On existence of holomorphic Lambert-Tsallis functions

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Abstract. In this paper, we study a new important class of special functions: the holomorphic Lambert-Tsallis functions, which are a two-parameter generalization of the Lambert function. A necessary and sufficient condition on the parameters is given for the existence of the main branch of the Lambert-Tsallis function.

Resumé: Dans cet article, nous étudions une nouvelle classe importante de fonctions spéciales: les fonctions holomorphes de Lambert-Tsallis qui sont une généralisation à deux paramètres de la fonction de Lambert. Une condition nécessaire et suffisante sur les paramètres est donnée pour l'existence de la branche principale de la fonction de Lambert-Tsallis.

Introduction

The Lambert W function is the multivariate inverse function of $f(z) = ze^z$, studied intensely in [4]. It plays an important role in various fields (see Corless et al [4] and references therein), including the study of asymptotics of Triangular Wishart Ensembles [3] in the Random Matrix Theory. In a previous paper [10], its two-parameter generalization $W_{\kappa,\gamma}$, called Lambert-Tsallis function, was introduced for some particular values of parameters κ, γ, with motivations and important applications in Random Matrix Theory and statistics.

The function $W_{\kappa,\gamma}$ is defined as a holomorphic extension of the inverse function of a function $f_{\kappa,\gamma}$, called a generalized Tsallis function, which generalizes f above and is defined by the product of a linear fraction of z and the Tsallis q-exponential function, see (1) for definition. We note that the Tsallis q-exponential function is now actively studied in information geometry [2, 12] and physics [11]. There were several attempts of generalizing the Lambert function in many directions. In Mezö and Baricz [8], z before e^z is replaced by rational function of z (see also [6, 7]) and in da Silva and Ramos [11], e^z is replaced by the Tsallis q-exponential function. The matrix Lambert W function is considered in [5] and [9].

In this note, generalized Tsallis exponential functions and Lambert-Tsallis functions are studied for the first time in the whole range of parameters κ, γ and a necessary and sufficient condition on parameters κ, γ is given for the existence of the main branch of the Lambert-Tsallis W function. More precisely, in the main Theorem 1.3, we characterize the parameters κ, γ such that there exists a complex domain Ω^+ containing zero on the boundary, which is mapped bijectively to the upper half plane $\mathbb{C}^+$ by the generalized Tsallis function $f_{\kappa,\gamma}$.

The proof of Theorem 1.3 is based on involved complex analysis. We omit some technical details.

1. Preliminaries and main result

For a non zero real number κ, we set

$$\exp_\kappa(z) := \left(1 + \frac{z}{\kappa}\right)^\kappa \quad (1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}),$$

where we take the main branch of the power function when κ is not integer. If $\kappa = \frac{1}{1-q}$, then it is exactly the so-called Tsallis *q-exponential function* (cf. [2, 12]). For the sake of simplicity, we use the expression $\exp_\kappa$ with parameter $\kappa \neq 0$. By virtue of $\lim_{\kappa\to\infty} \exp_\kappa(z) = e^z$, we define $\exp_\infty(z) = e^z$.

For two real numbers κ, γ such that $\kappa \neq 0$, we introduce a holomorphic function $f_{\kappa,\gamma}(z)$, which we call *generalized Tsallis function*, by

$$f_{\kappa,\gamma}(z) := \frac{z}{1+\gamma z} \exp_\kappa(z) \quad (1 + \frac{z}{\kappa} \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}). \tag{1}$$

Analogously to Tsallis q-exponential, we also consider $f_{\infty,\gamma}(z) = \frac{ze^z}{1+\gamma z}$ $(z \in \mathbb{C})$. In particular, $f_{\infty,0}(z) = ze^z$. Let $\mathcal{D}(f_{\kappa,\gamma})$ be the domain of $f_{\kappa,\gamma}$, that is, if κ is integer then $\mathcal{D}(f_{\kappa,\gamma}) = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$, and if κ is not integer, then $\mathcal{D}(f_{\kappa,\gamma}) = \mathbb{C} \setminus \left\{x \in \mathbb{R};\ 1 + \frac{x}{\kappa} \leq 0,\ \text{or } x = -\frac{1}{\gamma}\right\}$.

The purpose of this work is to study the inverse function to $f_{\kappa,\gamma}$ in detail. A multivariate inverse function of $f_{\infty,0}(z) = ze^z$ is called the Lambert W function and is studied intensively in [4]. Hence, we call a multivariate inverse function of $f_{\kappa,\gamma}$ the *Lambert–Tsallis W function.* We have

$$f'_{\kappa,\gamma}(z) = \frac{\gamma z^2 + \big(1 + 1/\kappa\big)z + 1}{(1+\gamma z)^2}\left(1 + \frac{z}{\kappa}\right)^{\kappa-1}, \tag{2}$$

so that $f'_{\kappa,\gamma}(0) \neq 0$. Consequently, in a neighborhood U of $z = 0$, the function $f_{\kappa,\gamma}(z)$ has an inverse function, denoted $w_{\kappa,\gamma}$:

$$w_{\kappa,\gamma} \circ f_{\kappa,\gamma}(z) = z \quad (z \in U),$$

and $w_{\kappa,\gamma}(0) = 0$. Let $z = x + yi \in \mathbb{C}$ and we set $\theta(x,y) := \operatorname{Arg}\big(1 + \frac{z}{\kappa}\big)$ for finite $\kappa \neq 0$, where $\operatorname{Arg}(w)$ stands for the principal argument of w; $-\pi < \operatorname{Arg}(w) \leq \pi$. Since we now take the main branch of the power function, we have for finite $\kappa \neq 0$

$$f_{\kappa,\gamma}(z) = \frac{|1 + \frac{z}{\kappa}|^2}{(1+\gamma x)^2 + \gamma^2 y^2}\begin{pmatrix} (x + \gamma x^2 + \gamma y^2)\cos(\kappa\theta(x,y)) - y\sin(\kappa\theta(x,y)) \\ \quad +i\big\{(x + \gamma x^2 + \gamma y^2)\sin(\kappa\theta(x,y)) + y\cos(\kappa\theta(x,y))\big\} \end{pmatrix}. \tag{3}$$

If $\kappa = \infty$, then we regard $\kappa\theta(x,y)$ as y because we have $\lim_{\kappa\to\infty} \exp_\kappa(z) = e^z = e^x(\cos y + i\sin y)$. Then, $f_{\kappa,\gamma}(z) \in \mathbb{R}$ implies

$$(x + \gamma x^2 + \gamma y^2)\sin(\kappa\theta(x,y)) + y\cos(\kappa\theta(x,y)) = 0. \tag{4}$$

If $\sin(\kappa\theta(x,y)) = 0$, then $\cos(\kappa\theta(x,y))$ does not vanish so that y needs to be zero. This means that if $z = x + yi \in f_{\kappa,\gamma}(\mathbb{R})$ with $y \neq 0$, then we have $\sin(\kappa\theta(x,y)) \neq 0$. Thus, the equation (4) for $y \neq 0$ can be rewritten as

$$F(x,y) := (x + \gamma x^2 + \gamma y^2) + y\cot(\kappa\theta(x,y)) = 0. \tag{5}$$

For $y = 0$, we set $F(x,0) := \lim_{y\to 0} F(x,y)$. If $x = 0$, then since $\theta(0,y) = \operatorname{Arctan}(\frac{y}{\kappa})$, we have $F(0,0) = \lim_{y\to 0} y\cot(\kappa\operatorname{Arctan}(\frac{y}{\kappa})) = 1$. Here, we take the main branch of $\operatorname{Arctan} x$, that is, we have $\operatorname{Arctan} 0 = 0$. Let us introduce a connected domain $\Omega = \Omega_{\kappa,\gamma}$ by

$$\Omega = \Omega_{\kappa,\gamma} := \{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma});\ F(x,y) > 0\}^\circ, \tag{6}$$

where A° denotes the connected component of an open set $A \subset \mathbb{C}$ containing $z = 0$. Note that since F is an even function in y, the domain Ω is symmetric with respect to the real axis. Set

$$\mathcal{S} := \mathbb{R} \setminus f_{\kappa,\gamma}\big(\Omega_{\kappa,\gamma} \cap \mathbb{R}\big). \tag{7}$$

Definition 1.1. If there exists a unique holomorphic extension $W_{\kappa,\gamma}$ of $w_{\kappa,\gamma}$ to $\mathbb{C} \setminus \overline{\mathcal{S}}$, then we call $W_{\kappa,\gamma}$ the *Lambert-Tsallis function.*

Remark 1.2. Strictly speaking, the Lambert–Tsallis function $W_{\kappa,\gamma}$ is the main branch of the multivalued Lambert-Tsallis W function (recall that $W_{\kappa,\gamma}(0) = 0$). In our terminology the Lambert-Tsallis W function is multivalued and the Lambert-Tsallis function $W_{\kappa,\gamma}$ is single-valued. In this paper, we only study $W_{\kappa,\gamma}$ among other branches of the Lambert-Tsallis W function.

Our goal is to prove the following theorem which is the main result of the paper.

Theorem 1.3. *There exists the main branch $W_{\kappa,\gamma}$ of the Lambert-Tsallis W function if and only if* (i) $0 < |\kappa| < 1$ *and* $\gamma \leq \min(0, \frac{1}{\kappa})$, *or* (ii) $|\kappa| \geq 1$ *and* $\gamma \leq \frac{1}{4}(1 + \frac{1}{\kappa})^2$, *or* (iii) $\kappa = \infty$ *and* $\gamma \leq \frac{1}{4}$. *Moreover, $W_{\kappa,\gamma}$ maps $\mathbb{C}^+$ onto $\Omega^+ := \Omega \cap \mathbb{C}^+$ bijectively.*

Remark 1.4. As we shall see in §2.3, in the case $\kappa > 1$ and $\gamma > \frac{1}{4}(1 + \frac{1}{\kappa})^2$, the function $f_{\kappa,\gamma}$ maps Ω to $\mathbb{C} \setminus \overline{\mathcal{S}}$ two-to-one, and hence a holomorphic extension of $w_{\kappa,\gamma}$ exists on a smaller domain in Ω which is mapped by $f_{\kappa,\gamma}$ bijectively to $\mathbb{C} \setminus \overline{\mathcal{S}}$, but it is not unique. In the case $0 < \kappa < 1$ and $\gamma > 0$, the function $f_{\kappa,\gamma}\colon \mathcal{D}(f_{\kappa,\gamma}) \to \mathbb{C} \setminus \overline{\mathcal{S}}$ is not surjective.

Remark 1.5. In the recent paper [10], we have proven Theorem 1.3 for the case $\gamma \leq \frac{1}{\kappa} \leq 1$ and $\gamma < 1$, and when $\kappa = \infty$ and $\gamma \leq 0$. Note that the case $\kappa < 0$ can be derived from the case $\kappa > 0$, see §2.3.5. Thus the sufficiency in the Theorem 1.3 is essentially new in the following three cases

(a) $0 < \kappa < 1$ and $\gamma < 0$, (b) $\kappa > 1$ and $\frac{1}{\kappa} < \gamma < \frac{1}{4}(1 + \frac{1}{\kappa})^2$, (c) $\kappa = \infty$ and $0 < \gamma < \frac{1}{4}$.

The necessity part in the Theorem 1.3 is also a new result.

2. Proof of Theorem 1.3

The proof of the Theorem 1.3 is done in two steps. More precisely, we first give an explicit expression of $\Omega = \Omega_{\kappa,\gamma}$, and then show that $f_{\kappa,\gamma}$ maps Ω to $\mathbb{C} \setminus \overline{\mathcal{S}}$ bijectively.

2.1. **The domain Ω for $0 < \kappa < +\infty$.** In this section, we shall determine the connected component Ω defined in (6) for the case $0 < \kappa < +\infty$. Let us change variables in (5) by

$$re^{i\theta} = 1 + \frac{z}{\kappa} \quad (r > 0,\ \theta \in (0,\pi)), \quad \text{or equivalently} \quad \begin{cases} x = \kappa(r\cos\theta - 1), \\ y = \kappa r \sin\theta. \end{cases} \tag{8}$$

Set $a := \gamma\kappa$ and

$$b(\theta) = (1-2a)\cos\theta + \sin\theta\cot(\kappa\theta) \quad (\theta \neq 0), \quad b(0) = 1 - 2a + \frac{1}{\kappa}. \tag{9}$$

Then, the equation (5) can be written as

$$ar^2 + b(\theta)r + a - 1 = 0. \tag{10}$$

If $\sin(\kappa\theta) \neq 0$, then (10) has a solution $r = r_\pm(\theta) = \big(-b(\theta) \pm \sqrt{b(\theta)^2 - 4a(a-1)}\big)/(2a)$. Thus, for each angle θ, there exists at most two points on $f_{\kappa,\gamma}^{-1}(\mathbb{R})$. Since the change (8) of variables is the polar transformation, we need to know whether $r_\pm(\theta)$ is positive real or not. To do so, we shall study the function

$$D(\theta) := b(\theta)^2 - 4a(a-1).$$

Since $D'(\theta) = 2b(\theta)b'(\theta)$, we first consider $b'(\theta)$. Set

$$G(\kappa) := \frac{2\kappa^2 + 3\kappa + 1}{6\kappa}.$$

Since $b(\theta) = 2(a-1)\sin\theta$ for $\kappa = 1, \frac{1}{2}$, we exclude these two cases.

Lemma 2.1. *Let us set $I_0 := (0, \min(\pi, \frac{\pi}{\kappa}))$.*

(1) *Suppose that $0 < \kappa < \frac{1}{2}$, or $\kappa > 1$. If $a > G(\kappa)$, then there exists a unique $\varphi_* \in I_0$ such that $b'(\varphi_*) = 0$, and one has $b'(\theta) > 0$ for $\theta \in (0, \varphi_*)$, and $b'(\theta) < 0$ otherwise. If $a < G(\kappa)$, then one has $b'(\theta) < 0$ for any $\theta \in I_0$.*

(2) *Suppose that $\frac{1}{2} < \kappa < 1$. If $a \leq G(\kappa)$, then there exists a unique $\varphi_* \in I_0$ such that $b'(\varphi_*) = 0$, and one has $b'(\theta) < 0$ for $\theta \in (0, \varphi_*)$, and $b'(\theta) > 0$ for $\theta \in (\varphi_*, \pi)$. If $a \geq G(\kappa)$, then one has $b'(\theta) > 0$ for any $\theta \in I_0$.*

Proof. Let us set for $\alpha, \kappa \in \mathbb{R}$

$$H_\alpha(x) := \sin(\alpha x) - \alpha\sin(x), \quad F_\kappa(x) := \tan x \cdot \cot(\kappa x), \quad J_\kappa(x) := -\frac{2x - 2\kappa - 1}{4\kappa x - 2\kappa - 1} \quad (x \in \mathbb{R}). \tag{11}$$

For $\kappa \neq 1, \frac{1}{2}$, we have

$$b(\theta) = \cos\theta + \sin\theta\cot(\kappa\theta) - 2a\cos\theta = \frac{\cos\theta\sin(\kappa\theta) + \sin\theta\cos(\kappa\theta)}{\sin(\kappa\theta)} - 2a\cos\theta,$$

and hence $b'(\theta)$ can be written as

$$b'(\theta) = \frac{H_{2\kappa+1}(\theta)}{2\sin^2(\kappa\theta)} + 2a\sin\theta.$$

Let us set

$$B(\theta) := 2\sin^2(\kappa\theta)b'(\theta) = H_{2\kappa+1}(\theta) + 4a\sin\theta\sin^2(\kappa\theta), \quad \ell(\kappa, a) := 4a\kappa - 2\kappa - 1. \tag{12}$$

Then, the signatures of $b'(\theta)$ and $B(\theta)$ are the same on the interval I_0, and hence we actually work with $B(\theta)$. We have $B(0) = 0$ for any $\kappa > 0$, and

$$B(\pi) = -\cot(\kappa\pi)\sin^2(\kappa\pi)\begin{cases} < 0 & (\text{if } 0 < \kappa < \frac{1}{2}), \\ > 0 & (\text{if } \frac{1}{2} < \kappa < 1) \end{cases} \quad B\Big(\frac{\pi}{\kappa}\Big) = -2\kappa\sin\frac{\pi}{\kappa} < 0 \quad (\kappa > 1). \tag{13}$$

In fact, we have $B(\frac{\pi}{\kappa}) = H_{2\kappa+1}(\frac{\pi}{\kappa}) = -2\kappa\sin\frac{\pi}{\kappa} < 0$ when $\kappa > 1$. The derivative of $B(\theta)$ is given as

$$B'(\theta) = \begin{cases} 2\ell(\kappa,a)\cos\theta\sin^2(\kappa\theta)\Big(F_\kappa(\theta) - J_\kappa(a)\Big) & (\ell(\kappa,a) \neq 0), \\ \dfrac{1-4\kappa^2}{\kappa}\cos\theta\sin^2(\kappa\theta) & (\ell(\kappa,a) = 0). \end{cases} \tag{14}$$

We note that the condition $G(\kappa) = a$ comes from a condition $F_\kappa(0) - J_\kappa(a) = 0$. According to a case classification with respect to $F_\kappa(\theta)$ (cf. Table 1), we need to divide cases into the following four cases: (i) $0 < \kappa < \frac{1}{2}$, (ii) $\frac{1}{2} < \kappa < 1$, (iii) $1 < \kappa < 2$, and (iv) $\kappa \geq 2$.

Let us first consider the equation

$$F_\kappa(\theta) - J_\kappa(a) = 0 \iff F_\kappa(\theta) = J_\kappa(a) \quad \text{on } I_0 = (0, \min(\pi, \tfrac{\pi}{\kappa})). \tag{15}$$

Since $J_\kappa(a)$ does not depend on θ, it is important to know where it meets end points or maximal/minimal values of $F_\kappa(\theta)$. By Table 1, for end points, they are $J_\kappa(a) = \frac{1}{\kappa}$ or 0, and its solutions on a are given as $a = G(\kappa)$ or $a = \kappa + 1/2$, respectively. If $\frac{1}{2} < \kappa < 2$ ($\kappa \neq 1$), then we need a solution of $J_\kappa(a) = F_\kappa(x_*)$. Since J_κ is bijective, there exists a unique solution a_* such that $J_\kappa(a_*) = F_\kappa(x_*)$.

The proof can be done by considering further cases according to the position of $J_\kappa(a)$. The calculations are delicate and tedious but elementary, and thus we only give a proof for the case $\frac{1}{2} < \kappa < 1$ and $0 \leq J_\kappa(a) \leq F_\kappa(x_*)$.

Assume that $\frac{1}{2} < \kappa < 1$. In this case, we have $B(\pi) > 0$ by (13). Recall that J_κ is monotonic decreasing in a in this situation. Let x_* be the maximal point of F_κ in I_0. Then, we have $0 < F_\kappa(x_*) < 1$. In fact, since $x_* \in (\frac{\pi}{2}, \pi)$, we have $\sin((1-\kappa)x_*) > 0$ and hence it implies that

$$\sin x_* \cos(\kappa x_*) - \cos x_* \sin(\kappa x_*) > 0 \iff F_\kappa(x_*) = \frac{\sin x_* \cos(\kappa x_*)}{\cos x_* \sin(\kappa x_*)} < 1.$$

Here we use $\cos x_* < 0$ and $\sin(\kappa x_*) > 0$. Let a_* be the unique solution of $F_\kappa(x_*) - J_\kappa(a) = 0$ in a. Then, we see that $a_* \in (1, \frac{1}{2} + \kappa)$ by $J_\kappa(1) = 1$. Table 1 tells us that we have three cases in the equation (15) as follows.

(a) It does not have a solution if $F_\kappa(x_*) < J_\kappa(a) < \frac{1}{\kappa}$, which is equivalent to the condition $G(\kappa) < a < a_*$. Since $F_\kappa(x_*) > 0$, we see that $\ell(\kappa, a) > 0$ in this situation.

(b) The equation (15) has a unique solution φ if $J_\kappa(a) \geq \frac{1}{\kappa}$ or $J_\kappa(a) < 0$. In the former case, the condition is equivalent to $\frac{1}{2} + \frac{1}{4\kappa} < a \leq G(\kappa)$, and we have $\varphi < \frac{\pi}{2}$ and $\ell(\kappa, a) > 0$. In the latter case, we have $\varphi > \frac{\pi}{2}$, and the condition is divided into two situations; one is $a > \frac{1}{2} + \kappa$, and the other is $a < \frac{1}{2} + \frac{1}{4\kappa}$.

(c) The equation (15) has two solutions $\varphi_1 \leq \varphi_2$ if $0 \leq J_\kappa(a) \leq F_\kappa(x_*)$, which is equivalent to the condition $a_* \leq a \leq \frac{1}{2} + \kappa$ so that $\ell(\kappa, a) > 0$. We note that we have $\varphi_i > \frac{\pi}{2}$, and if $J_\kappa(a) = F_\kappa(x_*)$ then we have $\varphi_1 = \varphi_2$. We only deal with the case (c).

Let us assume that $0 \leq J_\kappa(a) \leq F_\kappa(x_*)$, i.e., $a_* \leq a \leq \frac{1}{2} + \kappa$. In this situation, the signature of B' is negative in the interval $(0, \varphi_1)$ and (φ_2, π), and positive in (φ_1, φ_2). In order to show that $B(\varphi_i) > 0$ $(i = 1, 2)$, which implies $b'(\theta) > 0$ for any $\theta \in I_0$, we calculate $b'(\theta)$ in a different way as follow. By differentiating $b(\theta)$ by using expression (9), we obtain

$$b'(\theta) = (2a-1)\sin\theta + \cos\theta\cot(\kappa\theta) - \frac{\kappa \sin\theta}{\sin^2(\kappa\theta)},$$

and hence $B(\theta)$ can be also described as

$$B(\theta) = 2(2a - \kappa - 1)\sin\theta + 2\cos\theta\sin(\kappa\theta)\cos(\kappa\theta)((2a-1)F_\kappa(\theta) + 1). \tag{16}$$

Since φ_i $(i = 1, 2)$ satisfy $F_\kappa(\varphi_i) - J_\kappa(a) = 0$ by definition, we see that

$$(2a-1)F_\kappa(\varphi_i) + 1 = (2a-1)J_\kappa(a) + 1 = \frac{-(2a-1)(2a-2\kappa-1) + (4a\kappa - 2\kappa - 1)}{\ell(\kappa, a)} = \frac{4a(a-1)}{\ell(\kappa, a)}. \tag{17}$$

Now we assume $a_* < a < \frac{1}{2} + \kappa$ and we know $a_* > 1$ so that we have $2a - \kappa - 1 > 0$ and $(2a-1)F_\kappa(\varphi_i) + 1 > 0$ for $i = 1, 2$ by (17). Since $\varphi_i \in (\frac{\pi}{2\kappa}, \pi)$, we see that $\cos\varphi_i \sin(\kappa\varphi_i)\cos(\kappa\varphi_i) > 0$ and hence we obtain $B(\varphi_i) > 0$ $(i = 1, 2)$ by (16). Thus, we conclude $b'(\theta) > 0$ on the interval I_0.

The other cases can be done similarly. □

Now we are able to obtain explicit formulas of Ω. Note that, since $F(x, y)$ is a continuous function, the boundary $\partial\Omega$ is included in the set $\{z = x + yi \in \mathbb{C};\ F(x, y) = 0\} \subset f_{\kappa,\gamma}^{-1}(\mathbb{R})$.

Proposition 2.2. *Suppose that $\kappa = 1$.*

(1) *If $\gamma > 1$, then one has $\Omega = \mathbb{C} \setminus \{-\frac{1}{\gamma}\}$.*

(2) *If $0 < \gamma \leq 1$, then one has $\Omega = \left\{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma});\ \left(x + \frac{1}{\gamma}\right)^2 + y^2 > \frac{1-\gamma}{\gamma^2}\right\}$.*

(3) *If $\gamma = 0$, then one has $\Omega = \{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma});\ 1 + 2x > 0\}$.*

(4) *If $\gamma < 0$, then one has $\Omega = \left\{z = x + yi \in \mathcal{D}(f_{\kappa,\gamma});\ \left(x + \frac{1}{\gamma}\right)^2 + y^2 < \frac{1-\gamma}{\gamma^2}\right\}$, which is bounded.*

TABLE 1. Increasing/Decreasing table of $F_\kappa(\theta)$

(A) $0 < \kappa \le \frac{1}{2}$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	π
F'_κ		$+$	$\times$	$+$	
F_κ	$\frac{1}{\kappa}$	$\nearrow^{+\infty}$	$\times$	$_{-\infty}\nearrow$	0

(B) $\frac{1}{2} < \kappa < 1$, $F_\kappa(x_*) < 1$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	x_*	$\cdots$	π
F'_κ		$+$	$\times$	$+$	0	$-$	
F_κ	$\frac{1}{\kappa}$	$\nearrow^{+\infty}$	$\times$	$_{-\infty}\nearrow$	$F_\kappa(x_*)$	$\searrow$	0

(C) $1 < \kappa < 2$, $F_\kappa(x_*) > 1$

x	0	$\cdots$	$\frac{\pi}{2}$	$\cdots$	x_*	$\cdots$	$\frac{\pi}{\kappa}$
F'_κ		$-$	$\times$	$-$	0	$+$	
F_κ	$\frac{1}{\kappa}$	$\searrow_{-\infty}$	$\times$	$^{+\infty}\searrow$	$F_\kappa(x_*)$	$\nearrow^{+\infty}$	$\times$

(D) $\kappa \ge 2$

x	0	$\cdots$	$\frac{\pi}{\kappa}$
F'_κ		$-$	$\times$
F_κ	$\frac{1}{\kappa}$	$\searrow_{-\infty}$	$\times$

In the table, the symbol $\times$ means that the functions are not defined at that point. The symbol x_* denotes a maximal/minimal point in the interval I_0, if it exists.

Proof. In the case $\kappa = 1$, we have $\theta(x,y) = \mathrm{Arg}(1+z)$ so that $\tan(\theta(x,y)) = y/(1+x)$. Thus, we have

$$F(x,y) = 1 + 2x + \gamma x^2 + \gamma y^2 = \begin{cases} \gamma\left(x + \frac{1}{\gamma}\right)^2 + \gamma y^2 + \frac{\gamma - 1}{\gamma} & (\text{if } \gamma \neq 0), \\ 1 + 2x & (\text{if } \gamma = 0). \end{cases}$$

Note that $z = 0$ is contained in the set $\left\{z = x + yi \in \mathbb{C};\ 1 + 2x + \gamma x^2 + \gamma y^2 > 0\right\}$. Since it is connected, the proof is now completed. □

Set $\theta_0 := \frac{\pi}{\kappa}$ and $I_0 = (0, \min(\pi, \theta_0))$. Let $r_\pm(\theta)$ be the solutions of the equation (10) and let α_i, $i = 1, 2$ be the solutions of the equation $f_{\kappa,\gamma}(z) = 0$ which come from rational part of $f_{\kappa,\gamma}$. If α_i are real, then we assume that $\alpha_1 \le \alpha_2$, and if not, then we assume that $\mathrm{Im}\,\alpha_1 > \mathrm{Im}\,\alpha_2$. Recall that $a = \kappa\gamma$ and $D(0) = \left(1 + \frac{1}{\kappa}\right)^2 - 4\gamma$.

Proposition 2.3. *Let $\kappa > 0$ with $\kappa \neq 1$. For $z \in \mathcal{D}(f_{\kappa,\gamma})$, one sets $re^\theta = 1 + \frac{z}{\kappa}$. Then, Ω can be described as follows.*

(1) *If $a < 0$, then there exists a unique $\theta_* \in (0, \frac{\pi}{\kappa+1})$ such that $r_+(\theta_*) = r_-(\theta_*)$ and that $r_\pm(\theta)$ are both positive with $r_-(\theta) \le r_+(\theta)$ on $(0, \theta_*)$. Moreover, one has*

$$\Omega = \left\{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \theta_* \text{ and } r_-(\theta) < r < r_+(\theta)\right\}.$$

In particular, Ω is bounded. One has $\alpha_1, \alpha_2 \in \partial\Omega$ and $-\frac{1}{\gamma} \in \overline{\Omega}$, whereas $-\kappa \notin \overline{\Omega}$.

(2) *If $a = 0$, then one has $r(\theta) = r_\pm(\theta) = \frac{\sin(\kappa\theta)}{\sin((\kappa+1)\theta)}$ which is positive on the interval $(0, \frac{\pi}{\kappa+1})$. Moreover, one has*

$$\Omega = \left\{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \frac{\pi}{\kappa+1} \text{ and } r > r(\theta)\right\}.$$

Ω has an asymptotic line $y = \pm(\tan\frac{\pi}{\kappa+1})(x + \frac{\kappa^2}{\kappa+1})$. One has $\alpha_1 = \alpha_2 = -\frac{\kappa}{\kappa+1} \in \partial\Omega$ and $-\kappa \notin \overline{\Omega}$.

(3) *If $0 < a < 1$, then $r_+(\theta)$ is the only positive solution of (10) on I_0, and one has*

$$\Omega = \left\{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \min(\theta_0, \pi) \text{ and } r > r_+(\theta)\right\}.$$

One has $\alpha_2 \in \partial\Omega$, whereas $\alpha_1, -\frac{1}{\gamma}, -\kappa \notin \overline{\Omega}$. If $\kappa > 1$, then Ω has an asymptotic line $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$. Note that if $0 < \kappa < 1$ then $\Omega = \mathcal{D}(f_{\kappa,\gamma})$.

(4) *Suppose that $a = 1$ and $\kappa \neq 1$.*
   (a) *If $0 < \kappa < 1$, then one has $r_+(\theta) = 0$ and $r_-(\theta) = -b(\theta) < 0$ for $\theta \in I_0$, and $\Omega = \mathcal{D}(f_{\kappa,\gamma})$. One has $\alpha_1 = -1 \in \partial\Omega$ and $\alpha_2 = -\kappa = -\frac{1}{\gamma} \in \partial\Omega$.*
   (b) *If $\kappa > 1$, then one has $r_+(\theta) = -b(\theta) > 0$ and $r_-(\theta) = 0$ for $\theta \in I_0$, and one has*

$$\Omega = \left\{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \theta_0 \text{ and } r > r_+(\theta)\right\}.$$

One has $\alpha_2 = -1 \in \partial\Omega$, *while* $\alpha_1 = -\kappa = -\frac{1}{\gamma} \notin \overline{\Omega}$. *Moreover,* Ω *has an asymptotic line* $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$.

(5) *Suppose that* $a > 1$.

(a) *If* $\kappa > 1$ *with* $D(0) \geq 0$, *then* $r_\pm(\theta)$ *are both positive in* I_0 *with* $r_-(\theta) \leq r_+(\theta)$, *and one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \theta_0 \text{ and } r > r_+(\theta)\}.$$

Ω *has an asymptotic line* $y = \pm(\tan\theta_0)(x+\kappa-\frac{1}{a})$. *One has* $\alpha_2 \in \partial\Omega$, *while* $-\kappa, -\frac{1}{\gamma}, \alpha_1 \notin \overline{\Omega}$.

(b) *If* $\kappa > 1$ *and* $D(0) < 0$, *then there exists a unique* $\theta_* \in (0, \theta_0)$ *such that* $D(\theta_*) = 0$, *and* $r_\pm(\theta)$ *are both positive in the interval* (θ_*, θ_0). *Moreover, one has*

$$\Omega = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ |\theta| < \theta_0 \text{ and if } |\theta| \geq \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}.$$

In this case, α_i, $i = 1, 2$ *are both non-real numbers and one has* $\alpha_i, -\kappa \in \partial\Omega$ *and* $-\frac{1}{\gamma} \in \overline{\Omega}$. *Moreover,* Ω *has an asymptotic line* $y = \pm(\tan\theta_0)(x + \kappa - \frac{1}{a})$.

(c) *If* $0 < \kappa < 1$, *then there are no* $\theta \in I_0$ *such that* $D(\theta) > 0$, *and one has* $\Omega = \mathcal{D}(f_{\kappa,\gamma})$.

Proof. Since Ω is symmetric with respect to the real axis, we shall actually work with the boundary $\partial\Omega^+$ of $\Omega^+ = \Omega \cap \mathbb{C}^+$. The cases (1)–(3) are done in the previous paper [10], and thus we omit them.

(4) Assume that $a = 1$ and $\kappa \neq 1$. In this case, α_i, $i = 1, 2$ are given as $\alpha_i = -1$ or $-\frac{1}{\gamma} = -\kappa$, and the equation (10) reduces to $r^2 + b(\theta)r = 0$, whose solutions are $r(\theta) = 0$, $-b(\theta)$. Since $b(\theta)$ can be described as $b(\theta) = \sin((1-\kappa)\theta)/\sin(\kappa\theta)$ in this case, it is easily verified that, if $0 < \kappa < 1$ then $b(\theta) > 0$, and if $\kappa > 1$ then $b(\theta) < 0$ for any $\theta \in I_0$. This means that if $0 < \kappa < 1$ then the equation (10) does not have a positive solution, and hence we have

$$\Omega^+ = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ 0 < \theta < \pi,\ r > 0\} = \mathbb{C}^+,$$

which shows $\Omega = \mathcal{D}(f_{\kappa,\gamma})$. Since we have $\alpha_1 = -1$ and $\alpha_2 = -\kappa = -\frac{1}{\gamma}$, the assertion (4)-(a) is proved. On the other hand, if $\kappa > 1$, then we have $b'(\theta) < 0$ for $\theta \in I_0$ by Lemma 2.1 together with the fact $G(\kappa) > 1$ for any $\kappa > 1$. Since $b(0) = 1 + \frac{1}{\kappa} - 2 < 0$ and $\lim_{\theta\to\theta_0-0} b(\theta) = -\infty$, the function $D(\theta)$ is monotonic increasing on I_0 and hence we have $\lim_{\theta\to\theta_0-0} r_+(\theta) = +\infty$. Thus, the curve $r_+(\theta)$, $\theta \in I_0$ has an asymptotic line with gradient $\tan\theta_0$, which is determined later. Therefore, we have

$$\Omega^+ = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ 0 < \theta < \theta_0 \text{ and } r > r_+(\theta)\}.$$

In this case, we have $\alpha_1 = -\kappa = -\frac{1}{\gamma}$ and $\alpha_2 = -1$. The assertion (4)-(b) is now proved.

(5) Suppose that $a > 1$ and $\kappa \neq 1$. In this case, we have $b(0) = 1 + \frac{1}{\kappa} - 2a$ and

$$\lim_{\theta\to\theta_0-0} b(\theta) = -\infty \quad (\text{if } \kappa > 1), \quad b(\pi) = 2a - 1 > 0 \quad (0 < \kappa < 1).$$

Note that $b(0) > 0$ if $\kappa > 1$. Since $r_+(\theta) \cdot r_-(\theta) = \frac{a-1}{a} > 0$, two solutions $r_\pm(\theta)$ of (10) have the same signature if $r_\pm(\theta)$ are real.

(a) We first consider the case $\kappa > 1$ and $D(0) \geq 0$. Let us show that $D(\theta) > 0$ for $\theta \in I_0$. If we set $K(x) := \frac{x}{4}\left(1 + \frac{1}{x}\right)^2$, then the condition $D(0) \geq 0$ is equivalent to $a \leq K(\kappa)$, and hence we have $a \leq G(\kappa)$ because $G(x) - K(x) = (x^2 - 1)(12x) > 0$ if $x > 1$. By the assumption $\kappa > 1$, we see that $b'(\theta) < 0$ for any $\theta \in I_0$ by Lemma 2.1 so that b is monotonic decreasing in this interval. Since $b(0) < 0$, the function b is negative in I_0, and hence $D'(\theta) = 2b(\theta)b'(\theta) > 0$ so that $D(\theta)$ is monotonic increasing in the interval I_0, and in particular, it is positive on I_0. Thus, we see that $r_\pm(\theta)$ are real on I_0, and by $r_+(\theta) + r_-(\theta) = -b(\theta)/a > 0$ we have $r_\pm(\theta) > 0$ ($\theta \in I_0$). Since for $\varepsilon = \pm 1$

$$r'_\varepsilon(\theta) = \frac{1}{2a}\left(-b'(\theta) + \varepsilon\frac{2b(\theta)b'(\theta)}{2\sqrt{D(\theta)}}\right) = \frac{-\varepsilon b'(\theta)}{2a} \cdot \frac{-b(\theta) + \varepsilon\sqrt{D(\theta)}}{\sqrt{D(\theta)}} = -\varepsilon b'(\theta)\frac{r_\varepsilon(\theta)}{\sqrt{D(\theta)}}, \tag{18}$$

we have $r'_+(\theta) > 0$, and hence the function $r_+(\theta)$ is monotonic increasing, whereas $r_-(\theta)$ is monotonic decreasing because $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$. Note that we have $r_+(\theta) \to +\infty$ and $r_0(\theta) \to 0$ as $\theta \to \theta_0 - 0$. Thus, $r_+(\theta)$, $\theta \in I_0$ draws an unbounded curve connecting $z = \alpha_2$ to ∞ with an asymptotic line with slope $\tan\theta_0$, and $r_-(\theta)$, $\theta \in I_0$ draws a bounded curve connecting $z = \alpha_1$ to $z = -\kappa$. Since we have $-\kappa < -\frac{1}{\gamma} < \alpha_1 < \alpha_2 < 0$ and since Ω is the connected component including $z = 0$, we have

$$\Omega^+ = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ 0 < \theta < \theta_0,\ r > r_+(\theta)\}.$$

(b) Next, we assume that $\kappa > 1$ and $D(0) < 0$. According to Lemma 2.1 (1), we consider the function $b'(\theta)$ in two cases, that is, (i) $a \leq G(\kappa)$ and (ii) $a > G(\kappa)$.

(i) Assume that $a \le G(\kappa)$. Then, b' is negative and hence $b(\theta)$ is monotonic decreasing. Since $b(0) < 0$, we see that $b(\theta) < 0$ for any $\theta \in I_0$ and therefore $D'(\theta) = 2b(\theta)b'(\theta) > 0$ for any $\theta \in I_0$. Thus, $D(\theta)$ is monotonic increasing with $D(0) < 0$. Since $D(\theta) \to +\infty$ as $\theta \to \theta_0 - 0$, we see that there exists a unique θ_* such that $D(\theta_*) = 0$, and $r_\pm(\theta)$ are real for $\theta \in (\theta_*, \theta_0)$. In this interval, since $r_+(\theta) + r_-(\theta) = -b(\theta)/a > 0$, we see that $r_\pm(\theta)$ are both positive. By (18), we have $r'_+(\theta) > 0$ and thus the function $r_+(\theta)$ is monotonic increasing, whereas $r_-(\theta)$ is monotonic decreasing because $r_-(\theta) = \frac{a-1}{ar_+(\theta)}$. As $\theta \to \theta_0 - 0$, we have $r_+(\theta) \to +\infty$ and $r_-(\theta) \to 0$. This means that $r_+(\theta)$ draws an unbounded curve connecting $z = \alpha_1$ and $z = \infty$, and $r_-(\theta)$ draws a bounded curve connecting $z = \alpha_1$ and $z = -\kappa$, where α_1 is the complex solution of $f_{\kappa,\gamma}(z) = 0$ with positive imaginary part. Since we have $-\kappa < -\frac{1}{\gamma} < 0$, the domain Ω^+ is given as

$$\Omega^+ = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ 0 < \theta < \theta_0,\ \text{if } \theta \ge \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}\,.$$

(ii) Assume that $a > G(\kappa)$. In this case, we have $D(0) < 0$ and $D(\theta) \to +\infty$ as $\theta \to \theta_0 - 0$. Then, there are two possibilities on $b(\varphi_*)$, that is, it is positive or negative. If $b(\varphi_*) \le 0$, then $D(\theta)$ has a unique minimal point at $\theta = \varphi_*$. On the other hand, if $b(\varphi_*) > 0$, then there exist exactly two $\varphi_1 < \varphi_2$ in I_0 such that $b(\varphi_i) = 0$. Then, $D(\theta)$ has a unique maximal point at $\theta = \varphi_*$ whereas there are exactly two minimal point at $\theta = \varphi_1, \varphi_2$ in I_0. We note that $D(\varphi_i) = b(\varphi_i)^2 - 4a(a-1) < 0$. Since $r_+(\theta) + r_-(\theta) = -b(\theta)/a$, if $D(\varphi_*) > 0$, then $r_\pm(\theta)$ are both negative so that we need not deal with this case. Thus, regardless of whether $b(\varphi_*)$ is positive or not, there exists a unique $\theta_* \in I_0$ such that $D(\theta_*) = 0$ and $r_\pm(\theta) > 0$ for any $\theta \in (\theta_*, \theta_0)$. By (18), we see that $r_+(\theta)$ is monotonic increasing on (θ_*, θ_0), whereas $r_-(\theta)$ is monotonic decreasing. Moreover, we have $r_+(\theta) \to +\infty$ and $r_-(\theta) \to 0$ as $\theta \to \theta_0 - 0$, and hence the curves $r_\pm(\theta)$, $\theta \in (\theta_*, \theta_0)$ form an unbounded curve connecting $z = \alpha_1$ and $z = \infty$. Since $-\kappa < -\frac{1}{\gamma} < 0$, we have

$$\Omega^+ = \{z \in \mathcal{D}(f_{\kappa,\gamma});\ 0 < \theta < \theta_0,\ \text{and if } \theta \ge \theta_* \text{ then } 0 < r < r_-(\theta) \text{ or } r > r_+(\theta)\}\,.$$

(c) We finally assume that $0 < \kappa < 1$. In this case, we have $D(\pi) = (2a-1)^2 - 4a(a-1) = 1$. We note that $b(0) < 0$ implies $D(0) < 0$. In fact, $b(0) < 0$ means $1 + \frac{1}{\kappa} < 2a$ so that

$$D(0) = \Big(1 + \frac{1}{\kappa}\Big)^2 - \frac{4a}{\kappa} < 2a\Big(1 + \frac{1}{\kappa}\Big) - \frac{4a}{\kappa} = 2a\Big(1 - \frac{1}{\kappa}\Big) < 0. \tag{19}$$

Since $a > 1$, the signatures of $r_\pm(\theta)$ are the same, and they are the opposite to the signature of $b(\theta)$.

Let $I_D \subset I_0$ be the maximal interval such that D is positive on I_D. We shall show that there are no suitable solutions of (10), that is, $r_\pm(\theta) < 0$ for any $\theta \in I_D$, which yields that $\Omega^+ = \mathbb{C}^+$ and hence $\Omega = \mathcal{D}(f_{\kappa,\gamma})$. Let us recall Lemma 2.1.

(i) Assume that $0 < \kappa < \frac{1}{2}$ and $a > G(\kappa)$. In this case, $b(\theta)$ have a unique maximal point at $\theta = \varphi_*$, and we have $b(\pi) > 0$. If $b(0) \ge 0$, then we see that $b(\theta) > 0$ for any $\theta \in I_0$, which implies $r_\pm(\theta) < 0$ for any $\theta \in I_D$. If $b(0) < 0$, then there exists a unique $0 < \varphi < \varphi_*$ such that $b(\varphi) = 0$, and we have $D(0) < 0$ by (19). Hence, $D(\theta)$ have a unique maximal point $D(\varphi_*) > 0$ at $\theta = \varphi_*$ and a unique minimal point $D(\varphi) < 0$ at $\theta = \varphi$. Thus, I_D is included in (φ, π) and b is positive on I_D, whence $r_\pm(\theta) < 0$ for any $\theta \in I_D$.

(ii) Assume that $0 < \kappa < \frac{1}{2}$ and $a \le G(\kappa)$. Then, Lemma 2.1 tells us that $b'(\theta) < 0$ for any $\theta \in I_0$. Thus, $b(\theta)$ is monotonic decreasing on the interval I_0 with $b(\pi) > 0$, and hence $b(\theta) > 0$ for any $\theta \in I_0$. This means that $D'(\theta) < 0$ and $D(\theta)$ is monotonic decreasing on I_0. Since $D(\pi) = 1$, we see that $I_D = I_0$ and hence $r_\pm(\theta) < 0$ for any $\theta \in I_D$.

(iii) Assume that $\frac{1}{2} \le \kappa < 1$. In this case, we have we have $b'(\theta) > 0$ for any $\theta \in I_0$ so that $b(\theta)$ is monotonic increasing. In fact, if $\kappa > \frac{1}{2}$, then since $G(\kappa) < 1$ for $\frac{1}{2} < \kappa < 1$ and since $a > 1$, we always have $a > G(\kappa)$ so that $b'(\theta) > 0$ by Lemma 2.1, and if $\kappa = \frac{1}{2}$ then we have $b'(\theta) = 2(a-1)\sin\theta$ so that $b'(\theta) > 0$. If $b(0) \ge 0$, then we have $b(\theta) \ge 0$ for any $\theta \in I_0$ and hence we see that $r_\pm(\theta) < 0$ for $\theta \in I_D$. If $b(0) < 0$ then there exists a unique φ such that $b(\varphi) = 0$. Thus, $D(\theta)$ has a unique minimal point $D(\varphi) < 0$ with $D(0) < 0$ and $D(\pi) > 0$, and hence there are no $\theta \in I_0$ such that $D(\theta) > 0$ and $b(\theta) > 0$. Thus, we have $b(\theta) > 0$ for $\theta \in I_D$, which implies $r_\pm(\theta) < 0$.

We shall determine an asymptotic line with respect to Ω^+ when $r_+(\theta) \to +\infty$ as $\theta \to \frac{\pi}{\kappa}$ or $\frac{\pi}{\kappa+1}$. To calculate them in one scheme, we set $\vartheta = \frac{\pi}{\kappa}$ or $\frac{\pi}{\kappa+1}$, and denote its denominator by k. A line having gradient $\tan\vartheta$ can be written as $x\sin\vartheta - y\cos\vartheta = A$ with some constant A. Let us determine the constant A. Since $x = \kappa(r(\theta)\cos\theta - 1)$ and $y = \kappa r(\theta)\sin\theta$ as in (8), we have

$$x\sin\vartheta - y\cos\vartheta = \kappa\big\{\sin\vartheta(r(\theta)\cos\theta - 1) - \cos\vartheta r(\theta)\sin\theta\big\} = -\kappa\big\{r(\theta)\sin(\theta - \vartheta) + \sin\vartheta\big\}.$$

A simple calculation yields that $\frac{\sin(\theta-\vartheta)}{\sin(k\vartheta)} \to -\frac{1}{k}$ as $\theta \to \theta_0 - 0$, which implies

$$\lim_{\theta\to\vartheta-0} b(\theta)\sin(\theta-\vartheta) = \lim_{\theta\to\vartheta-0}\Big((1-2a)\cos\theta\sin(\theta-\vartheta) + \sin\theta\cos(k\theta)\frac{\sin(\theta-\vartheta)}{\sin(k\theta)}\Big) = \frac{\sin\vartheta}{ak}.$$

Since $r_+(\theta)$ can be described as

$$r_+(\theta) = \frac{-b(\theta)+\sqrt{b(\theta)^2-4a(a-1)}}{2a} = -\frac{b(\theta)}{2a}\left(1+\sqrt{1-\frac{4a(a-1)}{b(\theta)^2}}\right),$$

we have

$$\lim_{\theta\to\vartheta-0} r_+(\theta)\sin(\theta-\vartheta) = \lim_{\theta\to\vartheta-0} -\frac{b(\theta)\sin(\theta-\vartheta)}{2a}\left(1+\sqrt{1-\frac{4a(a-1)}{b(\theta)^2}}\right) = -\frac{\sin\vartheta}{ak}.$$

Thus, we have

$$A = \begin{cases} -\kappa(-\frac{\sin\theta_0}{a\kappa}+\sin\theta_0) = \left(\frac{1}{a}-\kappa\right)\sin\theta_0 & (\text{if } k=\kappa), \\ -\kappa(-\frac{\sin\theta_1}{\kappa+1}+\sin\theta_1) = -\frac{\kappa^2}{\kappa+1}\sin\theta_1 & (\text{if } k=\kappa+1), \end{cases}$$

and therefore the proof is now complete. □

2.2. **The domain Ω for $\kappa=\infty$.** In this section, we deal with the case $\kappa=\infty$. Since $\kappa\theta(x,y)$ is regarded as y in this case, the equation (5) can be written as

$$F(x,y) = x+\gamma x^2+\gamma y^2+y\cot y = 0. \tag{20}$$

Proposition 2.4. *Assume that $\kappa=\infty$ and let $x_i(y)$, $i=1,2$ be solutions of (20) with $x_1(y)\le x_2(y)$ if they are real.*

(1) *If $\gamma=0$, then one has $\Omega=\{z=x+yi;\ |y|<\pi$ and $x>-y\cot y\}$.*
(2) *Suppose that $\gamma>\frac{1}{4}$. Then, there exists a unique function $y=y(x)$ defined on $\mathbb{R}$ such that*

$$\Omega = \{z=x+yi\in\mathbb{C};\ x\in\mathbb{R} \text{ and } |y|<y(x)\}.$$

The function $y(x)$ has a unique minimal point and satisfies $\lim_{x\to\pm\infty} y(x)=\pi$.
(3) *If $0<\gamma\le\frac{1}{4}$, then $x_i(y)$, $i=1,2$ are both real for any $y\in(0,\pi)$, and one has*

$$\Omega = \{z=x+yi\in\mathbb{C};\ |y|<\pi \text{ and } x>x_2(y)\}.$$

(4) *Suppose that $\gamma<0$. Then, there exists a unique $y_0\in(0,\pi)$ such that $x_1(y_0)=x_2(y_0)$ and if $y\le y_0$ then $x_i(y)$, $i=1,2$ are real. Moreover, one has*

$$\Omega = \{z=x+yi\in\mathbb{C};\ |y|<y_0 \text{ and } x_1(y)<x<x_2(y)\}.$$

In particular, Ω is bounded.

Proof. Similarly to the proof of Proposition 2.3, we work with $\Omega^+=\Omega\cap\mathbb{C}^+$. Let us assume $y>0$.

(i) The case $\gamma\le 0$ is dealt in the previous paper [10], and hence we omit it. Note that The case $\gamma=0$ corresponds to the case of the classical Lambert function (cf. [4]).

(ii) Assume that $\gamma>0$. Since there are no solutions of (20) on the line $\mathbb{R}+i\pi$ and since Ω is symmetric with respect to the real axis, it is enough to consider y in the interval $(0,\pi)$. The equation (20) can be calculated as

$$x^2+\frac{x}{\gamma}+y^2+\frac{y\cot y}{\gamma}=0 \iff \Big(x+\frac{1}{2\gamma}\Big)^2 = \frac{1}{4\gamma^2}-y^2-\frac{y\cot y}{\gamma} =: h(y). \tag{21}$$

Let us set $g(y):=\cos y+2\gamma y\sin y$. Then, the function $h(y)$ satisfies

$$h(y)=\frac{1-g(y)^2}{4\gamma^2\sin^2 y},\quad h(0)=\lim_{y\to0}h(y)=\frac{1-4\gamma}{4\gamma^2} \quad\text{and}\quad \lim_{y\to\pi-0}|h(y)|=+\infty.$$

In order that the equation (21) has a real solution in x and y, the function $h(y)$ needs to be non-negative, and it is equivalent to the condition that the absolute value of the function $g(y)$ is less than or equal to 1. Thus we investigate the function $g(y)$. At first, we observe that $g(0)=1$ and $g(\pi)=-1$. Its derivative is calculated as

$$g'(y) = -\sin y+2\gamma(\sin y+y\cos y) = -(1-2\gamma)\sin y+2\gamma y\cos y = (2\gamma-1)\Big(\frac{2\gamma}{2\gamma-1}y+\tan y\Big)\cos y.$$

Set $c_\gamma=\frac{2\gamma}{2\gamma-1}$. Then, the signature of g' can be determined by the product of signatures of $2\gamma-1$, $\cos y$ and $c_\gamma y+\tan y$.

(ii-1) Assume that $\gamma > \frac{1}{4}$. If $\gamma > \frac{1}{2}$, then we have $c_\gamma > 1$ and hence there exists a unique $y_* \in (\frac{\pi}{2}, \pi)$ such that $c_\gamma y + \tan y = 0$. If $\frac{1}{4} < \gamma < \frac{1}{2}$, we have $c_\gamma < -1$ so that there exists a unique $y_* \in (0, \frac{\pi}{2})$ such that $c_\gamma y_* + \tan y_* = 0$. Thus, for both cases, $g(y)$ has a unique maximal point at $y = y_*$ with $g(0) = 1$ and $g(\pi) = -1$. If $\gamma = \frac{1}{2}$, then we have $g'(y) = y \cos y$, whence $g'(y) > 0$ for $y \in (0, \frac{\pi}{2})$ and $g'(y) < 0$ for $y \in (\frac{\pi}{2}, \pi)$ so that in this case $y_* = \frac{\pi}{2}$ is the only solution of $g(y) = 0$ for $y \in (0, \pi)$.

These observation shows that, if $\gamma > \frac{1}{4}$, then there exists one and only one $y_0 \in (y_*, \pi)$ such that $g(y_0) = 1$ and $g(y_0 - \varepsilon) > 1$ for $\varepsilon \in (0, y_0 - y_*)$. In this case, $h(y)$ is non-negative in the interval $[y_0, \pi)$, and $h(h_0 - \varepsilon) < 0$ for $\varepsilon \in (0, y_0 - y_*)$. Let $x_i(y)$, $i = 1, 2$ be the real solutions of the equation (21) with $x_1(y) \le x_2(y)$. Then, since we have $x_1(y_0) = x_2(y_0)$, the curves $x_i(y)$, $y \in (y_0, \pi)$ form a connected curve. Moreover, since the correspondence of $y \in (y_0, \pi)$ to $x_i(y)$ is one-to-one and since $x_1(y) \to -\infty$, and $x_2(y) \to +\infty$ as $y \to \pi - 0$, the function $y = y(x)$ connecting the two inverse functions of $x = x_i(y)$ is defined for any $x \in \mathbb{R}$ and its image is $[y_0, \pi)$. Thus, we have

$$\Omega^+ = \{z = x + yi \in \mathbb{C};\ x \in \mathbb{R} \text{ and } 0 < y < y(x)\}.$$

(ii-2) Assume that $0 < \gamma \le \frac{1}{4}$. Then, we have $-1 \le c_\gamma < 1$ and hence there are no $y \in (0, \pi)$ such that $c_\gamma y + \tan y = 0$. Thus, we obtain $g'(y) < 0$ for any $y \in (0, \pi)$ so that g is monotonic decreasing from $g(0) = 1$ to $g(\pi) = -1$. This shows that $h(y)$ is non-negative in the interval $(0, \pi)$. Let $x_i(y)$, $i = 1, 2$ be the solutions of the equation (21) with $x_1(y) \le x_2(y)$. Then, since $x_1(y) \to -\infty$ and $x_2(y) \to +\infty$ as $y \to \pi - 0$, we have by $x_1(0) < x_2(0) < 0$

$$\Omega^+ = \{z = x + yi \in \mathbb{C};\ 0 < y < \pi \text{ and } x > x_2(y)\},$$

which completes the proof. □

2.3. **Bijectivity of $f_{\kappa,\gamma}$.** We consider the domain $\Omega^+ := \Omega \cap \mathbb{C}^+$, and investigate whether $f_{\kappa,\gamma}$ maps D to $\mathbb{C}^+$ bijectively or not. Then, since $f_{\kappa,\gamma}(\bar{z}) = \overline{f_{\kappa,\gamma}(z)}$, the bijectivity of $f_{\kappa,\gamma}$ from Ω to $\mathbb{C} \setminus \overline{\mathcal{S}}$ is obtained at the same time.

The key tool is the argument principle (see [1, Theorem 18, p.152], for example).

Theorem 2.5 (**The argument principle**). *If $f(z)$ is meromorphic in a domain Ω with the zeros a_j and the poles b_k, then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles. Here, $n(\gamma, a)$ is the winding number of γ with respect to a.

We also use the following elementary property of holomorphic functions.

Lemma 2.6. *Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function. The implicit function $v(x, y) = 0$ has an intersection point at $z = x + yi$ only if $f'(z) = 0$.*

2.3.1. *The case $\kappa \ge 1$ and $\gamma \le 0$.* This case is done in the previous paper [10], and thus we omit it.

2.3.2. *The case $\kappa \ge 1$ and $\gamma > 0$.* In this case, Propositions 2.3, 2.2 and 2.4 tell us that Ω is unbounded. We first consider $\mathcal{S}$ defined in (7). Note that the case $(\kappa, \gamma) = (1, 1)$ is the trivial case $f_{\kappa,\gamma}(z) = z$.

Lemma 2.7. *Let α_i, $i = 1, 2$ be the solutions of* (10).

(1) *Assume that $\kappa > 1$ or $\kappa = \infty$ together with $D(0) \ge 0$. Then, α_i are both real and $\mathcal{S} = (-\infty, f_{\kappa,\gamma}(\alpha_2))$ with $f_{\kappa,\gamma}(\alpha_2) < 0$.*
(2) *Assume that $\kappa = 1$ and $0 < \gamma < 1$. Then, α_i are both real, and one has $\mathcal{S} = (f_{\kappa,\gamma}(\alpha_1), f_{\kappa,\gamma}(\alpha_2))$ with $f_{\kappa,\gamma}(\alpha_2) < 0$. Note that $0 < \gamma < 1$ is equivalent to $D(0) > 0$.*
(3) *Assume that $\kappa \ge 1$ or $\kappa = \infty$ together with $D(0) < 0$. Then, α_i are both non-real, and one has $\mathcal{S} = \emptyset$. On the other hand, one has $f_{\kappa,\gamma}(\partial\Omega \cap \mathbb{C}^+) = (-\infty, 0)$.*

Proof. The proof is elementary so we only deal with the latter assertion in (3). It is easily verified that $|f_{\kappa,\gamma}(z)| \to +\infty$ as $|z| \to +\infty$ by $\kappa > 1$. Propositions 2.3, 2.2 and 2.4 show that if a point $z = x + yi \in \mathbb{C}^+$ is on the curve $\partial\Omega \cap \mathbb{C}^+$, then we have $F(x, y) = 0$ and

$$f_{\kappa,\gamma}(z) = \frac{\left|1 + \frac{z}{\kappa}\right|^\kappa}{\left|1 + \gamma z\right|^2} \cdot \frac{-y}{\sin(\kappa\theta(x, y))} < 0. \tag{22}$$

This means that, if $z \in \mathbb{C}^+$ goes to ∞ along the path $\partial\Omega \cap \mathbb{C}^+$, then $f_{\kappa,\gamma}$ must tend to $-\infty$. On the other hand, taking a limit $y \to +0$ along $\partial\Omega \cap \mathbb{C}^+$ is equivalent to $z \to -\kappa$ in this case and hence $f_{\kappa,\gamma}(z) \to 0$. Since $f_{\kappa,\gamma}$ is continuous, the assertion is now proved. □

For $L > 0$, let Γ_L be the circle $-\kappa + Le^{i\theta}$ of origin $z = -\kappa$ with radius L. We divide cases according to this lemma.

The case (1). In this case, Ω is given in Propositions 2.3 or 2.4. Let us take a path $C = C(t)$, $t \in (0,1)$ in such a way that by starting from $z = \infty$, it goes to $z = \alpha_2$ along the curve r_+ describing $\partial\Omega \cap \mathbb{C}^+$ in the upper half plane, and then goes to $z = \infty$ along the real axis. Here, we can assume that $C'(t) \neq 0$ whenever $C(t) \neq \alpha_i$, $i = 1, 2$ because $f'_{\kappa,\gamma}(C(t)) \neq 0$ otherwise. We take an arc-length parameter t so that $C'(t)$ represents the direction of the tangent line at $x + yi = C(t)$. Lemma 2.6 tells us that $g(t) := f_{\kappa,\gamma}\big(C(t)\big)$ is a monotonic increasing function on $(0,1)$ such that $g(t) \to -\infty$ as $t \to 0$ and $g(t) \to +\infty$ as $t \to 1$.

We shall show that for any $w_0 \in \mathbb{C}^+$ there exists one and only one $z_0 \in \mathbb{C}^+$ such that $f_{\kappa,\gamma}(z_0) = w_0$. Recall that $\operatorname{Im} f'_{\kappa,\gamma}(z) > 0$ for any $z \in \mathbb{C}^+$. Let us take an $R > 0$ such that $|w_0| < R$. For an $L > 0$, let z_i, $i = 1, 2$ be two distinct intersection points of C and Γ_L. Note that we can take $z_1 = -\kappa + L \in \mathbb{R}$ and $z_2 \in \mathbb{C}^+$. Let $\widetilde{C}$ be a closed path obtained from C by connecting z_1 and z_2 via the arc A_L of Γ_L included in $\mathbb{C}^+$.

Let $\widetilde{\Omega}^+$ be the inside set of $\widetilde{C}$. Since $f_{\kappa,\gamma}$ is non-singular on the arc A, the curve $f_{\kappa,\gamma}(A)$ does not have a singular point so that it is homotopic to a semi-circle in $\mathbb{C}^+$. In particular, we can take an L such that its radius is larger than R so that the inside set $f_{\kappa,\gamma}(\widetilde{\Omega}^+)$ of the curve $f_{\kappa,\gamma}(\widetilde{C})$ is a bounded domain including $w_0 \in \mathbb{C}^+$. Since the winding number of the path $f_{\kappa,\gamma}(\widetilde{C})$ about $w = w_0$ is exactly one, we see that

$$\frac{1}{2\pi i}\int_{\widetilde{C}} \frac{f'_{\kappa,\gamma}(z)}{f_{\kappa,\gamma}(z) - w_0} dz = \frac{1}{2\pi i}\int_{f_{\kappa,\gamma}(\widetilde{C})} \frac{dw}{w - w_0} = 1.$$

Since $f_{\kappa,\gamma}$ does not have a pole on $\widetilde{D}$ be definition, the function $f_{\kappa,\gamma}(z) - w_0$ has the only one zero point, say z_0, by the argument principle in Theorem 2.5. Then, we obtain $f_{\kappa,\gamma}(z_0) = w_0$, and such z_0 is unique. Since we can take $w_0 \in \mathbb{C}^+$ arbitrary, we conclude that the map $f_{\kappa,\gamma}$ is a bijection from D to the upper half plane $\mathbb{C}^+$.

The case (2). In this case, Ω is given in Proposition 2.2 (2). As in the case (1), it is enough to show that an approximating domain $\widetilde{\Omega}^+$ of $\Omega^+ = \Omega \cap \mathbb{C}^+$ satisfies that the image of its boundary curve $\widetilde{C} = \partial\widetilde{\Omega}^+$ under $f_{\kappa,\gamma}$ has a winding number about $w = w_0$ being exactly one for any $w_0 \in \mathbb{C}^+$.

By the proof of Proposition 2.2, the boundary of Ω^+ can be described as a path $C(t)$, $t \in (0,1)$ defined in a such a way that by staring $z = \infty$, it goes to $z = \alpha_1$ along the real axis, and then it goes to $z = \alpha_2$ along the curve $\partial\Omega \cap \mathbb{C}^+$, which is a semi-circle of origin $z = -\frac{1}{\gamma}$ with radius $\sqrt{1-\gamma}/\gamma$, and then it goes to $z = \infty$ along the real axis. Notice that $f'_{\kappa,\gamma}(\widetilde{C}(t)) \neq 0$ for any $t \in (0,1)$ except for t_i such that $C(t_i) = \alpha_i$, $i = 1, 2$. In particular, Lemma 2.6 tells us that $f_{\kappa,\gamma}(C(t))$ is monotonic increasing in the interval (t_1, t_2), and hence the function $f_{\kappa,\gamma}(C(t))$ maps $(0,1)$ to $\mathbb{R}$ bijectively. Therefore, we can construct an approximating domain $\widetilde{\Omega}^+$ from $C(t)$ and Γ_L similar to the case (1), such that the image of its boundary $\widetilde{C}$ under $f_{\kappa,\gamma}$ has a winding number being exactly one, and hence we have shown the bijectivity of $f_{\kappa,\gamma}$ on Ω^+ to $\mathbb{C}^+$.

The case (3). In this case, we shall show that the approximating domain $\widetilde{\Omega}^+$ of $\Omega^+ = \Omega \cap \mathbb{C}^+$ satisfies that the image of its boundary curve $\widetilde{C} = \partial\widetilde{\Omega}^+$ under $f_{\kappa,\gamma}$ has a winding number about $w = w_0$ being exactly two for any $w_0 \in \mathbb{C}^+$, and hence the function $f_{\kappa,\gamma}$ maps Ω^+ to $\mathbb{C}^+$ two-to-one.

(i) Let us assume that $\kappa \geq 1$. The domain Ω is given in Proposition 2.3 (5-b). Let $C(t)$, $t \in (0,1)$ be a path defined in such a way that by starting from $z = \infty$, it goes to $z = -\kappa$ along the curve $\partial\Omega \cap \mathbb{C}^+$, and then goes to $z = +\infty$ along the real axis. Let $t_0 \in (0,1)$ such that $C(t_0) = -\kappa$. Then, Lemma 2.7 (3) tells us that $f_{\kappa,\gamma}(C(t))$ maps $(0, t_0)$ to $(-\infty, 0)$ bijectively, and hence we see that it maps $(0,1)$ to $\mathbb{R}$ in two-to-one correspondence. Let $\widetilde{\Omega}^+ = \Omega^+ \cap \Gamma_L$ and $\widetilde{C}$ its boundary. the image of its boundary curve $\widetilde{C} = \partial\widetilde{\Omega}^+$ under $f_{\kappa,\gamma}$ has a winding number about $w = w_0$ being exactly two for any $w_0 \in \mathbb{C}^+$, and hence the function $f_{\kappa,\gamma}$ maps Ω^+ to $\mathbb{C}^+$ two-to-one.

(ii) Assume that $\kappa = \infty$ together with $D(0) < 0$, i.e., $\gamma > \frac{1}{4}$. Then, Ω is described in Proposition 2.4 (2). Take a large $R > 0$ and a small $\varepsilon > 0$ such that $\varepsilon < |w_0| < R$. Then, we approximate Ω^+ by the curve $\widetilde{C}(t)$, $t \in [0,1]$ defined in such a way that, for $L > 0$ and $\delta > 0$, by starting from $\widetilde{C}(0) = -L$, it goes to $z = L$ along the real axis avoiding $z = -\frac{1}{\gamma}$ along a semicircle of radius $\delta > 0$ in $\mathbb{C}^+$, and next

to $z = L + y(L)i$ along the line parallel to the imaginary axis, and next to $z = -L + y(-L)i$ along $\partial\Omega \cap \mathbb{C}^+$ and finally goes back to $\widetilde{C}(1) = -L$ along the line parallel to the imaginary axis. Then, since $|f_{\infty,\gamma}(z)| = \left|\frac{z}{1+\gamma z}\right| e^x = \frac{e^x}{|\gamma + \frac{1}{z}|}$, we can choose $L > 0$ such that

$$|f_{\infty,\gamma}(-L + yi)| < \varepsilon \quad (y \in (0, y(-L))), \quad |f_{\infty,\gamma}(L + yi)| > R \quad (y \in (0, y(L))),$$

and since $|1 + \gamma z| = \gamma\delta$ and ze^z are bounded around $z = -\frac{1}{\gamma}$, we can choose $\delta > 0$ such that $\left|f_{\infty,\gamma}(-\frac{1}{\gamma} + \delta e^{i\theta})\right| > R$. Since $f_{\infty,\gamma}$ is non-singular on the lines $\pm L + yi$, $y \in (0, y(\pm L))$, the curve $f_{\infty,\gamma}(C'(t))$, $t \in [0,1]$ contains the domain obtained from a semicircle of radius R removing a semicircle of radius ε. In particular, it contains w_0. Since we can show that the winding number of the path $f_{\infty,\gamma}(\widetilde{C})$ with respect to $w = w_0$ is exactly two, we conclude that the function $f_{\kappa,\gamma}$ maps Ω^+ to $\mathbb{C}^+$ in two-to-one by a similar argument as in (i) above.

2.3.3. *The case* $0 < \kappa < 1$ *and* $\gamma \leq 0$. Although this case is out of the result stated in the previous paper [10], the same proof given there (see the supplemental material of [10]) is valid, and thus we omit it.

2.3.4. *The case* $0 < \kappa < 1$ *and* $\gamma > 0$. In this case, Proposition 2.3 tells us that if $0 < a = \kappa\gamma < 1$ then $\Omega \cap \mathbb{R} = (\alpha_2, +\infty)$, and if $a > 1$ then $\Omega \cap \mathbb{R} = \left\{x \in \mathbb{R};\ x > -\kappa \text{ and } x \neq -\frac{1}{\gamma}\right\}$. If $f_{\kappa,\gamma}$ maps Ω^+ to $\mathbb{C}^+$ bijectively, then it needs map $\partial\Omega^+$ to $\mathbb{R}$. However, if $x \in \mathbb{R}$ satisfies $x < \min(\alpha_1, -\kappa)$, then $\exp_\kappa(z)$ tends to $\left|1 + \frac{x}{\kappa}\right|^\kappa e^{i\kappa\pi}$ as $z \to x$ via the arc $re^{i\theta} = 1 + \frac{z}{\kappa}$. Since $0 < \kappa < 1$, then we see that $\lim_{z\to x} f_{\kappa,\gamma}(z)$ is not real and hence $f_{\kappa,\gamma}$ cannot map Ω^+ to $\mathbb{C}^+$ bijectively.

2.3.5. *The case of* $\kappa < 0$. We shall complete the proof of Theorem 1.3 by proving it for the case $\kappa < 0$. To do so, let us recall the homographic (linear fractional) action of $SL(2,\mathbb{R})$ on $\mathbb{C}$: $g \cdot z := \frac{az+b}{cz+d}$ for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ and $z \in \mathbb{C}^+$. For each $g \in SL(2,\mathbb{R})$, the corresponding homographic action map $\mathbb{C}^+$ to $\mathbb{C}^+$ bijectively. Let $\kappa = -\kappa'$ with positive $\kappa' > 0$. Consider the transformation

$$1 + \frac{z'}{\kappa'} = \left(1 + \frac{z}{\kappa}\right)^{-1}.$$

Notice that this transformation can be written as a homographic action as $z' = \left(\begin{smallmatrix} 1 & 0 \\ 1/\kappa & 1 \end{smallmatrix}\right) \cdot z$, and hence it maps $\mathbb{C}^+$ to $\mathbb{C}^+$ bijectively. Then, since

$$\frac{z}{1+\gamma z} = \frac{z'}{1 + (\gamma + 1/\kappa')z'} \quad \text{and} \quad \left(1 + \frac{z}{\kappa}\right)^\kappa = \left(\left(1 + \frac{z}{\kappa}\right)^{-1}\right)^{-\kappa} = \left(1 + \frac{z'}{\kappa'}\right)^{\kappa'}$$

(recall that we are taking the main branch so that $\log z = -\log(z^{-1})$), we obtain

$$f_{\gamma,\kappa}(z) = \frac{z}{1+\gamma z}\left(1 + \frac{z}{\kappa}\right)^\kappa = \frac{z'}{1 + (\gamma + 1/\kappa')z'}\left(1 + \frac{z'}{\kappa'}\right)^{\kappa'} = f_{\gamma+1/\kappa',\,\kappa'}(z').$$

Set $\gamma' = \gamma + 1/\kappa'$. Since homographic actions map $\mathbb{C}^+$ to $\mathbb{C}^+$ bijectively, there exists a domain Ω such that $f_{\kappa,\gamma}$ maps $\Omega^+ = \Omega \cap \mathbb{C}^+$ to $\mathbb{C}^+$ bijectively if and only if it holds for $f_{\gamma',\kappa'}$. Thus, the condition $\gamma' \leq 0$ is equivalent to $\gamma \leq \frac{1}{\kappa}$, and the condition $\kappa' > 1$ and $\gamma' > 0$ with $\gamma' \leq \frac{1}{4}(1 + \frac{1}{\kappa'})^2$ is equivalent to

$$\gamma > \frac{1}{\kappa} \quad \text{and} \quad \kappa < -1 \quad \text{with} \quad \gamma - \frac{1}{\kappa} \leq \frac{1}{4}\left(1 - \frac{1}{\kappa}\right)^2 \iff \gamma \leq \frac{1}{4}\left(1 + \frac{1}{\kappa}\right)^2.$$

This shows the case $\kappa < 0$, and therefore we have completed the proof of Theorem 1.3. □

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