On the concurrent normals conjecture for convex bodies

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Abstract

It is conjectured that any convex body in $\mathbb{R}^n$ has an interior point lying on normals through $2n$ distinct boundary points. This concurrent normals conjecture has been proved for $n = 2$ and $n = 3$ by E. Heil. For $n \geq 4$, it is only known that any convex body in $\mathbb{R}^n$ has an interior point lying on normals through six distinct boundary points. For $n \in \{3, 4\}$, we prove in this paper that any normal through a boundary point to any convex body K (with a smooth enough support function) in $\mathbb{R}^n$ passes arbitrarily close to the set of interior points of K lying on normals through at least 6 distinct points of ∂K. This study leads us to introduce and study new concepts for studying focals of closed convex hypersurfaces in $\mathbb{R}^{n+1}$. Finally, we prove that for some convex body K of $\mathbb{R}^4$, there are only 6 normal lines passing through the center of the minimal spherical shell.

Contents:

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1 Introduction and statements of main results

It is conjectured that any convex body in n-dimensional Euclidean space $\mathbb{R}^n$ has an interior point lying on normals through $2n$ distinct boundary points. Using the existence of a minimal spherical shell for any convex body and, a combination of Morse theory and approximation, E. Heil has proved this concurrent normals conjecture for $n = 2$ and $n = 3$ in [5, 6, 7]. For $n = 4$, the conjecture has been claimed to be proved by J. Pardon under a smoothness assumption on the boundary [17], but the proof is wrong and cannot be corrected: indeed, in Pardon's paper, the proof of the claim (Theorem 2) relies on a lemma (Lemma 3) in the statement of which it should be assumed that $A \cup B$ is the closure of K, condition which is not fulfilled in the proof of the claim. For $n \geq 4$, it is only known that any convex body in $\mathbb{R}^n$ has an interior point lying on normals through six distinct boundary points. However Zamfirescu has shown that, in the sense of Baire category based on the Hausdorff distance between convex sets, most interior points of most convex bodies lie on infinitely many normals [21]. For $n \in \{3, 4\}$, we prove in this paper that any normal through a boundary point to any convex body K (with a smooth enough support function) in $\mathbb{R}^n$ passes arbitrarily close to the set of interior points of K lying on normals through at least 6 distinct points of ∂K.

Our setting and tools

In this paper, we assume for the sake of simplicity of the presentation that the support function is C^∞ but our results remain true provided that the support function is smooth enough (say at least of class C^4). Actually, our main arguments essentially relies on the Morse lemma, which remains true for C^2-functions [15], and on elementary properties of the focal of the boundary.

In most of the papers on concurrent normals to a convex body K with a smooth boundary ∂K in $\mathbb{R}^{n+1}$, the focal (or evolute) of ∂K is regarded as the complement of the set of points $x \in \mathbb{R}^{n+1}$ such that the square of the distance function from x induces a Morse function on ∂K. In this paper, we will adopt another point of view. For any $x \in \mathbb{R}^{n+1}$, we will consider the support function of ∂K with respect to x, that is $h_x : \mathbb{S}^n \to \mathbb{R}$, $u \mapsto h(u) - \langle x, u \rangle$, where $h : \mathbb{S}^n \to \mathbb{R}$ is the support function of K, and we will regard the evolute of ∂K as the complement of set of points $x \in \mathbb{R}^{n+1}$ such that $h_x : \mathbb{S}^n \to \mathbb{R}$ is a Morse function.

We will also make intensive use of the so-called 'hedgehogs' of $\mathbb{R}^{n+1}$, which are the (possibly singular and self-intersecting) hypersurfaces of $\mathbb{R}^{n+1}$ that are parametrized by their Gauss map and parallel to some C^2 convex hypersurface in $\mathbb{R}^{n+1}$. Every $h \in C^2(\mathbb{S}^n; \mathbb{R})$ can be regarded as the support function of the hedgehog (hypersurface) $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ that is parametrized by the map $x_h : \mathbb{S}^n \to \mathcal{H}_h \subset \mathbb{R}^{n+1}$, $u \mapsto h(u)u + (\nabla h)(u)$, which may be interpreted as the inverse of the Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$ (see Section 2). Since all the hedgehogs that are parallel to a same C^2 convex hypersurface with support function $h \in C^2(\mathbb{S}^n; \mathbb{R})$ have the same evolute and support functions that only differ by a constant, we will denote this common evolute by $\mathcal{F}_{\nabla h}$, and place ourselves in the setting of hedgehogs.

For $h \in C^\infty(\mathbb{S}^n; \mathbb{R})$, and $u \in \mathbb{S}^n$, the normal line to the hedgehog $\mathcal{H}_h$ at $x_h(u)$ is defined to be the line passing through $x_h(u)$ and oriented by u; this normal line $\mathcal{N}_{\nabla h}(u) := \{\nabla h(u)\} + \mathbb{R}u$ is the perpendicular to the support hyperplane to $\mathcal{H}_h$ at $x_h(u)$. The focal set (or evolute) $\mathcal{F}_{\nabla h}$ of $\mathcal{H}_h$ can be defined as the locus of its centers of principal curvatures, or equivalently, as the envelope of its normal lines $(\mathcal{N}_{\nabla h}(u))_{u \in \mathbb{S}^n}$.

Our main statements

In the case that $n = 2$, we will prove the following result which will turn out to be a refinement of Heil's theorem.

Theorem 1. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^\infty(\mathbb{S}^2; \mathbb{R})$, and let $\mathcal{F}_{\nabla h}$ denote its focal surface. If $u \in \mathbb{S}^2$ is such that the normal line $\mathcal{N}_{\nabla h}(u)$ to $\mathcal{H}_h$ at $x_h(u)$ does not meet the singular locus of $\mathcal{F}_{\nabla h}$,*

then there exists some $r \in \mathbb{R}$ *such that* $x_{h-r}(u) = x_h(u) - ru \in \mathcal{N}_{\nabla h}(u)$ *is an (at least) double hyperbolic point of* $\mathcal{H}_{h-r}$.

Here, "$x_{h-r}(u) = x_h(u) - ru \in \mathcal{N}_{\nabla h}(u)$ is an (at least) double hyperbolic point of $\mathcal{H}_{h-r}$" means that there exists $v \in \mathbb{S}^2 \backslash \{u\}$, such that:

$$x_{h-r}(u) = x_{h-r}(v), \quad R_{h-r}(u) < 0, \quad \text{and } R_{h-r}(v) < 0$$

where R_{h-r} is the curvature function of $\mathcal{H}_{h-r}$ (that is, the inverse $1/\kappa_{h-r}$ of the Gauss curvature κ_{h-r} of $\mathcal{H}_{h-r}$).

Remark. It is worth to stress that the double hyperbolic point that Theorem 1 states to exist is not an arbitrary double hyperbolic point of $\mathcal{H}_{h-r}$ but a double hyperbolic point of $\mathcal{H}_{h-r}$ in the vicinity of which what we will call the 'absolute transverse orientation' of $\mathcal{H}_{h-r}$ is reversed. Such a double hyperbolic point $x = x_{h-r}(u) = x_{h-r}(v)$ of $\mathcal{H}_{h-r}$, for which we have $\{u, v\} \subset (h_x)^{-1}(\{0\})$, correspond to what we will call a pair $\{u, v\}$ of '*critical*' saddle points of the foliation $\mathcal{F}(\nabla h_x)$ of $\mathbb{S}^2$ by the level sets of the family of hedgehogs $(h_x - r)_{r \in \mathbb{R}}$. Here it is worth noting that *the criticality of such a pair* $\{u, v\}$ *of saddle points of* $\mathcal{F}(\nabla h_x)$ *depends on the global geometry of* $\mathcal{H}_{h-r}$.

We will deduce the following reformulation of Heil's theorem without making use of the notion of a minimal spherical shell.

Corollary 1 (Heil's theorem [5, 6]). *Let* $\mathcal{H}_h$ *be a hedgehog of* $\mathbb{R}^3$ *such that* $h \in C^\infty(\mathbb{S}^2; \mathbb{R})$. *There exists a point of* $\mathbb{R}^3$ *lying on infinitely many normals to* $\mathcal{H}_h$ *or an open set formed by points of* $\mathbb{R}^3 \setminus \mathcal{F}_{\nabla h}$ *lying on at least* 6 *normals to* $\mathcal{H}_h$.

From our proof of Corollary 1, we can in fact deduce the following stronger result.

Corollary 2 *Let* $\mathcal{H}_h$ *be a hedgehog of* $\mathbb{R}^3$ *such that* $h \in C^\infty(\mathbb{S}^2; \mathbb{R})$. *If there does not exist a point of* $\mathbb{R}^3$ *lying on infinitely many normals to* $\mathcal{H}_h$, *then infinitely many normals to* $\mathcal{H}_h$ *do no meet the singular locus of* $\mathcal{F}_{\nabla h}$, *and any one of them meets the closure of*

$$\left\{ x \in \mathbb{R}^3 \setminus \mathcal{F}_{\nabla h} \,|\, N_{\nabla h}(x) \geq 6 \right\},$$

where $N_{\nabla h}(x)$ *denotes the number of normal lines to* $\mathcal{H}_h$ *passing through* x.

The focal (or evolute) $\mathcal{F}_{\nabla h}$ of $\mathcal{H}_h$ is the singular hypersurface of $\mathbb{R}^{n+1}$ formed by all the principal centers of curvature of $\mathcal{H}_h$ and it consists of n sheets $\mathcal{F}^1_{\nabla h}, \ldots, \mathcal{F}^n_{\nabla h}$ corresponding respectively to the principal radii of

curvature $R_h^1, \ldots, R_h^n$ of $\mathcal{H}_h$, which we label so that $R_h^1 \leq R_h^2 \leq \ldots \leq R_h^n$. In Subsection 2.3, we will introduce for each $k \in [\![1, n]\!]$ the index of a point $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$ with respect to the k^{th} sheet $\mathcal{F}_{\nabla h}^k$ of $\mathcal{F}_{\nabla h}$, and the interior $Int\left(\mathcal{F}_{\nabla h}^k\right)$ of this sheet. Under the assumptions of Corollary 2, we will in fact prove that any normal to $\mathcal{H}_h$ that does not meet the singular locus of $\mathcal{F}_{\nabla h}$ meets the closure of the nonempty interiors of both sheets of the focal.

We will obtain the following result as a corollary of our Theorem 3 stated and proved in Subsect 4.2.

Theorem 2. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3; \mathbb{R})$, let $\mathcal{F}_{\nabla h}$ denote its focal surface, and let $\mathcal{F}_{\nabla h}^2$ be its second sheet. For all $x \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$, we have:*

$$\left(x \in Int\left(\mathcal{F}_{\nabla h}^2\right)\right) \Longleftrightarrow \begin{pmatrix} h_x : \mathbb{S}^3 \to \mathbb{R} \text{ admits a smooth} \\ \text{level surface with nonzero genus} \end{pmatrix}.$$

The theorem below is an adaptation of Theorem 1 to dimension 4.

Theorem 4. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3; \mathbb{R})$, and let $\mathcal{F}_{\nabla h}$ denote its focal surface. If $u \in \mathbb{S}^3$ is such that the normal line $\mathcal{N}_{\nabla h}(u)$ to $\mathcal{H}_h$ at $x_h(u)$ does not meet the singular locus of $\mathcal{F}_{\nabla h}$, then there exists $(r_1, r_2) \in \mathbb{R}^2$ such that, for each $i \in [\![1, 2]\!]$, $x_{h-r_i}(u) = x_h(u) - r_i u$ is an (at least) type i double hyperbolic point of $\mathcal{H}_{h-r_i}$.*

Corollary 3 *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3; \mathbb{R})$. If there does not exist a point of $\mathbb{R}^4$ lying on infinitely many normals to $\mathcal{H}_h$, then infinitely many normals to $\mathcal{H}_h$ do no meet the singular locus of $\mathcal{F}_{\nabla h}$ and any one of them meets the closure of*

$$\left\{x \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h} \,|\, N_{\nabla h}(x) \geq 6\right\},$$

where $N_{\nabla h}(x)$ denotes the number of normal lines to $\mathcal{H}_h$ passing through x; more precisely, any one of these normals meets the closures of

$$Int\left(\mathcal{F}_{\nabla h}^1\right) \cap Int\left(\mathcal{F}_{\nabla h}^2\right) \qquad \text{and} \qquad Int\left(\mathcal{F}_{\nabla h}^2\right) \cap Int\left(\mathcal{F}_{\nabla h}^3\right).$$

Finally, we will prove that it is not true that for any convex body K of $\mathbb{R}^4$, there are at least 8 normal lines passing through the center of the minimal spherical shell of K (Theorem 6).

2 Basics on hedgehogs and their evolutes

2.1 Background on hedgehogs

Classical hedgehogs can be regarded as the geometrical realizations of formal differences of convex bodies in the Euclidean vector space $\mathbb{R}^{n+1}$. The idea of considering the Minkowski differences of convex bodies may be traced back to some papers by A.D. Alexandrov [1] and H. Geppert [4] in the 1930's. Many classical notions for convex bodies extend to hedgehogs and quite a number of classical results find their counterparts. Of course, a few adaptations are necessary. In particular, volumes have to be replaced by their algebraic versions. Hedgehogs have proved useful for studying convex bodies (one of the main successes of the theory is the construction of counterexamples to an old conjectured characterization of the 2-sphere [10, 13, 16]), and for geometrizing analytical problems by considering functions as support functions. This section will provide the reader with the necessary background on hedgehogs in order to facilitate the understanding of the following sections.

The set $\mathcal{K}^{n+1}$ of all convex bodies of $(n+1)$-Euclidean vector space $(\mathbb{R}^{n+1}, \langle .,. \rangle)$ is usually equipped with Minkowski addition and multiplication by nonnegative real numbers, which are respectively defined by:

$$(i) \quad \forall (K,L) \in \left(\mathcal{K}^{n+1}\right)^2, \, K+L = \{x+y \,|x \in K, y \in L\};$$
$$(ii) \quad \forall \lambda \in \mathbb{R}_+, \forall K \in \mathcal{K}^{n+1}, \, \lambda.K = \{\lambda x \,|x \in K\}.$$

It does not constitute a vector space since there is no subtraction in $\mathcal{K}^{n+1}$: not for every pair $(K,L) \in \left(\mathcal{K}^{n+1}\right)^2$ does there exist an $X \in \mathcal{K}^{n+1}$ such that $L+X=K$. Now, in the same way as we construct the group $\mathbb{Z}$, of integers from the monoid $\mathbb{N}$ of nonnegative integers, we can construct the vector space $\mathcal{H}^{n+1}$ of formal differences of convex bodies from $\mathcal{K}^{n+1}$. We can then regard $\mathcal{K}^{n+1}$ as a cone of $\mathcal{H}^{n+1}$ that spans the entire space. Hedgehog theory simply consists in:

1. considering each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a geometrical object in $\mathbb{R}^{n+1}$, called a *hedgehog* (see below);
2. extending the mixed volume $V : \left(\mathcal{K}^{n+1}\right)^{n+1} \to \mathbb{R}$ to a symmetric $(n+1)$-linear form on $\mathcal{H}^{n+1}$;
3. extending certain parts of the Brunn-Minkowski theory to $\mathcal{H}^{n+1}$.

For $n \leq 2$, it goes back to a paper by H. Geppert [4] who introduced hedgehogs under the German names *stützbare Bereiche* $(n=1)$ and *stützbare Flächen* $(n=2)$.

2.1.1 The C^2 case

Here we follow more or less [9]. As is well-known, every convex body $K \subset \mathbb{R}^{n+1}$ is determined by its support function $h_K : \mathbb{S}^n \longrightarrow \mathbb{R}$, where $h_K(u)$ is defined by $h_K(u) = \sup\{\langle x, u\rangle \,|x \in K\}$, $(u \in \mathbb{S}^n)$, that is, as the signed distance from the origin to the support hyperplane with normal vector u. In particular, every closed convex hypersurface of class C^2_+ (i.e., C^2-hypersurface with positive Gauss curvature) is determined by its support function h (which must be of class C^2 on $\mathbb{S}^n$ [19, p. 111]) as the envelope $\mathcal{H}_h$ of the family of hyperplanes with equation $\langle x, u\rangle = h(u)$. This envelope $\mathcal{H}_h$ is described analytically by the following system of equations

$$\begin{cases} \langle x, u\rangle = h(u) \\ \langle x, \,.\,\rangle = dh_u(.) \end{cases}.$$

The second equation is obtained from the first by performing a partial differentiation with respect to u. From the first equation, the orthogonal projection of x onto the line spanned by u is $h(u)\,u$, and from the second one, the orthogonal projection of x onto $u^\perp$ is the gradient of h at u (see Figure 1). Therefore, for each $u \in \mathbb{S}^n$, $x_h(u) = h(u)u + (\nabla h)(u)$ is the unique solution of this system.

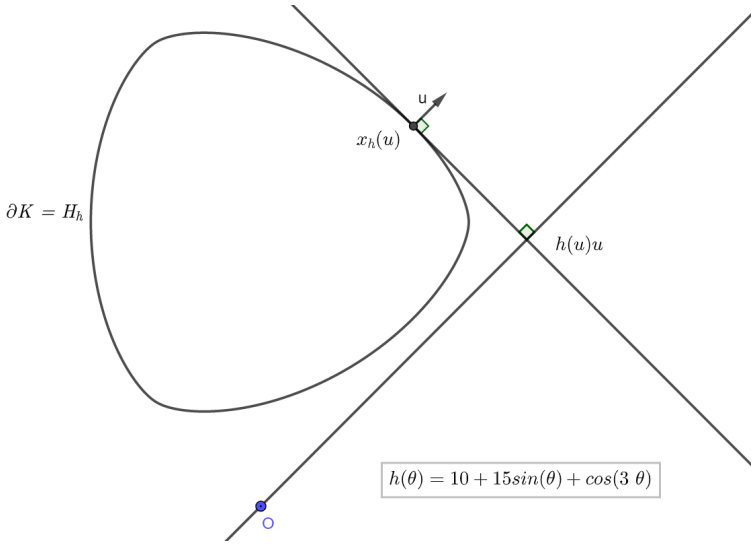


Figure 1. Envelope parametrized by its Gauss map

Now, for any C^2-function h on $\mathbb{S}^n$, the envelope $\mathcal{H}_h$ is in fact well-defined (even if h is not the support function of a convex hypersurface). Its natural parametrization $x_h : \mathbb{S}^n \to \mathcal{H}_h, u \mapsto h(u)u + (\nabla h)(u)$ can be

interpreted as the inverse of its Gauss map, in the sense that: at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$. We say that $\mathcal{H}_h$ is the hedgehog with support function h (see Figure 2). Note that x_h depends linearly on h.

Since $x_h : \mathbb{S}^n \to \mathcal{H}_h$ can be regarded as the inverse of the Gauss map, the Gauss curvature of $\mathcal{H}_h$ at $x_h(u)$ is given by $K_h(u) = 1/\det[T_u x_h]$, where $T_u x_h$ is the tangent map of x_h at u. Singular points of $\mathcal{H}_h$ are the points at which the the so-called 'curvature function' $R_h(u) := \det[T_u x_h]$ is equal to 0 (that is, loosely speaking, the points at which the Gauss curvature κ_h is infinite). For every $u \in \mathbb{S}^n$, the tangent map of x_h at the point u is $T_u x_h = h(u)\, Id_{T_u\mathbb{S}^n} + L_u\left(\nabla^2 h\right)$, where $L_u\left(\nabla^2 h\right)$ is the symmetric endomorphism associated with the Hessian $\left(\nabla^2 h\right)_u$ of h at u. In particular, the so-called 'curvature function' $R_h(u) := \det[T_u x_h]$ is given by $R_h(u) = \det\left[h(u)\, Id_{T_u\mathbb{S}^n} + L_u\left(\nabla^2 h\right)\right]$ for all $u \in \mathbb{S}^n$.

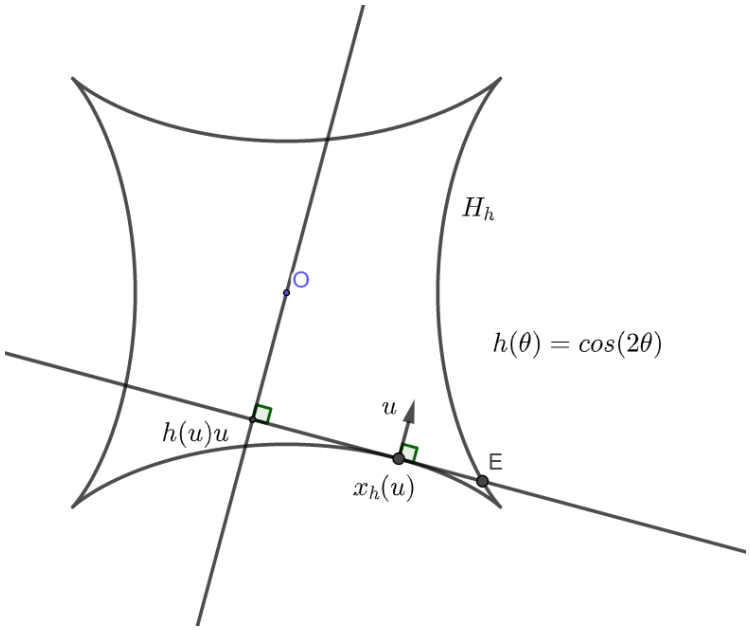


Figure 2. Plane hedgehog with C^2-support function

Hedgehogs with a C^2-support function can be regarded as Minkowski differences of convex hypersurfaces of class C^2_+. Indeed, given any $h \in C^2(\mathbb{S}^n;\mathbb{R})$, for all large enough real constants r, the functions $h + r$ and r are support functions of convex hypersurfaces of class C^2_+ such that $h = (h + r) - r$.

2.1.2 Generic singularities

Hedgehogs of $\mathbb{R}^{n+1}$ with a smooth support function can be regarded as Legendrian fronts [14, pp. 340-341], and by regarding the natural parametrizations $x_h : \mathbb{S}^n \to \mathbb{R}$ as Legendrian maps, Arnold's works can be used to classify their generic singularities for $n \leq 5$ [9]. In particular, generic singularities of smooth hedgehogs of $\mathbb{R}^3$ are cuspidal edges and swallowtails. Elliptic and hyperbolic regions, which are defined by the sign of the Gauss curvature $\kappa_h = 1/R_h$, are separated by cuspidal edges on which the curvature function R_h is equal to 0 (or, loosely speaking, on which the Gauss curvature κ_h is infinite): see Figure 3 (a). Swallowtails are the cusp points of cuspidal edges and we can distinguish two types of swallowtails (negative or positive) according to the sign of the Gauss curvature on the tail: see Figure 3 (a) and (b). More precisely, there exists an open dense subset $\mathcal{U}$ of $C^\infty(\mathbb{S}^2;\mathbb{R})$ in the C^4-topology, such that: for all $h \in \mathcal{U}$, the singularities of $\mathcal{H}_h$ are all equivalent to one of the three models of singularities represented in Figure 3.

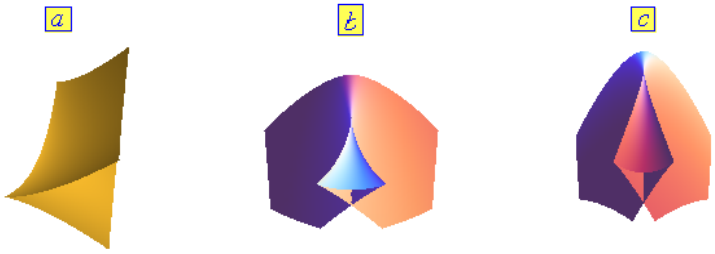


Figure 3. Generic singularities of hedgehogs in $\mathbb{R}^3$

2.1.3 The general case

In [11], the author extended the notion of a hedgehog by regarding hedgehogs as Minkowski differences of arbitrary convex bodies. The trick is to define hedgehogs inductively as collections of lower-dimensional 'support hedgehogs'. More precisely, the definition of general hedgehogs is based on the three following remarks.
(i) In $\mathbb{R}$, every convex body K is determined by its support function h_K as the segment $[-h_K(-1), h_K(1)]$, where $-h_K(-1) \leq h_K(1)$, so that the difference $K - L$ of two convex bodies K, L can be defined as an oriented segment of $\mathbb{R}$: $K - L := [-(h_K - h_L)(-1), (h_K - h_L)(1)]$.
(ii) If K and L are two convex bodies of $\mathbb{R}^{n+1}$ then for all $u \in \mathbb{S}^n$, their support sets with unit normal u, say K_u and L_u, can be identified with convex bodies K_u and L_u of the n-dimensional Euclidean vector space

$u^{\perp} \simeq \mathbb{R}^n$.
(iii) Addition of two convex bodies $K, L \subset \mathbb{R}^{n+1}$ corresponds to that of their support sets with same unit normal vector: $(K+L)_u = K_u + L_u$ for all $u \in \mathbb{S}^n$; therefore, the difference $K - L$ of two convex bodies $K, L \subset \mathbb{R}^{n+1}$ must be defined in such a way that $(K-L)_u = K_u - L_u$ for all $u \in \mathbb{S}^n$.

A natural way of defining geometrically general hedgehogs as differences of arbitrary convex bodies is therefore to proceed by induction on the dimension by extending the notion of *support set with normal vector* u to a notion of *support hedgehog with normal vector* u. Let us give an example in $\mathbb{R}^2$. Let K and L be the convex bodies of $\mathbb{R}^2$ with support function $h_K(x) = |\langle x, e_1\rangle| + |\langle x, e_2\rangle|$ and $h_L(x) = |\langle x, e_3\rangle| + |\langle x, e_4\rangle|$, where $\langle .,. \rangle$ is the standard inner product on $\mathbb{R}^2$, (e_1, e_2) the canonical basis of $\mathbb{R}^2$ and $e_3, e_4 \in \mathbb{R}^2$ the unit vectors given by $e_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $e_4 = \frac{1}{\sqrt{2}}(e_1 - e_2)$. These convex bodies are two squares whose formal difference $K - L$ can be realized geometrically as the hedgehog with support function $h = h_K - h_L$, which is a regular octagram constructed by connecting every third consecutive vertex of a regular octogon (i.e., a regular star polygon with Schläfli symbol $\{8/3\}$): see Figure 4.

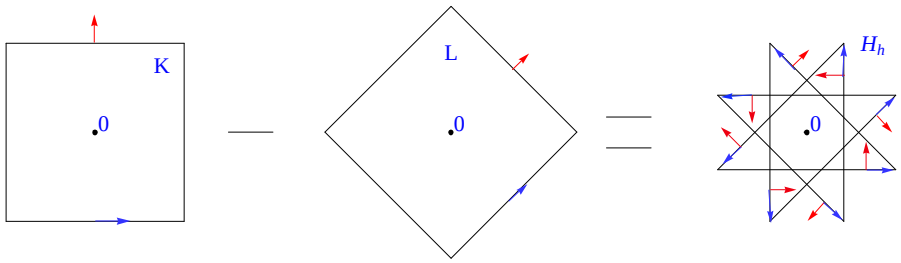


Figure 4. Octagram as the difference of two squares

2.2 Focal (or evolute) of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$

Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^{n+1}$ such that $h \in C^{\infty}(\mathbb{S}^n; \mathbb{R})$. For all $u \in \mathbb{S}^n$, the normal line to $\mathcal{H}_h$ at $x_h(u)$ is defined to be the line passing through $x_h(u)$ and oriented by u; this normal line $\mathcal{N}_{\nabla h}(u) := \{\nabla h(u)\} + \mathbb{R}u$ is the perpendicular to the support hyperplane to $\mathcal{H}_h$ at $x_h(u)$. The focal set (or evolute) of $\mathcal{H}_h$ is the locus of its centers of principal curvatures, or equivalently, the envelope of its normal lines. This set is also the union of all the singular points of all the parallel hedgehogs $\mathcal{H}_{h+r}$, $(r \in \mathbb{R})$. Since it only depends on the gradient of h, we will denote it by $\mathcal{F}_{\nabla h}$. The focal (or evolute) $\mathcal{F}_{\nabla h}$ of $\mathcal{H}_h$ is the singular hypersurface of $\mathbb{R}^{n+1}$ formed by all the principal centers of curvature of $\mathcal{H}_h$ and it consists of n

sheets $\mathcal{F}^1_{\nabla h}, \ldots, \mathcal{F}^n_{\nabla h}$ corresponding respectively to the principal radii of curvature $R^1_h, \ldots, R^n_h$ of $\mathcal{H}_h$, which we label so that $R^1_h \leq R^2_h \leq \ldots \leq R^n_h$. For all $k \in [\![1, n]\!]$, the sheet $\mathcal{F}^k_{\nabla h}$ can be parametrized by

$$c^k_{\nabla h} : \mathbb{S}^n \to \mathbb{R}^{n+1}, u \mapsto c^k_{\nabla h}(u) = x_h(u) - R^k_h(u)\, u = \nabla h(u) - \lambda^k_{\nabla h}(u)\, u,$$

since $x_h(u) = \nabla h(u) + h(u)\, u$ and $R^i_h(u) = \left(\lambda^k_{\nabla h} + h\right)(u)$, where $\nabla h(u)$ is the gradient of h at u, and $\lambda^1_{\nabla h}(u), \ldots, \lambda^n_{\nabla h}(u)$ are the respective eigenvalues of the Hessian of $\mathcal{H}_h$ at u.

In most of the papers on concurrent normals to a convex body K with a smooth boundary ∂K in $\mathbb{R}^{n+1}$, the evolute of ∂K is also regarded as the complement of the set of points $x \in \mathbb{R}^{n+1}$ such that the square of the distance function from x induces a Morse function on ∂K:

$$\begin{aligned} d_x : \partial K &\to \mathbb{R} \\ y &\mapsto \|x - y\|^2, \end{aligned}$$

where $\|.\| : \mathbb{R}^{n+1} \to \mathbb{R}_+$ is the Euclidean norm.
In this paper, we will adopt another point of view. For any $x \in \mathbb{R}^{n+1}$, we will consider the support function of $\mathcal{H}_h$ with respect to x, that is $h_x : \mathbb{S}^n \to \mathbb{R}$, $u \mapsto h(u) - \langle x, u \rangle$, and we will regard the evolute $\mathcal{F}_{\nabla h}$ as the complement of set of points $x \in \mathbb{R}^{n+1}$ such that $h_x : \mathbb{S}^n \to \mathbb{R}$ is a Morse function. In other words, we will regard the evolute of $\mathcal{H}_h$ as the subset $\mathcal{F}_{\nabla h}$ of $\mathbb{R}^{n+1}$ on which the number and nature of the critical points of h_x change. We will make use of singularity theory and Morse theory viewpoints in order to describe and summarize these changes.

2.3 Singularity and Morse theories viewpoints

In this subsection, we essentially follow [18, 9.1]. Given $h \in C^\infty(\mathbb{S}^n; \mathbb{R})$, we consider the following family of functions

$$\begin{aligned} F_h : U\mathbb{R}^{n+1} := \mathbb{R}^{n+1} \times \mathbb{S}^n &\to \mathbb{R} \\ (x, u) &\mapsto h_x(u) = h(u) - \langle x, u \rangle. \end{aligned}$$

The critical set (or catastrophe manifold) of F_h is the set

$$\mathcal{C}_{\nabla h} = \left\{ (x, u) \in U\mathbb{R}^{n+1} \,\middle|\, \frac{\partial F_h}{\partial u}(x, u) = 0 \quad i.e. \quad \nabla(h_x)(u) = 0 \right\}.$$

The catastrophe map $\chi := \chi_{\nabla h}$ is the restriction to the critical set $\mathcal{C}_{\nabla h}$ of the projection $\pi : U\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, $(x, u) \mapsto x$. The singularity set

$\mathcal{S}_{\nabla h}$ is the set of points of $\mathcal{C}_{\nabla h}$ at which the catastrophe map χ has rank less than $n+1$:

$$\begin{aligned}
\mathcal{S}_{\nabla h} &:= \left\{(x,u) \in \mathcal{C}_{\nabla h} \left| rk\left[T_{(x,u)}\chi\right] < N+1 \right.\right\} \\
&= \left\{(x,u) \in \mathcal{C}_{\nabla h} \left| \exists k \in [\![1,n]\!], x = c^k_{\nabla h}(u) \right.\right\} \\
&= \left\{(x,u) \in \mathcal{C}_{\nabla h} \left| \exists k \in [\![1,n]\!],\ 0 = c^k_{\nabla h}(u) - x = c^k_{\nabla(h_x)}(u) \right.\right. \\
&\qquad\qquad \left.\left. = \underbrace{\nabla(h_x)(u)}_{=0} - \lambda^k_{\nabla(h_x)}(u)u \right\}\right. \\
&= \left\{(x,u) \in \mathcal{C}_{\nabla h} \left| \exists k \in [\![1,n]\!], \lambda^k_{\nabla(h_x)}(u) = 0 \right.\right\} \\
&= \left\{(x,u) \in \mathcal{C}_{\nabla h} \left| \left(\det \circ \nabla^2\right)(h_x)(u) = 0 \right.\right\},
\end{aligned}$$

where ∇^2 is the Hessian operator and thus $M = \det \circ \nabla^2$ the Monge-Ampère one. It is thus the set of points $(x,u) \in \mathcal{C}_{\nabla h}$ at which $F_h : U\mathbb{R}^{n+1} \to \mathbb{R}$ has a degenerate critical point. The bifurcation set of $F_h : U\mathbb{R}^{n+1} \to \mathbb{R}$ is defined to be the image of the singularity set $\mathcal{S}_{\nabla h}$ under the catastrophe map χ. It is none other than the focal $\mathcal{F}_{\nabla h}$ of the hedgehog $\mathcal{H}_h$ in $\mathbb{R}^{n+1}$ (and of any hedgehog that is parallel to $\mathcal{H}_h$, that is of the form $\mathcal{H}_{h+\lambda}$, where $\lambda \in \mathbb{R}$):

$$\chi\left(\mathcal{S}_{\nabla h}\right) = \mathcal{F}_{\nabla h} = \bigcup_{k=1}^{n} \mathcal{F}^k_{\nabla h},$$

where

$$\begin{aligned}
\mathcal{F}^k_{\nabla h} &:= c^k_{\nabla h}\left(\mathbb{S}^n\right) = \left\{x \in \mathbb{R}^{n+1} \left| \exists u \in \mathbb{S}^n,\, x = c^k_{\nabla h}(u) \right.\right\} \\
&= \left\{x \in \mathcal{F}_{\nabla h} \left| \exists u \in \mathbb{S}^n,\, \nabla(h_x)(u) = 0 \text{ and } \lambda^k_{\nabla(h_x)}(u) = 0 \right.\right\}.
\end{aligned}$$

Note that

$$\mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h} = \left\{x \in \mathbb{R}^{n+1} \left| h_x : \mathbb{S}^n \to \mathbb{R} \text{ is a Morse function} \right.\right\},$$

while the bifurcation set $\mathcal{F}_{\nabla h} = \chi\left(\mathcal{S}_{\nabla h}\right)$ is the subset of $\mathbb{R}^{n+1}$ on which the number and nature of the critical points of $h_x : \mathbb{S}^n \to \mathbb{R}$ change (for by structural stability of Morse functions such a change can only occur passing through a degenerate critical point). This can be of course be checked by a direct computation.

2.3.1 Index of a point with respect to the focal

Of course, every x belonging to the unbounded connected component of $\mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$ lies on exactly two normal lines to $\mathcal{H}_h$. Note that, for every $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$, the number of normal lines to $\mathcal{H}_h$ passing through x is given by:

$$N_{\nabla h}(x) = \#\left\{u \in \mathbb{S}^n \,|\nabla\left(h_x(u)\right) = 0\right\}.$$

This number $N_{\nabla h}(x)$ is constant and even on any connected component of $\mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$ [7, Theorem 4], and it suddenly changes by two units every time x transversally crosses $\mathcal{F}_{\nabla h}$ at a simple regular point, so that $N_{\nabla h}(x)$ is even for any $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$. It is thus natural to define, for every $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$, the index $i_{\mathcal{F}_{\nabla h}}(x)$ of x with respect to $\mathcal{F}_{\nabla h}$ by putting:

$$i_{\mathcal{F}_{\nabla h}}(x) := 1 - \frac{1}{2} N_{\nabla h}(x),$$

This index induces a transverse orientation of the regular part of $\mathcal{F}_{\nabla h}$ (and thus of any of its n sheets): $\mathcal{F}_{\nabla h}$ is transversely oriented so that the number of normal lines to $\mathcal{H}_h$ passing through x increases by two units when x transversally crosses $\mathcal{F}_{\nabla h}$ at a simple regular point in the direction of the transverse orientation.

Let $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$ so that $h_x : \mathbb{S}^n \to \mathbb{R}$ is a Morse function. Denote by $C_m(x)$ and $C_M(x)$ the respective numbers of critical points of index 0 and n of this Morse function (that is, the respective numbers of its local minima and maxima), and, for every $i \in [\![1, n-1]\!]$, denote by $S_i(x)$ the number of its critical points of index i. Note that the number of normal lines to $\mathcal{H}_h$ passing through x is then given by:

$$\begin{aligned} N_{\nabla h}(x) &= \#\left(\left\{u \in \mathbb{S}^n \,|\nabla\left(h_x\right)(u) = 0\right\}\right) \\ &= C_m(x) + \sum_{i=1}^{n-1} S_i(x) + C_M(x). \end{aligned} \tag{1}$$

Furthermore, recall that, by virtue of the Morse-Euler relationship, we have:

$$C_m(x) + \sum_{i=1}^{n-1} (-1)^i S_i(x) + (-1)^n C_M(x) = \chi\left(\mathbb{S}^n\right), \tag{2}$$

where $\chi\left(\mathbb{S}^n\right)$ is the Euler characteristic of $\mathbb{S}^n$, i.e. $1 + (-1)^n$.

From (1) and (2), we can immediately deduce that:

$$i_{\mathcal{F}_{\nabla h}}(x) := \begin{cases} -\sum_{l=1}^{p} S_{2l-1}(x) \quad \textit{if} \quad n = 2p \in 2\mathbb{Z}^* \\ 1 - \left(C_m + \sum_{l=1}^{p} S_{2l} \right)(x), \textit{ or equivalently,} \\ 1 - \left(C_M + \sum_{l=0}^{p-1} S_{2l+1} \right)(x) \quad \textit{if} \quad n-1 = 2p \in 2\mathbb{Z}^*, \end{cases} \tag{3}$$

2.3.2 Index decomposition

For any $k \in [\![1, n]\!]$, assume that x moves and transversally crosses the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$ at a simple regular point of $\mathcal{F}^k_{\nabla h}$ in the direction of the transverse orientation. We know this crossing of $\mathcal{F}^k_{\nabla h}$ results in a two-units increase in the number of normal lines to $\mathcal{H}_h$ passing through x (i.e. in the number of critical points of $h_x : \mathbb{S}^n \to \mathbb{R}$). From the Morse-Euler relationship, the two new critical points of h_x cannot be of the same index (since the two indices must have different parities). Since, moreover, $\mathcal{F}^k_{\nabla h}$ is given by

$$\mathcal{F}^k_{\nabla h} = \left\{ x \in \mathcal{F}_{\nabla h} \,\middle|\, \exists u \in \mathbb{S}^n,\ \nabla(h_x)(u) = 0 \text{ and } \lambda^k_{\nabla(h_x)}(u) = 0 \right\},$$

it appears that the two critical points of $h_x : \mathbb{S}^n \to \mathbb{R}$ that arise at the moment of the crossing of $\mathcal{F}^k_{\nabla h}$ have adjacent indices $k-1$ and k, the sign of the function $\lambda^k_{\nabla(h_x)} : \mathbb{S}^n \to \mathbb{R}$ being different at these two points. Therefore, this crossing of $\mathcal{F}^k_{\nabla h}$ has the effect of transforming the vector $v(x) = (C_m(x), S_1(x), \ldots, S_{n-1}(x), C_M(x))$ into the vector $v(x) + e_{k-1} + e_k$, where $(e_0, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^{n+1}$.

Now, let us distinguish two cases according to the parity of $n-1$. In both cases, and for every $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$, we will split the index $i_{\mathcal{F}_{\nabla h}}(x)$ of x with respect to the focal $\mathcal{F}_{\nabla h}$ into the sum over $k \in [\![1, n]\!]$ of the (appropriately defined) index $i_{\mathcal{F}^k_{\nabla h}}(x)$ of x with respect to the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$.

In the case that $n = 2p \in 2\mathbb{Z}^*$, define the indices $i_{\mathcal{F}^k_{\nabla h}}(x), (k \in [\![1, 2p]\!])$, by

$$\begin{cases} i_{\mathcal{F}^1_{\nabla h}}(x) = 1 - C_m(x) \\ \\ i_{\mathcal{F}^2_{\nabla h}}(x) = (C_m - S_1)(x) - 1 \\ \quad \vdots \\ i_{\mathcal{F}^p_{\nabla h}}(x) = (-1)^{p-1}\left(1 - \left(C_m - S_1 + \ldots + (-1)^{p-1} S_{p-1}(x)\right)\right) \\ \\ i_{\mathcal{F}^{p+1}_{\nabla h}}(x) = (-1)^{p-1}\left(1 - \left(C_M - S_{2p-1} + \ldots + (-1)^{p-1} S_{p+1}(x)\right)\right) \\ \quad \vdots \\ i_{\mathcal{F}^{2p-1}_{\nabla h}}(x) = (C_M - S_{2p-1})(x) - 1 \\ \\ i_{\mathcal{F}^{2p}_{\nabla h}}(x) = 1 - C_M(x) \end{cases}$$

for all $x \in \mathbb{R}^{2p+1} \setminus \mathcal{F}_{\nabla h}$, so that when x moves and transversally crosses the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$ at a simple regular point of $\mathcal{F}^k_{\nabla h}$ in the direction of the transverse orientation, then the index $i_{\mathcal{F}^k_{\nabla h}}(x)$ decreases by one unit. Thus, for every $x \in \mathbb{R}^{2p+1} \setminus \mathcal{F}_{\nabla h}$, $i_{\mathcal{F}^k_{\nabla h}}(x)$ do can be interpreted as the index of x with respect to $\mathcal{F}^k_{\nabla h}$ equipped with its transverse orientation.

Note that, for every $x \in \mathbb{R}^{2p+1} \setminus \mathcal{F}_{\nabla h}$, the $2p$ strong Morse inequalities, which give lower bounds for the number of critical points of each index of the Morse function $h_x : \mathbb{S}^{2p} \to \mathbb{R}$ in terms of the Betti numbers of $\mathbb{S}^{2p}$, can simply be rewritten as: $\forall k \in [\![1, 2p]\!]$, $i_{\mathcal{F}^k_{\nabla h}}(x) \leq 0$. In other words, these $2p$ indices are nonpositive, and, for every $k \in [\![1, 2p]\!]$, the index $i_{\mathcal{F}^k_{\nabla h}}(x)$ negatively measures how far from equality we are in the k^{th} strong Morse inequality when considering the Morse function $h_x : \mathbb{S}^{2p} \to \mathbb{R}$.

Similarly, in the case that $n - 1 = 2p \in 2\mathbb{Z}^*$, define the indices $i_{\mathcal{F}^k_{\nabla h}}(x), (k \in [\![1, 2p+1]\!])$, by:

$$\begin{cases} i_{\mathcal{F}^1_{\nabla h}}(x) = 1 - C_m(x) \\ \\ i_{\mathcal{F}^2_{\nabla h}}(x) = (C_m - S_1)(x) - 1 \\ \quad \vdots \\ i_{\mathcal{F}^{p+1}_{\nabla h}}(x) = (-1)^p \left(1 - \left(C_m + \sum_{l=1}^{p} (-1)^l S_l(x)\right)\right). \\ \quad \vdots \\ i_{\mathcal{F}^{2p}_{\nabla h}}(x) = (C_M - S_{2p-1})(x) - 1 \\ \\ i_{\mathcal{F}^{2p+1}_{\nabla h}}(x) = 1 - C_M(x). \end{cases}$$

for all $x \in \mathbb{R}^{2p+2} \setminus \mathcal{F}_{\nabla h}$, so that when x moves and transversally crosses the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$ at a simple regular point of $\mathcal{F}^k_{\nabla h}$ in the direction of the transverse orientation, then the index $i_{\mathcal{F}^k_{\nabla h}}(x)$ decreases by one unit. Thus, for every $x \in \mathbb{R}^{2p+2} \setminus \mathcal{F}_{\nabla h}$, $i_{\mathcal{F}^k_{\nabla h}}(x)$ do can be interpreted as the index of x with respect to $\mathcal{F}^k_{\nabla h}$ equipped with its transverse orientation.

Again, for every $x \in \mathbb{R}^{2p+2} \setminus \mathcal{F}_{\nabla h}$, the $2p+1$ strong Morse inequalities, which give lower bounds for the number of critical points of each index of the Morse function $h_x : \mathbb{S}^{2p+1} \to \mathbb{R}$ in terms of the Betti numbers of $\mathbb{S}^{2p+1}$, can simply be rewritten as: $\forall k \in [\![1, 2p+1]\!]$, $i_{\mathcal{F}^k_{\nabla h}}(x) \leq 0$. These $2p+1$ indices are nonpositive, and, for every $k \in [\![1, 2p+1]\!]$, the index $i_{\mathcal{F}^k_{\nabla h}}(x)$ negatively measures how far from equality we are in the k^{th} strong Morse inequality when considering the Morse function $h_x : \mathbb{S}^{2p+1} \to \mathbb{R}$.

In both cases (n even or odd), using the Morse-Euler relationship we can verify that:

Proposition 1. *For every $x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h}$, we have*

$$i_{\mathcal{F}_{\nabla h}}(x) = \sum_{k=1}^{n} i_{\mathcal{F}^k_{\nabla h}}(x)\,,$$

and, for every $k \in [\![1, n]\!]$, $i_{\mathcal{F}^k_{\nabla h}}(x)$ can be interpreted as the index of x with respect to the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$ equipped with its transverse orientation; the index $i_{\mathcal{F}^k_{\nabla h}}(x)$ negatively measures how far from equality we are in the k^{th} strong Morse inequality when considering the Morse function $h_x : \mathbb{S}^n \to \mathbb{R}$, ($k \in [\![1, n]\!]$).

We define the interior and the body of the focal $\mathcal{F}_{\nabla h}$ by setting, respectively,

$$Int\left(\mathcal{F}_{\nabla h}\right) := \left\{x \in \mathbb{R}^{n+1} \setminus \mathcal{F}_{\nabla h} \,|i_{\mathcal{F}_{\nabla h}}(x) < 0\right\},$$

and

$$\mathcal{K}\left(\mathcal{F}_{\nabla h}\right) := \mathcal{F}_{\nabla h} \cup Int\left(\mathcal{F}_{\nabla h}\right).$$

Of course, for every $k \in [\![1, n]\!]$ we can also define the interior and the body of the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of the focal by setting, respectively,

$$Int\left(\mathcal{F}^k_{\nabla h}\right) := \left\{x \in \mathbb{R}^{n+1} \setminus \mathcal{F}^k_{\nabla h} \,\middle|\, i_{\mathcal{F}^k_{\nabla h}}(x) < 0\right\},$$

where $i_{\mathcal{F}^k_{\nabla h}}(x)$ is defined as indicated above, and

$$\mathcal{K}\left(\mathcal{F}_{\nabla h}^{k}\right):=\mathcal{F}_{\nabla h}^{k}\cup Int\left(\mathcal{F}_{\nabla h}^{k}\right).$$

It has been proved that arbitrarily close to the center of the minimal spherical shell of a convex body $K\subset\mathbb{R}^{n+1}$ with support function $h\in C^{\infty}\left(\mathbb{S}^{n};\mathbb{R}\right)$, there exists points $x\in\mathbb{R}^{n+1}\setminus\mathcal{F}_{\nabla h}$ such that $C_{m}\left(x\right)\geq 2$ and $C_{M}\left(x\right)\geq 2$ [7, Lemma 3], so that $i_{\mathcal{F}_{\nabla h}^{1}}(x)=1-C_{m}\left(x\right)\leq -1$ and $i_{\mathcal{F}_{\nabla h}^{n}}(x)=1-C_{M}\left(x\right)\leq -1$. In other words, we have:

$$Int\left(\mathcal{F}_{\nabla h}^{1}\right)\cap Int\left(\mathcal{F}_{\nabla h}^{n}\right)\neq\varnothing.$$

2.4 Singular locus of the focal $\mathcal{F}_{\nabla h}$ of $\mathcal{H}_h\subset\mathbb{R}^{n+1}$

Here $\mathcal{H}_h$ is a hedgehog of $\mathbb{R}^{n+1}$ such that $h\in C^{\infty}\left(\mathbb{S}^{n};\mathbb{R}\right)$. When two principal radii of curvature R_h^k, R_h^l of $\mathcal{H}_h$ coincide at a point $u\in\mathbb{S}^n$, then the corresponding sheets $\mathcal{F}_{\nabla h}^{k}$, $\mathcal{F}_{\nabla h}^{l}$ of $\mathcal{F}_{\nabla h}$ intersect at $c_{\nabla h}^{k}\left(u\right)=c_{\nabla h}^{l}\left(u\right)$, and this point is a singular point of both sheets $\mathcal{F}_{\nabla h}^{k}$, $\mathcal{F}_{\nabla h}^{l}$. When all the principal radii of curvature are pairwise distinct at $u\in\mathbb{S}^n$, the focal $\mathcal{F}_{\nabla h}$ is locally the union of n disjoint patches of hypersurfaces, parametrized by the maps $c_{\nabla h}^{k}:\mathbb{S}^{n}\rightarrow\mathbb{R}^{n+1}$, $u\mapsto x_{h}\left(u\right)-R_{h}^{k}\left(u\right)u$, $(k\in[\![1,n]\!])$. Let us examine the regularity of these patches at their point corresponding to u. In the case that the principal radii of curvature are pairwise distinct at $u\in\mathbb{S}^n$, there exists an orthonormal basis of $T_u\mathbb{S}^n$ consisting of eigenvectors $v_1,\ldots,v_n$ of the tangent map T_ux_h associated respectively with $R_h^1\left(u\right),\ldots,R_h^n\left(u\right)$: $\forall k\in[\![1,n]\!]$, $\left(T_ux_h\right)\left(v_k\right)=R_h^k\left(u\right)v_k$. For every $k\in[\![1,n]\!]$, we have then:

$$\frac{\partial c_{\nabla h}^{k}}{\partial v_{k}}\left(u\right)=\left(T_{u}x_{h}\right)\left(v_{k}\right)-\left(\frac{\partial R_{h}^{k}}{\partial v_{k}}\left(u\right)u+R_{h}^{k}\left(u\right)v_{k}\right)=-\frac{\partial R_{h}^{k}}{\partial v_{k}}\left(u\right)u,$$

and for every $l\in[\![1,n]\!]\setminus\{k\}$,

$$\frac{\partial c_{\nabla h}^{k}}{\partial v_{l}}(u)=\left(T_{u}x_{h}\right)(v_{l})-\left(\frac{\partial R_{h}^{k}}{\partial v_{l}}(u)\,u+R_{h}^{k}\left(u\right)v_{l}\right)$$

$$=\left(R_{h}^{l}-R_{h}^{k}\right)(u)\,v_{l}-\frac{\partial R_{h}^{k}}{\partial v_{l}}(u)\,u.$$

Denote by $\mathcal{B}$ the orthonormal system $(u,v_1,\ldots,\widehat{v_k},\ldots,v_n)$, where the hat over the term v_k means that it must be omitted. A straightforward computation shows that

$$\det_{\mathcal{B}}\left[\left(\frac{\partial c_{\nabla h}^{k}}{\partial v_{k}},\frac{\partial c_{\nabla h}^{k}}{\partial v_{1}},\ldots,\widehat{\frac{\partial c_{\nabla h}^{k}}{\partial v_{k}}},\ldots,\frac{\partial c_{\nabla h}^{k}}{\partial v_{n}}\right)(u)\right],$$

where the hat means again that the corresponding term must be omitted, is equal to

$$-\frac{\partial R_h^k}{\partial v_k}(u) \prod_{\substack{1 \leq l \leq n \\ l \neq k}} \left(R_h^l - R_h^k\right)(u).$$

Thus:

Proposition 2. *For every $k \in [\![1, n]\!]$ and every $u \in \mathbb{S}^n$, $c_{\nabla h}^k(u)$ is a singular point of $\mathcal{F}_{\nabla h}^k$ if, and only if, one of the following two conditions is satisfied:*

· $R_h^k(u)$ is an eigenvalue of multiplicity at least 2 of the tangent map $T_u x_h$;

· $R_h^k(u)$ is a simple eigenvalue of $T_u x_h$, and $\frac{\partial R_h^k}{\partial v_k}(u) = 0$, v_k being a unit eigenvector of $T_u x_h$ associated with $R_h^k(u)$.

For every $k \in [\![1, n]\!]$, the singular locus of the k^{th} sheet $\mathcal{F}_{\nabla h}^k$ of $\mathcal{F}_{\nabla h}$ is thus given by:

$$\mathcal{S}ing\left(\mathcal{F}_{\nabla h}^k\right) = c_{\nabla h}^k(S_k),$$

where S_k is the set of point $u \in \mathbb{S}^n$ that satisfy one of the two conditions of the above proposition; and the singular locus of the focal $\mathcal{F}_{\nabla h}$ is of course defined by:

$$\mathcal{S}ing\left(\mathcal{F}_{\nabla h}\right) := \bigcup_{k=1}^{n} \mathcal{S}ing\left(\mathcal{F}_{\nabla h}^k\right).$$

The three-dimensional case

Corollary. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^\infty(\mathbb{S}^2; \mathbb{R})$, and let $u \in \mathbb{S}^2$ be such that*

$$R_h^1(u) = 0, \quad R_h^2(u) \neq 0 \quad \text{and} \quad \frac{\partial R_h^1}{\partial v_1}(u) \neq 0$$

$$\left(\text{resp.} \quad R_h^2(u) = 0, \quad R_h^1(u) \neq 0 \quad \text{and} \quad \frac{\partial R_h^2}{\partial v_1}(u) \neq 0\right),$$

where (v_1, v_2) is an orthonormal basis of $T_u\mathbb{S}^2$ made of eigenvectors v_1, v_2 of $T_u x_h$ associated respectively with $R_h^1(u)$, $R_h^2(u)$. Then, $x_h(u)$ is equal to $c_{\nabla h}^1(u)$ (resp. $c_{\nabla h}^2(u)$), which is a regular point of $\mathcal{F}_{\nabla h}^1$ (resp. $\mathcal{F}_{\nabla h}^2$),

and a regular point of a cuspidal edge of $\mathcal{H}_h$.

Proof. The first part of the corollary is a straightforward consequence of the proposition. Now, let $(i,j)=(1,2)$ (resp. $(i,j)=(2,1)$) so that:

$$R_h^i(u)=0, \quad R_h^j(u)\neq 0 \quad \text{and} \quad \frac{\partial R_h^i}{\partial v_i}(u)\neq 0.$$

Since $R_h(u)=0$ and $\nabla R_h(u)\neq 0$, the level set R_h can be parametrized as a regular smooth curve Γ in a neighborhood of u on $\mathbb{S}^2$. Let $\gamma: I\to\mathbb{S}^2$, $t\mapsto\gamma(t)$ be this regular parametrization of Γ, and let $t_0\in I$ be such that $\gamma(t_0)=u$ and $\gamma'(t_0)=\lambda_1 v_1+\lambda_2 v_2$, where $(\lambda_1,\lambda_2)\in\mathbb{R}^2\setminus\{(0,0)\}$. Since $R_h\circ\gamma$ is identically equal to zero on I, we have $(R_h\circ\gamma)'(t_0)=\langle\nabla R_h(u),\gamma'(t_0)\rangle=0$. On the other hand, $\frac{\partial R_h^i}{\partial v_i}(u):=\langle\nabla R_h(u),v_i\rangle\neq 0$. Therefore $\gamma'(t_0)$ is not colinear to v_i, and thus $\lambda_j\neq 0$. As a result $(x_h\circ\gamma)'(t_0)=(T_u x_h)(\gamma'(t_0))=\lambda_i R_h^i(u)\,v_i+\lambda_j R_h^j(u)\,v_j\neq 0$, that is $u=\gamma(t_0)$ is a regular point of the cuspidal edge $x_h(\Gamma)$.

□

Note in passing that any cuspidal edge of $\mathcal{H}_h$ is locally separating a hyperbolic region from an elliptic one.

2.5 Volume of the focal $\mathcal{F}_{\nabla h}$ of $\mathcal{H}_h\subset\mathbb{R}^{n+1}$

Here again $\mathcal{H}_h$ is a hedgehog of $\mathbb{R}^{n+1}$ such that $h\in C^\infty(\mathbb{S}^n;\mathbb{R})$. We can define the (absolute) volume of its focal $\mathcal{F}_{\nabla h}$, and, for any $k\in[\![1,n]\!]$ the one of the k^{th} sheet $\mathcal{F}_{\nabla h}^k$ of $\mathcal{F}_{\nabla h}$, to be respectively

$$v(\mathcal{F}_{\nabla h}):=-\int_{\mathbb{R}^{n+1}} i_{\mathcal{F}_{\nabla h}}(x)\,dx, \quad \text{and} \quad v\left(\mathcal{F}_{\nabla h}^k\right):=-\int_{\mathbb{R}^{n+1}} i_{\mathcal{F}_{\nabla h}^k}(x)\,dx,$$

where the integrals are with respect to the Lebesgue measure on $\mathbb{R}^{n+1}$. By the index decomposition, we thus have:

$$v(\mathcal{F}_{\nabla h})=\sum_{k=1}^{n} v\left(\mathcal{F}_{\nabla h}^k\right).$$

In the case where $\mathcal{H}_h\subset\mathbb{R}^3$, the definition of the volume of $\mathcal{F}_{\nabla h}$ has already been introduced by the author in [12], where the following is proved.

Theorem. *Let* $\mathcal{H}_h$ *be a hedgehog of* $\mathbb{R}^3$ *such that* $h\in C^\infty(\mathbb{S}^2;\mathbb{R})$. *The volume of its focal* $\mathcal{F}_{\nabla h}$ *is given by*

$$v\left(\mathcal{F}_{\nabla h}\right)=\frac{1}{6}\int_{\mathbb{S}^2}\left|R_1-R_2\right|^3 d\sigma,$$

where σ is the spherical Lebesgue measure on $\mathbb{S}^2$, and $R_1(u)$, $R_2(u)$ are the principal radii of curvature of $\mathcal{H}_h$ at $x_h(u)$. Besides, the map

$$v:\mathcal{H}:=\left\{\nabla h\,\middle|\,h\in C^4\left(\mathbb{S}^2;\mathbb{R}\right)\right\}\to\mathbb{R}_+,\ \nabla h\longmapsto v\left(\mathcal{F}_{\nabla h}\right)^{\frac{1}{3}}$$

is a norm on the real vector space of families of parallel hedgehogs with support function of class C^4 in $\mathbb{R}^3$. Moreover, we have:

$$4L\left(x_h\right)^3\leq 9\pi v\left(\mathcal{F}_{\nabla h}\right)^2$$

where $L\left(x_h\right):=\frac{1}{4}\int_{\mathbb{S}^2}\left(R_1-R_2\right)^2 d\sigma$ is the Laguerre area of $\mathcal{H}_h$.

The opposite $-L\left(x_h\right)$ of the Laguerre area of $\mathcal{H}_h$ can be interpreted as the mixed surface area $s\left(h,\partial^2 h\right)$ of $\mathcal{H}_h$ and of its mean evolute $\mathcal{H}_{\partial^2 h}$, which is the envelope of the family of planes parallel to the support planes to $\mathcal{H}_h$ and passing through the midpoints of the focal segments (i.e. the segments joining the two centers of principal curvature of $\mathcal{H}_h$ at the same point in $\mathbb{R}^3$); this mean evolute of $\mathcal{H}_h$ is the hedgehog $\mathcal{H}_{\partial^2 h}$ with support function $\partial^2 h=\frac{1}{2}\Delta h$, where Δ is the Hodge Laplacian on $\mathbb{S}^2$. Here, $\sqrt{2}\,\partial$ stands for the Hodge-Dirac operator $D=d+\delta$ on $\mathbb{S}^2$, where d is the exterior differentiation and $\delta=-*d*$ the codifferential, so that $\partial h=dh/\sqrt{2}$ and $\partial^2 h=\frac{1}{2}\delta\left(dh\right)=\frac{1}{2}\Delta h(u)$. Besides, the opposite $-L\left(x_h\right)$ of the Laguerre area can also be regarded as the signed surface area of the 'co-evolute' $\mathcal{H}_{\partial h}$ of $\mathcal{H}_h$, which is a 'marginally trapped hedgehog' in the 4-dimensional Lorentz-Minkowski space $\mathbb{L}^4$[14].

3 The three-dimensional case

3.1 Usual index and transverse orientation

Any hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ is a (possibly singular and self-intersecting) parametrized hypersurface $x_h:\mathbb{S}^n\to\mathcal{H}_h\subset\mathbb{R}^{n+1}$ that is equipped with the transverse orientation defined as follows: at each regular point $x_h(u)$ of $\mathcal{H}_h$, the usual transverse orientation of $\mathcal{H}_h$ is given by the normal vector $sign\left[R_h(u)\right]u$, where sign is the sign function and $R_h:=1/\kappa_h$ the curvature function of $\mathcal{H}_h$ (κ_h denoting the Gauss curvature of $\mathcal{H}_h$). The Kronecker index $i_h(x)$ of a point $x\in\mathbb{R}^{n+1}\setminus\mathcal{H}_h$ with respect to $\mathcal{H}_h$ can be defined as the degree of the map

$$\mathcal{U}_{(h,x)}:\mathbb{S}^n\to\mathbb{S}^n,\ u\longmapsto\frac{x_h(u)-x}{\left\|x_h(u)-x\right\|},$$

and interpreted as the algebraic intersection number of an oriented half-line with origin x with the hypersurface $\mathcal{H}_h$ equipped with its usual transverse orientation (number independent of the oriented half-line for an open dense set of directions). The usual transverse orientation and the Kronecker index are thus mutually associated. It is worth noting that if we let $\widetilde{h}(u) = -h(-u)$ for all $u \in \mathbb{S}^n$, where $h \in C^2(\mathbb{S}^n; \mathbb{R})$, then the hedgehogs $\mathcal{H}_h = x_h(\mathbb{S}^n)$ and $\mathcal{H}_{\widetilde{h}} = x_{\widetilde{h}}(\mathbb{S}^n)$ are identical as hypersurfaces of $\mathbb{R}^{n+1}$ except that they have opposite transverse orientations when $n+1$ is odd. Indeed

$$x_{\widetilde{h}}(-u) = x_h(u) \text{ for all } u \in \mathbb{S}^n,$$

but

$$sign\left[R_{\widetilde{h}}(-u)\right](-u) = (-1)^{n+1} sign\left[R_h(u)\right] u,$$

and thus

$$i_{\widetilde{h}}(x) = (-1)^{n+1} i_h(x) \text{ for all } x \in \mathbb{R}^{n+1} \setminus \mathcal{H}_h.$$

In the three-dimensional case, we proved more precisely the following result in [13]:

Theorem. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^2(\mathbb{S}^2; \mathbb{R})$. Then, for all $x \in \mathbb{R}^3 \setminus \mathcal{H}_h$, we have:*

$$i_h(x) = r_h^+(x) - r_h^-(x),$$

where $r_h^-(x)$ (resp. $r_h^+(x)$) denotes the number of connected components of $\mathbb{S}^2 \setminus h_x^{-1}(\{0\})$ on which h_x is negative (resp. positive).

Since the usual transverse orientation does depend on the orientation of normal lines to $x_h(\mathbb{S}^2) = x_{\widetilde{h}}(\mathbb{S}^2)$ in $\mathbb{R}^3$, we will then rename it '*the relative transverse orientation*' of $\mathcal{H}_h$ in $\mathbb{R}^3$, so as to distinguish it from the one we define below. We refer the reader to [13] for more information on this subject.

3.2 New index and transverse orientation

In [13], the author introduces the following notion of index of a point $x \in \mathbb{R}^3 \setminus \mathcal{H}_h$ with respect to $\mathcal{H}_h$:

$$j_h(x) := 1 - c_h(x),$$

where $c_h(x)$ denotes the number of connected components of $(h_x)^{-1}(\{0\})$ on $\mathbb{S}^2$, that is the number of closed spherical curves formed by points $u \in \mathbb{S}^2$ such that x belongs to the support hyperplane of $\mathcal{H}_h$ at $x_h(u)$.

Of course, the index $j_h : x \mapsto j_h(x)$ remains constant on each connected component of $\mathbb{R}^3 \setminus \mathcal{H}_h$. In particular, j_h is equal to 0 on the unbounded component of $\mathbb{R}^3 \setminus \mathcal{H}_h$. It is worth noting that the value of $j_h(x)$ must obviously decreases as x transversally crosses $\mathcal{H}_h$ at a simple elliptic point from locally convex to locally concave side. Thus, if $\mathcal{H}_h$ is the boundary of a convex body K of which x is an interior point, we must have $j_h(x) = 1$, whereas $i_h(x) = -1$ or $i_h(x) = 1$ depending on whether u points inward or outward from K at $x_h(u) \in \mathcal{H}_h = \partial K$, $(u \in \mathbb{S}^2)$.

Now, the j_h-index corresponds to the transverse orientation of $\mathcal{H}_h$ that is such that whenever $x_h(u)$ is a simple regular point of $\mathcal{H}_h$, then the normal line to $\mathcal{H}_h$ at $x_h(u)$, is oriented in the direction that j_h decreases by one unit. Contrary to the usual transverse orientation of $\mathcal{H}_h$, it is clear from its definition that this transverse orientation of $\mathcal{H}_h$ does not depend on the choice of the orientation of normal lines to $x_h(\mathbb{S}^2) = x_{\widetilde{h}}(\mathbb{S}^2)$. We call it '*the absolute transverse orientation*' of $\mathcal{H}_h$. From the above, this absolute transverse orientation cannot change on an elliptic region (i.e., a region on which the Gauss curvature of $\mathcal{H}_h$ remains positive): the absolute transverse orientation is then simply given by the direction of convexity. For our present study, the crucial point will be that a hedgehog of $\mathbb{R}^3$ may admit reversals of its absolute transverse orientation along certain of its self-intersection curves formed by double hyperbolic points Such a reversal is possible along a self-intersection curve formed by double hyperbolic point, but not necessary: it depends on the global geometry of the hedgehog [13].

Absolute body of $\mathcal{H}_h$ in $\mathbb{R}^3$

We will call absolute body of $\mathcal{H}_h \subset \mathbb{R}^3$, and we will denote by $\mathcal{K}_h$, the set

$$\mathcal{K}_h := \mathcal{H}_h \cup Int(\mathcal{H}_h),$$

where $Int(\mathcal{H}_h) := \{x \in \mathbb{R}^3 \setminus \mathcal{H}_h \,|\, j_h(x) \neq 0\}$.

Comparison to the body of $\mathcal{F}_{\nabla h}$ in $\mathbb{R}^3$

Proposition 3. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^\infty(\mathbb{S}^2;\mathbb{R})$. We have $i_{\mathcal{F}_{\nabla h}} \leq j_h$ on $\mathbb{R}^3 \setminus (\mathcal{H}_h \cup \mathcal{F}_{\nabla h})$.*

Proof. Let $x \in \mathbb{R}^3 \setminus (\mathcal{H}_h \cup \mathcal{F}_{\nabla h})$. Let A be a connected component of $\mathbb{S}^2 \setminus (h_x)^{-1}(\{0\})$, and denote by b_A the number of connected component of its boundary in $\mathbb{S}^2$. Now, denote by $C_A(x)$ (resp. $S_A(x)$) the number of local extrema (resp. saddle points) of h_x in A. We know that:

$$C_A(x) - S_A(x) = 2 - b_A,$$

since the Euler characteristic of A is given by $\chi(A) = 2 - b_A$. Thus the number $N_A(x) = C_A(x) + S_A(x)$ of critical points of h_x in A is such that $N_A(x) = b_A + 2(C_A(x) - 1) \geq b_A$. Therefore $N_{\nabla h}(x) = \sum_A N_A(x) \geq \sum_A b_A = 2c_h(x)$, where the sums are taken over the set of connected components of $\mathbb{S}^2 \setminus (h_x)^{-1}(\{0\})$. Thus:

$$i_{\mathcal{F}_{\nabla h}}(x) = 1 - \frac{N_{\nabla h}(x)}{2} \leq 1 - c_h(x) = j_h(x).$$

□

Corollary. *If the mean of $h \in C^\infty(\mathbb{S}^2;\mathbb{R})$ on $\mathbb{S}^2$ is equal to 0, then the absolute body of $\mathcal{H}_h$ is included in the body $\mathcal{K}(\mathcal{F}_{\nabla h})$ of its focal.*

3.3 Proofs of Theorem 1 and Corollary 1

Proof of Theorem 1. We have assumed that $\mathcal{N}_{\nabla h}(u)$ does not meet the singular locus of $\mathcal{F}_{\nabla h}$, and we want to prove that there exists some $r \in \mathbb{R}$ such that $x_{h-r}(u)$ is an at least double hyperbolic point of $\mathcal{H}_{h-r}$. We thus consider the family $(\mathcal{H}_{h-r})_{r\in\mathbb{R}}$ of the hedgehogs that are parallel to $\mathcal{H}_h$. Note that the point $x_{h-r}(u) = \nabla h(u) + (h(u) - r)u$ describes the entire normal line $\mathcal{N}_{\nabla h}(u)$ when r describes the entire real line. Our proof relies on the comparative evolution of the absolute and relative transverse orientations of $\mathcal{H}_{h-r}$ at $x_{h-r}(u)$ when r describes the entire real line.

Define a function $\varepsilon_h^u : \mathbb{R} \to \{-1, 0, 1\}$ as follows: put $\varepsilon_h^u(r) = 1$ if both transverse orientations of $\mathcal{H}_{h-r}$ are well-defined and identical at $x_{h-r}(u)$; put $\varepsilon_h^u(r) = -1$ if both transverse orientations of $\mathcal{H}_{h+r}$ are well-defined and opposite at $x_{h-r}(u)$; finally put $\varepsilon_h^u(r) = 0$ if one the two transverse orientations cannot be defined at $x_{h-r}(u)$. If $x_{h-r}(u)$ is a simple elliptic point of $\mathcal{H}_{h-r}$, the relative transverse orientation of $\mathcal{H}_{h-r}$ at $x_{h-r}(u)$ is given by u, whereas the absolute one is given by the direction of convexity at $x_{h-r}(u)$, that is by $\left(R_{h-r}^1 + R_{h-r}^2\right)(u)\,u$, with $R_{h-r}^1(u)$, $R_{h-r}^2(u)$ being the principal radii of curvature of $\mathcal{H}_{h-r}$ at $x_{h-r}(u)$. In other words, we have in this case:

$$\varepsilon_h^u(r) = sign\left[\left(R_{h-r}^1 + R_{h-r}^2\right)(u)\right] = sign\left[\left(R_h^1 + R_h^2\right)(u) - 2r\right].$$

In fact this definition makes sense for any $r \in \mathbb{R}$ such that $R_{h-r}(u) > 0$ (i.e., such that $x_{h-r}(u)$ is an elliptic point of $\mathcal{H}_{h-r}$). Note we have thus

$$\varepsilon_h^u(r) = \begin{cases} 1 & \text{if} \quad r < R_h^1(u) \\ \\ -1 & \text{if} \quad r > R_h^2(u) \end{cases}$$

(since $R_{h-r}(u) = \left(R_{h-r}^1 R_{h-r}^2\right)(u) = \left(R_h^1(u) - r\right)\left(R_h^2(u) - r\right) > 0$ if $r \notin [R_h^1(u), R_h^2(u)]$), so that $\varepsilon_h^u : \mathbb{R} \to \{-1, 0, 1\}$ has a sign change on $[R_h^1(u), R_h^2(u)]$.

If $r \in \{R_h^1(u), R_h^2(u)\}$, then $R_{h-r}(u) = 0$ so that x_{h-r} is a singular point of $\mathcal{H}_{h-r}$, and thus a point of $\mathcal{F}_{\nabla h}$ (namely, $c_{\nabla h}^1(u)$ or $c_{\nabla h}^2(u)$). Since $\mathcal{N}_{\nabla h}(u)$ does not contain any singular point of $\mathcal{F}_{\nabla h}$, $x_{h-r}(u)$ is then a regular point of a cuspidal edge $x_{h-r}(\Gamma)$ (see Subsect. 2.4), which is separating a hyperbolic region of $\mathcal{H}_{h-r}$ from an elliptic one, and a sign change of ε_h^u cannot occur at such a point. In fact, we still could define $\varepsilon_h^u(r)$ by $\varepsilon_h^u(r) = sign\left[\left(R_{h-r}^1 + R_{h-r}^2\right)(u)\right]$ in order to be consistent with the changes of the i_{h-r} and j_{h-r} indices at the points of the adjacent elliptic region.

Finally, let us come to the case where $r \in]R_h^1(u), R_h^2(u)[$. We then have $R_{h-r}^1(u) = R_h^1(u) - r < 0$ and $R_{h-r}^2(u) = R_h^2(u) - r > 0$ so that $x_{h-r}(u)$ is a hyperbolic point of $\mathcal{H}_{h-r}$: $R_{h-r}(u) = R_{h-r}^1(u) R_{h-r}^2(u) < 0$. If $x_{h-r}(u)$ is moreover a single point of $\mathcal{H}_{h-r}$, then for any ρ that is close enough to r in $]R_h^1(u), R_h^2(u)[$, $x_{h-\rho}(u)$ is a hyperbolic point of $\mathcal{H}_{h-\rho}$, and the configuration of $h_{x_{h-\rho}(u)}(\mathbb{R}_-)$, $h_{x_{h-\rho}(u)}(\{0\})$ and $h_{x_{h-\rho}(u)}(\mathbb{R}_+)$ on $\mathbb{S}^2$ is qualitatively the same that the one of $h_{x_{h-r}(u)}(\mathbb{R}_-)$, $h_{x_{h-r}(u)}(\{0\})$ and $h_{x_{h-r}(u)}(\mathbb{R}_+)$, so that there is no sign change of ε_h^u at r in such a case (see the relationships between $i_h(x)$, $j_h(x)$ and $r_h^-(x)$, $r_h^+(x)$, $c_h(x)$ that we gave in Section 3 for all $x \in \mathbb{R}^3 \setminus \mathcal{H}_h$). Therefore, if a sign change of ε_h^u occurs at $r \in]R_h^1(u), R_h^2(u)[$, then the hyperbolic point $x_{h-r}(u)$ of $\mathcal{H}_{h-r}$ is a multiple point of $\mathcal{H}_{h-r}$. But, a sign change of ε_h^u at $r \in]R_h^1(u), R_h^2(u)[$ cannot be due to the fact that $x = x_{h-r}(u)$ is also an elliptic point $x_{h-r}(v)$ or a singular point $x_{h-r}(v)$ of $\mathcal{H}_{h-r}$, which must be a regular point of a cuspidal edge of $\mathcal{H}_{h-r}$ since by assumption $\mathcal{N}_{\nabla h}(u)$ does not meet $\mathcal{S}ing(\mathcal{F}_{\nabla h})$. Indeed, if $x = x_{h-r}(u)$ is such a point $x_{h-r}(v)$, then there exists a a neighborhood $\mathcal{V}$ of v in $\mathbb{S}^2$ such that the absolute transverse orientation of $x_{h-r}(\mathcal{V})$ is fully determined by the direction of convexity on its elliptic part, so that no change of this absolute transverse orientation can be due to the crossing with the image under x_{h-r} of a neighborhood of u in $\mathbb{S}^2$. Thus, in the present case, if a sign change of ε_h^u occurs at r then $x_{h-r}(u)$ is an (at least double) hyperbolic point of $\mathcal{H}_{h-r}$, which achieves the proof.

□

In our proof of Corollary 1, we will make use of the following immediate remark.

Remark. Let $\mathcal{H}_h$ be a hedgehog such that $h \in C^\infty(\mathbb{S}^n;\mathbb{R})$. For any $u \in \mathbb{S}^2$ such that $x = x_h(u) \in \mathbb{R}^3 \setminus \mathcal{H}_h$, denote by $\mathcal{F}(\nabla h_x)$ the foliation of $\mathbb{S}^2$ by the level lines of $h_x : \mathbb{S}^2 \to \mathbb{R}$. For any such $u \in \mathbb{S}^2$, we have:

$$(u \text{ is a saddle point of } \mathcal{F}(\nabla h_x)) \Longleftrightarrow (x = x_h(u) \text{ is a hyperbolic point of } \mathcal{H}_h).$$

We will deduce the following reformulation of Heil's theorem without making use of the notion of a minimal spherical shell.

Proof of Corollary 1. Assume first that there exists some $u \in \mathbb{S}^2$ such that $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$. We then know from Theorem 1 that there exists some $r \in \mathbb{R}$ such that $x_{h-r}(u) = x_h(u) - ru \in \mathcal{N}_{\nabla h}(u)$ is an (at least) double hyperbolic point of $\mathcal{H}_{h-r}$: that is, there exists $v \in \mathbb{S}^2 \setminus \{u\}$, such that:

$$x_{h-r}(u) = x_{h-r}(v), \quad R_{h-r}(u) < 0, \quad \text{and } R_{h-r}(v) < 0.$$

If this point $x = x_{h-r}(u)$ belongs to $\mathbb{R}^3 \setminus \mathcal{F}_{\nabla h}$, then

$$i_{\mathcal{F}_{\nabla h}}(x) = -S(x) \leq -2,$$

where $S(x)$ is the number of saddle points of $\mathcal{F}(\nabla h_x)$ (see (3) and the above remark), and thus $N_{\nabla h}(x) = 2(1 - i_{\mathcal{F}_{\nabla h}}(x)) \geq 6$. In that case, knowing that $N_{\nabla h}$ remains constant on each component of $\mathbb{R}^3 \setminus \mathcal{H}_h$, there thus exists an open set formed by points of $\mathbb{R}^3 \setminus \mathcal{H}_h$ lying on at least 6 normal lines to $\mathcal{H}_h$. To reach the same conclusion in the case where $x = x_{h-r}(u) \in \mathcal{F}_{\nabla h}$, it suffices to prove that, arbitrarily close to x, there exists points $y \in \mathbb{R}^3 \setminus \mathcal{F}_{\nabla h}$ such that $i_{\mathcal{F}_{\nabla h}}(y) = -S(y) \leq -2$, and thus $N_{\nabla h}(y) = 2(1 - i_{\mathcal{F}_{\nabla h}}(y)) \geq 6$. Since u, $v \in \mathbb{S}^2$ are such that $u \neq v$, $R_{h-r}(u) < 0$ and $R_{h-r}(v) < 0$, there exist $\mathcal{U} \times I$, $\mathcal{V} \times J$ neighborhoods of respectively (u, r) and (v, r) in $\mathbb{S}^2 \times \mathbb{R}$, such that $\mathcal{U} \cap \mathcal{V} \neq \varnothing$ and $R_{h-\rho}(\omega) < 0$ for all (ω, ϱ) in $(\mathcal{U} \times I)$ or $(\mathcal{V} \times \mathcal{J})$. Besides, there exists a neighborhood $\mathcal{Y}$ of $x = x_{h-r}(u) = x_{h-r}(v)$ in $\mathbb{R}^3$ such that any point $y \in \mathcal{Y} \cap (\mathbb{R}^3 \setminus \mathcal{H}_h)$ can be written under the form $x_{_{h-\rho}}(\omega)$ with $(\omega, \varrho) \in \mathcal{U} \times I$ (resp. $(\omega, \varrho) \in \mathcal{V} \times \mathcal{J}$). Therefore, any $y \in \mathcal{Y} \cap (\mathbb{R}^3 \setminus \mathcal{H}_h)$ is such that $i_{\mathcal{F}_{\nabla h}}(y) = -S(y) \leq -2$, and thus $N_{\nabla h}(y) = 2(1 - i_{\mathcal{F}_{\nabla h}}(y)) \geq 6$. Indeed, the foliation of $\mathbb{S}^2$ by the level lines of $h_y : \mathbb{S}^2 \to \mathbb{R}$ has at least two distinct saddle points: one in $\mathcal{U}$ and the other in $\mathcal{V}$. Thus, in that case also, there exists an open set formed by points of $\mathbb{R}^3 \setminus \mathcal{H}_h$ lying on at least 6 normal lines to $\mathcal{H}_h$.

Therefore, it only remains to consider the case where all the normal lines to $\mathcal{H}_h$ meet the singular locus $\mathcal{S}ing\left(\mathcal{F}_{\nabla h}\right)$ of its focal $\mathcal{F}_{\nabla h}$, that is: $\forall u \in \mathbb{S}^2$, $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing\left(\mathcal{F}_{\nabla h}\right) \neq \varnothing$. But in this last case, there must exist a point of $\mathbb{R}^3$ lying on infinitely many normal lines to $\mathcal{H}_h$.

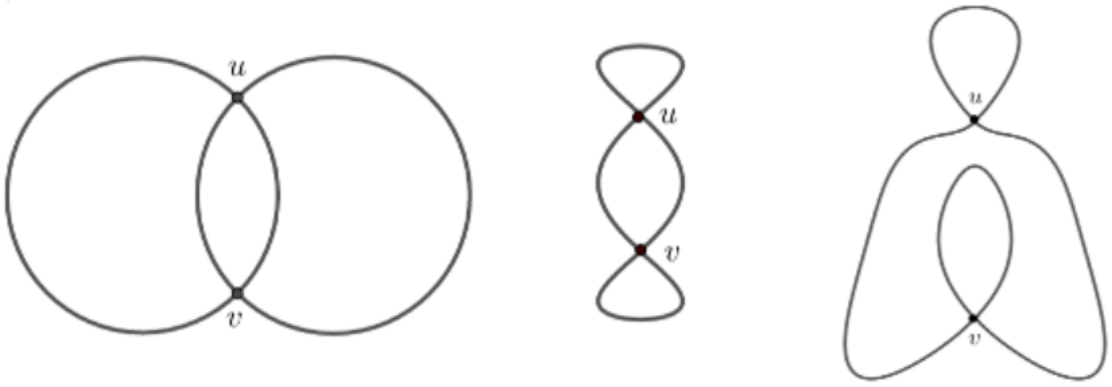
□

3.4 Critical saddle points

It is worth to stress that $x = x_{h-r}(u) = x_{h-r}(v)$, the double hyperbolic point of $\mathcal{H}_{h-r}$ that Theorem 1 states to exist, is not an arbitrary double hyperbolic point of $\mathcal{H}_{h-r}$ but a double hyperbolic point of $\mathcal{H}_{h-r}$ in the vicinity of which the absolute transverse orientation of $\mathcal{H}_{h-r}$ is reversed. This property can be read on the foliation $\mathcal{F}(\nabla h_x)$ of $\mathbb{S}^2$ by the level lines of $h_x : \mathbb{S}^2 \to \mathbb{R}$. Saying that $x = x_{h-r}(u) = x_{h-r}(v)$ is a double hyperbolic point of $\mathcal{H}_{h-r}$ is equivalent to saying that u and v are saddle points of $\mathcal{F}(h_x)$ lying on $(h_x)^{-1}(\{r\})$. Now, saying that x is a double hyperbolic point of $\mathcal{H}_{h-r}$ in the vicinity of which the absolute transverse orientation of $\mathcal{H}_{h-r}$ is reversed is equivalent to saying that these saddle points u, v of $\mathcal{F}(\nabla h_x)$ lying on $(h-r)_x^{-1}\{0\} = (h_x)^{-1}(\{r\})$ are 'critical saddle points' in the following sense.

Definition. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^\infty(\mathbb{S}^2;\mathbb{R})$, and let $u \in \mathbb{S}^2$ be such that $x = x_h(u)$ is a hyperbolic point of $\mathcal{H}_h$. Note that $u \in \mathbb{S}^2$ is such that any $v \in \mathbb{S}^2$ that is sufficiently close to u is a saddle point of $\mathcal{F}(\nabla h_y)$, where $y = x_h(v)$. We say that such a point v disconnects $(h_y)^{-1}(\mathbb{R}_-)$ (resp. $(h_y)^{-1}(\mathbb{R}_+)$) if $(h_y)^{-1}(\mathbb{R}_-)$ and $(h_y)^{-1}(\mathbb{R}_-)\setminus\{v\}$ (resp. $(h_y)^{-1}(\mathbb{R}_+)$ and $(h_y)^{-1}(\mathbb{R}_+)\setminus\{v\}$) does not have the same number of connected components. Now, the saddle point u of $\mathcal{F}(\nabla h_x)$ is said to be critical if the disconnecting or nondisconnecting character of v with respect to $(h_y)^{-1}(\mathbb{R}_-)$ and $(h_y)^{-1}(\mathbb{R}_+)$ changes in every neighborhood of u in the region of $(R_h)^{-1}(\mathbb{R}_-^*)$ containing it.*

By considering carefully the following three types of pairs of saddle points of $\mathcal{F}(\nabla h_x)$ shown in Figure 5, we check that u and v are critical saddle points of $\mathcal{F}(\nabla h_x)$ in the first case (a), and only in this case.



(a) (b) (c)

Figure 5

Recall that the critical and singular sets, $\mathcal{C}_{\nabla h}$ and $\mathcal{S}_{\nabla h}$, can respectively be defined by

$$\begin{aligned}\mathcal{C}_{\nabla h} &:= \{(x,u) \in U\mathbb{R}^3 \,|\nabla (h_x)(u) = 0\} \\ &= \{(x,u) \in U\mathbb{R}^3 \,|x \in \mathcal{N}_{\nabla h}(u)\},\end{aligned}$$

where $\mathcal{N}_{\nabla h}(u) := \{\nabla h(u)\} + \mathbb{R}u$ denotes the normal line to $\mathcal{H}_h$ at $x_h(u)$, and

$$\begin{aligned}\mathcal{S}_{\nabla h} &:= \left\{(x,u) \in \mathcal{C}_{\nabla h} \,\middle|\exists k \in [\![1,2]\!], x = c^k_{\nabla h}(u)\right\} \\ &= \left\{(x,u) \in \mathcal{C}_{\nabla h} \,\middle|\left(\det \circ \nabla^2\right)(h_x)(u) = 0\right\}.\end{aligned}$$

The following subset of $\mathcal{C}_{\nabla h} \setminus \mathcal{S}_{\nabla h}$ played an important role in our proof of Theorem 1:

$$\Sigma^c_{\nabla h} =: \{(x,u) \in \mathcal{C}_{\nabla h} \,|u \text{ is a critical saddle point of } \mathcal{F}(\nabla h_x)\}.$$

More precisely, our proof relies on the fact that under the assumption $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$, the focal segment $[c^1_{\nabla h}(u), c^2_{\nabla h}(u)]$ contains some interior point x such that $(x,u) \in \Sigma^c_{\nabla h}$. Indeed, for every $x \in \mathcal{N}_{\nabla h}(u)$, $(x,u) \in \Sigma^c_{\nabla h}$ if, and only if, there exist some $r \in \mathbb{R}$ such that $x = x_{h-r}(u)$ is a hyperbolic point of $\mathcal{H}_{h-r}$ and a neighborhood $\mathcal{U}$ of u on $\mathbb{S}^2$ such that the absolute transverse orientation of $x_{h-r}(\mathcal{U})$ is reversed in the vicinity of $x_{h-r}(u)$. From this remark we can deduce the following refinement of Corollary 2.

Proposition 4. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^3$ such that $h \in C^\infty(\mathbb{S}^2;\mathbb{R})$. If there does not exist a point of $\mathbb{R}^3$ lying on infinitely many normal*

lines to $\mathcal{H}_h$, then there exist infinitely many normal lines to $\mathcal{H}_h$ that do no meet the singular locus of $\mathcal{F}_{\nabla h}$ and any one of these lines meets the closure of $Int\left(\mathcal{F}^1_{\nabla h}\right) \cap Int\left(\mathcal{F}^2_{\nabla h}\right)$.

Proof Let u be any point of $\mathbb{S}^2$ such that $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing\left(\mathcal{F}_{\nabla h}\right) = \varnothing$. For such a point u, we know from our proof of Theorem 1 that there exists some $r \in \mathbb{R}$ such that u is a critical saddle point of $\mathcal{F}\left(\nabla h_x\right)$, where $x = x_{h-r}(u)$. By the very definition of a critical saddle point, this shows that $C_m(x) \geq 2$ and $C_M(x) \geq 2$. If $x \in \mathbb{R}^3 \backslash \mathcal{F}_{\nabla h}$, this exactly means that $x \in Int\left(\mathcal{F}^1_{\nabla h}\right) \cap Int\left(\mathcal{F}^2_{\nabla h}\right)$. Now if $x \in \mathcal{F}_{\nabla h}$, we know that arbitrarily close to x, there are points $y \in \mathbb{R}^3 \backslash \mathcal{F}_{\nabla h}$, and that if $y \in \mathbb{R}^3 \backslash \mathcal{F}_{\nabla h}$ is closed enough to x, then $C_m(y) \geq 2$ and $C_M(y) \geq 2$ (the arguments are the same as those used by E. Heil in [5]), so that $y \in Int\left(\mathcal{F}^1_{\nabla h}\right) \cap Int\left(\mathcal{F}^2_{\nabla h}\right)$. Therefore, in any case, $x \in \overline{Int\left(\mathcal{F}^1_{\nabla h}\right) \cap Int\left(\mathcal{F}^2_{\nabla h}\right)}$.

□

In higher dimension, our idea is to adopt the same type of approach.

4 The four-dimensional case

In order to follow a similar approach in the four dimensional case, we must overcome a series of difficulties.

First of all, when considering a hedgehog $\mathcal{H}_h$ in $\mathbb{R}^4$, we have to deal with two types of 'hyperbolic points', that is of points $x = x_h(u)$ in which the principal radii of curvature are non zero and not all of the same sign: a hyperbolic point $x = x_h(u)$ is of type 1 if $R_h(u) < 0$, and then it corresponds to a type 1 saddle point u of h_x (i.e. to a critical point u of index 1 of h_x); and a hyperbolic point $x = x_h(u)$ is of type 2 if $R_h(u) > 0$, and then it corresponds to a type 2 saddle point u of h_x (i.e. to a critical point u of index 2 of h_x). We will often say simply 'i-saddle of $\mathcal{F}\left(\nabla h_x\right)$' instead of 'type i saddle point u of h_x', ($i \in [\![1, 2]\!]$).

Besides, in the four dimensional case the usual transverse orientation of $x_h\left(\mathbb{S}^3\right) = x_{\widetilde{h}}\left(\mathbb{S}^3\right)$ is no longer relative but absolute due to the even parity of the dimension of $\mathbb{R}^4$: it does not depend on the choice between h and $\widetilde{h}$ as the support function (i.e. it does not depends on the choice of the orientation of the normal lines to the hypersurface). On the other hand, if the index r_h defined by

$$r_h(x) = r_h^+(x) - r_h^-(x) \quad \text{for all } x \in \mathbb{R}^{n+1} \setminus \mathcal{H}_h,$$

where $r_h^-(x)$ (resp. $r_h^+(x)$) is the number of connected components of $\mathbb{S}^n \setminus h_x^{-1}(\{0\})$ on which $h_x : \mathbb{S}^n \to \mathbb{R}$ is negative (resp. positive), is none other that the usual index i_h for $n+1 = 3$ (see above and [13]), it provides

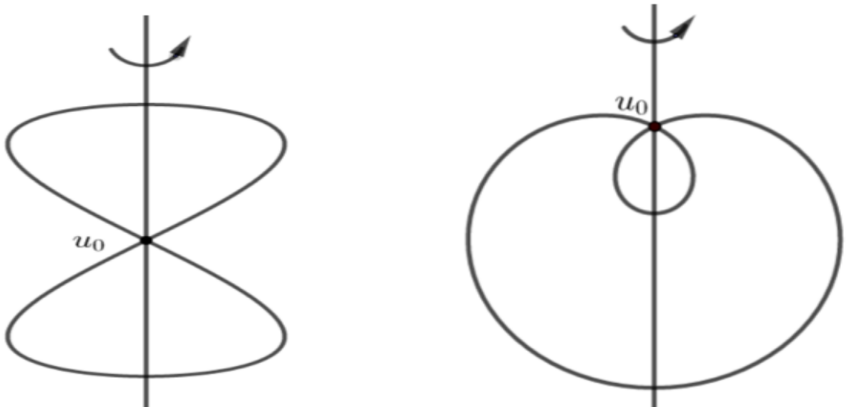
us with a new index and a new transverse orientation which is relative (to the choice of the orientation of the normal lines to $x_h(\mathbb{S}^3) = \widetilde{x_h}(\mathbb{S}^3)$) for $n+1=4$. We will call it '*the relative transverse orientation*'.

4.1 New index and transverse orientation

Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3;\mathbb{R})$. Above, we defined the index $r_h : \mathbb{R}^4 \setminus \mathcal{H}_h \to \mathbb{N}$ as follows:

$$r_h(x) = r_h^+(x) - r_h^-(x) \quad \text{for all } x \in \mathbb{R}^4 \setminus \mathcal{H}_h,$$

where $r_h^-(x)$ (resp. $r_h^+(x)$) denotes the number of connected components of $\mathbb{S}^3 \setminus h_x^{-1}(\{0\})$ on which $h_x : \mathbb{S}^3 \to \mathbb{R}$ is negative (resp. positive). Of course, the index r_h remains constant on each connected component of $\mathbb{R}^4 \setminus \mathcal{H}_h$. In particular, r_h is equal to 0 on the unbounded component of $\mathbb{R}^4 \setminus \mathcal{H}_h$. It worth noting that the value of $r_h(x)$ must obviously decrease by one unit as x orthogonally crosses $\mathcal{H}_h$ at a simple elliptic point $x_0 = x_h(u_0)$ in the direction of $u_0 \in \mathbb{S}^3$. When the crossing occurs at a simple hyperbolic point $x_0 = x_h(u_0)$, it is necessary to distinguish two cases depending on whether the saddle point u_0 of $\mathcal{F}(\nabla h_{x_0})$ disconnects the connected component, say $\mathcal{L}_{u_0}$, of $(h_{x_0})^{-1}(\{0\})$ containing it or not. We can imagine an example of each of these two configurations by rotating the two figures shown inf Figure 6 around a vertical axis passing through the saddle point u_0 in $\mathbb{R}^3 \subset \mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$. If $\mathcal{L}_{u_0} \setminus \{u_0\}$ is disconnected, as in the first example, we can check that the value of $r_h(x)$ decreases by one unit as x orthogonally crosses $\mathcal{H}_h$ at $x_0 = x_h(u_0)$ in the direction of $-u_0$. We say then that x_0 is an active hyperbolic point of $\mathcal{H}_h$, and that u_0 is an active saddle point of $\mathcal{F}(\nabla h_{x_0})$. If $\mathcal{L}_{u_0} \setminus \{u_0\}$ is connected, as in the second example where $\mathcal{L}_{u_0}$ is a torus pinched at u_0, we can check that the value of $r_h(x)$ does not change as x orthogonally crosses $\mathcal{H}_h$ at $x_0 = x_h(u_0)$. We say then that x_0 is a neutral hyperbolic point of $\mathcal{H}_h$, and that u_0 is a neutral saddle point of $\mathcal{F}(\nabla h_{x_0})$. In short, the 'relative transverse orientation' of $\mathcal{H}_h$ (that is, the one that is associated with the r_h-index) is such that whenever $x_h(u)$ is a simple regular point of $\mathcal{H}_h$, then the normal line to $\mathcal{H}_h$ at $x_h(u)$ is oriented in the direction of u (resp. $-u$) if $x_h(u)$ is elliptical (resp. hyperbolic and active), and is not oriented if $x_h(u)$ is a neutral hyperbolic point.



Pairing saddle Neutral saddle

Figure 6

4.2 Effect of a transverse crossing of $\mathcal{F}_{\nabla h}$ and $i_{\mathcal{F}^2_{\nabla h}}$

Let $k \in [\![1,3]\!]$. As noticed above in 2.3.2., when $x \in \mathbb{R}^4$ transversally crosses the k^{th} sheet $\mathcal{F}^k_{\nabla h}$ of $\mathcal{F}_{\nabla h}$ at a simple regular point of $\mathcal{F}^k_{\nabla h}$ in the direction of the transverse orientation, this crossing of $\mathcal{F}^k_{\nabla h}$ results in a two-units increase in the number of critical points of $h_x : \mathbb{S}^3 \to \mathbb{R}$, and the two critical points created have consecutive indices $k-1$ and k. Of course, the same crossing but in the opposite direction results in the elimination of such a pair of critical points. Mathematically, such a transition can be regarded as a path in the vector space $C^\infty(\mathbb{S}^3;\mathbb{R})$ that crosses a 'codimension 1 non-Morse stratum' transversely at one point. We know from 'parametrized Morse theory' that: (i) at the very moment of the transition, $h_x : \mathbb{S}^3 \to \mathbb{R}$ admits a so-called 'birth-death singularity'; (ii) the transition itself can locally be described by a smooth family of functions $f_t : u \mapsto f_t(u)$ parametrized by t, $(t \in \mathbb{R})$, of the form

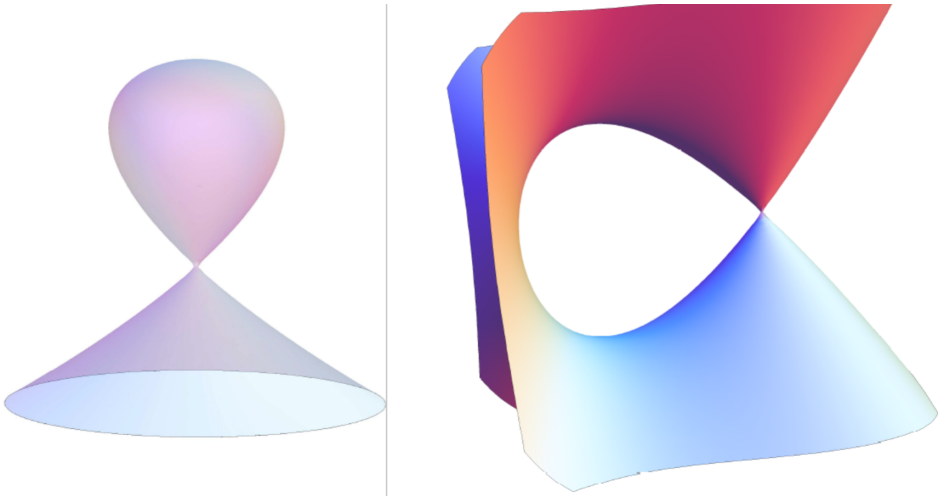
$$f_t(u_1,u_2,u_3) = u_1^3 - tu_1 + \varepsilon_2 u_2^2 + \varepsilon_3 u_3^2 + g(t),$$

where $\varepsilon_2, \varepsilon_3 \in \{\pm 1\}$, with respect to some local coordinates (u_1,u_2,u_3); when $t<0$, $f_t : u \mapsto f_t(u)$ is a Morse function without critical point; $f_0 : u \mapsto f_0(u)$ is a generalized Morse function with a birth-death singularity at $u=(0,0,0)$; when $t>0$, $f_t : u \mapsto f_t(u)$ is a Morse function with two critical points of consecutive indices (these points were 'born' at $t=0$).

When $x \in \mathbb{R}^4$ transversally crosses $\mathcal{F}_{\nabla h}$ at a simple regular point in the direction of the transverse orientation, we are thus in one of the following three cases:

Case	Sheet crossed by x	Pair of critical points that are born
$\varepsilon_2 = \varepsilon_3 = 1$	$\mathcal{F}^1_{\nabla h}$	1 local minimal and 1 type 1 saddle
$\varepsilon_2 \neq \varepsilon_3$	$\mathcal{F}^2_{\nabla h}$	1 type 1 saddle and 1 type 2 saddle
$\varepsilon_2 = \varepsilon_3 = -1$	$\mathcal{F}^3_{\nabla h}$	1 local maxima and 1 type 2 saddle

In the first (resp. third) case, a 'trivial bubble' (a 'trivial centre-saddle pairing' of $\mathcal{F}(\nabla h_x)$) was born as shown in Figure 7 (a): the centre of this trivial centre-saddle pairing is of course located inside the bubble, and the corresponding saddle point is obviously active. In the second case, two saddle points of distinct types of $\mathcal{F}(\nabla h_x)$ were born. In the vicinity of each of them, the level set of the corresponding saddle point looks qualitatively like the piece of surface shown in Figure 7 (b). Moreover, the two level sets corresponding to these two saddle points are in fact two pinched torus that are 'linked somehow like a Hopf link' [20]. In particular, these two saddle points are neutral.



(a) (b)

Figure 7

Let $x \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$. For any regular value d of $h_x : \mathbb{S}^3 \to \mathbb{R}$, denote by $g_{h_x}(d)$ the genus of the (not necessarily connected) surface $(h_x)^{-1}(\{d\})$. Now, for any critical value c of h_x, put

$$g_{h_x}(c) := \min_{\varepsilon>0} \left(\max\left(\{g_{h_x}(d) \,|d \text{ regular value of } h_x \text{ in }]c-\varepsilon, d+\varepsilon[\}\right)\right),$$

and then

$$G_h(x) = \frac{1}{2} \sum_{c \in C_h(x)} g_{h_x}(c), \text{ where } C_h(x) = h_x\left(\left\{u \in \mathbb{S}^3 \,|\nabla h_x(u) = 0\right\}\right).$$

From the above study, when x transversally crosses $\mathcal{F}_{\nabla h}$ at a simple regular point $x_0 \in \mathcal{F}_{\nabla h}$, the value of $G_h(x)$ changes if, and only if $x_0 \in \mathcal{F}^2_{\nabla h}$. In this case, the value of $G_h(x)$ increases (resp. decreases) by one unit if the crossing occurs in the direction of the transverse orientation of $\mathcal{F}_{\nabla h}$ (resp. in the opposite direction). Knowing that $G_h(x) = 0$ when x is far from $\mathcal{H}_h$, we deduce the following result.

Theorem 3. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3;\mathbb{R})$, let $\mathcal{F}_{\nabla h}$ denote its focal surface, and let $\mathcal{F}^2_{\nabla h}$ be the second sheet of $\mathcal{F}_{\nabla h}$. We have:*

$$\forall x \in \mathbb{R}^4 \setminus \mathcal{F}^2_{\nabla h}, \quad i_{\mathcal{F}^2_{\nabla h}}(x) = -G_h(x).$$

As a corollary we obtain that: $\forall x \in \mathbb{R}^4 \setminus \mathcal{F}^2_{\nabla h}$,

$$Int\left(\mathcal{F}^2_{\nabla h}\right) = \left\{x \in \mathbb{R}^4 \setminus \mathcal{F}^2_{\nabla h} \,|G_h(x) \neq 0\right\};$$

and thus:

$$\left(x \in Int\left(\mathcal{F}^2_{\nabla h}\right)\right) \Longleftrightarrow \begin{pmatrix} h_x : \mathbb{S}^3 \to \mathbb{R} \text{ admits a smooth} \\ \text{level surface with nonzero genus} \end{pmatrix};$$

which is the statement of theorem 2.

4.3 Critical saddle points

As in dimension 3, our idea will be to focus our attention, for each $u \in \mathbb{S}^3$ such that $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$, on the values of r such that $x = x_{h-r}(u)$ is a (type 1 or type 2) hyperbolic point of $\mathcal{H}_{h-r}$ (and thus, u a saddle point of $\mathcal{F}(\nabla h_x)$) at which the relative transverse orientation of $\mathcal{H}_{h-r}$ switches (here, between the active and neutral modes): we will say then that u is a critical (type 1 or type 2) saddle point of $\mathcal{F}(\nabla h_x)$, where $x = x_{h-r}(u)$.

Definition. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^4$ such that $h \in C^\infty(\mathbb{S}^3;\mathbb{R})$, and let $x = x_h(u)$ be a hyperbolic point of type i of $\mathcal{H}_h$, $(i \in [\![1,2]\!])$. Then, any $v \in \mathbb{S}^3$ that is sufficiently close to u on $\mathbb{S}^3$ is a i-saddle point of $\mathcal{F}(\nabla h_y)$, where $y = x_h(v)$. We say that such an v disconnects*

$(h_y)^{-1}(\{0\})$ *if* $(h_y)^{-1}(\{0\})$ *and* $(h_y)^{-1}(\{0\})\backslash\{v\}$ *does not have the same number of connected components. Now, the saddle point* u *of* $\mathcal{F}(\nabla h_x)$ *is said to be critical if the disconnecting or nondisconnecting character of* v *changes in every neighborhood of* u *in the region of* $(R_h)^{-1}(\mathbb{R}^*)$ *containing it.*

Return briefly to singularity and Morse theories viewpoints. Recall that the critical and singular sets, $\mathcal{C}_{\nabla h}$ and $\mathcal{S}_{\nabla h}$, can respectively defined by

$$\begin{aligned}\mathcal{C}_{\nabla h} &:= \{(x,u) \in U\mathbb{R}^4 \,|\nabla(h_x)(u) = 0\} \\ &= \{(x,u) \in U\mathbb{R}^4 \,|x \in \mathcal{N}_{\nabla h}(u)\},\end{aligned}$$

where $\mathcal{N}_{\nabla h}(u) := \{\nabla h(u)\} + \mathbb{R}u$ denotes the normal line to $\mathcal{H}_h$ at $x_h(u)$, and

$$\begin{aligned}\mathcal{S}_{\nabla h} &:= \left\{(x,u) \in \mathcal{C}_{\nabla h} \,\middle|\, \exists k \in [\![1,3]\!], x = c^k_{\nabla h}(u)\right\} \\ &= \left\{(x,u) \in \mathcal{C}_{\nabla h} \,\middle|\, \left(\det \circ \nabla^2\right)(h_x)(u) = 0\right\}.\end{aligned}$$

Consider the following subsets of $\mathcal{C}_{\nabla h} \setminus \mathcal{S}_{\nabla h}$ for each $i \in [\![1,2]\!]$:

$$\Sigma_i(\nabla h) := \{(x,u) \in \mathcal{C}_{\nabla h} \setminus \mathcal{S}_{\nabla h} \,|u \text{ is a } i\text{-saddle point of } \mathcal{F}(\nabla h_x)\}$$

and

$$\Sigma^c_i(\nabla h) := \{(x,u) \in \Sigma_i(\nabla h) \mid u \text{ is a critical } i\text{-saddle point of } \mathcal{F}(\nabla h_x)\}.$$

For each $i \in [\![1,2]\!]$, $\Sigma_i(\nabla h) \setminus \Sigma^c_i(\nabla h)$ is the following union of disjoint open subsets:

$$\Sigma_i(\nabla h) \setminus \Sigma^c_i(\nabla h) = \Sigma^a_i(\nabla h) \bigsqcup \Sigma^n_i(\nabla h),$$

where:

$$\Sigma^a_i(\nabla h) := \{(x,u) \in \Sigma_i(\nabla h) \mid u \text{ is an active } i\text{-saddle point of } \mathcal{F}(\nabla h_x)\}$$

and

$$\Sigma^n_i(\nabla h) := \{(x,u) \in \Sigma_i(\nabla h) \mid u \text{ is a neutral } i\text{-saddle point of } \mathcal{F}(\nabla h_x)\}.$$

More precisely, for each $i \in [\![1,2]\!]$, $\Sigma^c_i(\nabla h)$ separates $\Sigma^a_i(\nabla h)$ and $\Sigma^n_i(\nabla h)$ in $\Sigma_i(\nabla h)$. We are now prepared to prove our Theorem 4.

4.4 Proofs of Theorem 4 and Corollary 3

Proof of Theorem 4. Since $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$, the 3 centers of principal curvature $c^1_{\nabla h}(u)$, $c^2_{\nabla h}(u)$ and $c^3_{\nabla h}(u)$ are pairwise distinct. When $r \in]R^1_h(u), R^2_h(u)[$ tends to $R^1_h(u)$, u is an active 1-saddle point of $\mathcal{F}(\nabla h_x)$, where $x = x_{h-r}(u)$, whereas when $r \in]R^1_h(u), R^2_h(u)[$ tends to $R^2_h(u)$, u is a neutral 1-saddle point of $\mathcal{F}(\nabla h_x)$, where $x = x_{h-r}(u)$. More precisely, $(x_{h-r}(u), u) \in \Sigma^a_i(\nabla h)$ (resp. $(x_{h-r}(u), u) \in \Sigma^n_i(\nabla h)$) when $r \in]R^1_h(u), R^2_h(u)[$ tends to $R^1_h(u)$ (resp. $R^2_h(u)$). From this it follows that there exists some $r_1 \in]R^1_h(u), R^2_h(u)[$ such that u is a critical 1-saddle point of $\mathcal{F}(\nabla h_{x_1})$, where $x_1 = x_{h-r_1}(u)$. Since u is a critical 1-saddle point of $\mathcal{F}(\nabla h_{x_1})$, there exists some $v \in \mathbb{S}^3 \setminus \{u\}$ that is a critical point of $\mathcal{F}(\nabla h_{x_1})$ lying on the connected component, say $\mathcal{L}_u$, of the level set of h_{x_1} that contains u. Such an v is of course such that $x_1 = x_{h-r_1}(v)$. Since $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$, if v is a degenerate critical point of $(h - r_1)_{x_1}$, then v is such that $R_{h-r_1}(v) = 0$ (since v is degenerate), $\nabla R_{h-r_1}(v) \neq 0$ and $rk[T_v x_{h-r_1}] = 2$ (otherwise x_1 would belong to $\mathcal{S}ing(\mathcal{F}_{\nabla h})$). It follows that under our assumptions, $\mathcal{H}_{h-r_1}$ has then a cusp singularity at $x_1 = x_{h-r_1}(v)$; that is, the image under x_{h-r_1} of a neighborhood $\mathcal{V}$ of v is diffeomorphic to a neighborhood of 0 in $\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|x_2^2 = x_1^3\}$. But in such a configuration, the relative transverse orientation of $x_{h-r_1}(v)$ is fully determined in the vicinity of x_1 by the index of the principal radius of curvature that vanishes at v: indeed, if $R^2_{h-r_1}(v) = 0$, then the hyperbolic points of $x_{h-r_1}(\mathcal{V})$ are neutral in the vicinity of x_1, and if $R^1_{h-r_1}(v) = 0$ (resp. $R^3_{h-r_1}(v) = 0$) the relative transverse orientation of $x_{h-r_1}(\mathcal{V})$ at a regular point $x_{h-r_1}(w)$ that is close enough to $x_1 = x_{h-r_1}(v)$ is given by:

$$sgn[R_{h-r_1}(w)]\, w \quad (\text{resp. } -sgn[R_{h-r_1}(w)]\, w).$$

Thus if v is a degenerate critical point of $(h - r_1)_{x_1}$, then no switch of the relative transverse orientation (between the active and neutral modes) at u can be due to the crossing of $x_{h-r_1}(\mathcal{V})$ with the image under x_{h-r_1} of a neighborhood of u in $\mathbb{S}^3$. Therefore, v is a nondegenerate critical point of $(h - r_1)_{x_1}$ and thus a 1-saddle point of $\mathcal{F}(\nabla h_{x_1})$ by the way it is obtained. Thus $r_1 \in]R^1_h(u), R^2_h(u)[$ is such that $x_{h-r_1}(u) = x_h(u) - r_1 u$ is an (at least) type 1 double hyperbolic point of $\mathcal{H}_{h-r_1}$.4

Finally, we can adapt the above arguments to prove that there exists some $r_2 \in]R^2_h(u), R^3_h(u)[$ such that $x_{h-r_2}(u) = x_h(u) - r_2 u$ is an (at least) type 2 double hyperbolic point of $\mathcal{H}_{h-r_2}$.

□

Proof of Corollary 3. Let $u \in \mathbb{S}^3$ such that $\mathcal{N}_{\nabla h}(u) \cap \mathcal{S}ing(\mathcal{F}_{\nabla h}) = \varnothing$.

For such a point u, we know from our proof of Theorem 4 that there exists $(r_1, r_2) \in \mathbb{R}^2$ such that, for each $i \in [\![1, 2]\!]$, $x_{h-r_i}(u) = x_h(u) - r_i u$ is an (at least) type i double hyperbolic point of $\mathcal{H}_{h-r_i}$, and u a critical saddle point of $\mathcal{F}(\nabla h_{x_i})$, where $x_i = x_{h-r_i}(u)$.

Assume first that $x_1 = x_h(u) - r_1 u$ is in $\mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$. Since x_1 is an (at least) double type 1 hyperbolic point, we then have $S_1(x_1) \geq 2$. Now (see Subsubsection 2.3.2)

$$\sum_{k=1}^{2} i_{\mathcal{F}^k_{\nabla h}}(x_1) = (1 - C_m(x_1)) + (C_m(x_1) - S_1(x_1) - 1) = -S_1(x_1),$$

and consequently

$$i_{\mathcal{F}_{\nabla h}}(x_1) = \sum_{k=1}^{3} i_{\mathcal{F}^k_{\nabla h}}(x_1) = -S_1(x_1) + i_{\mathcal{F}^3_{\nabla h}}(x_1) \leq -S_1(x_1).$$

Therefore $i_{\mathcal{F}_{\nabla h}}(x_1) \leq -2$, and thus $N_{\nabla h}(x_1) = 2(1 - i_{\mathcal{F}_{\nabla h}}(x_1)) \geq 6$.

Now, if $x_1 \in \mathcal{F}_{\nabla h}$, we can proceed as in the proof of Corollary 1 to show that arbitrarily close to x_1, there exists some $y_1 \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$ such that $S_1(y_1) \geq 2$, and thus $N_{\nabla h}(y_1) = 2(1 - i_{\mathcal{F}_{\nabla h}}(y_1)) \geq 6$. Therefore, in any case, $x_1 = x_h(u) - r_1 u$ is in the closure of $\{x \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h} \,|\, N_{\nabla h}(x) \geq 6\}$.

But, in fact, we can be more precise. Since u a critical saddle point of $\mathcal{F}(\nabla h_{x_1})$, we know that arbitrarily close to x_1, there exists some $y_1 \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$ that is a neutral hyperbolic point and thus a point of $Int(\mathcal{F}^2_{\nabla h})$. Furthermore, since u is a critical saddle point of $\mathcal{F}(\nabla h_{x_1})$, we have $C_m(x_1) \geq 2$, and we know [5] that arbitrarily close to x_1, there are points $y_1 \in \mathbb{R}^3 \setminus \mathcal{F}_{\nabla h}$, and that if $y_1 \in \mathbb{R}^3 \setminus \mathcal{F}_{\nabla h}$ is closed enough to x, then $C_m(y_1) \geq 2$ and thus $y_1 \in Int(\mathcal{F}^1_{\nabla h})$. Therefore, arbitrarily close to x_1, there exists some $y_1 \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h}$ such that $y_1 \in Int(\mathcal{F}^1_{\nabla h}) \cap Int(\mathcal{F}^2_{\nabla h})$.

Of course, we can prove in the same way that $x_2 = x_h(u) - r_2 u$ is also in the closure of $\{x \in \mathbb{R}^4 \setminus \mathcal{F}_{\nabla h} \,|\, N_{\nabla h}(x) \geq 6\}$, and more precisely in the closure of $Int(\mathcal{F}^2_{\nabla h}) \cap Int(\mathcal{F}^3_{\nabla h})$.

□

5 Further results and remarks

5.1 Minimal spherical shell of a convex body

As we said in introduction, E. Heil proved the concurrent normal conjecture in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) using of the existence of a minimal spherical shell for any convex body of $\mathbb{R}^2$(resp. $\mathbb{R}^3$) which had been established by T. Bonnesen [3] (resp. N. Kritikos [8]). In 1988, I. Bárány extended

this existence to higher dimensions [2]. Given a convex body K in $\mathbb{R}^{n+1}$, he defined

$$r(x) := \max(\{r \in \mathbb{R}_+ \,|B(x,r) \subseteq K\}),$$

$$\text{and} \quad R(x) := \min(\{R \in \mathbb{R}_+ \,|K \subseteq B(x,r)\}),$$

where $B(x,r)$ denotes the closed ball with center x and radius r, and proved the following:

Theorem (I. Bárány, 1988). *There exists a unique point x_0 in K at which the function $R - r : K \to \mathbb{R}_+$, $x \mapsto R(x) - r(x)$ attains its minimum value.*

The set $C(x_0; r(x_0), R(x_0)) := \{x \in \mathbb{R}^{n+1} \,|r(x_0) \leq \|x - x_0\| \leq R(x_0)\}$ is called the minimal spherical shell of K. On each sphere bounding $C(x_0; r(x_0), R(x_0))$ there are at least two points of K. More precisely, $C(x_0; r(x_0), R(x_0))$ can be characterized as follows.

Theorem (I. Bárány, 1988). *The point $x_0 \in K$ is the center of the minimal spherical shell of K if, and only if, there exist $(u_1, \ldots, u_p)\,(\mathbb{S}^n)^p$ and $(v_1, \ldots, v_q) \in (\mathbb{S}^n)^q$, $(p, q \geq 1)$, such that:*

$$(i)\ \ Conv(\{u_1, \ldots, u_p\}) \cap Conv(\{v_1, \ldots, v_q\}) \neq \varnothing,$$

where Conv denotes the convex hull;

$$(ii)\ \ x_0 + r(x_0)\, u_i \in \partial K \ \text{ and } \ x_0 + R(x_0)\, v_j \in \partial K \ \text{ for all } \ (i,j) \in [\![1,p]\!] \times [\![1,q]\!],$$

where ∂K denotes the boundary of K.

5.2 Extension to hedgehogs

Any hedgehog $\mathcal{H}_h$ of $\mathbb{R}^{n+1}$ that is such that $h \in C^2(\mathbb{S}^n; \mathbb{R})$ can be regarded as a parallel hypersurface to the boundary of some convex body of class C^2_+, say K, in $\mathbb{R}^{n+1}$ (see 2.1.1): there exists $\rho \in \mathbb{R}$ such that $h = h_K - \rho$, where h_K is the support function of K. We can thus define the minimal spherical shell of $\mathcal{H}_h$ to be the transversely oriented shell that is bounded by the transversely oriented spheres or point spheres

$$S(x_0; r(x_0) - \rho) \quad \text{and} \quad S(x_0; R(x_0) - \rho),$$

where $C(x_0; r(x_0), R(x_0))$ is the minimal spherical shell of K, and where $S(a; r))$ denotes the sphere of radius $|r|$ centered at a that is transversely oriented by its outward (resp. inward) pointing normals if $r > 0$ (resp. $r < 0$) holds, and the point sphere $\{a\}$ if $r = 0$ holds. This

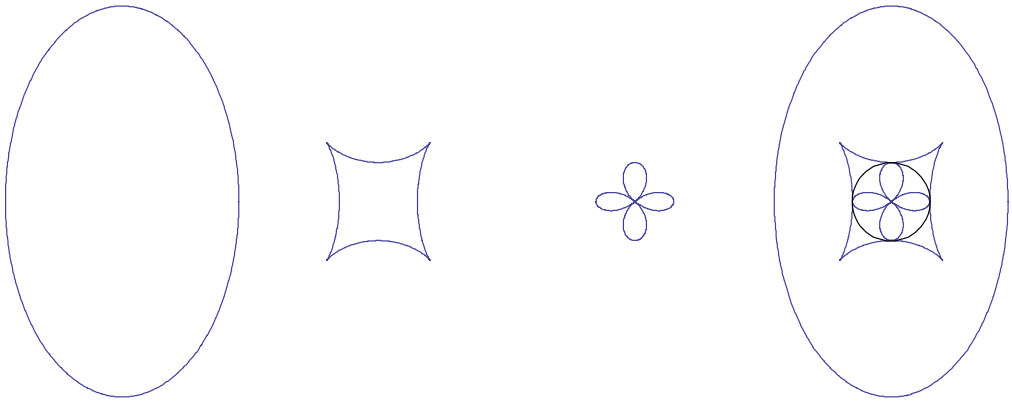
definition is of course independent from the choice of the convex body K of class C_+^2 the boundary of which is parallel to $.\mathcal{H}_h$.

Remark. All these minimal spherical shells can be interpreted in terms of pedal hypersurfaces. Recall that the pedal hypersurface of the hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ with respect to a point $x \in \mathbb{R}^{n+1}$ is parametrized by

$$\begin{aligned} &\mathbb{S}^n \to \mathbb{R}^{n+1} \\ &u \longmapsto x + h_x(u)\, u, \end{aligned}$$

where $h_x(u) = h(u) - \langle x, u\rangle$. In other words, to any $u \in \mathbb{S}^n$ is associated the foot of the perpendicular line from x to the support hyperplane $\mathcal{H}_h$ at $x_h(u)$. If $C(x_0; r(x_0), R(x_0))$ is the minimal spherical shell of some convex body K with support function $h \in C^2(\mathbb{S}^n; \mathbb{R})$ in $\mathbb{R}^{n+1}$, we can easily check that: (i) the pedal hypersurface of the hedgehog $\mathcal{H}_{h-(r(x_0)+R(x_0))/2}$ with respect to x_0 is contained in the closed ball $B\left(x_0; \frac{R(x_0)-r(x_0)}{2}\right)$; (ii) no other pedal hypersurface of $\mathcal{H}_h$ is contained in a smaller closed ball of $\mathbb{R}^{n+1}$; (iii) Properties (i) and (ii) permit to characterize the minimal spherical shell of K. Moreover, this interpretation can be extended to hedgehogs.

Let us give an example. Figure 8 shows: (a) the ellipse with support function $\theta \in \mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z} \mapsto h(\theta) := \sqrt{9\cos^2\theta + 25\sin^2\theta}$, the minimal spherical shell of which is $C(0_{\mathbb{R}^2}; 3, 5)$ where $0_{\mathbb{R}^2}$ is the origin of $\mathbb{R}^2$; (b) the hedgehog with support function $h - (r(0_{\mathbb{R}^2}) + R(0_{\mathbb{R}^2}))/2 = h - 4$, which is parallel to the ellipse ; (c) the pedal $\mathcal{P}_0(\mathcal{H}_{h-4})$ of $\mathcal{H}_{h-4}$ with respect to the origin; (d) all these curves together with the circle $C(0_{\mathbb{R}^2}; 1)$ with center $0_{\mathbb{R}^2}$ and radius $\frac{1}{2}(5-3) = 1$. Note that in this example, the minimal spherical shell $C(0_{\mathbb{R}^2}; -1, 1)$ of $\mathcal{H}_{h-4}$ is formed by two circles that are the same but with opposite transverse orientations. Figure 9 shows the hedgehog with support function $h - \frac{7}{2}$, its pedal with respect to the origin, and both curves together with the minimal spherical shell (abbreviated into m.s.s) of $\mathcal{H}_{h-\frac{7}{2}}$.



(a) The ellipse (b) $\mathcal{H}_{h-4}$ (c) $\mathcal{P}_0\left(\mathcal{H}_{h-4}\right)$; (d)These curves and $C\left(0_{\mathbb{R}^2};1\right)$

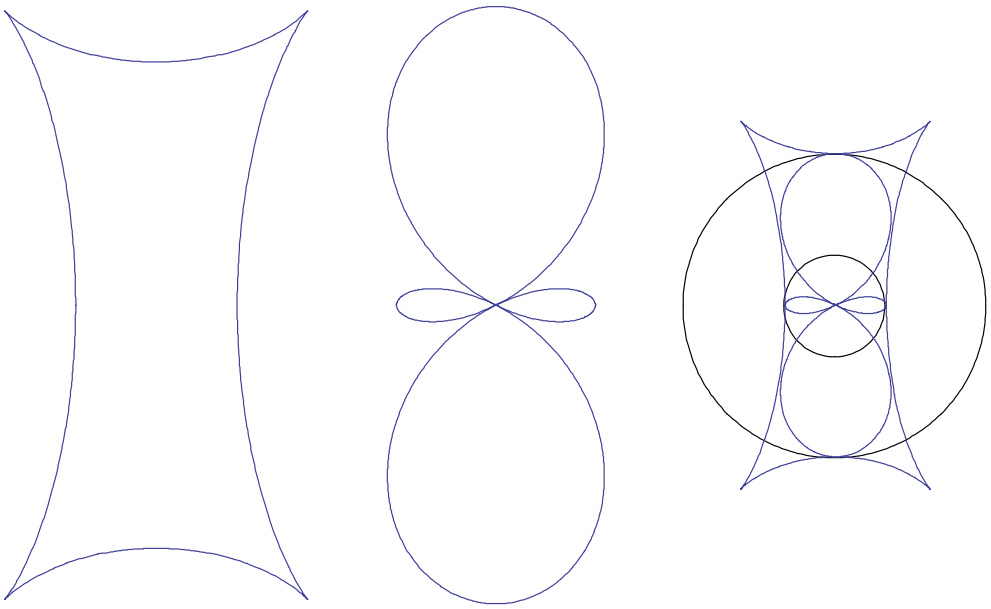


Figure 9. $\mathcal{H}_{h-\frac{7}{2}}$, $\mathcal{P}_0\left(\mathcal{H}_{h-\frac{7}{2}}\right)$, and these curves with the m.s.s. of $\mathcal{H}_{h-\frac{7}{2}}$

Bárány's characterization theorem of the minimal spherical shell can of course be adapted to hedgehogs:

Theorem 5. *Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^{n+1}$ such that $h \in C^2\left(\mathbb{S}^n;\mathbb{R}\right)$, and let $x \in \mathbb{R}^{n+1}$ and $(r,R) \in \mathbb{R}^2$ be such that $r \leq h_x \leq R$, where $h_x\left(u\right) = h\left(u\right) - \langle x,u\rangle$, $(u \in \mathbb{S}^n)$. Then $C\left(x;r,R\right)$ is the minimal spherical shell of $\mathcal{H}_h$ if, and only if, there exist $\left(u_1,\ldots,u_p\right)\left(\mathbb{S}^n\right)^p$ and $\left(v_1,\ldots,v_q\right) \in$*

$(\mathbb{S}^n)^q$, $(p, q \geq 1)$, *such that:*

$$(i)\ \ Conv\left(\{u_1, \ldots, u_p\}\right) \cap Conv\left(\{v_1, \ldots, v_q\}\right) \neq \varnothing,$$

where Conv denotes the convex hull;

$$(ii)\ \ h_x\left(u_i\right) = r,\ h_x\left(v_j\right) = R,\ and\ \nabla h_x\left(u_i\right) = \nabla h_x\left(v_j\right) = 0\ .$$

for all $(i, j) \in [\![1, p]\!] \times [\![1, q]\!]$.

Now, it is easy to see that there exists $h \in C^2\left(\mathbb{S}^3; \mathbb{R}\right)$ such that: $(i)\,|h| \leq 1$; (ii) the set $\left\{u \in \mathbb{S}^3 \,|h\left(u\right) = -1\right\}$ has exactly two distinct elements u_1, u_2, and the set $\left\{u \in \mathbb{S}^3 \,|h\left(u\right) = 1\right\}$ has exactly two distinct elements v_1, v_2; (iii) $[u_1u_2] \cap [v_1v_2] \neq \varnothing$; (iv) h has exactly six critical points: 2 minima, 2 maxima and two saddle points. For such a $h \in C^2\left(\mathbb{S}^3; \mathbb{R}\right)$, $C\left(0_{\mathbb{R}^4}; -1, 1\right)$ is the minimal spherical shell of $\mathcal{H}_h$, and there are only six normal lines to $\mathcal{H}_h$ passing through $0_{\mathbb{R}^4}$. Therefore:

Theorem 6. *For $n = 4$, it is not true that for any convex body K of $\mathbb{R}^n$, there are at least $2n$ normal lines passing through the center of the minimal spherical shell of K.*

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