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ISS LYAPUNOV STRICTIFICATION VIA OBSERVER DESIGN AND INTEGRAL ACTION CONTROL FOR A KORTEWEG-DE VRIES EQUATION

ISMAILA BALOGOUN*, SWANN MARX* , AND DANIELE ASTOLFI †

Abstract. The article deals with the output regulation of a Korteweg-de-Vries (KdV) system subject to a distributed disturbance. The control input and the output are located at the boundary. To achieve this objective, we follow a Lyapunov approach. For that, inspired by the strictification methodology proposed in [42] in the finite-dimensional context, we construct an ISS-Lyapunov functional for the KdV equation thanks to the use of an observer designed via the backstepping approach. Then, thanks to this Lyapunov functional, we apply the forwarding method in order to solve the desired output regulation problem.

Key words. Input-to-state stability, integral controller, Korteweg-de Vries equation, backstepping, regulation, forwarding.

1. Introduction. This paper deals with the output regulation of a Korteweg-de Vries (KdV) equation. The KdV equation is a mathematical model of waves on shallow water surfaces (see e.g., [7] for a survey). This equation has been studied in [44, 9, 12] in the controllability context, in [8, 13, 31, 46] in terms of stabilization, and in [45, 11] where some asymptotic analysis of the equilibrium point 0 is given. We may also mention [33, 34] where *input-to state stability* (ISS, in short) properties are obtained via feedback stabilization in presence of a saturated damping (we refer to [43, 21, 22] or the recent survey [36] for ISS Lyapunov functional in the infinite-dimensional context).

Roughly speaking, the aim of the output regulation consists in designing a feedback-law such that the output converges asymptotically towards a desired reference and such that disturbances are rejected, possibly in spite of some “small” model uncertainties. Following the celebrated *internal-model principle*, a solution to such a problem exists when references and disturbance (denoted generically as exosignals) are generated by a known autonomous dynamical system (denoted as exosystem), and a copy of such a system is embedded in the controller dynamics. A well known example is the use of integral action for tracking and rejecting constant references and disturbances. Output regulation is an old topic in the finite-dimensional context, but many results remain to be found in the context of nonlinear systems (see e.g., [2] for a recent article in this field), and many further research lines have to be followed when dealing with time-varying references. See, for instance, [1] where a finite-dimensional system is regulated by adding a transport equation for the case of periodic exosignals.

For infinite-dimensional systems, even if one can mention some old results such as [16], the topic is still very active. A generalization of internal-model principle has been proposed in [37], but the use of integral action to achieve output regulation in the presence of constant references/perturbations for infinite dimensional systems has been initiated early in [40]. Since then, several methods to design integral action have been developed following for instance a spectral approach in [41, 49, 37], by using operator and semi-group methods in [25, 50], based on frequency domain methods with Laplace transform in [4, 14] or by relying on Lyapunov techniques in [23, 19, 48]. We may also mention [17, 18] which proposes to regulate an output towards time-varying references that are generated by a known linear dynamic or [24] which extends the sliding mode methodology for hyperbolic systems to reject time-varying disturbances.

Among all these techniques, we are particularly interested by Lyapunov techniques in this article. Indeed, such a methodology has been proved to be efficient for nonlinear systems. Among these techniques, we aim at using the forwarding method that has been first designed for finite-dimensional systems in cascade form [35, 2] and then extended for some hyperbolic systems [47] in the regulation context, and for abstract systems [29] in the stabilization context. In [47], it is shown that a *strict* Lyapunov functional¹ is needed for open-loop stable systems that we aim at regulating. In other words, before adding an integral action, we should be able to show that a strict Lyapunov functional for the open-loop dynamics does exist. Such Lyapunov functionals are known for hyperbolic systems [3], but it is not the case for the KdV equation. In addition to the existence of this Lyapunov functional, some ISS properties are needed to apply the forwarding method.

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¹Strict Lyapunov functionals are Lyapunov functionals whose time-derivative is bounded by a negative function depending on the full-state.

In the perspective of addressing an output regulation problem for KdV, we therefore establish new results that may have their own interest. In particular, we first study the following KdV equation

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ w_x(t, L) = d_2(t), \\ w(0, x) = w_0(x), \end{cases} \quad (1.1)$$

where $(t, x) \in \mathbb{R}_+ \times [0, L]$, $L > 0$, d_1 and d_2 denote external inputs that might be seen, for instance, as disturbances, and its associated linearized system around the origin described by

$$\begin{cases} w_t + w_x + w_{xxx} = d_1(t, x), \\ w(t, 0) = w(t, L) = 0, \\ w_x(t, L) = d_2(t), \\ w(0, x) = w_0(x). \end{cases} \quad (1.2)$$

where $(t, x) \in \mathbb{R}_+ \times [0, L]$.

We will first show that the KdV equations (1.1) and (1.2) satisfy an ISS property with respect to the disturbances d_1, d_2 by explicitly constructing a strict Lyapunov functional. Note that there is no systematic method to build strict Lyapunov functionals either for nonlinear ordinary differential equations or (linear or nonlinear) partial differential equations. However, in many situations, a *weak* Lyapunov functional, i.e., a Lyapunov functional whose time derivative is nonpositive, exists. Often, it also coincides with the energy of the system. It is however difficult to deduce any quantitative robustness properties from a weak Lyapunov functional, and in particular, ISS properties are hard to obtain from such functions. For this reason, in the finite-dimensional context, a lot of attention has been put in the *strictification* of such weak Lyapunov functionals, namely the existence of systematic procedures to modify such a weak Lyapunov functional in order to make it strict. See, for instance, [27, 42]. To the best of our knowledge, in the infinite-dimensional context, such an approach has been applied only to certain classes of hyperbolic systems [43].

The first contribution of this paper, that might be seen thus of independent interest with respect to the context of output regulation, is the construction of an ISS-Lyapunov functional for our KdV system via a strictification procedure. The methodology we propose is inspired on [42] and is based on the design of an observer, which is also a new result in the KdV context and therefore consists in the second main contribution of this article. Let us illustrate it. Consider system (1.1) with no inputs, namely $d_1 = d_2 = 0$. A formal computation shows that the time derivative of the energy E defined as

$$E(w) := \int_0^L w(t, x)^2 dx \quad (1.3)$$

yields along solutions

$$\dot{E}(w) := \frac{d}{dt} \int_0^L w(t, x)^2 dx = -|w_x(t, 0)|^2. \quad (1.4)$$

Although Lyapunov stability can be concluded, nothing can be said about the attractivity and ISS-properties of the plant. In other words, the energy E is a weak-Lyapunov functional. Since $w_x(t, 0)$ is an exactly observable output as soon as $L \notin \mathcal{N}$ with

$$\mathcal{N} := \left\{ 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}} : k, l \in \mathbb{N} \right\},$$

then, following [42], our strategy consists in designing an observer with the output $w_x(t, 0)$. Such an observer is designed following the *backstepping* approach (see, e.g., [26]) and Fredholm operator (see, e.g., [13] or [20]). The proposed observer differs from the works in [30, 31, 46] in the same context of KdV equations because a different measured output is considered. Finally, by combining the Lyapunov functional derived from the observer analysis and the energy E , we obtain a strict Lyapunov functional, that will be used to establish the desired ISS properties for systems (1.1) and (1.2) with respect to the inputs d_1 and d_2 .

The third contribution of this article consists in addressing an output regulation problem. We suppose that a control is acting at the boundary $w_x(t, L)$ of the KdV equations (1.1) and (1.2) and that we want to regulate the output $w_x(t, 0)$ to a desired reference r . As a consequence, we extend the plant with an integral action processing the error $w_x(t, 0) - r$ and we show how to design an output-feedback law. The gain of the controller is obtained via the forwarding technique which is employed to construct a Lyapunov functional built upon the ISS Lyapunov functional obtained in the first part of this article. Note that the results of [47, Theorems 1 & 2] cannot be used of the shelf because in our article we consider a control input acting at the boundary (to be more precise, with an unbounded operator). Nevertheless, since we now an ISS-Lyapunov function we can apply the proposed methodology, similarly to what has been done for the hyperbolic equations in [47, Theorem 3].

This paper is organized as follows. In Section 2, we formulate the problem and state the results about the construction of ISS Lyapunov functional. In Section 3, an observer is designed using a Fredholm operator. Section 4 contains the proofs of the ISS results of the KdV systems under consideration. Section 5 states and proves some regulation results for the KdV equation. Finally, Section 6 collects concluding remarks and discuss some remaining open problems.

Notation: Set $\mathbb{R}_+ = [0, \infty)$. The term w_t stands for the partial derivative of the function w with respect to t . The term w_x (resp. w_{xx}, w_{xxx}) stands for the first (resp. second and third) order partial derivative of the function w with respect to x . Given $L > 0$, we set $\Omega := \mathbb{R}_+ \times [0, L]$. The functional space $L^2(0, L)$ denotes the set of (Lebesgue) measurable functions f such that $\int_0^L |f(x)|^2 dx < +\infty$. The associated norm is $\|f\|_{L^2}^2 := \int_0^L |f(x)|^2 dx$. We define the functional space $C^2([0, T])$ as the class of continuous functions on $[0, T]$, which have continuous derivatives of order two on $[0, T]$ and the functional space $H^k(0, L)$ denotes the Sobolev spaces equipped with the Sobolev-norm. Finally, we set $C_0^2([0, T]) := \{d \in C^2([0, T]) : d(0) = 0\}$, $H_L^3(0, L) := \left\{w \in H^3(0, L) : w(0) = w(L) = w'(L) = 0\right\}$ and $H_0^3(0, L) := \left\{w \in H^3(0, L) : w(0) = w(L) = w'(0) = 0\right\}$.

A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of a class \mathcal{K} , $\alpha \in \mathcal{K}$ if it is continuous, strictly increasing and satisfies $\alpha(0) = 0$. A function $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{K}_∞ , if it is of class \mathcal{K} and unbounded. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is of class \mathcal{KL} , $\beta \in \mathcal{KL}$, if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class \mathcal{K} , and, for each fixed $r \geq 0$, $\beta(r, \cdot)$ is decreasing and satisfies $\lim_{t \rightarrow \infty} \beta(r, t) = 0$. A Lyapunov functional $V : L^2(0, L) \rightarrow \mathbb{R}_+$ is said to be a strict Lyapunov functional, when its time derivative along the trajectories of the system is negative definite, and weak, when its time derivative along the trajectories is negative semi-definite [27, §2.1]. For compactness, we denote its time derivative as $\dot{V} := \frac{d}{dt} V(w(t, \cdot))$.

2. Construction of a ISS Lyapunov functional. As mentioned in the introduction, ISS properties are crucial in order to apply the forwarding method in the regulation context (see e.g., [47]). We will then focus on (1.1), which is a KdV equation subject to disturbances. In addition to the ISS properties, we also need a strict Lyapunov functional, and the latter will be built through a strictification technique inspired by [42]. Note however that the ISS properties that we establish are of independent interest: one may indeed use these results in a different context than the regulation.

We now state the definitions of the functional spaces where we will prove our results. We recall therefore some definitions and results about the solutions of the KdV equation. First, we have the following well-posedness property established in [9, Theorem 9] concerning the KdV equations (1.1) and (1.2).

PROPOSITION 2.1. *For any $T, L > 0$, for any $w_0 \in L^2(0, L)$, for any $d_1 \in L^1([0, T]; L^2(0, L))$ and $d_2 \in L^2(0, T)$, systems (1.1) and (1.2) admit a unique solution*

$$w \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L)).$$

We state also the next well-posedness result of the linearized system (1.2) provided in [15, Proposition 7] and [44, Propositions 3.2 and 3.7], which deals with more regular initial data. Indeed, this regularity will be essential when computing time derivative of Lyapunov functionals.

PROPOSITION 2.2. *For any $T, L > 0$, for any initial condition $w_0 \in H_L^3(0, L)$ and for any $d_1 \in C^1([0, T]; L^2(0, L))$, $d_2 \in C_0^2([0, T])$, system (1.2) admits a unique solution satisfying*

$$w \in C([0, T]; H_L^3(0, L)) \cap C^1([0, T]; L^2(0, L)).$$

Next, we give some definitions of ISS properties for systems (1.1) and (1.2).

DEFINITION 2.3. *System (1.1) (resp. (1.2)) is said to be input-to-state stable (ISS) with respect to d_1 and d_2 if there exist functions $\beta \in \mathcal{KL}$ and $\rho_1, \rho_2 \in \mathcal{K}_\infty$ such that, for any initial condition $w_0 \in L^2(0, L)$, any solution to (1.1) (respectively to (1.2)) satisfies, for all $t \geq 0$*

$$\|w(t, \cdot)\|_{L^2} \leq \beta(\|w_0\|_{L^2}, t) + \rho_1\left(\sup_{s \in [0, t]} \|d_1(s, \cdot)\|_{L^2}\right) + \rho_2\left(\sup_{s \in [0, t]} |d_2(s)|\right). \quad (2.1)$$

If (2.1) holds only for all $\|w_0\|_{L^2} \leq \delta$, $\|d_1(t)\|_{L^2} \leq \delta$, $|d_2(t)| \leq \delta$ for all $t \geq 0$ and for some $\delta > 0$, then the system (1.1) (resp. (1.2)) is said to be locally input-to-state stable (LISS) with respect to d_1 and d_2 .

In some cases, it is easier to prove such a property by using a Lyapunov functional. Recalling that, according to [36, Theorem 3], the existence of an ISS Lyapunov functional is sufficient to establish the ISS properties of Definition 2.3, we provide now a definition of ISS Lyapunov functional.

DEFINITION 2.4. *A function $V : L^2(0, L) \rightarrow \mathbb{R}$ is said to be an ISS Lyapunov functional for the system (1.1) (resp. (1.2)), if there exist $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma_1, \sigma_2 \in \mathcal{K}_\infty$ such that:*

(i) *For all $w \in L^2(0, L)$,*

$$\underline{\alpha}(\|w\|_{L^2}) \leq V(w) \leq \bar{\alpha}(\|w\|_{L^2}). \quad (2.2)$$

(ii) *The time derivative of V along the trajectories of (1.1) (resp. (1.2)) satisfies*

$$\dot{V} \leq -\alpha(\|w(t, \cdot)\|_{L^2}) + \sigma_1(\|d_1(t, \cdot)\|_{L^2}) + \sigma_2(|d_2(t)|). \quad (2.3)$$

If (ii) holds only for all $\|w(t, \cdot)\|_{L^2} \leq \delta$, $\|d_1(t)\|_{L^2} \leq \delta$, $|d_2(t)| \leq \delta$ for all $t \geq 0$ for some $\delta > 0$, then V is said to be a locally ISS Lyapunov functional for the system (1.1) (resp. (1.2)).

As explained in the introduction, for any $L \notin \mathcal{N}$, the energy function in L^2 -norm defined in (1.3) is a weak Lyapunov functional for the system (1.1) (resp. (1.2)) in view of (1.4). Indeed, on the left right side of the inequality, we have a function which depends only on a part the state $w(t, x)$, i.e. $w_x(t, 0)$. From (1.4), one can deduce that the origin of the system of (1.1) with $d_1 = d_2 = 0$ is Lyapunov stable. In order to show also the exponential stability properties of the origin, one can follow [44, Proposition 3.3], by using the fact that $w_x(t, 0)$ is exactly observable as soon as $L \notin \mathcal{N}$: indeed, using the related observability inequality, and integrating (1.4) between 0 and T , this property can be shown as illustrated in [7, §4.1.]. But nothing can be easily said in the non-nominal case, i.e. in the presence of disturbances. As a consequence, in order to show the ISS properties of the system (1.1) (resp. (1.2)), we follow a different approach here: we aim at constructing a strict ISS Lyapunov functional, which has not been found yet to the best of our knowledge. Using the observability of the output $w_x(t, 0)$, we can follow the methodology described in [42] and that can be decomposed as follows: we first design an observer for the output $w_x(t, 0)$; we second consider the sum of the Lyapunov functional coming from the latter observer design and the natural energy, and we prove that this sum boils down to be a strict Lyapunov functional. Finally, thanks to this strict Lyapunov functional, we deduce ISS properties for systems (1.2) and (1.1). These properties are written more precisely in the following theorem, that is our first main result.

THEOREM 2.5. *Suppose that $L \notin \mathcal{N}$. Then, there exists a function $W : L^2(0, L) \rightarrow \mathbb{R}$ such that, the function $V(w) := W(w) + E(w)$ with E being the energy in L^2 -norm defined in (1.3), is*

(a) *an ISS Lyapunov functional for the system (1.2);*

(b) *a locally ISS Lyapunov functional for the system (1.1).*

Moreover, the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma_1$ and σ_2 are quadratic in their arguments.

The proof of Theorem 2.5 is postponed to Section 4. In particular, in the next section, we will first show how to design an ISS observer for the linearized system (1.2) by means of the output $w_x(t, 0)$. The proposed design is based on the backstepping method, see, e.g., [26] and on some Fredholm transformation, see, e.g., [20, 13]. Then, in Section 4, we will use the ISS-Lyapunov functional associated to such an observer to build the function W claimed in the statement of Theorem 2.5.

3. Observer design for Linear KDV. In this section, we design an observer for system (1.2) with $y(t) = w_x(t, 0)$ defined as the output function.

Following [26, Chapter 5], we consider the linearized system around the origin

$$\begin{cases} w_t + w_x + w_{xxx} = d_1, & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = d_2, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (3.1)$$

and we design an observer with a distributed correction term of the form

$$\begin{cases} \hat{w}_t + \hat{w}_x + \hat{w}_{xxx} + p(x)[y(t) - \hat{w}_x(t, 0)] = 0, & (t, x) \in \Omega, \\ \hat{w}(t, 0) = \hat{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}_x(t, L) = 0, & t \in \mathbb{R}_+, \\ \hat{w}(0, x) = \hat{w}_0(x), & x \in [0, L], \end{cases} \quad (3.2)$$

where p is the output injection gain to be selected in order to guarantee the convergence of the observer. The well-posedness (3.2) can be proved by following the same approach as in [30].

We define the estimation error coordinates as follows

$$\hat{w} \mapsto \tilde{w} := w - \hat{w}$$

mapping system (3.2) into

$$\begin{cases} \tilde{w}_t + \tilde{w}_x + \tilde{w}_{xxx} - p(x)\tilde{w}_x(t, 0) = d_1, & (t, x) \in \Omega, \\ \tilde{w}(t, 0) = \tilde{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \tilde{w}_x(t, L) = d_2, & t \in \mathbb{R}_+, \\ \tilde{w}(0, x) = \tilde{w}_0(x), & x \in [0, L]. \end{cases} \quad (3.3)$$

In nominal conditions ($d_1 = 0, d_2 = 0$), the origin of the estimation error \tilde{w} with error-dynamics (3.3) has to be stable so that to guarantee the convergence of the estimate \hat{w} to state w . In the presence of disturbances, we would like such an error-dynamics \tilde{w} to possess some bounded-input bounded-output properties so that for small perturbations the estimated state \hat{w} doesn't deviate too much from the real value of the state w . In other words, we are interested in guaranteeing the estimation error dynamics to be ISS with respect to the perturbations d_1, d_2 . This is shown in the next result showing that it is always possible to select the gain p so that to satisfy the desired ISS-properties of the observer (3.2).

THEOREM 3.1. *Suppose that $L \notin \mathcal{N}$. For any $\lambda > 0$, there exists a function $p \in L^2(0, L)$, a Lyapunov functional $U : L^2(0, L) \rightarrow \mathbb{R}$ and some positive constants \underline{c}, \bar{c} satisfying the following properties.*

(i) For all $\tilde{w} \in L^2(0, L)$

$$\underline{c}\|\tilde{w}\|_{L^2}^2 \leq U(\tilde{w}) \leq \bar{c}\|\tilde{w}\|_{L^2}^2. \quad (3.4)$$

(ii) The time derivative of U along the trajectories of (3.3) satisfies

$$\dot{U} \leq -\lambda U + \frac{1}{\lambda}\|d_1\|_{L^2}^2 + |d_2|^2 \quad (3.5)$$

for all $d_1 \in L^1([0, T]; L^2(0, L))$, $d_2 \in L^2(0, T)$ and $\tilde{w} \in C^0([0, T]; L^2(0, L)) \cap L^2(0, T; H^1(0, L))$.

Proof: Inspired by [13], consider the change of coordinates

$$\tilde{w} \mapsto \gamma := \Pi^{-1}\tilde{w} \quad (3.6)$$

where the function Π is defined thanks to the following Fredholm integral transformation

$$\tilde{w}(x) := \Pi(\gamma)(x) = \gamma(x) - \int_0^L P(x, z)\gamma(z)dz, \quad (3.7)$$

for all $x \in [0, L]$. The state γ is the solution to the following system:

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = 0, & (t, x) \in \Omega, \\ \gamma(t, 0) = \gamma(t, L) = \gamma_x(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L], \end{cases} \quad (3.8)$$

where λ is a positive constant. Using integration by parts and the boundary conditions of (3.8), we get

$$\frac{d}{dt} \int_0^L |\gamma(t, x)|^2 dx \leq -2\lambda \int_0^L |\gamma(t, x)|^2 dx$$

from which it is straightforward to deduce the exponential stability of the L^2 -norm of γ . As a consequence, the main idea of the proof consists in selecting the function p so that to show that the candidate ISS Lyapunov functional

$$U(\tilde{w}) := \|\gamma\|_{L^2}^2 = \|\Pi^{-1}(\tilde{w})\|_{L^2}^2 \quad (3.9)$$

satisfy the conditions (3.4), and (3.5).

To this end, we first need to find the kernel P such that $\tilde{w}(t, x) = \Pi(\gamma)(t, x)$ satisfies (3.3) when $d_1 = 0$ and $d_2 = 0$. We have also to ensure that the corresponding transformation is invertible and continuous. By differentiating formally with respect to the time (3.7) one obtains

$$\tilde{w}_t(t, x) = \gamma_t(t, x) - \int_0^L P(x, z) \gamma_t(t, z) dz,$$

which yields, using (3.8) to

$$\tilde{w}_t(t, x) = \gamma_t(t, x) + \int_0^L P(x, z) \left(\lambda\gamma(t, z) + \gamma_z(t, z) + \gamma_{zzz}(t, z) \right) dz,$$

and finally, after some integrations by parts, to

$$\begin{aligned} \tilde{w}_t(t, x) &= \gamma_t(t, x) - P(x, 0)\gamma(t, 0) + P(x, L)\gamma(t, x) + P(x, L)\gamma_{xx}(t, L) - P(x, 0)\gamma_{xx}(t, 0) + P_z(x, 0)\gamma_x(t, 0) \\ &\quad - \int_0^L \left(-\lambda P(x, z) + P_z(x, z) + P_{zzz}(x, z) \right) \gamma(t, z) dz - P_z(x, L)\gamma_x(t, L) + P_{zz}(x, L)\gamma(t, L) \\ &\quad - P_{zz}(x, 0)\gamma(t, 0). \end{aligned}$$

Then, using the Leibniz rule, we obtain

$$\tilde{w}_x(t, x) = \gamma_x(t, x) - \int_0^L P_x(x, z) \gamma(t, z) dz, \quad (3.10)$$

and

$$\tilde{w}_{xxx}(t, x) = \gamma_{xxx}(t, x) - \int_0^L P_{xxx}(x, z) \gamma(t, z) dz.$$

Identifying (3.3) and (3.8), we have

$$\begin{aligned} \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) - p(x)\tilde{w}_x(t, 0) &= \gamma_t(t, x) + \gamma_x(t, x) + \gamma_{xxx}(t, x) + \lambda\gamma(t, x) \\ &\quad - \int_0^L \left(-\lambda P + P_z + P_{zzz} + P_{xxx} + P_x \right) \gamma(t, z) dz \\ &\quad - \lambda\gamma(t, x) + P(x, L)\gamma_{xx}(t, L) + P_z(x, 0)\gamma_x(t, 0) \\ &\quad - P(x, 0)\gamma_{xx}(t, 0) - p(x) \left[\gamma_x(t, 0) - \int_0^L P_x(0, z) \gamma(t, z) dz \right], \end{aligned}$$

where some arguments are omitted for compactness when clear from the context. Then, using the identity

$$-\lambda\gamma(t, x) = \int_0^L \lambda\delta(x - z)\gamma(t, z)dz,$$

where $\delta(x - z)$ denotes the Dirac measure on the diagonal of the square $[0, L] \times [0, L]$, previous equation gives

$$\begin{aligned} \tilde{w}_t(t, x) + \tilde{w}_x(t, x) + \tilde{w}_{xxx}(t, x) - p(x)\tilde{w}_x(t, 0) &= \gamma_t(t, x) + \gamma_x(t, x) + \gamma_{xxx}(t, x) + \lambda\gamma(t, x) \\ &\quad - \int_0^L \left(-\lambda P + P_z + P_{zzz} + P_x + P_{xxx} - \lambda\delta(x - z) \right) \gamma(t, z) dz \\ &\quad - P(x, 0)\gamma_{xx}(t, 0) + P(x, L)\gamma_{xx}(t, L) + \int_0^L P_x(0, z)\gamma(t, z) dz \\ &\quad - \gamma_x(t, 0)[p(x) - P_z(x, 0)]. \end{aligned} \quad (3.11)$$

From equation (3.11), we obtain the following conditions.

(a) The following identity is satisfied for all $(x, z) \in [0, L] \times [0, L]$:

$$-\lambda P + P_z + P_{zzz} + P_x + P_{xxx} = \lambda\delta(x - z).$$

(b) Three boundary conditions are satisfied for $(x, z) \in [0, L] \times [0, L]$:

$$P(x, 0) = P(x, L) = P_x(0, z) = 0.$$

(c) An appropriate choice of p is given by

$$p(x) := P_z(x, 0).$$

Moreover, note also that the following.

(d) By setting $x = 0$ and $x = L$ in (3.7), we obtain:

$$P(0, z) = P(L, z) = 0, \quad \forall z \in [0, L],$$

(e) By setting $x = L$ in (3.10), we obtain:

$$P_x(L, z) = 0.$$

Therefore, combining conditions (a)-(e), we impose that the function P satisfies the following PDE:

$$\begin{cases} -\lambda P + P_z + P_{zzz} + P_x + P_{xxx} = \lambda\delta(x - z), \\ P(x, 0) = P(x, L) = 0, \\ P(L, z) = P(0, z) = 0, \\ P_x(L, z) = P_x(0, z) = 0, \end{cases} \quad (3.12)$$

where $(x, z) \in [0, L] \times [0, L]$ and $\delta(x - z)$ denotes the Dirac measure on the diagonal of the square $[0, L] \times [0, L]$. Let us make the following change of variable:

$$\begin{pmatrix} z \\ x \end{pmatrix} \mapsto \begin{pmatrix} \bar{x} \\ \bar{z} \end{pmatrix} := \begin{pmatrix} L - z \\ L - x \end{pmatrix},$$

and define $G(\bar{x}, \bar{z}) := -P(x, z)$. From (3.12) it is obtained

$$\begin{cases} \lambda G + G_{\bar{z}} + G_{\bar{z}\bar{z}\bar{z}} + G_{\bar{x}} + G_{\bar{x}\bar{x}\bar{x}} = \lambda\delta(\bar{x} - \bar{z}), \\ G(\bar{x}, 0) = G(\bar{x}, L) = 0, \\ G(L, \bar{z}) = G(0, \bar{x}) = 0, \\ G_{\bar{z}}(\bar{x}, 0) = G_{\bar{z}}(\bar{x}, L) = 0, \end{cases} \quad (3.13)$$

with (\bar{x}, \bar{z}) belonging to $[0, L] \times [0, L]$. Note that in [13, Lemma 2.1], it has been proved that, for any $L \notin \mathcal{N}$, the system (3.13) admits a unique solution $G \in H_0^1((0, L) \times (0, L))$. Therefore, we can conclude that the kernel P exists. Then according to [13, Lemma 3.1], the transformation Π is invertible and continuous on $L^2(0, L)$ and its inverse is also continuous. Moreover there exist two positive constants C_0 and C_1 such that, for all $t \geq 0$, we have

$$\begin{aligned} \|w(t, \cdot)\|_{L^2}^2 &\leq C_0 \|\Pi^{-1}w(t, \cdot)\|_{L^2}^2, \\ \|\Pi^{-1}w(t, \cdot)\|_{L^2}^2 &\leq C_1 \|w(t, \cdot)\|_{L^2}^2. \end{aligned} \quad (3.14)$$

As a consequence, we deduce that the function U defined in (3.9) is well defined in $L^2(0, L)$ and inequality (3.4) is satisfied with $\underline{c} = C_0^{-1}$ and $\bar{c} = C_1$. Note that the function $w \in L^2(0, L) \mapsto U(w) \in \mathbb{R}_+$ is equivalent to the standard norm on the space $L^2(0, L)$ according to (3.14).

Then, we need to show inequality (3.5). To this end, since $H_L^3(0, L)$, $C_0^2([0, T])$ and $C^1([0, T], L^2(0, L))$ are dense in $L^2(0, L)$, $L^2(0, T)$ and $L^1([0, T]; L^2(0, L))$, respectively, we show inequality (3.5) for $\tilde{w}_0 \in H_L^3(0, L)$, $d_2 \in C_0^2([0, T])$ and $d_1 \in C^1([0, T], L^2(0, L))$. The result follows for all $w_0 \in L^2(0, L)$, $d_1 \in L^1([0, T]; L^2(0, L))$ and $d_2 \in L^2(0, T)$, by a standard density argument (see e.g., [28, Lemma 1] for more details).

Now, we apply the change of coordinates (3.6) to system (3.3). It gives

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = d_1, & (t, x) \in \Omega, \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (3.15)$$

The time derivative of U along the trajectory of (3.3), that is on the trajectory of (3.15), yields

$$\begin{aligned} \dot{U} &= -2 \int_0^L \gamma(t, x) (\gamma_{xxx} + \gamma_x + \lambda\gamma - d_1) dx \\ &= -2\lambda \int_0^L |\gamma(t, x)|^2 dx + 2 \int_0^L \gamma_x(t, x) \gamma_{xx}(t, x) dx + 2 \int_0^L d_1(t, x) \gamma(t, x) dx \\ &\leq -2\lambda \|\gamma(t, \cdot)\|_{L^2}^2 + 2 \left| \int_0^L d_1(t, x) \gamma(t, x) dx \right| + d_2(t)^2 - \gamma_x(t, 0)^2, \end{aligned} \quad (3.16)$$

where, in the second equation, we have used an integration by parts to compute

$$2 \int_0^L \gamma_x(t, x) \gamma_{xx}(t, x) dx = \left[\gamma_x^2(t, x) \right]_0^L = d_2(t)^2 - \gamma_x(t, 0)^2.$$

Since the term $-\gamma_x(t, 0)^2$ is always negative, using first Cauchy-Schwarz's inequality and then Young's inequality², from (3.16) we finally obtain

$$\begin{aligned} \dot{U} &\leq -2\lambda \|\gamma(t, \cdot)\|_{L^2}^2 + 2 \|\gamma(t, \cdot)\|_{L^2} \|d_1(t, \cdot)\|_{L^2} + |d_2(t)|^2 \\ &\leq -\lambda \|\gamma(t, \cdot)\|_{L^2}^2 + \frac{1}{\lambda} \|d_1(t, \cdot)\|_{L^2}^2 + |d_2(t)|^2, \end{aligned}$$

showing inequality (3.5), which completes the proof. \square

From the existence of the ISS Lyapunov functional established in Theorem 3.1, one can immediately deduce the following property for the observer (3.2).

COROLLARY 3.2. *For any $\mu > 0$, there exists a function $p \in L^2(0, L)$ so that the observer (3.2) is an ISS exponential convergent observer for system (3.1) with rate μ . Namely the following inequality holds*

$$\|w(t, \cdot) - \hat{w}(t, \cdot)\|_{L^2} \leq C e^{-\mu t} \|w_0 - \hat{w}_0\|_{L^2} + \rho_1 \sup_{s \in [0, t]} \|d_1(s, \cdot)\|_{L^2} + \rho_2 \sup_{s \in [0, t]} |d_2(s)|$$

²In particular, $2ab \leq \nu a^2 + \frac{1}{\nu} b^2$, for any $\nu > 0$.

for any initial conditions $w_0, \widehat{w}_0 \in L^2(0, L)$ and all $t \geq 0$, and for some $C, \rho_1, \rho_2 > 0$.

Proof: The proof can be directly inherited from Theorem 3.1 by selecting p so that inequality (3.5) holds with $\lambda = 2\mu$. Then, using equations (3.4), and the triangular inequality $\sqrt{a^2 + b^2} \leq a + b$, it directly yields the inequality of the corollary by selecting $C = \sqrt{\bar{c}/\underline{c}}$, $\rho_1 = 1/\sqrt{2\underline{c}\mu}$, $\rho_2 = 1/\sqrt{\bar{c}}$, with \bar{c}, \underline{c} given by Theorem 3.1. \square

4. Proof of Theorem 2.5. Since $H_L^3(0, L)$, $C_0^2([0, T])$ and $C^1([0, T], L^2(0, L))$ are dense in $L^2(0, L)$, $L^2(0, T)$ and $L^1([0, T]; L^2(0, L))$, respectively, we prove the statement of the Theorem 2 for $w_0 \in H_L^3(0, L)$, $d_2 \in C_0^2([0, T])$ and $d_1 \in C^1([0, T], L^2(0, L))$. The result follows for all $w_0 \in L^2(0, L)$, $d_1 \in L^1([0, T]; L^2(0, L))$ and $d_2 \in L^2(0, T)$, by a standard density argument (see e.g., [28, Lemma 1] for more details).

Proof of item (a) of Theorem 2.5. Fix $\lambda = 1$ and consider the function p and U given by Theorem 3.1. The statement of the theorem holds by selecting

$$W(w) := \frac{1}{2\bar{p}}U(w), \quad \bar{p} := \|p\|_{L^2}^2 \quad (4.1)$$

and with inequalities (2.2), (2.3) satisfied with the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma_1, \sigma_2$ given as

$$\underline{\alpha}(s) := (1 + \underline{c})s^2, \quad \bar{\alpha}(s) := (1 + \bar{c})s^2, \quad (4.2)$$

$$\alpha(s) := \frac{\underline{c}}{2\bar{p}}s^2, \quad \sigma_1(s) = \frac{1}{\bar{p}}s^2, \quad \sigma_2(s) := \frac{1}{2\bar{p}}s^2. \quad (4.3)$$

To show this, consider system (1.2) and add and subtract the term $p(x)w_x(t, 0)$ in the first equation, where p is the gain of observer (3.2) established in Theorem 3.1. We obtain

$$\begin{cases} w_t + w_x + w_{xxx} - p(x)w_x(t, 0) = -p(x)w_x(t, 0) + d_1(t, x), & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L]. \end{cases} \quad (4.4)$$

Using the Fredholm integral transformation defined in (3.7), on the state w , and following similar computations as in the proof of Theorem 3.1, we obtain the following system

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = -p(x)w_x(t, 0) + d_1(t, x), & (t, x) \in \Omega, \\ \gamma(t, 0) = \gamma(t, L) = 0, & t \in \mathbb{R}_+, \\ \gamma_x(t, L) = d_2(t), & t \in \mathbb{R}_+, \\ \gamma(0, x) = \gamma_0(x), & x \in [0, L]. \end{cases} \quad (4.5)$$

Consider the Lyapunov functional $U(w) := \|\gamma\|_{L^2}^2$ as in (3.9). Applying inequality (3.5) along solutions to (4.5), with $\lambda = 1$, gives

$$\begin{aligned} \dot{U} &\leq -U + \|d_1 - p(\cdot)w_x(t, 0)\|_{L^2}^2 + |d_2|^2 \\ &\leq -U + 2\|d_1\|_{L^2}^2 + 2\|p(\cdot)w_x(t, 0)\|_{L^2}^2 + |d_2|^2. \end{aligned} \quad (4.6)$$

Hence, select $V = E + W$, with E defined as in (1.3) and W defined as in (4.1). From the bound (3.4) it straightforward to obtain inequality (2.2) with the bounds in (4.2). Then, computing the time derivative of V using (1.4), (3.4) and (4.6) yields

$$\dot{V} \leq -|w_x(t, 0)|^2 - \frac{\underline{c}}{2\bar{p}}\|w\|_{L^2}^2 + \frac{1}{\bar{p}}\|p(\cdot)w_x(t, 0)\|_{L^2}^2 + \frac{1}{\bar{p}}\|d_1\|_{L^2}^2 + \frac{1}{2\bar{p}}|d_2|^2. \quad (4.7)$$

Note that

$$-|w_x(t, 0)|^2 + \frac{1}{\bar{p}}\|p(\cdot)w_x(t, 0)\|_{L^2}^2 = 0. \quad (4.8)$$

As a consequence, inequality (2.3) is directly obtained from (4.7) with the definitions (4.3). This concludes the proof of item (a) of Theorem 2.5.

Proof of the item (b) of Theorem 2. Now, let us consider w the solution to the nonlinear equation (1.1). Then, $\gamma = \Pi^{-1}(w)$ satisfies

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \lambda\gamma = -p(x)w_x(t,0) + d_1(t,x) \\ \quad - \left(\gamma(t,x) - \int_0^L P(x,z)\gamma(t,z)dz \right) \left(\gamma_x(t,x) - \int_0^L P_x(x,z)\gamma(t,z)dz \right), & (t,x) \in \Omega \\ \gamma(t,0) = \gamma(t,L) = 0, & t \in \mathbb{R}_+ \\ \gamma_x(t,L) = d_2(t), & t \in \mathbb{R}_+ \\ \gamma(0,x) = \gamma_0(x), & x \in [0,L]. \end{cases} \quad (4.9)$$

Now, we consider the Lyapunov functional $U(w) := \|\gamma\|_{L^2}^2$ as in (3.9). According to (3.16), the time derivative of this function along the trajectory of (4.9) satisfies

$$\dot{U} \leq -2\lambda\|\gamma(t,\cdot)\|_{L^2}^2 + d_2(t)^2 - \gamma_x(t,0)^2 + 2 \left| \int_0^L (d_1(t,x) - p(x)w_x(t,0) - F(t,x))\gamma(t,x)dx \right| \quad (4.10)$$

where F is given by

$$\begin{aligned} F(t,x) := & \left(\int_0^L P_x(x,z)\gamma(t,z)dz \right) \left(\int_0^L P(x,z)\gamma(t,z)dz \right) - \gamma_x(t,x) \int_0^L P(x,z)\gamma(t,z)dz \\ & - \gamma(t,x) \int_0^L P_x(x,z)\gamma(t,z)dz. \end{aligned} \quad (4.11)$$

Since the term $-\gamma_x(t,0)^2$ is always negative, using first Cauchy-Schwarz's inequality and then Young's inequality, from (4.10) we finally obtain

$$\dot{U} \leq -\lambda\|\gamma(t,\cdot)\|_{L^2}^2 + \frac{2}{\lambda}\|d_1(t,\cdot)\|_{L^2}^2 + d_2(t)^2 + \frac{2}{\lambda}\|p(\cdot)w_x(t,0)\|_{L^2}^2 + 2 \left| \int_0^L F(t,x)\gamma(t,x)dx \right|. \quad (4.12)$$

By using the same argument as in [13, Proof of Theorem 1.2], we can prove the existence of a positive constant M , that depends on the function P , such that

$$2 \left| \int_0^L F(t,x)\gamma(t,x)dx \right| \leq M\|\gamma\|_{L^2}^3. \quad (4.13)$$

Then, we have, using the bound given in (3.14)

$$\begin{aligned} \dot{U} & \leq -\lambda\|\gamma(t,\cdot)\|_{L^2}^2 + \frac{2}{\lambda}\|d_1(t,\cdot)\|_{L^2}^2 + d_2(t)^2 + \frac{2}{\lambda}\|p(\cdot)w_x(t,0)\|_{L^2}^2 + M\|\gamma\|_{L^2}^3 \\ & \leq -\lambda c\|w(t,\cdot)\|_{L^2}^2 + \frac{2}{\lambda}\|d_1(t,\cdot)\|_{L^2}^2 + d_2(t)^2 + \frac{2}{\lambda}\|p(\cdot)w_x(t,0)\|_{L^2}^2 + M\bar{c}^{\frac{3}{2}}\|w(t,\cdot)\|_{L^2}^3. \end{aligned} \quad (4.14)$$

Given any positive value T , according to [15, Proposition 14] we know that there exist $\delta > 0$ and $C > 0$ such that, if

$$\|d_1\|_{L^1([0,T];L^2(0,L))} + \|d_2\|_{L^2([0,T])} + \|w_0\|_{L^2(0,L)} \leq \delta, \quad (4.15)$$

then

$$\|w\|_{\mathcal{B}} \leq C\delta, \quad \forall t \in [0,T]. \quad (4.16)$$

The functional space \mathcal{B} being equipped with the following norm

$$\|w\|_{\mathcal{B}} := \sup_{t \in [0,T]} \|w(t,\cdot)\|_{L^2} + \left(\int_0^T \|w(t,\cdot)\|_{H^1(0,L)}^2 dt \right)^{\frac{1}{2}}, \quad (4.17)$$

one can deduce that, for all $t \in [0, T]$

$$\|w(t, \cdot)\|_{L^2} \leq C\delta \quad (4.18)$$

and thus, for all $t \in [0, T]$, we obtain

$$\|w(t, \cdot)\|_{L^2}^3 \leq C\delta \|w(t, \cdot)\|^2. \quad (4.19)$$

Plugging this latter inequality in (4.14) leads to the following estimate

$$\dot{U} \leq -\left(\lambda \underline{c} - C\delta M \bar{c}^{\frac{3}{2}}\right) \|w(t, \cdot)\|_{L^2}^2 + \frac{2}{\lambda} \|d_1(t, \cdot)\|_{L^2}^2 + \frac{2}{\lambda} \|p(\cdot)w_x(t, 0)\|_{L^2}^2 + \|w(t, \cdot)\|_{L^2}^3 + d_2(t)^2. \quad (4.20)$$

Without loss of generality we fix $\lambda = 1$ and we choose the constant δ sufficiently small such that

$$K = \underline{c} - C\delta M \bar{c}^{\frac{3}{2}} > 0.$$

Hence, select $V = E + W$, with E defined as in (1.3) and W defined as in (4.1). Following the same arguments employed in the proof of the item (a), from the bound (3.4) we obtain inequality (2.2) with the bounds in (4.2). Then, computing the time derivative of V using (1.4), (3.4) and (4.14) yields

$$\dot{V} \leq -|w_x(t, 0)|^2 - \frac{K}{2\bar{p}} \|w\|_{L^2}^2 + \frac{1}{\bar{p}} \|p(\cdot)w_x(t, 0)\|_{L^2}^2 + \frac{1}{\bar{p}} \|d_1\|_{L^2}^2 + \frac{1}{2\bar{p}} |d_2|^2. \quad (4.21)$$

Then, according to (4.8), inequality (2.3) is directly obtained from (4.21) with the definitions (4.3) where $\alpha(s) := \frac{K}{2\bar{p}} s^2$. This concludes the proof of Item (b) of Theorem 2.

5. Adding an integral action. In this section we consider the regulation problem of a KdV equation in which the disturbance d_2 is considered as a control input acting at the boundary condition, and the output $y(t) = w_x(t, 0)$ has to be regulated at a certain desired constant reference r in presence of unknown distributed constant disturbances d_1 . We aim at showing that such a problem can be solved by means of an integral action and an output feedback control law. The proposed design is based on the forwarding method (see e.g., [47] or [29]). Note that in Section 5.1, we focus on the linearized version of the system. Then, in Section 5.2 we will show that our control design allows to regulate the trajectories towards the equilibrium points of the nonlinear system as in [8] or [16] for instance.

5.1. Regulation of linear KdW by means of the forwarding method. Consider the following system

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = u(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L] \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+ \end{cases} \quad (5.1)$$

where $d \in L^2(0, L)$ is a constant perturbation, $u \in \mathbb{R}$ is the control input and $y \in \mathbb{R}$ is the output to be regulated to a certain desired reference r . To this end, we follow the standard set-up of output regulation [2] and we extend system (5.1) with an integral action processing the desired error to be regulated. In other words, we consider a dynamical feedback law of the form

$$\dot{\eta} = w_x(t, 0) - r, \quad u(t) = k\eta(t), \quad t \in \mathbb{R}_+ \quad (5.2)$$

where $\eta \in \mathbb{R}$ is the state of the controller, $r \in \mathbb{R}$ is the desired reference and $k \neq 0$ is a positive constant to be selected small enough, as shown later. The closed-loop system (5.1), (5.2) can be seen as an augmented system, i.e. a PDE system (whose state is w) coupled with an ODE (whose state is η), which reads

$$\begin{cases} w_t + w_x + w_{xxx} = d(x), & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = w_x(t, 0) - r, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (5.3)$$

System (5.3) can be rewritten in the abstract form as follows

$$\begin{cases} \frac{d}{dt}\zeta &= \mathcal{A}\zeta + D, \\ \zeta(0) &= \zeta_0, \end{cases} \quad (5.4)$$

where $D = [-r \quad d]^\top$, $\zeta = [\eta \quad w]^\top$ and where the operator \mathcal{A} is defined by

$$\mathcal{A}(\eta, w) = [w'(0) \quad -w''' - w']^\top. \quad (5.5)$$

The domain of \mathcal{A} is $D(\mathcal{A}) := \{(\eta, w) \in \mathbb{R} \times H^3(0, L) \mid w(0) = w(L) = 0, w'(L) = k\eta\}$. Moreover, consider $X := \mathbb{R} \times L^2(0, L)$, that is the state space of (5.3). It is a Hilbert space as the Cartesian product of two Hilbert spaces.

In this section, we aim at proving the well-posedness and the asymptotic stability of an equilibrium of (5.3). To do so, we will consider a change of coordinates, defined thanks to such a nonzero equilibrium state of (5.3). Its existence and uniqueness is ensured by the following lemma.

LEMMA 5.1. *For any $k \neq 0$ and $(d, r) \in L^2(0, L) \times \mathbb{R}$ there exist a unique equilibrium state $(\eta_\infty, w_\infty) \in X$ to system (5.3).*

Proof: Let $(d, r) \in L^2(0, L) \times \mathbb{R}$. We consider the following boundary value problem

$$\begin{cases} w_\infty''(x) + w_\infty'''(x) = d(x), & x \in [0, L], \\ w_\infty(0) = w_\infty(L) = 0, \\ w_\infty'(0) = r, \end{cases}$$

which represents the nonzero equilibrium state of (5.3). Consider the function $\phi(x) = \frac{rx(L-x)}{L}$. Note that it is smooth. It satisfies the boundary conditions,

$$\phi(0) = \phi(L) = 0, \quad \phi'(0) = r.$$

Let set $\psi = w_\infty - \phi$. Then ψ satisfies the following system

$$\begin{cases} \psi'(x) + \psi'''(x) = j(x), & x \in [0, L], \\ \psi(0) = \psi(L) = 0, \\ \psi'(0) = 0, \end{cases}$$

where $j(x) = d(x) - \phi'(x)$. Let \mathcal{S} denote the operator associated with the linear KdV equation

$$\mathcal{S}\psi = -\psi' - \psi'''$$

with the domain $D(\mathcal{S}) \subset L^2(0, L)$ defined as $D(\mathcal{S}) := H_L^3(0, L)$. The adjoint operator \mathcal{S}^* is defined by $\mathcal{S}^*\psi = \psi''' + \psi'$ with domain $D(\mathcal{S}^*) := H_0^3(0, L)$. Following [32, Lemma 4], we can prove that the canonical embedding from $D(\mathcal{S}^*)$, equipped with the graph norm, into $L^2(0, L)$, is compact. Then, according to [10, Proposition 4.24], \mathcal{S}^* is an operator with compact resolvent. This implies that its spectrum consists only of eigenvalues. Moreover, 0 is not an eigenvalue of \mathcal{S}^* . Hence, there exists a unique solution ψ_∞ to the equation $\mathcal{S}^*\psi = j$. The equilibrium (η_∞, w_∞) can then be computed as

$$\begin{cases} w_\infty(x) &= \psi_\infty + \phi(x), & x \in [0, L], \\ \eta_\infty &= \frac{w_\infty'(L)}{k}, \end{cases}$$

with the function ϕ defined at the beginning of the proof. This concludes the proof of Lemma 5.1. \square

Next, we show the following well-posedness result for the closed-loop system (5.3).

THEOREM 5.2. *There exist $k^* > 0$ such that for any $k \in (0, k^*)$, for any $(d, r) \in L^2(0, L) \times \mathbb{R}$ and for any initial condition $(\eta_0, w_0) \in X$ (resp. $D(\mathcal{A})$), there exists a unique weak solution $(\eta, w) \in C^0(\mathbb{R}_+; X)$ (resp. strong solution in $C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$) to system (5.3).*

Proof: Given $(d, r) \in L^2(0, L) \times \mathbb{R}$ and k , let (η_∞, w_∞) the corresponding equilibrium to (5.3) computer according to Lemma 5.1. Consider the following change of coordinates

$$(w, \eta) \mapsto (\bar{w}, \bar{\eta}) := (w - w_\infty, \eta - \eta_\infty).$$

The $(\bar{w}, \bar{\eta})$ -dynamics are given by

$$\begin{cases} \bar{w}_t + \bar{w}_x + \bar{w}_{xxx} = 0, & (t, x) \in \Omega, \\ \bar{w}(t, 0) = \bar{w}(t, L) = 0, & t \in \mathbb{R}_+, \\ \bar{w}_x(t, L) = k\bar{\eta}(t), & t \in \mathbb{R}_+, \\ \dot{\bar{\eta}}(t) = \bar{w}_x(t, 0), & t \in \mathbb{R}_+, \\ \bar{w}(0, x) = \bar{w}_0(x), \bar{\eta}(0) = \bar{\eta}_0, & x \in [0, L], \end{cases} \quad (5.6)$$

where $\bar{w}_0(x) = w_0(x) - w_\infty(x)$ and $\bar{\eta}_0 = \eta_0 - \eta_\infty$.

Note that the systems (5.3) and (5.6) are equivalent. Then, if one proves that the operator \mathcal{A} defined in (5.5) is a m -dissipative operator on $(X, \|\cdot\|_X)$, one can apply the result provided by [5, Theorem 3.1], and conclude that the statement of Theorem 2.1 holds. For that, we look for an equivalent norm and a related scalar product coming from a Lyapunov functional. We will prove then the dissipativity with respect with this scalar product. This Lyapunov functional is built following the forwarding approach (see e.g [47]). To this end, we define the operator $\mathcal{M} : L^2(0, 1) \rightarrow \mathbb{R}$ as follows

$$\mathcal{M}\mathcal{S}w = \mathcal{C}w, \quad \forall w \in D(\mathcal{S}), \quad (5.7)$$

where $\mathcal{C} : f \in H_0^1(0, L) \mapsto f'(0) \in \mathbb{R}$. Note that this operator equation is a Sylvester equation. Since the strongly continuous semigroup generated by the operator \mathcal{S} is exponentially stable, (5.7) admits a unique solution, see [39, Lemma 22]. As in [47], we look \mathcal{M} in the form $\mathcal{M}w = \int_0^L M(x)w(x)dx$. In order to obtain an explicit solution, we can write equation (5.7) in the explicit form

$$w'(0) = - \int_0^L M(x)[w'(x) + w'''(x)]dx \quad \forall w \in D(\mathcal{S}).$$

Using integration by parts we obtain

$$w'(0) = \int_0^L w(x)[M'(x) + M'''(x)]dx + M(0)w''(0) - M(L)w''(L) - M'(0)w'(0),$$

for all $w \in D(\mathcal{S})$. From the latter equation, we obtain the following boundary value problem

$$\begin{cases} M''' + M' = 0, \\ M(0) = M(L) = 0, \\ M'(0) = -1. \end{cases} \quad (5.8)$$

It can be verified that the function

$$M : x \in \mathbb{R} \mapsto \frac{-2 \sin(\frac{x}{2}) \sin(\frac{L-x}{2})}{\sin(\frac{L}{2})}$$

is a solution to (5.8). Computations are omitted for space reasons. Moreover, it is the unique solution to (5.8) and the operator \mathcal{M} defined above is the unique solution to the Sylvester equation (5.7). Then, the operator $\mathcal{M} : L^2(0, L) \rightarrow \mathbb{R}$ can be expressed as

$$\mathcal{M}\varphi = \int_0^L M(x)\varphi(x)dx.$$

Using the Lyapunov functional V built in Theorem 2.5, we consider the candidate Lyapunov functional $\mathcal{V} : X \rightarrow \mathbb{R}$ defined as

$$\mathcal{V}(\eta, w) = V(w) + q(\eta - \mathcal{M}w)^2, \quad (5.9)$$

where $q > 0$ is a positive parameter to be defined. Notice that the Lyapunov functional (5.9) is equivalent to the standard norm on the space X for any $q > 0$. More precisely, there exists a positive constant ξ (that depends on q) such that the following inequality holds

$$\frac{1}{\xi} \|(\eta, w)\|_X^2 \leq \mathcal{V}(\eta, w) \leq \xi \|(\eta, w)\|_X^2, \quad \forall (\eta, w) \in X. \quad (5.10)$$

To show (5.10), note that, according to [47, Proof of Proposition 4], we have, for all $\rho \in]0, 1[$ and for all $(\eta, w) \in X$

$$(\eta - \mathcal{M}w)^2 \geq \rho \left(\frac{1}{2} \eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right).$$

On the other hand, using the Young's inequality, we have, for all

$$(\eta - \mathcal{M}w)^2 \leq 2(\eta^2 + M\|_{L^2}^2 \|w\|_{L^2}^2).$$

Furthermore, according to Theorem 2.5, we know that V satisfies the inequality (2.2). Then we have

$$\rho q \left(\frac{1}{2} \eta^2 - \|M\|_{L^2}^2 \|w\|_{L^2}^2 \right) + \underline{\alpha}(\|w\|_{L^2}) \leq \mathcal{V}(w) \leq 2q(\eta^2 + M\|_{L^2}^2 \|w\|_{L^2}^2) + \bar{\alpha}(\|w\|_{L^2}). \quad (5.11)$$

Therefore, by selecting ρ sufficiently small, inequality (5.10) holds. From Lyapunov functional \mathcal{V} defined in (5.9), one finally deduces a scalar product, that we define as follows

$$\left\langle [\eta_1 \quad w_1]^\top, [\eta_2 \quad w_2]^\top \right\rangle_{\mathcal{V}} := q(\eta_1 - \mathcal{M}w_1)(\eta_2 - \mathcal{M}w_2) + \langle w_1, w_2 \rangle_{L^2} + \frac{1}{2\bar{p}} \langle \Pi^{-1}w_1, \Pi^{-1}w_2 \rangle_{L^2}, \quad (5.12)$$

that is equivalent to the usual scalar product in X .

Now, we are in position to prove that \mathcal{A} is m -dissipative according to [38]. For this, we first prove that \mathcal{A} is dissipative. Second, we show that \mathcal{A} is maximal.

Step 1: \mathcal{A} is dissipative. To prove this result, we use the scalar product given in (5.12). Let $\zeta = (\eta, w) \in D(\mathcal{A})$. One therefore has, for all $\zeta \in D(\mathcal{A})$

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &= q(w'(0) + \mathcal{M}(w''' + w'))(\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} - \frac{1}{2\bar{p}} \langle \Pi^{-1}(w''' + w'), \Pi^{-1}w \rangle_{L^2} \\ &= q \left(w'(0) + \int_0^L M(x)[w'(x) + w'''(x)] dx \right) (\eta - \mathcal{M}w) - \langle w' + w''', w \rangle_{L^2} \\ &\quad - \frac{1}{2\bar{p}} \langle \Pi^{-1}(w''' + w'), \Pi^{-1}w \rangle_{L^2}. \end{aligned} \quad (5.13)$$

Now, after some integrations by parts, we have, for all $\zeta \in D(\mathcal{A})$

$$\int_0^L M(x)[w'(x) + w'''(x)] dx = -k\eta - w'(0). \quad (5.14)$$

On the other hand, according to Section 4 and in particular inequality (4.7), we have, for all $\zeta \in D(\mathcal{A})$

$$- \langle w' + w''', w \rangle_{L^2} - \frac{1}{2\bar{p}} \langle \Pi^{-1}(w''' + w'), \Pi^{-1}w \rangle_{L^2} \leq \frac{1}{2} \left(-\frac{c}{2\bar{p}} \|w\|_{L^2}^2 + \frac{k^2}{2\bar{p}} \eta^2 \right). \quad (5.15)$$

Combining inequality (5.14) and (5.15) in (5.13) leads to the following inequality, for all $\zeta \in D(\mathcal{A})$

$$\langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} \leq -kqz(\eta - \mathcal{M}w) + \frac{1}{2} \left(-\frac{c}{2\bar{p}} \|w\|_{L^2}^2 + \frac{k^2}{2\bar{p}} \eta^2 \right). \quad (5.16)$$

Using first Young's inequality and then Cauchy-Schwarz's inequality, we finally obtain that, for all $\zeta \in D(\mathcal{A})$

$$\begin{aligned} \langle \mathcal{A}\zeta, \zeta \rangle_{\mathcal{V}} &\leq \frac{1}{2} \left(-\frac{\underline{c}}{4\bar{p}} \|w\|_{L^2}^2 + \frac{k^2}{4\bar{p}} \eta^2 \right) - kq\eta^2 + kq \left(\frac{\eta^2}{2} + \frac{\|M\|_{L^2}^2}{2} \|w\|_{L^2}^2 \right) \\ &\leq \frac{1}{2} \left(-\frac{\underline{c}}{2\bar{p}} + kq\|M\|_{L^2}^2 \right) \|\bar{w}\|_{L^2}^2 + \frac{1}{2} \left(\frac{k^2}{2\bar{p}} - kq \right) \bar{\eta}^2. \end{aligned} \quad (5.17)$$

Note that if

$$kq\|M\|_{L^2}^2 < \frac{\underline{c}}{2\bar{p}} \quad \text{and} \quad k < 2q\bar{p},$$

then the operator \mathcal{A} is dissipative. This is obtained for all $0 < k < k^*$ where k^* and q are selected as

$$k^* = \frac{\sqrt{\underline{c}}}{\|M\|_{L^2}} \quad \text{and} \quad \frac{k}{2\bar{p}} < q < \frac{\underline{c}}{2k\bar{p}\|M\|_{L^2}^2}. \quad (5.18)$$

This concludes the proof of dissipativity of \mathcal{A} .

Step 2: \mathcal{A} is a maximal operator. According to Lümer–Phillips theorem [38, Theorem 4.3], proving that \mathcal{A} is maximal reduces to show that for all $\lambda_0 \geq k^*$ and for all $\zeta \in X$, there exists $\tilde{\zeta} \in D(\mathcal{A})$ such that

$$(\lambda_0 I_X - \mathcal{A})\tilde{\zeta} = \zeta.$$

Let $(\eta, w) \in X$. We look for a $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$ satisfying

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = k\tilde{\eta}, \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta, \end{cases} \quad (5.19)$$

namely

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0} (\eta + \tilde{w}'(0)), \\ \lambda_0 \tilde{\eta} - \tilde{w}'(0) = \eta. \end{cases}$$

Now, we consider the following boundary value problem

$$\begin{cases} \tilde{w}''' + \tilde{w}' + \lambda_0 \tilde{w} = w, & x \in [0, L], \\ \tilde{w}(0) = \tilde{w}(L) = 0, \\ \tilde{w}'(L) = \frac{k}{\lambda_0} (\eta + \tilde{w}'(0)), \end{cases}$$

and the smooth function $\tilde{\phi}(x) = \frac{k\eta x^2(x-L)}{\lambda_0 L^2}$ satisfying the boundary conditions

$$\tilde{\phi}(0) = \tilde{\phi}(L) = \tilde{\phi}'(0) = 0, \quad \tilde{\phi}'(L) = \frac{k}{\lambda_0} \eta.$$

We set $\tilde{\psi} = \tilde{w} - \tilde{\phi}$. Then $\tilde{\psi}$ satisfies the following boundary value problem

$$\begin{cases} \tilde{\psi}' + \tilde{\psi}''' + \lambda_0 \tilde{\psi} = \tilde{j}(x), & x \in [0, L], \\ \tilde{\psi}(0) = \tilde{\psi}(L) = 0, \\ \tilde{\psi}'(L) = \frac{k}{\lambda_0} \tilde{\psi}'(0), \end{cases} \quad (5.20)$$

where $\tilde{j}(x) = w(x) - \tilde{\phi}'(x) - \tilde{\phi}'''(x) - \lambda_0 \tilde{\phi}$. Now, let $\hat{\mathcal{A}}$ be the operator defined by

$$\hat{\mathcal{A}}\psi = -\psi' - \psi'''$$

with the domain $D(\widehat{\mathcal{A}}) \subset L^2(0, L)$ defined by

$$D(\widehat{\mathcal{A}}) := \left\{ \psi \in H^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(L) = \frac{k}{\lambda_0} \psi'(0) \right\}.$$

We define also the adjoint operator $\widehat{\mathcal{A}}^*$ as

$$\widehat{\mathcal{A}}^* \psi = \psi''' + \psi'$$

with domain

$$D(\widehat{\mathcal{A}}^*) := \left\{ \psi \in H^3(0, L) : \psi(0) = \psi(L) = 0, \psi'(0) = \frac{k}{\lambda_0} \psi'(L) \right\}.$$

Note that $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{A}}^*$ are dissipative. Indeed, since $\lambda_0 > k$, we have

$$\begin{aligned} \int_0^L \psi \widehat{\mathcal{A}} \psi dx &= \left(\frac{k}{\lambda_0} - 1 \right) \psi'(0)^2 < 0, & \psi \in D(\widehat{\mathcal{A}}), \\ \int_0^L \psi \widehat{\mathcal{A}}^* \psi dx &= \left(\frac{k}{\lambda_0} - 1 \right) \psi'(L)^2 < 0, & \psi \in D(\widehat{\mathcal{A}}^*). \end{aligned}$$

Moreover, $\widehat{\mathcal{A}}$ is closed and $D(\widehat{\mathcal{A}})$ is dense in $L^2(0, L)$. Then, according to [38, Theorem 4.3 and Corollary 4.4] $\widehat{\mathcal{A}}$ is m -dissipative operator.

Finally, since $\widehat{\mathcal{A}}$ is a m -dissipative operator then the system (5.20) admits a solution $\tilde{\psi}$ in $D(\widehat{\mathcal{A}})$. As a consequence, there exist $(\tilde{\eta}, \tilde{w}) \in D(\mathcal{A})$ solution of (5.19). This proves that \mathcal{A} is maximal and concludes the proof of Theorem 2.1. \square

The next result deals with the exponential stability of equilibrium state (η_∞, w_∞) and with the related output regulation.

THEOREM 5.3 (Stabilization and regulation). *Consider system (5.3). Let $k \in (0, k^*)$ with k^* given by Theorem 5.2. Then, for any $(d, r) \in L^2(0, L) \times \mathbb{R}$ and for any initial condition $(\eta_0, w_0) \in X$ (resp. $D(\mathcal{A})$), the following holds, the solutions to system (5.3) satisfies the following.*

1. *There exist positive constants ν and C such that, for all $(\eta_0, w_0) \in X$, and for all $t \geq 0$*

$$\|(\eta, w) - (\eta_\infty, w_\infty)\|_X \leq C e^{-\nu t} \|(\eta_0, w_0) - (\eta_\infty, w_\infty)\|_X. \quad (5.21)$$

2. *Moreover, the output y is regulated towards the reference r . In other words, for all $(\eta_0, w_0) \in D(\mathcal{A})$*

$$\lim_{t \rightarrow +\infty} |w_x(t, 0) - r| = 0. \quad (5.22)$$

for any strong solution to (5.3).

Proof: We prove each item of the statement separately.

Step 1: Exponential stability of equilibrium state (η_∞, w_∞) . We prove Item 1 of the theorem for initial conditions $(\eta_0, w_0) \in D(\mathcal{A})$. The result follows for all initial conditions in X by a standard density argument (see e.g. [28, Lemma 1]). Moreover, this is enough to prove that the origin is an exponentially stable equilibrium point for system (5.6). To this end, consider the Lyapunov functional \mathcal{V} defined in (5.9). According to the proof of dissipativity of \mathcal{A} , the time derivative of \mathcal{V} along the strong solution to (5.6) yields

$$\dot{\mathcal{V}} \leq \left(-\frac{c}{2\bar{p}} + kq \|M\|_{L^2}^2 \right) \|\bar{w}\|_{L^2}^2 + \left(\frac{k^2}{2\bar{p}} - kq \right) \bar{\eta}^2 \quad (5.23)$$

with k and q satisfying (5.18). As a consequence, from (5.11) and Grönwall's lemma, there exist positive constants ν and C such that, for all $(\eta_0, w_0) \in D(\mathcal{A})$,

$$\|(\bar{\eta}, \bar{w})\|_X \leq C e^{-\nu t} \|(\bar{\eta}_0, \bar{w}_0)\|_X \quad (5.24)$$

i.e for all $(\eta_0, w_0) \in D(\mathcal{A})$,

$$\|(\eta, w) - (\eta_\infty, w_\infty)\|_X \leq C e^{-\nu t} \|(\eta_0, w_0) - (\eta_\infty, w_\infty)\|_X. \quad (5.25)$$

Then, by using the density of $D(\mathcal{A})$ in X , we conclude that the first statement of Theorem 5.3 holds.

Step 2: Output regulation. At this point, it only remains to prove the second statement of Theorem 5.3 to complete this proof. For this, note that if $(\eta_0, w_0) \in D(\mathcal{A})$, then $(\bar{\eta}_0, \bar{w}_0) \in D(\mathcal{A})$. Then $(\bar{\eta}, \bar{w}) \in C^1(\mathbb{R}_+; X) \cap C^0(\mathbb{R}_+; D(\mathcal{A}))$. Now, let us introduce the new variables \bar{v}, \bar{b} defined as follows

$$\bar{v} := \bar{w}_t \quad \text{and} \quad \bar{b} := \dot{\bar{\eta}}.$$

The dynamics of (\bar{v}, \bar{b}) is given as

$$\begin{cases} \bar{v}_t + \bar{v}_x + \bar{v}_{xxx} = 0, & (t, x) \in \Omega, \\ \bar{v}(t, 0) = \bar{v}(t, L) = 0, & t \in \mathbb{R}_+ \\ \bar{v}_x(t, L) = k\bar{b}(t), & t \in \mathbb{R}_+ \\ \dot{\bar{b}}(t) = \bar{v}_x(t, 0), & t \in \mathbb{R}_+ \\ \bar{v}(0, x) = -\bar{w}'_0(x) - \bar{w}'''_0(x), \bar{b}(0) = \bar{w}'_0(0), & x \in [0, L]. \end{cases} \quad (5.26)$$

Since $(\bar{v}(0, \cdot), \bar{b}(0)) \in X$, then, according to the Theorem 2.1 and the first statement of Theorem 5.3, we have $(\bar{v}, \bar{b}) \in C^0(\mathbb{R}_+; X)$ and

$$\|(\bar{b}, \bar{v})\|_X \leq C e^{-\nu t} \|(\bar{b}(0), \bar{v}(0, \cdot))\|_X, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

In particular, we obtain

$$\|\bar{w}_t\|_{L^2} \leq C e^{-\nu t} \|\bar{w}_t(0, \cdot)\|_{L^2}, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}). \quad (5.27)$$

Now, by multiplying the first equation of (5.6) by \bar{w} and integrating by parts, we get after some computations

$$k^2 \bar{\eta}(t)^2 - \bar{w}_x(t, 0)^2 = \int_0^L \bar{w}(t, x) \bar{w}_t(t, x) dx.$$

Using Cauchy-Schwarz's inequality, from (5.24) and (5.27) we finally obtain

$$|\bar{w}_x(t, 0)|^2 \leq \|\bar{w}(t, \cdot)\|_{L^2} \|\bar{w}_t(t, \cdot)\|_{L^2} + k^2 |\bar{\eta}(t)|^2 \xrightarrow{t \rightarrow \infty} 0, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

From the previous inequality we can deduce

$$\lim_{t \rightarrow \infty} |\bar{w}_x(t, 0)|^2 = 0 \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

and therefore,

$$\lim_{t \rightarrow +\infty} |w_x(t, 0) - r| = 0, \quad \forall (\eta_0, w_0) \in D(\mathcal{A}).$$

This concludes the proof of Theorem 5.3. \square

5.2. Equilibrium points of the nonlinear system. In this section, we consider the regulation problem for a nonlinear KdV (1.1). In particular, we consider the system

$$\begin{cases} w_t + w_x + w_{xxx} + w w_x = d_*(x), & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = u(t), & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), & x \in [0, L], \\ y(t) = w_x(t, 0), & t \in \mathbb{R}_+, \end{cases} \quad (5.28)$$

where $d_* \in L^2(0, L)$ is a constant perturbation, $u \in \mathbb{R}$ is the control input and $y(t) \in \mathbb{R}$ is the output to be regulated. Following the results of the previous section, we consider the integral control as follows

$$\dot{\eta} = w_x(t, 0) - r_*, \quad u(t) = k\eta(t), \quad t \in \mathbb{R}_+, \quad (5.29)$$

where $r \in \mathbb{R}$ is a constant reference and $k > 0$ is the control gain. Then, the closed-loop system reads

$$\begin{cases} w_t + w_x + w_{xxx} + ww_x = d_*(x), & (t, x) \in \Omega, \\ w(t, 0) = w(t, L) = 0, & t \in \mathbb{R}_+, \\ w_x(t, L) = k\eta(t), & t \in \mathbb{R}_+, \\ \dot{\eta}(t) = w_x(t, 0) - r_*, & t \in \mathbb{R}_+, \\ w(0, x) = w_0(x), \eta(0) = \eta_0, & x \in [0, L]. \end{cases} \quad (5.30)$$

In this article, we do not prove any well-posedness of (5.30), but we limit ourselves to establish the existence of a nonzero equilibrium state (η_*, w_*) to (5.30). However, we believe that existence and uniqueness for this system hold using some fixed-point strategy. Indeed, our goal is to change d in system (5.3) such that the solution to the linear system (5.3) converges exponentially to the equilibrium state (η_*, w_*) in the state space X . To do this, we first prove the existence of a unique equilibrium state of system (5.30), as stated in the following lemma.

LEMMA 5.4. *There exist $\varepsilon > 0$, and $R > 0$ such that, for $(d_*, r_*) \in L^2(0, L) \times \mathbb{R}$ satisfying $\|d_*\|_{L^2} \leq \varepsilon$ and $|r_*| \leq R$, there exists a unique equilibrium state $(\eta_*, w_*) \in X$ to system (5.30).*

Proof: Let $(d_*, r_*) \in L^2(0, L) \times \mathbb{R}$ such that $\|d_*\|_{L^2} \leq \varepsilon$ and $|r_*| \leq R$ with ε and R to be chosen later. Then, we want to prove the existence of a unique solution $(\eta_*, w_*) \in D(\mathcal{A})$ to the following boundary value problem

$$\begin{cases} w_*'(x) + w_*'''(x) + w_*(x)w_*'(x) = d_*(x), & x \in [0, L], \\ w_*(0) = w_*(L) = 0, \\ w_*'(L) = k\eta_*, \\ w_*'(0) = r_*, \end{cases}$$

where (η_*, w_*) corresponds to the equilibrium of system (5.30), if it exists. Note that, it is enough to prove the existence of a unique solution $w_* \in H^3(0, L)$ to the following boundary value problem

$$\begin{cases} w_*'(x) + w_*'''(x) + w_*(x)w_*'(x) = d_*(x), & x \in [0, L], \\ w_*(0) = w_*(L) = 0, \\ w_*'(0) = r_*, \end{cases} \quad (5.31)$$

and then choose afterwards $\eta_* = \frac{w_*'(L)}{k}$. Hence, we prove that there exists a solution to system (5.31) by following a fixed-point strategy. We set

$$H_r^3(0, L) := \left\{ w \in H^3(0, L) : w(0) = w(L) = w'(0) = r_* \right\}.$$

We consider the operator $\Gamma : H_r^3(0, L) \rightarrow H_r^3(0, L)$ defined by $\Gamma(w) = \bar{\varphi}$ where $\bar{\varphi}$ is the solution to

$$\begin{cases} \bar{\varphi}'(x) + \bar{\varphi}'''(x) = d_1(x) - w_*(x)w_*'(x), & x \in [0, L], \\ \bar{\varphi}(0) = \bar{\varphi}(L) = 0, \\ \bar{\varphi}'(0) = r_*, \end{cases} \quad (5.32)$$

Note that the function $\|\cdot\|_{H_r^3} : w \in H^3(0, L) \mapsto \|w' + w'''\|_{L^2} \in \mathbb{R}_+$ is a semi-norm on the space $H^3(0, L)$. Furthermore, $H_r^3(0, L) \subset H_0^1(0, L)$. Then, according to the Poincaré's inequality, the semi-norm $\|\cdot\|_{H_r^3}$ is a norm on the space $H_r^3(0, L)$ which is equivalent to the standard norm induced by $H^3(0, L)$. In other words, there exists a positive constant C such that

$$\|w_*\|_{H_r^3} \leq \|w_*\|_{H^3(0, L)} \leq C\|w_*\|_{H_r^3}, \quad \forall w_* \in H_r^3(0, L). \quad (5.33)$$

Now, we have, for all

$$\begin{aligned} \|\Gamma(w_*)\|_{H_r^3} &= \|d_* - w_*w_*'\|_{L^2} \\ &\leq \|d_*\|_{L^2} + \|w_*w_*'\|_{L^2} \\ &\leq \|d_*\|_{L^2} + \|w_*\|_{L^\infty}\|w_*'\|_{L^2}, \end{aligned}$$

for all $w_* \in H_r^3(0, L)$. Denoting with the constant K the norm of the embedding $H^3(0, L)$ in $L^\infty(0, L)$, according to the Rellich-Kondrachov Theorem see [6, Theorem 9.16], we have

$$\begin{aligned} \|\Gamma(w_*)\|_{H_r^3} &\leq \|d_*\|_{L^2} + K\|w_*\|_{H^3(0, L)}\|w_*'\|_{L^2} \\ &\leq \|d_*\|_{L^2} + KC\|w_*\|_{H_r^3}^2 \\ &\leq \varepsilon + KC\|w_*\|_{H_r^3}^2, \end{aligned}$$

$w_* \in H_r^3(0, L)$. Moreover, we have for all $w_1, w_2 \in H_r^3(0, L)$

$$\begin{aligned} \|\Gamma(w_1) - \Gamma(w_2)\|_{H_r^3} &= \|w_1 w_1' - w_2 w_2'\|_{L^2} \\ &\leq \|(w_1 - w_2)w_1'\|_{L^2} + \|w_2(w_1' - w_2')\|_{L^2} \\ &\leq KC\|w_1 - w_2\|_{H_r^3}\|w_1\|_{H_r^3} + KC\|w_1 - w_2\|_{H_r^3}\|w_2\|_{H_r^3} \\ &\leq KC(\|w_2\|_{H_r^3} + \|w_2\|_{H_r^3})\|w_1 - w_2\|_{H_r^3}. \end{aligned}$$

We consider now the operator Γ defined as in (5.32), restricted on the closed ball

$$B_{\bar{R}} := \left\{ w \in H_r^3(0, L) : \|w\|_{H_r^3} \leq \bar{R} \right\}$$

with \bar{R} to be chosen later. Then, we have

$$\begin{aligned} \|\Gamma(w)\|_{H_r^3} &\leq \varepsilon + KC\bar{R}^2, \\ \|\Gamma(w_1) - \Gamma(w_2)\|_{H_r^3} &\leq 2KC\bar{R}\|w_1 - w_2\|_{H_r^3}, \end{aligned}$$

for all $w \in B_{\bar{R}}$. We choose \bar{R} such that the following conditions hold

$$\varepsilon < \frac{1}{4KC} \quad \text{and} \quad \frac{1 - \sqrt{1 - 4\varepsilon KC}}{2KC} \leq \bar{R} < \frac{1}{2KC}.$$

Then, we can apply the Banach fixed point theorem [6, Theorem 5.7] and prove that the operator Γ admits a unique fixed point, concluding the proof of existence of the unique solution $w_* \in B_{\bar{R}}$ to (5.31).

Now, given \bar{R} , we deduce the value of r . Indeed, since $w_* \in H^3(0, L)$ then we have $w_*' \in H^2(0, L)$. Then, according to the embedding of $H^2(0, L)$ in $C^1([0, L])$, we have $w_*' \in C^1([0, L])$. Therefore, according to [51, Lemma 1] we have

$$(w_*'(0))^2 \leq \frac{2}{L}\|w_*'\|_{L^2}^2 + L\|w_*''\|_{L^2}^2,$$

which implies

$$(w_*'(0))^2 \leq \left(\frac{2}{L} + L \right) \|w_*\|_{H^3(0, L)}^2. \quad (5.34)$$

Since $w_* \in B_{\bar{R}}$, then according to (5.33) and (5.34), we obtain

$$r_*^2 \leq C \left(\frac{2}{L} + L \right) \bar{R}^2.$$

Finally, we can choose $R = \bar{R}\sqrt{C \left(\frac{2}{L} + L \right)}$. This concludes the proof of Lemma 5.4. \square

Then, the main result of this section is stated as follows. In particular, we show that the equilibrium (η_*, w_*) of the nonlinear KdV equation (5.30) is an equilibrium for its linearized equation, given by (5.3), and for some perturbation $d \in L^2(0, L)$, with possibly $d \neq d_*$.

THEOREM 5.5. *Let $(d_*, r_*) \in L^2(0, L) \times \mathbb{R}$ be selected according to Lemma 5.4 and let (η_*, w_*) be the corresponding equilibrium of system (5.30). Then, there exists $d \in L^2(0, L)$ such that, for any $k \in (0, k^*)$ with*

k^* given by Theorem 5.2, and for all initial condition $(\eta_0, w_0) \in X$, the solution to system (5.3), with such a d and $r = r_*$, converges exponentially to the equilibrium state (η_*, w_*) in the state space X . In other words, there exist positive constants ν and C such that for all $(\eta_0, w_0) \in X$ and for all $t \geq 0$

$$\|(\eta, w) - (\eta_*, w_*)\|_X \leq C e^{-\nu t} \|(\eta_0, w_0) - (\eta_*, w_*)\|_X.$$

Proof: Let $(d_*, r_*) \in L^2(0, L) \times \mathbb{R}$ such that $\|d_*\|_{L^2} \leq \varepsilon$ and $(\eta_0, w_0) \in X$ as in the statement of Lemma 5.4. Let us introduce the new variables

$$(w, \eta) \mapsto (\omega, v) := (w - w_*, \eta - \eta_*)$$

where (η, w) is the solution to (5.3). The dynamics of ω, v is given by

$$\begin{cases} \omega_t + \omega_x + \omega_{xxx} = d - d_* - w_* w_*', & (t, x) \in \Omega, \\ \omega(t, 0) = \omega(t, L) = 0, & t \in \mathbb{R}_+, \\ \omega_x(t, L) = kv(t), & t \in \mathbb{R}_+, \\ \dot{v}(t) = \omega_x(t, 0), & t \in \mathbb{R}_+, \\ \omega(0, x) = \omega_0(x), v(0) = v_0, & x \in [0, L], \end{cases} \quad (5.35)$$

where $\omega_0(x) = w_0(x) - w_*(x)$ and $v_0 = \eta_0 - \eta_*$. Then, select d as

$$d = d_* + w_* w_*'. \quad (5.36)$$

System (5.35) with the perturbation d selected as in (5.36) reads

$$\begin{cases} \omega_t + \omega_x + \omega_{xxx} = 0, & (t, x) \in \Omega, \\ \omega(t, 0) = \omega(t, L) = 0, & t \in \mathbb{R}_+, \\ \omega_x(t, L) = kv(t), & t \in \mathbb{R}_+, \\ \dot{v}(t) = \omega_x(t, 0), & t \in \mathbb{R}_+, \\ \omega(0, x) = \omega_0(x), v(0) = v_0, & x \in [0, L]. \end{cases} \quad (5.37)$$

Then, according to Theorem 2.1 and the first statement of Theorem 5.3, we have $(v, \omega) \in C^0(\mathbb{R}_+; X)$ and

$$\|(v, \omega)\|_X \leq C e^{-\nu t} \|(v_0, \omega_0)\|_X, \quad \forall (v_0, \omega_0) \in X.$$

Therefore, by linearity, we have $(\eta, w) \in C^0(\mathbb{R}_+; X)$ and

$$\|(\eta, w) - (\eta_*, w_*)\|_X \leq C e^{-\nu t} \|(\eta_0, w_0) - (\eta_*, w_*)\|_X, \quad \forall (v_0, \omega_0) \in X,$$

concluding the proof of Theorem 5.5. □

6. Conclusion. In this article, we have solved the output regulation problem by means of an integral action for a Korteweg-de-Vries (KdV) equation controlled at the boundary and subject to some distributed disturbance. For this, we have followed a Lyapunov approach. We have first designed an ISS Lyapunov functional which is obtained by strictifying the energy associated to the system. In particular, the energy is modified by adding a second term which is obtained from the design of an observer built with the backstepping technique. Then, thanks to this ISS Lyapunov functional, we have applied the forwarding method to achieve our goal.

Future extensions include the application of our methodology on the nonlinear KdV equation (5.30). We are also interested in establishing ISS properties for a nonlinear KdV when $L \in \mathcal{N}$. Finally, we believe that the proposed strictification approach can be extended also to other classes of PDEs for which a strict Lyapunov function is not yet known.

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