\bar{\delta}$ .

**5. Adding an integral action.** In this section we consider the regulation problem of a KdV equation in which the disturbance  $d_2$  is considered as a control input acting at the boundary condition, and the output  $y(t) = w_x(t, 0)$  has to be regulated at a certain desired constant reference  $r$  in presence of unknown distributed constant disturbances  $d_1$ . We aim at showing that such a problem can be solved by means of an integral action and an output feedback control law. The proposed design is based on the forwarding method (see e.g., [56] or [35]). Note that in Section 5.1, we focus on the linearized version of the KdV model (1.2). Then, in Section 5.2, we will show a local result for the nonlinear system (1.1).

**5.1. Regulation of linear KdV equation by means of the forwarding method.** Consider the following system

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+ \\ w_x(t, L) = u(t), & t \in \mathbb{R}_+ \\ w(0, x) = w_0(x), & x \in [0, L] \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+ \end{cases} \quad (5.1)$$

where  $d \in L^2(0, L)$  is a constant perturbation,  $u \in \mathbb{R}$  is the control input, and  $y \in \mathbb{R}$  is the output to be regulated at a certain desired constant reference  $r$ . We define the regulated output error  $e = y - r$  and we defined our regulation objective as

$$\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} y(t) - r = 0. \quad (5.2)$$

To this end, we follow the standard set-up of output regulation [2, 56] and we extend system (5.1) with an integral action processing the desired error to be regulated. In other words, we consider a dynamical feedback law of the form

$$\dot{\eta} = y - r, \quad u = k\eta, \quad (5.3)$$

where  $\eta \in \mathbb{R}$  is the state of the controller and  $k$  is a positive constant to be selected small enough, as shown later. The closed-loop system (5.1), (5.3) can be seen as an augmented system, i.e. a PDE system (whose state is  $w$ ) coupled with an ODE (whose state is  $\eta$ ), which reads

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = w_x(t, 0) - r, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \quad \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (5.4)$$

We define the space  $X := \mathbb{R} \times L^2(0, L)$ , that is the state space of (5.4). It is a Hilbert space as the Cartesian product of two Hilbert spaces. In the rest of the section, we will show the following properties for the closed-loop system (5.4): it is well posed, it admits a unique equilibrium which is exponentially stable, and the regulation objective (5.2) is achieved when considering sufficiently regular solutions.

To this end, we introduce now the following two linear operators  $\mathcal{S}$  and  $\mathcal{A}$  that will be used in the rest of the section. In particular, we denote with  $\mathcal{S}$  the operator associated with the linear KdV equation (1.2). The operator  $\mathcal{S}$  and its domain  $D(\mathcal{S}) \subset L^2(0, L)$  are defined as

$$\mathcal{S}w = -w' - w''', \quad D(\mathcal{S}) := \{w \in H^3(0, L) : w(0) = w(L) = w'(L) = 0\}. \quad (5.5)$$

Then, we define the operator  $\mathcal{A}$  in order to describe the closed-loop system (5.4) in the following abstract form

$$\frac{d}{dt}\zeta = \mathcal{A}\zeta + \Gamma, \quad \zeta(0) = \tilde{\zeta}_0, \quad \zeta := \begin{pmatrix} \eta \\ w \end{pmatrix}, \quad \mathcal{A}(\eta, w) := \begin{bmatrix} w'(0) \\ -w' - w''' \end{bmatrix}, \quad \Gamma := \begin{bmatrix} -r \\ d \end{bmatrix}, \quad (5.6)$$

with the domain of  $\mathcal{A}$  defined as  $D(\mathcal{A}) := \{(\eta, w) \in \mathbb{R} \times H^3(0, L) \mid w(0) = w(L) = 0, w'(L) = k\eta\} \subset X$ . We start by proving the existence and uniqueness of an equilibrium for system (5.4) in the following lemma.

LEMMA 5.1. *For any  $k \neq 0$  and  $(d, r) \in L^2(0, L) \times \mathbb{R}$  there exist a unique equilibrium state  $(\eta_\infty, w_\infty) \in X$  to system (5.4).*

**Proof:** Consider the following boundary value problem

$$\begin{cases} w'_\infty(x) + w'''_\infty(x) = d(x), & x \in [0, L], \\ w_\infty(0) = w_\infty(L) = 0, \\ w'_\infty(0) = r, \end{cases}$$

which represents the nonzero equilibrium state of (5.4), together with  $\eta_\infty = \frac{w'_\infty(L)}{k}$ . Consider the smooth function  $\phi(x) = \frac{rx(L-x)}{L}$ . It satisfies the boundary conditions  $\phi(0) = \phi(L) = 0$  and  $\phi'(0) = r$ . We set  $\psi = w_\infty - \phi$ . Then  $\psi$  satisfies the following system

$$\begin{cases} \psi'(x) + \psi'''(x) = j(x), & x \in [0, L], \\ \psi(0) = \psi(L) = 0, \\ \psi'(0) = 0, \end{cases}$$

where  $j(x) = d(x) - \phi'(x)$ . This system can be written in the operator form as  $\mathcal{S}^*\psi = j$ , where  $\mathcal{S}^*$ , is the adjoint operator of  $\mathcal{S}$  defined in (5.5). In particular,  $\mathcal{S}^*$ , is defined as  $\mathcal{S}^*\psi = \psi''' + \psi'$  with domain  $D(\mathcal{S}^*) := \{w \in H^3(0, L) : w(0) = w(L) = w'(0) = 0\}$ . Following [38, Lemma 4], we can prove that the canonical embedding from  $D(\mathcal{S}^*)$ , equipped with the graph norm, into  $L^2(0, L)$ , is compact. Then, according to [11, Proposition 4.24],  $\mathcal{S}^*$  is an operator with compact resolvent. This implies that its spectrum consists only of eigenvalues. Moreover, 0 is not an eigenvalue of  $\mathcal{S}^*$ . Hence, there exists a unique solution  $\psi_\infty$  to the equation  $\mathcal{S}^*\psi = j$ . The equilibrium  $(\eta_\infty, w_\infty)$  can then be computed as  $w_\infty(x) = \psi_\infty + \phi(x)$  for all  $x \in [0, L]$  and  $\eta_\infty = \frac{w'_\infty(L)}{k}$ , with  $\phi$  being the function defined at the beginning of the proof.  $\square$

Next, we show the following well-posedness result for the closed-loop system (5.4). In the proof, we will also introduce a strict Lyapunov functional for the closed-loop system (5.4). Such a Lyapunov functional is obtained via the forwarding methodology similarly to [56, 35] and it is based on the ISS-Lyapunov established in Theorem 2.3.

LEMMA 5.2. *Let  $L \notin \mathcal{N}$ . There exist  $k_0^* > 0$  such that for any  $k \in (0, k_0^*)$ , for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  and for any initial condition  $(\eta_0, w_0) \in X$  (resp.  $D(\mathcal{A})$ ), there exists a unique weak solution  $(\eta, w) \in C^0(\mathbb{R}_+; X)$  (resp. strong solution in  $C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$ ) to system (5.4).*

**Proof:** Given  $(d, r) \in L^2(0, L) \times \mathbb{R}$  let  $(\eta_\infty, w_\infty)$  the corresponding equilibrium to (5.4) computed according to Lemma 5.1. Consider the following change of coordinates

$$(w, \eta) \mapsto (\tilde{w}, \tilde{\eta}) := (w - w_\infty, \eta - \eta_\infty). \quad (5.7)$$

The  $(\tilde{w}, \tilde{\eta})$ -dynamics is given by

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \tilde{w}_x(t, L) = k\tilde{\eta}(t), & t \in \mathbb{R}_+, \\ \dot{\tilde{\eta}}(t) = \tilde{w}_x(t, 0), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), \tilde{\eta}(0) = \tilde{\eta}_0, & x \in [0, L], \end{cases} \quad (5.8)$$

where  $\tilde{w}_0(x) = w_0(x) - w_\infty(x)$  and  $\tilde{\eta}_0 = \eta_0 - \eta_\infty$ . System (5.8) can be rewritten, in the operator form, as

$$\frac{d}{dt} \tilde{\zeta} = \mathcal{A} \tilde{\zeta}, \quad \tilde{\zeta} = \zeta_0, \quad \tilde{\zeta} := \begin{pmatrix} \tilde{\eta} \\ \tilde{w} \end{pmatrix}$$

with  $\mathcal{A}$  and its domain  $D(\mathcal{A})$  defined as in (5.6). As a consequence, systems (5.4) and (5.8) are equivalent. Then, if one proves that the operator  $\mathcal{A}$  defined in (5.6) is a  $m$ -dissipative operator on  $(X, \|\cdot\|_X)$ , one can apply the result provided by [6, Theorem 3.1], and conclude that the statement of Lemma 5.2 holds. For that,

we look for an equivalent norm and a related scalar product coming from a Lyapunov functional. We will prove then the dissipativity with respect to such a scalar product. This Lyapunov functional is built following the forwarding approach (see e.g [56]). To simplify the notation, in the rest of this proof, we will write  $(\eta, w)$  instead of  $(\tilde{\eta}, \tilde{w})$ .

Now, by recalling the definition of the operator  $\mathcal{S}$  in given in (5.5), we define the operator  $\mathcal{M} : L^2(0, 1) \rightarrow \mathbb{R}$  as solution to the following Sylvester equation

$$\mathcal{M}\mathcal{S}w = \mathcal{C}w, \quad \forall w \in D(\mathcal{S}), \quad (5.9)$$

where  $\mathcal{C} : f \in H_0^1(0, L) \mapsto f'(0) \in \mathbb{R}$ . Since the strongly continuous semigroup generated by the operator  $\mathcal{S}$  is exponentially stable, the Sylvester equation (5.9) admits a unique solution, see [47, Lemma 22]. Moreover, since  $\mathcal{M}$  is a linear form, according to Riesz representation theorem [7, Theorem 4.11], the operator  $\mathcal{M}$  is uniquely defined as  $\mathcal{M}w = \int_0^L M(x)w(x)dx$ . In order to obtain an explicit solution, we write equation (5.9) in the explicit form

$$w'(0) = - \int_0^L M(x)[w'(x) + w'''(x)]dx \quad \forall w \in D(\mathcal{S}).$$

Using integration by parts we obtain

$$w'(0) = \int_0^L w(x)[M'(x) + M'''(x)]dx + M(0)w''(0) - M(L)w''(L) - M'(0)w'(0),$$

for all  $w \in D(\mathcal{S})$ . From the latter equation, we obtain the following boundary value problem

$$\begin{cases} M''' + M' = 0, \\ M(0) = M(L) = 0, \\ M'(0) = -1. \end{cases} \quad (5.10)$$

It can be verified that the function

$$M : x \in \mathbb{R} \mapsto \frac{-2 \sin(\frac{x}{2}) \sin(\frac{L-x}{2})}{\sin(\frac{L}{2})} \quad (5.11)$$

is a solution to (5.10). Computations are omitted for space reasons. Moreover, it is the unique solution to (5.10) and the operator  $\mathcal{M}$  defined above is the unique solution to the Sylvester equation (5.9). Then, the operator  $\mathcal{M} : L^2(0, L) \rightarrow \mathbb{R}$  can be expressed as  $\mathcal{M}\varphi = \int_0^L M(x)\varphi(x)dx$ .

With the operator  $\mathcal{M}$  so defined, consider the candidate Lyapunov functional  $\mathcal{V} : X \rightarrow \mathbb{R}$  defined as

$$\mathcal{V}(\eta, w) = V(w) + (\eta - \mathcal{M}w)^2, \quad (5.12)$$

where  $V$  is the Lyapunov functional given by Theorem 2.3. By construction, the Lyapunov functional  $\mathcal{V}$  is equivalent to the standard norm on the space  $X$ , and in particular, there exist positive constants  $\underline{\nu}, \bar{\nu}$  such that the following holds

$$\underline{\nu}\|(\eta, w)\|_X^2 \leq \mathcal{V}(\eta, w) \leq \bar{\nu}\|(\eta, w)\|_X^2, \quad \forall (\eta, w) \in X. \quad (5.13)$$

To show this fact, note that, following similar arguments used in the proof of Proposition 4 of [56], for any  $\rho \in ]0, 1[$  we have

$$\rho \left( \frac{1}{2}\eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right) \leq (\eta - \mathcal{M}w)^2 \leq 2(\eta^2 + \|M\|_{L^2}^2 \|w\|_{L^2}^2)$$

for all  $(\eta, w) \in X$ . Furthermore, according to Theorem 2.3, we know that  $V$  satisfies the inequality (2.6). Then we have

$$\rho \left( \frac{1}{2}\eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right) + \underline{\alpha}\|w\|_{L^2}^2 \leq \mathcal{V}(w) \leq 2(\eta^2 + \|M\|_{L^2}^2 \|w\|_{L^2}^2) + \bar{\alpha}\|w\|_{L^2}^2.$$

Therefore, by selecting  $\rho$  sufficiently small, inequality (5.13) holds for some  $\bar{\nu} > \underline{\nu} > 0$ . By recalling that the function  $V$  established in Theorem 2.3 is of the form  $V = E + W$ , where  $E$  and  $W$  are quadratic forms of the  $L^2$  norm of  $w$ , from the Lyapunov functional  $\mathcal{V}$  defined in (5.12), we can also deduce a scalar product, that we define as follows

$$\left\langle \begin{bmatrix} \eta_1 & w_1 \end{bmatrix}^\top, \begin{bmatrix} \eta_2 & w_2 \end{bmatrix}^\top \right\rangle_{\mathcal{V}} := (\eta_1 - \mathcal{M}w_1)(\eta_2 - \mathcal{M}w_2) + \langle w_1, w_2 \rangle_{L^2} + \langle \Pi w_1, \Pi w_2 \rangle_{L^2}, \quad (5.14)$$

with  $\Pi$  being the linear operator given by Theorem 2.3. It is equivalent to the usual scalar product in  $X$ .

Now, we are in position to prove that  $\mathcal{A}$  is  $m$ -dissipative according to [46]. For this, we need to show that  $\mathcal{A}$  is dissipative and maximal. We begin with showing the dissipative properties. To this end, we use the scalar product given in (5.14). By using the definition of  $\mathcal{A}$  given in (5.6), we obtain, for all  $\zeta \in D(\mathcal{A})$ ,

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &= (w'(0) + \mathcal{M}(w''' + w'))(\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} - \langle \Pi(w''' + w'), \Pi w \rangle_{L^2} \\ &= \left( w'(0) + \int_0^L M(x)[w'(x) + w'''(x)]dx \right) (\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} - \langle \Pi(w''' + w'), \Pi w \rangle_{L^2}. \end{aligned} \quad (5.15)$$

For the first term, it can be shown, after some integrations by parts, that

$$\int_0^L M(x)[w'(x) + w'''(x)]dx = -k\eta - w'(0) \quad (5.16)$$

for all  $\zeta \in D(\mathcal{A})$ . Then, for the second term, we recall the ISS properties of the function  $V$  stated in Theorem 2.3. In particular, applying the inequality (2.7) to the system (5.4), in which  $d$  is the distributed disturbance (thus having the role of  $d_1$ ) and  $k\eta$  is seen as a disturbance acting at the boundary condition (thus having the role of  $d_2$ ), we obtain

$$-2\langle w' + w''', w \rangle_{L^2} - 2\langle \Pi(w''' + w'), \Pi w \rangle_{L^2} \leq -\alpha\|w\|_{L^2}^2 + \sigma_2 k^2 \eta^2. \quad (5.17)$$

for all  $\zeta \in D(\mathcal{A})$ . Hence, combining inequalities (5.15) with (5.16) and (5.17), we obtain

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &\leq -k\eta(\eta - \mathcal{M}w) - \frac{\alpha}{2}\|w\|_{L^2}^2 + \frac{\sigma_2}{2}k^2\eta^2 \\ &\leq -k\left(1 - \left(\frac{\sigma_2}{2} + \frac{\|M\|_{L^2}}{4\alpha}\right)k\right)\eta^2 - \frac{\alpha}{4}\|w\|_{L^2}^2 \end{aligned}$$

for all  $\zeta \in D(\mathcal{A})$ , where the second inequality has been obtained by using Young's inequality. As a consequence, we can select

$$k_0^* = \left( \frac{\sigma_2}{2} + \frac{\|M\|_{L^2}}{4\alpha} \right)^{-1}.$$

This implies that for any  $k \in (0, k_0^*)$  there exists  $\varepsilon > 0$  such that we have

$$\langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} \leq -\varepsilon(|\eta|^2 + \|w\|_{L^2}^2) \quad (5.18)$$

for all  $\zeta \in D(\mathcal{A})$ , which shows that the operator  $\mathcal{A}$  is dissipative.

Now, we want to show that  $\mathcal{A}$  is a maximal operator. According to Lümer–Phillips theorem [46, Theorem 4.3], proving that  $\mathcal{A}$  is maximal reduces to show that there exists a positive  $\lambda_0$  such that for all  $\zeta \in X$ , there exists  $\tilde{\zeta} \in D(\mathcal{A})$  such that  $(\lambda_0 I_X - \mathcal{A})\tilde{\zeta} = \zeta$ . Let  $(\eta, w) \in X$ . We look for a  $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$  satisfying

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = k\tilde{\eta}, \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta, \end{cases} \quad (5.19)$$



namely

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0}(\eta + \tilde{w}'(0)), \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta. \end{cases}$$

Now, we consider the following boundary value problem

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0}(\eta + \tilde{w}'(0)), \end{cases}$$

and the smooth function  $\tilde{\phi}(x) = \frac{k\eta x^2(x-L)}{\lambda_0 L^2}$  satisfying the boundary conditions

$$\tilde{\phi}(0) = \tilde{\phi}(L) = \tilde{\phi}'(0) = 0, \quad \tilde{\phi}'(L) = \frac{k}{\lambda_0}\eta.$$

We set  $\tilde{\psi} = \tilde{w} - \tilde{\phi}$ . Then  $\tilde{\psi}$  satisfies the following boundary value problem

$$\begin{cases} \tilde{\psi}' + \tilde{\psi}''' + \lambda_0 \tilde{\psi} = \tilde{j}(x), & x \in [0, L], \\ \tilde{\psi}(0) = \tilde{\psi}(L) = 0, \\ \tilde{\psi}'(L) = \frac{k}{\lambda_0}\tilde{\psi}'(0), \end{cases} \quad (5.20)$$

where  $\tilde{j}(x) = w(x) - \tilde{\phi}'(x) - \tilde{\phi}'''(x) - \lambda_0 \tilde{\phi}$ . Now, we define the operator  $\widehat{\mathcal{S}}$  and its domain  $D(\widehat{\mathcal{S}}) \subset L^2(0, L)$  as

$$\widehat{\mathcal{S}}\psi = -\psi' - \psi''', \quad D(\widehat{\mathcal{S}}) := \left\{ \psi \in H^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(L) = \frac{k}{\lambda_0}\psi'(0) \right\}.$$

We define also its adjoint operator  $\widehat{\mathcal{S}}^*$  and its domain  $D(\widehat{\mathcal{S}}^*)$  as

$$\widehat{\mathcal{S}}^*\psi = \psi''' + \psi', \quad D(\widehat{\mathcal{S}}^*) := \left\{ \psi \in H^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(0) = \frac{k}{\lambda_0}\psi'(L) \right\}.$$

Note that  $\widehat{\mathcal{S}}$  and  $\widehat{\mathcal{S}}^*$  are dissipative. Indeed, by selecting  $\lambda_0 > k$ , we have

$$\begin{aligned} \int_0^L \psi \widehat{\mathcal{S}}\psi dx &= \left( \frac{k}{\lambda_0} - 1 \right) \psi'(0)^2 < 0, & \psi \in D(\widehat{\mathcal{S}}), \\ \int_0^L \psi \widehat{\mathcal{S}}^*\psi dx &= \left( \frac{k}{\lambda_0} - 1 \right) \psi'(L)^2 < 0, & \psi \in D(\widehat{\mathcal{S}}^*). \end{aligned}$$

Moreover,  $\widehat{\mathcal{S}}$  is closed and  $D(\widehat{\mathcal{S}})$  is dense in  $L^2(0, L)$ . Then, according to [46, Theorem 4.3 and Corollary 4.4]  $\widehat{\mathcal{S}}$  is  $m$ -dissipative operator. Finally, since  $\widehat{\mathcal{S}}$  is a  $m$ -dissipative operator then the system (5.20) admits a solution  $\tilde{\psi}$  in  $D(\widehat{\mathcal{S}})$ . As a consequence, there exist  $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$  solution of (5.19). This proves that  $\mathcal{A}$  is maximal and concludes the proof of Lemma 5.2.  $\square$

Finally, the next result deals with the exponential stability of equilibrium state  $(\eta_\infty, w_\infty)$  and with the related output regulation objective (5.2).

**THEOREM 5.3 (Stabilization and regulation).** *Let  $L \notin \mathcal{N}$  and consider system (5.4). For any  $k \in (0, k_0^*)$ , with  $k_0^*$  given by Lemma 5.2, there exist  $b_0, \nu_0 > 0$ , and for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  there exists  $(\eta_\infty, w_\infty) \in X$ , computed according to Lemma 5.1, such that any solution to system (5.4) with initial condition  $(\eta_0, w_0) \in X$  satisfies*

$$\|(\eta(t), w(t, \cdot)) - (\eta_\infty, w_\infty)\|_X \leq b_0 e^{-\nu_0 t} \|(\eta_0, w_0) - (\eta_\infty, w_\infty)\|_X. \quad (5.21)$$

for all  $t \geq 0$ . Moreover, for any strong solution to (5.4), and in particular, for any  $(\eta_0, w_0) \in D(\mathcal{A})$ , the output  $y$  is asymptotically regulated at the reference  $r$ , namely (5.2) is satisfied.

**Proof:** The first part of the proof is proved for any initial condition  $(\eta_0, w_0) \in D(\mathcal{A})$ . The result follows for all initial conditions in  $X$  by a standard density argument (see e.g. [34, Lemma 1]). Consider the equilibrium  $(\eta_\infty, w_\infty)$ , recall the change of coordinates defined in (5.7) and consider the error system (5.8). We show now that its origin is exponentially stable. To this end, consider the Lyapunov functional  $\mathcal{V}$  defined in (5.12). According to the proof of dissipativity of  $\mathcal{A}$  of Lemma (5.2), for any  $k \in (0, k^*)$  the time derivative of  $\mathcal{V}$  along the strong solution to (5.8) satisfies (5.18). As a consequence, from (5.13) and Grönwall's lemma, there exist positive constants  $b_0, \nu_0$  such that, for all  $(\eta_0, w_0) \in D(\mathcal{A})$  and for all  $t \geq 0$

$$\|(\tilde{\eta}(t), \tilde{w}(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X. \quad (5.22)$$

By using the density of  $D(\mathcal{A})$  in  $X$ , and the change of coordinates (5.7), we conclude that (5.21) holds.

Now, we need to show that the regulation objective (5.2) is achieved for strong solutions. For this, note that if  $(\eta_0, w_0) \in D(\mathcal{A})$ , then  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$ . Then  $(\tilde{\eta}, \tilde{w}) \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$ . Now, let us introduce the new variables  $v, \xi$  defined as follows

$$(\tilde{w}, \tilde{\eta}) \mapsto (v, \xi) := (\tilde{w}_t, \tilde{\eta}). \quad (5.23)$$

The dynamics of  $(v, \xi)$  is given as

$$\begin{cases} v_t + v_x + v_{xxx} = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ v(t, 0) = v(t, L) = 0, & t \in \mathbb{R}_+ \\ v_x(t, L) = k\xi(t), & t \in \mathbb{R}_+ \\ \dot{\xi}(t) = v_x(t, 0), & t \in \mathbb{R}_+ \\ v(0, x) = v_0(x), \xi(0) = \xi_0, & x \in [0, L]. \end{cases} \quad (5.24)$$

with

$$v_0(x) = -\tilde{w}'_0(x) - \tilde{w}'''_0(x), \quad x \in [0, L], \quad \xi_0 = \tilde{w}'_0(0). \quad (5.25)$$

Since  $(v(0, \cdot), \xi(0)) \in X$ , then, according to the Lemma 5.2 and the first statement of Theorem 5.3, we have  $(v, \xi) \in C^0(\mathbb{R}_+; X)$  and

$$\|(\xi(t), v(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(\xi(0), v(0, \cdot))\|_X, \quad \forall (v_0, \xi_0) \in X.$$

By definition of  $v$  and  $\xi$  and using (5.25), one can see that, once one considers  $(v_0, \xi_0) \in X$ , then this implies that  $(\eta_0, w_0) \in D(\mathcal{A})$ . Then, using the definition of the change of coordinates (5.23) and (5.25) we obtain

$$\|\tilde{w}_t(t, \cdot)\|_{L^2} \leq \|(\dot{\tilde{\eta}}(t), \tilde{w}_t(t, \cdot))\|_X \leq b_0 e^{-\nu_0 t} \|(-\tilde{w}'_0(x) - \tilde{w}'''_0(x), \tilde{w}'_0(0))\|_X, \quad \forall w_0 \in D(\mathcal{A}). \quad (5.26)$$

Now, by multiplying the first equation of (5.8) by  $\tilde{w}$  and integrating by parts, we get after some computations

$$k^2 \tilde{\eta}(t)^2 - \tilde{w}_x(t, 0)^2 = \int_0^L \tilde{w}(t, x) \tilde{w}_t(t, x) dx.$$

Using Cauchy-Schwarz's inequality, from (5.22) and (5.26) we finally obtain

$$|\tilde{w}_x(t, 0)|^2 \leq \|\tilde{w}(t, \cdot)\|_{L^2} \|\tilde{w}_t(t, \cdot)\|_{L^2} + k^2 |\tilde{\eta}(t)|^2 \xrightarrow{t \rightarrow \infty} 0, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

From the previous inequality we obtain  $\lim_{t \rightarrow \infty} |\tilde{w}_x(t, 0)| = \lim_{t \rightarrow \infty} |w_x(t, 0) - r| = 0$  for all  $(\eta_0, w_0) \in D(\mathcal{A})$ , and therefore (5.2), concluding the proof.  $\square$

**5.2. Regulation of nonlinear KdV equation by means of the forwarding method.** In this section, we consider the regulation problem for a nonlinear KdV equation (1.1). In particular, we consider the system

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = u(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (5.27)$$

where  $d \in L^2(0, L)$  is a constant perturbation,  $u \in \mathbb{R}$  is the control input and  $y(t) \in \mathbb{R}$  is the output to be regulated to a constant reference  $r$  as in (5.2). Following the design proposed in Section 5.1 for the linear model (1.2), we consider the same output-feedback integral control (5.3) and we compactly write the closed-loop system (5.27), (5.3) as

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = d(x), & (t, x) \in \mathbb{R}_+ \times [0, L], \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = w_x(t, 0) - r, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (5.28)$$

In the following, we will show that for sufficiently small perturbations  $d$  and references  $r$  the closed-loop system (5.28) is well posed and it admits a unique equilibrium which is locally exponentially stable. Furthermore, for solutions which are sufficiently regular, the regulation objective (5.2) is satisfied. We start by showing the existence and uniqueness of an equilibrium.

**LEMMA 5.4.** *There exist  $\bar{d} > 0$  and  $\bar{r} > 0$  such that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq \bar{d}$  and  $|r| \leq \bar{r}$ , there exists a unique equilibrium state  $(\eta_\infty, w_\infty) \in X$  to system (5.28). Furthermore there exists  $\bar{w} > 0$  such that for any  $\bar{w}_0 \in (0, \bar{w}]$ , there exists  $d_0 > 0$  and  $r_0 > 0$  so that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq d_0$  and  $|r| \leq r_0$  then  $\|w_\infty\|_{H^3} \leq \bar{w}_0$ .*

**Proof:** Consider the following boundary value problem

$$\begin{cases} w'_\infty(x) + w'''_\infty(x) + w_\infty(x)w'_\infty(x) = d(x), & x \in [0, L], \\ w_\infty(0) = w_\infty(L) = 0, \\ w'_\infty(0) = r, \end{cases} \quad (5.29)$$

which represents the nonzero equilibrium state of (5.4), with  $\eta_\infty = \frac{w'_\infty(L)}{k}$ . We prove that there exists a solution to system (5.29) by following a fixed-point strategy. We set

$$H_r^3(0, L) := \left\{ w \in H^3(0, L) : w(0) = w(L) = 0, w'(0) = r \right\},$$

and we introduce the operator  $\mathcal{T}_0 : H_r^3(0, L) \rightarrow H_r^3(0, L)$  defined by  $\mathcal{T}_0(w) = \varphi$  where  $\varphi$  is the solution to

$$\begin{cases} \varphi'(x) + \varphi'''(x) = d(x) - w_\infty(x)w'_\infty(x), & x \in [0, L], \\ \varphi(0) = \varphi(L) = 0, \\ \varphi'(0) = r, \end{cases} \quad (5.30)$$

Note that the function  $\|\cdot\|_{H_r^3} : w \in H^3(0, L) \mapsto \|w' + w'''\|_{L^2} \in \mathbb{R}_+$  is a semi-norm on the space  $H^3(0, L)$ . Furthermore,  $H_r^3(0, L) \subset H_0^1(0, L)$ . Then, according to the Poincaré's inequality, the semi-norm  $\|\cdot\|_{H_r^3}$  is a norm on the space  $H_r^3(0, L)$  which is equivalent to the standard norm induced by  $H^3(0, L)$ . In other words, there exists a positive constant  $\kappa$  such that

$$\|w\|_{H_r^3} \leq \|w\|_{H^3(0, L)} \leq \kappa \|w\|_{H_r^3}, \quad \forall w \in H_r^3(0, L). \quad (5.31)$$

Now, we have,

$$\begin{aligned} \|\mathcal{T}_0(w)\|_{H_r^3} &= \|d - ww'\|_{L^2} \\ &\leq \|d\|_{L^2} + \|ww'\|_{L^2} \\ &\leq \|d\|_{L^2} + \|w\|_{L^\infty} \|w'\|_{L^2}, \end{aligned}$$

for all  $w \in H_r^3(0, L)$ . Denoting with the constant  $\ell$  the norm of the embedding  $H^3(0, L)$  in  $L^\infty(0, L)$ , according to the Rellich-Kondrachov Theorem (see [7, Theorem 9.16]), we have

$$\begin{aligned}\|\mathcal{T}_0(w)\|_{H_r^3} &\leq \|d\|_{L^2} + \ell\|w\|_{H^3(0,L)}\|w'\|_{L^2} \\ &\leq \|d\|_{L^2} + \kappa\ell\|w\|_{H_r^3}^2 \\ &\leq \bar{d} + \kappa\ell\|w\|_{H_r^3}^2,\end{aligned}$$

for all  $w \in H_r^3(0, L)$  and all  $d$  satisfying  $\|d\|_{L^2} \leq \bar{d}$ . Moreover, we have for all  $w_1, w_2 \in H_r^3(0, L)$

$$\begin{aligned}\|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{H_r^3} &= \|w_1w'_1 - w_2w'_2\|_{L^2} \\ &\leq \|(w_1 - w_2)w'_1\|_{L^2} + \|w_2(w'_1 - w'_2)\|_{L^2} \\ &\leq \kappa\ell\|w_1 - w_2\|_{H_r^3}\|w_1\|_{H_r^3} + \kappa\ell\|w_1 - w_2\|_{H_r^3}\|w_2\|_{H_r^3} \\ &\leq \kappa\ell(\|w_2\|_{H_r^3} + \|w_2\|_{H_r^3})\|w_1 - w_2\|_{H_r^3}.\end{aligned}$$

We consider now the operator  $\mathcal{T}_0$  defined as in (5.30), restricted on the closed ball

$$B_{\bar{w}} := \left\{ w \in H_r^3(0, L) : \|w\|_{H_r^3} \leq \bar{w} \right\}$$

with  $\bar{w}$  to be chosen later. Then, collecting all the previous inequality we have

$$\begin{aligned}\|\mathcal{T}_0(w)\|_{H_r^3} &\leq \bar{d} + \kappa\ell\bar{w}^2, \\ \|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{H_r^3} &\leq 2\kappa\ell\bar{w}\|w_1 - w_2\|_{H_r^3},\end{aligned}$$

for all  $w, w_1, w_2 \in B_{\bar{w}}$ . Finally, we select  $\bar{d}$  and  $\bar{w}$  such that the following conditions hold

$$\bar{d} < \frac{1}{4\kappa\ell} \quad \text{and} \quad \frac{1 - \sqrt{1 - 4\bar{d}\kappa\ell}}{2\kappa\ell} \leq \bar{w} < \frac{1}{2\kappa\ell}. \quad (5.32)$$

With such a choice, we obtain  $\|\mathcal{T}_0(w)\|_{H_r^3} \leq \bar{w}$  for all  $w \in B_{\bar{w}}$  and  $\|\mathcal{T}_0(w_1) - \mathcal{T}_0(w_2)\|_{H_r^3} < \|w_1 - w_2\|_{H_r^3}$ , for all  $w_1, w_2 \in B_{\bar{w}}$ . This shows that the operator  $\mathcal{T}_0$  is an operator of contraction. Applying the Banach fixed point theorem [7, Theorem 5.7] we deduce that the operator  $\mathcal{T}_0$  admits a unique fixed point, and therefore that there exists a unique solution  $w_\infty \in B_{\bar{w}}$  to (5.29). Now, given  $\bar{w}$ , we deduce the value of  $\bar{r}$ . Indeed, since  $w_\infty \in H^3(0, L)$  then we have  $w'_\infty \in H^2(0, L)$ . Then, according to the embedding of  $H^2(0, L)$  in  $C^1([0, L])$ , we have  $w'_\infty \in C^1([0, L])$ . Therefore, according to [61, Lemma 1] we have

$$(w'_\infty(0))^2 \leq \frac{2}{L}\|w'_\infty\|_{L^2}^2 + L\|w''_\infty\|_{L^2}^2 \leq \left(\frac{2}{L} + L\right)\|w_\infty\|_{H^3(0,L)}^2. \quad (5.33)$$

Since  $w_\infty \in B_{\bar{w}}$ , then according to (5.31) and (5.33), and to the definition of  $H_r^3(0, L)$ , we obtain

$$r^2 = (w'_\infty(0))^2 \leq \kappa\left(\frac{2}{L} + L\right)\bar{w}^2. \quad (5.34)$$

Finally, we can choose  $\bar{r} = \bar{w}\sqrt{\kappa\left(\frac{2}{L} + L\right)}$ . Therefore, according to (5.32) and (5.34), we deduce that for any  $\bar{w}_0 \in (0, \bar{w}]$ , there exists  $d_0 > 0$  and  $\bar{r}_0 > 0$  so that, for any  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying  $\|d\|_{L^2} \leq \bar{d}_0$  and  $|r| \leq \bar{r}_0$  then  $\|w_\infty\|_{H^3} \leq \bar{w}_0$ . This concludes the proof of Lemma 5.4.  $\square$

Now, given  $(d, r) \in L^2(0, L) \times \mathbb{R}$  satisfying the assumptions of Lemma 5.4, let  $(\eta_\infty, w_\infty)$  be the corresponding equilibrium to system (5.28) and consider the following change of coordinates

$$(w, \eta) \mapsto (\tilde{w}, \tilde{\eta}) := (w - w_\infty, \eta - \eta_\infty).$$

The  $(\tilde{w}, \tilde{\eta})$ -dynamics is given by

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \tilde{w}\tilde{w}_x = -w'_\infty \tilde{w} - w_\infty \tilde{w}_x, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \tilde{w}_x(t, L) = k\tilde{\eta}(t), & t \in \mathbb{R}_+, \\ \dot{\tilde{\eta}}(t) = \tilde{w}_x(t, 0), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), \tilde{\eta}(0) = \tilde{\eta}_0, & x \in [0, L], \end{cases} \quad (5.35)$$

where  $\tilde{w}_0(x) = w_0(x) - w_\infty(x) \in H^3(0, L)$  and  $\tilde{\eta}_0 = \eta_0 - \eta_\infty \in \mathbb{R}$ . In the new coordinates, the regulation objective (5.2) for system (5.35) reads

$$\lim_{t \rightarrow \infty} e(t) = \lim_{t \rightarrow \infty} \tilde{w}_x(t, 0) = 0 \quad (5.36)$$

Note that showing the well-posedness of system (5.35) is equivalent to prove the well-posedness of system (5.28) in the original coordinates  $(w, \eta)$ . As a consequence, in the rest of the section, we will focus on the system (5.35) in the new coordinates  $(\tilde{w}, \tilde{\eta})$ .

**LEMMA 5.5.** *For any  $\bar{w}_\infty$ , there exists  $k_1^* > 0$  such that, for any  $k \in (0, k_1^*]$ , for any  $w_\infty \in H^3(0, L)$  satisfying  $\|w_\infty\|_{H^3} \leq \bar{w}_\infty$ , and any initial condition  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$ , there exists  $\tau > 0$  such that the Cauchy problem (5.35) is well-posed in the space  $C^1(0, \tau) \times (C([0, \tau]; H^3(0, L)) \cap L^2([0, \tau]; H^4(0, L)))$ .*

**Proof:** First, by writing the explicit solution of  $\tilde{\eta}$  along solutions, that is  $\tilde{\eta}(t) = \tilde{\eta}_0 + \int_0^t \tilde{w}_x(s, 0)ds$ , we rewrite system (5.35) as follows

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} + \tilde{w}\tilde{w}_x + (\tilde{w}w_\infty)_x = 0, & (t, x) \in \mathbb{R}_+ \times [0, L], \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \tilde{w}_x(t, L) = k \left( \tilde{\eta}_0 + \int_0^t \tilde{w}_x(s, 0)ds \right), & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), & x \in [0, L]. \end{cases} \quad (5.37)$$

Now, given  $(s, \tau) \in \mathbb{N} \times \mathbb{R}$ , we introduce the space  $S^s(\tau) := C([0, \tau]; H^s(0, L)) \cap L^2([0, \tau]; H^{s+1}(0, L))$  equipped with the norm defined as

$$\|w\|_{S^s(\tau)}^2 := \|w\|_{C([0, \tau]; H^s(0, L))}^2 + \|w\|_{L^2([0, \tau]; H^{s+1}(0, L))}^2 + \|w_x\|_{C([0, \tau]; L^2(0, L))}^2.$$

We consider the operator  $\mathcal{T}_1 : S^3(\tau) \rightarrow S^3(\tau)$  defined by  $\mathcal{T}_1(\tilde{w}) = \varphi$  where  $\varphi$  is the solution of

$$\begin{cases} \varphi_t + \varphi_x + \varphi_{xxx} + (w_\infty \varphi)_x = -\tilde{w}\tilde{w}_x, & (t, x) \in [0, \tau] \times [0, L], \\ \varphi(t, 0) = \varphi(t, L) = 0, & t \in [0, \tau], \\ \varphi_x(t, L) = k \left( \tilde{\eta}_0 + \int_0^t \tilde{w}_x(s, 0)ds \right), & t \in [0, \tau], \\ \varphi(0, x) = \tilde{w}_0(x), & x \in [0, L] \end{cases} \quad (5.38)$$

with  $\tau > 0$  and  $k > 0$  to be chosen later. With the operator  $\mathcal{T}_1$  so defined, we deduce that if  $\tilde{w}$  is a fixed point of  $\mathcal{T}_1$  then  $\tilde{w} \in S^3(\tau)$  is a solution of (5.37). To this end we will apply the Banach fixed-point Theorem. According to [5, Proposition 5.1], for any  $\bar{w}_\infty$  there exists  $C > 0$  that such that

$$\begin{aligned} \|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 &\leq C \left( \|\tilde{w}_0\|_{H^3(0, L)}^2 + k^2 \left( \tau |\tilde{\eta}_0|^2 + \int_0^\tau \left| \int_0^t \tilde{w}_x(s, 0)ds \right|^2 dt + \|w_x(\cdot, 0)\|_{L^2(0, \tau)}^2 \right) \right. \\ &\quad \left. + \|\tilde{w}\tilde{w}_x\|_{H^1(0, \tau; H^1(0, L))}^2 \right) \end{aligned}$$

for all  $\tilde{w} \in S^3(\tau)$  and any  $\|w_\infty\|_{H^3} \leq \bar{w}_\infty$ . On the other hand, we have

$$\int_0^\tau \left| \int_0^t \tilde{w}_x(s, 0)ds \right|^2 dt \leq \int_0^\tau \int_0^t |\tilde{w}_x(s, 0)|^2 ds \leq \tau \|\tilde{w}_x(\cdot, 0)\|_{L^2(0, \tau)}^2.$$

Since,  $\tilde{w} \in S^3(\tau)$ , then for all  $t \in [0, \tau]$ ,  $\tilde{w}(t, \cdot) \in H^3(0, L)$ , which imply  $\tilde{w}_x(t, \cdot) \in H^2(0, L)$  for all  $t \in [0, \tau]$ . Then, according to the embedding of  $H^2(0, L)$  in  $C^1([0, L])$ , we have  $\tilde{w}_x(t, \cdot) \in C^1([0, L])$  for all  $t \in [0, \tau]$ . Therefore, according to [61, Lemma 1], we obtain

$$(\tilde{w}_x(t, 0))^2 \leq \left(\frac{2}{L} + L\right) \|\tilde{w}(t, \cdot)\|_{H^3(0, L)}^2.$$

for all  $t \in [0, \tau]$ , which implies

$$\|\tilde{w}_x(\cdot, 0)\|_{L^2(0, \tau)}^2 \leq \left(\frac{2}{L} + L\right) \|\tilde{w}\|_{L^2(0, \tau; H^3(0, L))}^2 \leq \left(\frac{2}{L} + L\right) \|\tilde{w}\|_{S^3(\tau)}^2. \quad (5.39)$$

Then, we have

$$\int_0^\tau \left| \int_0^t \tilde{w}_x(s, 0) ds \right|^2 dt \leq \tau \left(\frac{2}{L} + L\right) \|\tilde{w}\|_{L^2(0, \tau; H^3(0, L))}^2 \leq \tau \left(\frac{2}{L} + L\right) \|\tilde{w}\|_{S^3(\tau)}^2. \quad (5.40)$$

Also, since  $\tilde{w} \in S^3(\tau)$ , then according to [5, Lemma 3.1], we deduce the existence of  $C > 0$  such that

$$\|\tilde{w}\tilde{w}_x\|_{H^1(0, \tau; H^1(0, L))}^2 = \|\tilde{w}\tilde{w}_x\|_{L^2(0, \tau; H^1(0, L))}^2 + \|(\tilde{w}\tilde{w}_x)_t\|_{L^2(0, \tau; H^1(0, L))}^2 \leq C(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}\|_{S^3(\tau)}^4. \quad (5.41)$$

As a consequence, from (5.39), (5.40) and (5.41), there exists a positive constant  $C > 0$  such that

$$\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 \leq C \left( \|\tilde{w}_0\|_{H^3(0, L)}^2 + k^2 \tau |\tilde{\eta}_0|^2 + \left( (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}\|_{S^3(\tau)}^2 + \left(\frac{2}{L} + L\right) (k^2 \tau + k^2) \right) \|\tilde{w}\|_{S^3(\tau)}^2 \right). \quad (5.42)$$

Moreover, from (5.39), (5.40) and (5.41), we obtain

$$\begin{aligned} & \|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)}^2 \leq \\ & \leq C \left( k^2 \int_0^\tau \left| \int_0^t (\tilde{w}_x^1(s, 0) - \tilde{w}_x^2(s, 0)) ds \right|^2 dt + \|w_x^1(\cdot, 0) - w_x^2(\cdot, 0)\|_{L^2(0, \tau)}^2 \right. \\ & \quad \left. + \|\tilde{w}^2(\tilde{w}_x^2 - \tilde{w}_x^1) + \tilde{w}_x^1(\tilde{w}^2 - \tilde{w}^1)\|_{H^1(0, \tau; H^1(0, L))}^2 \right) \\ & \leq C \left( \left(\frac{2}{L} + L\right) (k^2 \tau + k^2) + (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}^1\|_{S^3(\tau)}^2 + (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \|\tilde{w}^2\|_{S^3(\tau)}^2 \right) \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}^2 \end{aligned}$$

for all  $\tilde{w}^1, \tilde{w}^2 \in S^3(\tau)$ . We consider  $\mathcal{T}_1$  restricted to the closed ball  $B_\rho = \{\tilde{w} \in S^3(\tau) : \|\tilde{w}\|_{S^3(\tau)} \leq \rho\} \subset S^3(\tau)$  with  $\rho$  to be chosen later. Then

$$\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)}^2 \leq C \left( \|\tilde{w}_0\|_{H^3(0, L)}^2 + k^2 \tau |\tilde{\eta}_0|^2 + \rho^2 \left(\frac{2}{L} + L\right) k^2 + \left( (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 + \left(\frac{2}{L} + L\right) k^2 \tau \right) \rho^2 \right)$$

and

$$\|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)}^2 \leq C \left( k^2 \left(\frac{2}{L} + L\right) + \left(\frac{2}{L} + L\right) k^2 \tau + 2(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 \right) \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}^2.$$

Finally, we select the constant  $\rho, k_1$  and  $\tau$  so that to obtain a contractive operator. For instance, we can select  $\rho = \sqrt{3C} \|\tilde{w}_0\|_{H^3(0, L)}$  and

$$k_1^* = \sqrt{\frac{1}{6C} \left( \frac{L}{2 + L^2} \right)}$$

and  $\tau > 0$  such that the following inequalities are satisfied

$$\begin{aligned} & \tau (k_1^*)^2 |\tilde{\eta}_0|^2 < \|\tilde{w}_0\|_{H^3(0, L)}^2, \\ & \left( (\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 + \left(\frac{2}{L} + L\right) (k_1^*)^2 \tau \right) \rho^2 < \frac{1}{2} \|\tilde{w}_0\|_{H^3(0, L)}^2, \\ & \left(\frac{2}{L} + L\right) (k_1^*)^2 \tau + 2(\tau^{\frac{1}{2}} + \tau^{\frac{1}{3}})^2 \rho^2 < \frac{1}{2}. \end{aligned}$$

It follows that, for any  $k \in (0, k_1^*]$ ,  $\|\mathcal{T}_1(\tilde{w})\|_{S^3(\tau)} \leq \rho$  for any  $\tilde{w} \in B_\rho$  and  $\|\mathcal{T}_1(\tilde{w}^1) - \mathcal{T}_1(\tilde{w}^2)\|_{S^3(\tau)} < \|\tilde{w}^1 - \tilde{w}^2\|_{S^3(\tau)}$  for any  $\tilde{w}^1, \tilde{w}^2 \in B_\rho$ . Then,  $\mathcal{T}_1$  is a contraction operator from  $B_\rho$  to  $B_\rho$ . According to the Banach fixed-point theorem,  $\mathcal{T}_1$  admits a unique fixed point. Its unique fixed point is the desired solution of (5.37) for  $0 \leq t \leq \tau$ . This shows that  $\tilde{w}$  in (5.38) has a unique solution in  $S^3(\tau)$ . Since  $\tilde{w}_x(t, 0)$  is continuous on  $[0, \tau]$  then  $\tilde{\eta}$  is in  $C^1(\tau)$  by definition of solution of an ODE. This concludes the proof of Lemma (5.5).  $\square$

Note that we have established the existence of unique classical solution of (5.35) locally in time. However, the Lyapunov functional introduced in the Section 5.1 needed to establish Lemma 5.2 and Theorem 5.3 can be used to deduce the existence of unique solution global in time. Indeed, since the time derivative of the Lyapunov functional will be proved to be non-increasing, this shows that the solution cannot explode for large time, proving thus that the solution exists for any positive time, as soon as the initial conditions are small enough. The next result deals with the local exponential stability of the origin of system (5.35).

**THEOREM 5.6 (Local Exponential Stability).** *There exist positive real number  $k_2^*, \bar{w}_\infty$ , such that, for any  $k \in (0, k_2^*)$  there exist positive real numbers  $\Delta, \nu_1, b_1$ , such that for any solution to system (5.35) with  $w_\infty$  satisfying  $\|w_\infty\|_{H^3} \leq \bar{w}_\infty$  and initial conditions  $(\tilde{\eta}_0, \tilde{w}_0) \in D(\mathcal{A})$  satisfying  $|\tilde{\eta}_0| + \|\tilde{w}_0\|_{L^2} \leq 2\Delta$ , the following inequality holds  $\|(\tilde{\eta}(t), \tilde{w}(t))\|_X \leq b_1 e^{-\nu_1 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X$  for all  $t \geq 0$ . Moreover the regulation objective defined in (5.36) is satisfied.*

**Proof:** The main idea of this proof is to extend the analysis developed for the linear KdV model in Section 5.1. In particular, following the main steps of the proof of Lemma 5.2, we aim at building a Lyapunov functional for the overall closed-loop system (5.28) by relying on Corollary 2.4. Indeed, setting  $a = -w'_\infty$ ,  $b = -w_\infty$  and  $d_2 = k\eta$ , system (5.35) is in the form (2.8). As a consequence, there exist  $\delta > 0$  and a Lyapunov functional  $V$  such that, for any  $\|w'_\infty\|_\infty \leq \bar{a}$  and  $\|w_\infty\|_{W^{1,\infty}} \leq \bar{b}$ , the derivative of  $V$  along the trajectories of system (5.35) satisfies

$$\dot{V}(\tilde{w}) \leq -\alpha \|\tilde{w}\|_{L^2}^2 + \sigma_1 k^2 \tilde{\eta}^2 \quad \forall (\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A}), \quad (5.43)$$

with  $D_\delta(\mathcal{A}) := \{(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A}) : \|(\tilde{\eta}, \tilde{w})\|_X \leq \delta\}$ . Now, we consider the Lyapunov functional  $\mathcal{V}$  defined in (5.12). We want to show the local exponential stability of the origin of the system (5.35) with the functional  $\mathcal{V}$ . First, note that  $\mathcal{V}$  is uniformly bounded by the norm in the space  $X$  of  $(\tilde{\eta}, \tilde{w})$ , similar to inequality (5.13). Then, using (5.43), its derivative along solutions to (5.28) is given by, for any  $(\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A})$ .

$$\dot{\mathcal{V}}(\tilde{\eta}, \tilde{w}) \leq -\alpha \|\tilde{w}\|_{L^2}^2 + \sigma_1 k^2 \tilde{\eta}^2 + 2F(\tilde{\eta}, \tilde{w}) \quad (5.44)$$

with  $F(\tilde{\eta}, \tilde{w}) := (\dot{\tilde{\eta}} - \mathcal{M}\tilde{w})(\dot{\tilde{\eta}} - \mathcal{M}\tilde{w}_t)$ . After some integrations by parts, and recalling the property of  $\mathcal{M}$  in (5.16), we obtain

$$\dot{\tilde{\eta}} - \mathcal{M}\tilde{w}_t = -k\tilde{\eta}(t) + \int_0^L M(x)\tilde{w}(x)w'_\infty(x)dx - \frac{1}{2} \int_0^L M'(x)\tilde{w}(x)^2 dx - \int_0^L (M(x)w_\infty(x))' \tilde{w}(x)dx,$$

from which we obtain

$$F(\tilde{\eta}, w) = -k\tilde{\eta}^2 + \tilde{\eta}k\mathcal{M}\tilde{w} + \tilde{\eta}\Phi(\tilde{w}) - \mathcal{M}\tilde{w}\Phi(\tilde{w})$$

with

$$\Phi(\tilde{w}) = \int_0^L M(x)\tilde{w}(x)w'_\infty(x)dx - \frac{1}{2} \int_0^L M'(x)\tilde{w}(x)^2 dx - \int_0^L (M(x)w_\infty(x))' \tilde{w}(x)dx$$

According to (5.11),  $M \in C^\infty([0, L])$ . Therefore  $M'$  is bounded on  $[0, L]$ . Then, using first Cauchy-Schwarz's inequality and then Young's inequality, we bound the terms in  $F$  as follows:

$$\begin{aligned} |k\tilde{\eta}\mathcal{M}\tilde{w}| &\leq \frac{2k^2\|M\|_{L^2}^2}{\alpha} \tilde{\eta}^2 + \frac{\alpha}{8} \|\tilde{w}\|_{L^2}^2, \\ |\tilde{\eta}\Phi(\tilde{w})| &\leq \frac{k}{2} \tilde{\eta}^2 + \frac{1}{2k} \left( 4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \|\tilde{w}\|_{L^2}^2, \\ |\mathcal{M}\tilde{w}\Phi(\tilde{w})| &\leq \frac{\alpha}{8} \|\tilde{w}\|_{L^2}^2 + \frac{2\|M\|_{L^2}^2}{\alpha} \left( 4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \|\tilde{w}\|_{L^2}^2. \end{aligned}$$

As a consequence, combining the previous bounds we further obtain

$$\begin{aligned}
F(\tilde{\eta}, \tilde{w}) &\leq \left( \frac{\alpha}{4} + \left( \frac{2\|M\|_{L^2}^2}{\alpha} + \frac{1}{2k} \right) \left( 4\|Mw'_\infty\|_{L^2}^2 + 4\|(Mw_\infty)'\|_{L^2}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \right) \|\tilde{w}\|_{L^2}^2 \\
&\quad - \frac{k}{2} \left( 1 - 4k \frac{\|M\|_{L^2}^2}{\alpha} \right) \tilde{\eta}^2 \\
&\leq \left( \frac{\alpha}{4} + \left( \frac{2\|M\|_{L^2}^2}{\alpha} + \frac{1}{2k} \right) \left( 8\|M\|_{W^{1,\infty}}^2 \|w_\infty\|_{H^3}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \right) \|\tilde{w}\|_{L^2}^2 \\
&\quad - \frac{k}{2} \left( 1 - 4k \frac{\|M\|_{L^2}^2}{\alpha} \right) \tilde{\eta}^2
\end{aligned}$$

where, in the second inequality, we have used  $\|Mw'_\infty\|_{L^2}^2 + \|(Mw_\infty)'\|_{L^2}^2 \leq 2\|M\|_{W^{1,\infty}}^2 \|w_\infty\|_{H^3}^2$ . Using the previous inequality together with (5.44) yields

$$\begin{aligned}
\dot{\mathcal{V}}(\tilde{\eta}, \tilde{w}) &\leq \left( -\frac{\alpha}{2} + \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right) \left( 8\|M\|_{W^{1,\infty}}^2 \|w_\infty\|_{H^3}^2 + \|M'\|_\infty^2 \|\tilde{w}\|_{L^2}^2 \right) \right) \|\tilde{w}\|_{L^2}^2 \\
&\quad + \left( \sigma_1 k^2 - k \left( 1 - 4k \frac{\|M\|_{L^2}^2}{\alpha} \right) \right) \tilde{\eta}^2.
\end{aligned} \tag{5.45}$$

for all  $(\tilde{\eta}, \tilde{w}) \in D_\delta(\mathcal{A})$ . As a consequence, we can finally fix all the parameters. In particular, we select

$$k_2^* = \min \left\{ k_0^*, k_1^*, \left( \frac{\alpha}{\alpha\sigma_1 + 4\|M\|_{L^2}^2} \right) \right\}$$

with  $k_0^*$  given by Lemma 5.2, and  $k_1^*$  given by Lemma 5.5. Moreover, given any  $k \in (0, k_2^*)$ , select

$$\bar{w}_\infty = \min \left\{ \bar{a}, \bar{b}, \sqrt{\frac{\alpha}{64\|M\|_{W^{1,\infty}}^2 \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right)}} \right\},$$

with  $\bar{a}, \bar{b}$  given by Corollary 2.4. Moreover, let us define  $\bar{\delta} \in (0, \delta)$  satisfying

$$\bar{\delta}^2 \leq \frac{\alpha}{8\|M'\|_\infty^2} \left( \frac{4\|M\|_{L^2}^2}{\alpha} + \frac{1}{k} \right)^{-1}.$$

Using all these bounds we can finally conclude the existence of a positive constant  $\varepsilon$  such that

$$\dot{\mathcal{V}} \leq -\varepsilon(\|\tilde{w}\|_{L^2}^2 + \tilde{\eta}^2) \quad \forall (\tilde{\eta}, \tilde{w}) \in D_{\bar{\delta}}(\mathcal{A}) \tag{5.46}$$

Finally, standard Lyapunov arguments briefly recalled here allows to conclude the result of the proof. In particular, consider a  $c > 0$  small enough such that  $\Omega_c := \{(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A}) : \mathcal{V}(\tilde{\eta}, \tilde{w}) \leq c\} \subset D_{\bar{\delta}}(\mathcal{A})$ . Now consider any solution to (5.28) starting inside  $\Omega_c$ . By Lemma 5.4 there exists  $\tau > 0$  such that such a solution exists on  $[0, \tau]$ . Let  $T \geq \tau$  be its maximal interval of time of existence. In view of (5.46), the derivative of  $\mathcal{V}$  is always negative, showing that such the solution cannot escape the level set  $\Omega_c$ . Hence, its maximal interval of existence is  $[0, \infty)$ . Moreover, we can conclude the existence of a positive constant  $\Delta \in (0, \bar{\delta})$  such that the set  $D_\Delta(\mathcal{A})$  is included in the domain of attraction of the origin of system (5.35). Combining the Fréchet derivative (2.3) and the Grönwall's lemma with (5.46) one can show the first part of the statement, that is  $\|(\tilde{\eta}(t), \tilde{w}(t))\|_X \leq b_1 e^{-\nu_1 t} \|(\tilde{\eta}_0, \tilde{w}_0)\|_X$ , for all  $t \geq 0$  and for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$ .

Finally, to prove the second part of the statement, we can use the same argument as in the proof of [53, Proposition 3.9] to deduce that there exists a continuous nonnegative function  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  and positive constants  $C, \mu$  such that, for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$

$$\|\tilde{w}_t(t, \cdot)\|_{L^2} \leq C e^{-\mu t} \chi(\|\tilde{w}_0\|_{L^2}) \|\tilde{w}_t(0, \cdot)\|_{L^2}, \quad \forall t \geq 0. \tag{5.47}$$



By multiplying the first equation of (5.35) by  $\tilde{w}$  and integrating by parts, we get after some computations

$$\begin{aligned} -k^2\tilde{\eta}(t)^2 + \tilde{w}_x(t,0)^2 &= -2 \int_0^L \tilde{w}(t,x)\tilde{w}_t(t,x)dx + \int_0^L |\tilde{w}(t,x)|^2 w'_\infty(x)dx \\ &\leq 2\|\tilde{w}(t,\cdot)\|_{L^2}\|\tilde{w}_t(t,\cdot)\|_{L^2} + \ell\|\tilde{w}(t,\cdot)\|_{L^2}^2\|\tilde{w}_\infty\|_{H^3(0,L)}. \end{aligned}$$

where  $\ell$  is the constant of the embedding of  $H^3(0,L)$  in  $L^\infty(0,L)$ . Then, we can deduce  $\lim_{t \rightarrow \infty} |\tilde{w}_x(t,0)| = \lim_{t \rightarrow \infty} |w_x(t,0) - r| = 0$  for all  $(\tilde{\eta}_0, \tilde{w}_0) \in D_\Delta(\mathcal{A})$ , concluding the proof.  $\square$

Finally, by combining the statement of Lemma 5.4 and Theorem 5.6 we have the following output regulation result for the system (5.28) in the original coordinates  $w, \eta$ . The proof is omitted for space reasons.

**COROLLARY 5.7 (Output Regulation).** *There exist positive real number  $k_2^*, \bar{d}, \bar{r}$ , such that, for any disturbance  $d$  and reference  $r$  satisfying  $\|d\|_{L^2} \leq \bar{d}$  and  $|r| \leq \bar{r}$  and for any  $k \in (0, k_2^*)$  the output  $y$  is asymptotically regulated at the reference  $r$ , namely (5.2) is satisfied, for any solution to system (5.28), with initial conditions  $(\eta_0, w_0) \in D(\mathcal{A})$  sufficiently small in the norm  $\mathbb{R} \times L^2(0, L)$ .*

**6. Conclusion.** In this article, we have solved the output regulation problem by means of an integral action for a Korteweg-de-Vries (KdV) equation controlled at the boundary and subject to some distributed constant disturbances so that to regulate a boundary output to a given constant reference. For this, we have followed a Lyapunov approach. We have first designed an ISS Lyapunov functional which is obtained by strictifying the energy associated to the system. In particular, the energy is modified by adding a second term which is obtained from the design of an observer built with the backstepping technique. Then, thanks to this ISS Lyapunov functional, we have applied the forwarding method to achieve our goal in the context of output regulation. In particular, we extended the system with an integral action and we designed an output feedback controller acting at the boundary. We show that if the selected gain is sufficiently small then the solutions of the closed-loop system converge to an equilibrium. Furthermore, for strong solutions, point-wise convergence of the regulated output is achieved. Similar results hold locally for the nonlinear model of the KdV, namely in presence of small references and perturbations and with a local domain of attraction.

For future work, we wish to study the case in which  $L \in \mathcal{N}$ . We believe furthermore that the proposed strictification approach can be extended also to other classes of PDEs for which a strict Lyapunov functional is not yet known.

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