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► **To cite this version:**

Antoine Mhanna. TRIGONOMETRIC IDENTITIES AND URQUHART'S THEOREM. *International Journal of Computer Discovered Mathematics*, 2023, 8, pp.9-14. <hal-03291910v2>

HAL Id: hal-03291910

<https://hal.science/hal-03291910v2>

Submitted on 7 Apr 2023

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TRIGONOMETRIC IDENTITIES AND URQUHART'S THEOREM

ANTOINE MHANNA^{1*}

ABSTRACT. The purpose of this note is to represent a general analytic geometry problem implying Urquhart's Theorem. The proofs are elementary and rely on some calculations and trigonometric identities.

1. INTRODUCTION

We start by stating this simple problem: In the Euclidean plane (O, \vec{i}, \vec{j}) , take two concentric circles (A) respectively (B) of radius a respectively b centered at O . Let $A \in (A)$ with $A(a \cos(\alpha); a \sin(\alpha))$ and $B \in (B)$ with $B(b \cos(\beta); b \sin(\beta))$ for some reals α and β in $[-\pi, \pi]$. For $C(r, 0)$, $r > 0$ the perpendicular bisector of $[AC]$ resp. $[BC]$ intersect (OA) at A' resp. (OB) at B' . Assuming that no opposite sides of the quadrilateral $OA'CB'$ are parallel, let $I = (A'C) \cap (OB')$ and $J = (B'C) \cap (OA')$. We fix all parameters except the radius b considered as a variable $b \geq 0$ and want to find b for which

$$(1.1) \quad |IO \pm IC| = |JO \pm JC|.$$

In this context we state the known Urquhart's theorem (see also [4]) as follows:

Theorem 1 ([1, 2]). *For $\alpha\beta \leq 0$, $a \geq r$ and $b = a$ we have $IO + IC = JO + JC$ (see Figure 1).*

Date: 2022

* Corresponding author.

2010 *Mathematics Subject Classification.* 51N20; 33B10; 26B99.

Key words and phrases. Urquhart's Theorem; Trigonometric Identities.

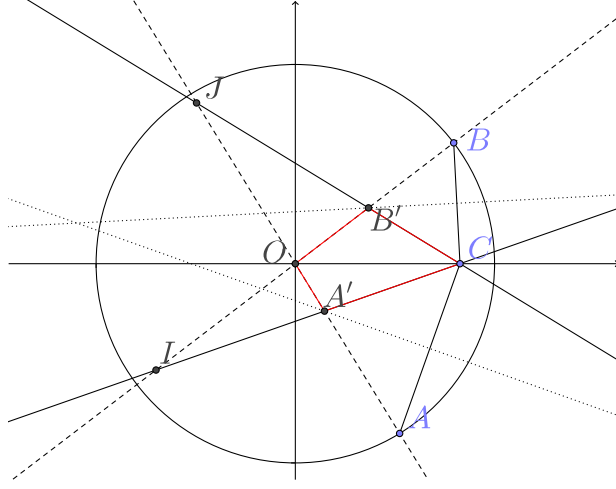


FIGURE 1. Urquhart's theorem

2. MAIN RESULTS

Considering the line (OA) resp. (OB) of equation $y = \tan(\alpha)x$ resp. $y = \tan(\beta)x$; by straight calculations we have:

$$A' \left(\frac{\cos(\alpha)(a^2 - r^2)}{2a - 2r \cos(\alpha)}, \frac{\sin(\alpha)(a^2 - r^2)}{2a - 2r \cos(\alpha)} \right) \text{ and thus } B' \left(\frac{\cos(\beta)(b^2 - r^2)}{2b - 2r \cos(\beta)}, \frac{\sin(\beta)(b^2 - r^2)}{2b - 2r \cos(\beta)} \right).$$

We also verify that the line (CA') is:

$$y = \left(\frac{\sin(\alpha)(a^2 - r^2)}{(r^2 + a^2) \cos(\alpha) - 2ra} \right) x - \frac{r \sin(\alpha)(a^2 - r^2)}{(r^2 + a^2) \cos(\alpha) - 2ar}.$$

The points I and J are:

$$I \begin{cases} x_I = \frac{r \cos(\beta) \sin(\alpha)(a^2 - r^2)}{\cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar)} \\ y_I = \frac{r \sin(\beta) \sin(\alpha)(a^2 - r^2)}{\cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar)} \end{cases}$$

and by switching $\alpha \leftrightarrow \beta$, $a \leftrightarrow b$:

$$J \begin{cases} x_J = \frac{r \cos(\alpha) \sin(\beta)(b^2 - r^2)}{\cos(\alpha) \sin(\beta)(b^2 - r^2) - \sin(\alpha)((r^2 + b^2) \cos(\beta) - 2br)} \\ y_J = \frac{r \sin(\alpha) \sin(\beta)(b^2 - r^2)}{\cos(\alpha) \sin(\beta)(b^2 - r^2) - \sin(\alpha)((r^2 + b^2) \cos(\beta) - 2br)}. \end{cases}$$

Hereafter we assume that $\sin(\alpha)\sin(\beta) \neq 0$ since otherwise the configuration is trivial. The next lemma collects some identities that can be obtained at www.dcode.fr/math-expression-factor:

Lemma 1. Let α , β , r and a be reals numbers, set $s = \frac{\alpha+\beta}{2}$ and $d = \frac{\alpha-\beta}{2}$:

$$(2.1) \quad (\sin(\alpha)(a^2 - r^2))^2 + ((r^2 + a^2) \cos(\alpha) - 2ar)^2 = (a^2 + r^2 - 2ar \cos(\alpha))^2.$$

$$(2.2) \quad \begin{aligned} & r \sin(\alpha)(a^2 - r^2) + r \sin(\beta)(a^2 + r^2 - 2ar \cos(\alpha)) \\ & = 2r(a \sin(s) + r \sin(d))(a \cos(d) - r \cos(s)). \end{aligned}$$

$$(2.3) \quad \begin{aligned} & r \sin(\alpha)(a^2 - r^2) - r \sin(\beta)(a^2 + r^2 - 2ar \cos(\alpha)) \\ & = 2r(r \sin(s) + a \sin(d))(a \cos(s) - r \cos(d)). \end{aligned}$$

$$(2.4) \quad \begin{aligned} & \cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar) \\ & = 2(a \sin(d) + r \sin(s))(a \cos(d) - r \cos(s)). \end{aligned}$$

Lemma 2. *With the given definitions we have:*

- $OA' = \frac{|a^2 - r^2|}{|2a - 2r \cos(\alpha)|}, \quad OB' = \frac{|b^2 - r^2|}{|2b - 2r \cos(\beta)|}$
- $CA' = \frac{a^2 + r^2 - 2ar \cos(\alpha)}{|2a - 2r \cos(\alpha)|}$ and $CB' = \frac{b^2 + r^2 - 2br \cos(\beta)}{|2b - 2r \cos(\beta)|}$
- $OI = \frac{|r \sin(\alpha)(a^2 - r^2)|}{|\cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar)|}$
- $OJ = \frac{|r \sin(\beta)(b^2 - r^2)|}{|\cos(\alpha) \sin(\beta)(b^2 - r^2) - \sin(\alpha)((r^2 + b^2) \cos(\beta) - 2br)|}$
- $CI = \frac{r |\sin(\beta)|(a^2 + r^2 - 2ar \cos(\alpha))}{|\cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar)|}$
- $CJ = \frac{r |\sin(\alpha)|(b^2 + r^2 - 2br \cos(\beta))}{|\cos(\alpha) \sin(\beta)(b^2 - r^2) - \sin(\alpha)((r^2 + b^2) \cos(\beta) - 2br)|}$
- $\frac{CB'}{OB'} \cdot \frac{OI}{CI} = \frac{CJ}{OJ} \cdot \frac{OA'}{CA'}$ (see also [5] page 63).

Proof. A straight application of previous calculations. □

For any reals α, β, r and a we define $F_{\alpha, \beta}^{\pm}(a)$ whenever the denominator is non zero by:

$$(2.5) \quad F_{\alpha, \beta}^{\pm}(a) = \frac{r \sin \alpha (a^2 - r^2) \pm r \sin \beta (a^2 + r^2 - 2ar \cos(\alpha))}{\cos(\beta) \sin(\alpha)(a^2 - r^2) - \sin(\beta)((r^2 + a^2) \cos(\alpha) - 2ar)},$$

(with the same sign for \pm on both sides). By Lemma 1 we know that

$$F_{\alpha, \beta}^+(a) = \frac{r(a \sin(s) + r \sin(d))}{a \sin(d) + r \sin(s)} \quad \text{and} \quad F_{\alpha, \beta}^-(a) = \frac{r(a \cos(s) - r \cos(d))}{a \cos(d) - r \cos(s)}.$$

In order to get identities of the form (1.1) we need to solve in b , $F_{\alpha, \beta}^{\pm}(a) = \pm F_{\beta, \alpha}^{\pm}(b)$. This is summarized in the next theorems.

Theorem 2. *Under previous notations assuming $\sin(\alpha) \sin(\beta) \neq 0$:*

(1) $F_{\alpha,\beta}^+(a) = F_{\beta,\alpha}^+(b)$ if and only if

$$b = r \frac{a(1 - \cos(\alpha) \cos(\beta)) + r(\cos(\beta) - \cos(\alpha))}{r(1 - \cos(\alpha) \cos(\beta)) + a(\cos(\beta) - \cos(\alpha))}.$$

(2) $F_{\alpha,\beta}^+(a) = -F_{\beta,\alpha}^+(b)$ if and only if $b = -a$.

(3) $F_{\alpha,\beta}^+(a) = F_{\beta,\alpha}^-(b)$ if and only if $b = r \frac{a \cos(\alpha) - r}{a - r \cos(\alpha)}$.

(4) $F_{\alpha,\beta}^+(a) = -F_{\beta,\alpha}^-(b)$ if and only if $b = r \frac{a \cos(\beta) + r}{r \cos(\beta) + a}$.

(5) $F_{\alpha,\beta}^-(a) = F_{\beta,\alpha}^-(b)$ if and only if $b = a$.

(6) $F_{\alpha,\beta}^-(a) = -F_{\beta,\alpha}^-(b)$ if and only if

$$b = r \frac{a(1 + \cos(\alpha) \cos(\beta)) - r(\cos(\beta) + \cos(\alpha))}{a(\cos(\beta) + \cos(\alpha)) - r(1 + \cos(\alpha) \cos(\beta))}.$$

(7) $F_{\alpha,\beta}^-(a) = F_{\beta,\alpha}^+(b)$ if and only if $b = r \frac{r - a \cos(\beta)}{r \cos(\beta) - a}$.

(8) $F_{\alpha,\beta}^-(a) = -F_{\beta,\alpha}^+(b)$ if and only if $b = r \frac{r - a \cos(\alpha)}{a - r \cos(\alpha)}$.

Theorem 3. Take α and β in $]0; \pi[$, $a \geq 0$ with $r > 0$:

1) For $b = r \frac{a(1 - \cos(\alpha) \cos(\beta)) + r(\cos(\beta) - \cos(\alpha))}{r(1 - \cos(\alpha) \cos(\beta)) + a(\cos(\beta) - \cos(\alpha))}$,

- if $\alpha \geq \beta$ then
 - for $a \leq r$; $b \leq r$ and $|IO - IC| = |JO - JC|$.
 - for $a \geq r$; $b \geq r$ and $IO + IC = JO + JC$,
- if $\alpha < \beta$ then
 - for $\frac{r(\cos(\alpha) - \cos(\beta))}{1 - \cos(\alpha) \cos(\beta)} \leq a \leq r$; $0 \leq b \leq r$ and $|IO - IC| = |JO - JC|$.
 - for $r \leq a < \frac{r(1 - \cos(\alpha) \cos(\beta))}{(\cos(\alpha) - \cos(\beta))}$; $b \geq r$ and $IO + IC = JO + JC$.

2) For $b = \frac{r(a \cos(\alpha) - r)}{a - r \cos(\alpha)}$,

- if $0 < \alpha < \frac{\pi}{2}$ then
 - for $0 \leq a < r \cos(\alpha) \leq r$; $b \geq r$ and $|IO - IC| = |JO - JC|$,
 - for $a \geq \frac{r}{\cos(\alpha)} \geq r$; $b \leq r$ and $IO + IC = JO + JC$.

3) For $b = \frac{r(a \cos(\beta) + r)}{a + r \cos(\beta)}$,

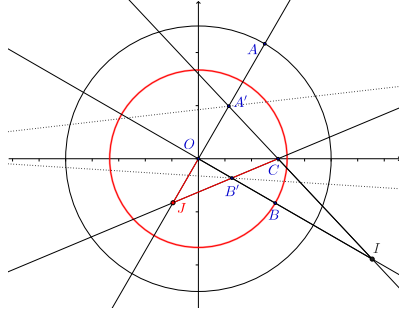
- if $0 < \beta \leq \frac{\pi}{2}$ then
 - for $a \geq r$; $b \leq r$ and $IO + IC = JO + JC$,
 - for $a \leq r$; $b \geq r$ and $|IO - IC| = |JO - JC|$.
- if $\frac{\pi}{2} < \beta < \pi$ then
 - for $-r \cos(\beta) < a \leq r$; $b \geq r$ and $|IO - IC| = |JO - JC|$,
 - for $r \leq a \leq \frac{-r}{\cos(\beta)}$; $b \leq r$ and $IO + IC = JO + JC$.

- 4) For $b = a$,
- for $a \geq r$; $|IO - IC| = |JO - JC|$,
 - for $a \leq r$; $IO + IC = JO + JC$.
- 5) For $b = r \frac{a(1 + \cos(\alpha) \cos(\beta)) - r(\cos(\beta) + \cos(\alpha))}{a(\cos(\beta) + \cos(\alpha)) - r(1 + \cos(\alpha) \cos(\beta))}$,
- if $\alpha + \beta \leq \pi$ then
 - for $0 \leq a \leq \frac{r(\cos(\alpha) + \cos(\beta))}{1 + \cos(\alpha) \cos(\beta)} \leq r$; $b \leq r$ and $IO + IC = JO + JC$,
 - for $r \leq \frac{r(1 + \cos(\alpha) \cos(\beta))}{\cos(\alpha) + \cos(\beta)} < a$; $b \geq r$ and $|IO - IC| = |JO - JC|$.
- 6) For $b = \frac{r(r - a \cos(\beta))}{r \cos(\beta) - a}$,
- if $0 < \beta < \frac{\pi}{2}$ then
 - for $0 \leq a < r \cos(\beta) \leq r$; $b \geq r$ and $IO + IC = JO + JC$,
 - for $a \geq \frac{r}{\cos(\beta)} \geq r$; $b \leq r$ and $|IO - IC| = |JO - JC|$.
- 7) For $b = \frac{r(r - a \cos(\alpha))}{a - r \cos(\alpha)}$,
- if $0 < \alpha < \frac{\pi}{2}$ then
 - for $r \cos(\alpha) < a \leq r$; $b \geq r$ and $IO + IC = JO + JC$,
 - for $r \leq a \leq \frac{r}{\cos(\alpha)}$; $b \leq r$ and $|IO - IC| = |JO - JC|$.
 - if $\frac{\pi}{2} \leq \alpha < \pi$ then
 - for $a \geq r$; $b \leq r$ and $|IO - IC| = |JO - JC|$,
 - for $a \leq r$; $b \geq r$ and $IO + IC = JO + JC$.

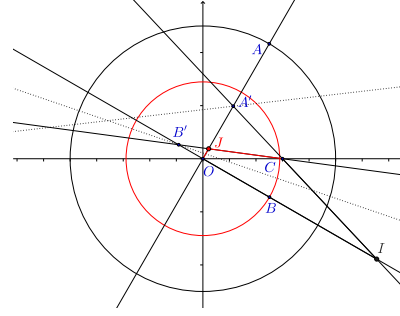
Proof. The expressions of b are direct computations from Theorem 2. The bounds on (α, β, a) are to verify that b is nonnegative and the bounds on b can be deduced by taking the derivative of b as a function in a . \square

The case $\alpha \in]0; \pi[$ and $\beta \in]-\pi; 0[$ can be deduced from the previous theorem by replacing (verbatim) β with $|\beta|$ and switching identities that is $IO + IC = JO + JC \leftrightarrow |IO - IC| = |JO - JC|$ and vice versa. This gives for example Theorem 1 as a consequence of case 4) which also appeared in [3].

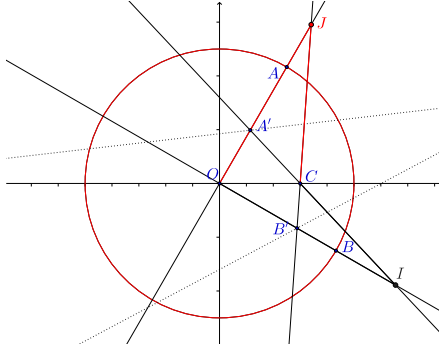
Example 1. In this example we illustrate the proven identities for $a = 5$, $r = 3$, $\alpha = \frac{\pi}{3}$ and $\beta = -\frac{\pi}{6}$.



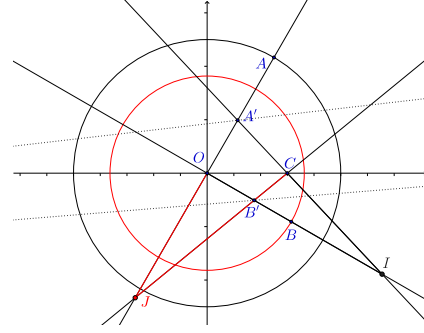
$$1) b = \frac{288\sqrt{3}-21}{143}, |IO - IC| = |JO - JC|$$



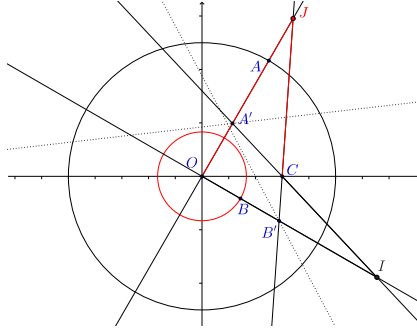
$$3) b = \frac{96\sqrt{3}+45}{73}, |IO - IC| = |JO - JC|$$



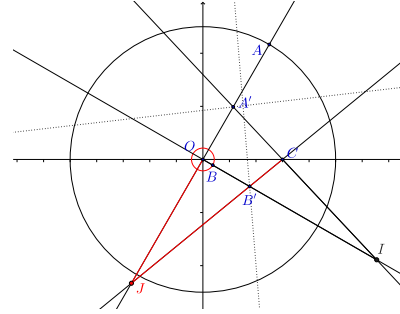
$$4) b = 5, IO + IC = JO + JC$$



$$5) b = \frac{288\sqrt{3}+21}{143}, IO + IC = JO + JC$$



$$6) b = \frac{96\sqrt{3}-45}{73}, IO + IC = JO + JC$$



$$7) b = \frac{3}{7}, IO + IC = JO + JC.$$

We point out at last that if the coordinates of J are taken as real functions, the following couple values of b give the same point J :

$$\left(a, r \frac{r - a \cos \beta}{r \cos(\beta) - a} \right), \left(-a, r \frac{r + a \cos \beta}{r \cos(\beta) + a} \right),$$

$$\left(r \frac{a \cos(\alpha) - r}{a - r \cos(\alpha)}, r \frac{a(1 - \cos(\alpha) \cos(\beta)) + r(\cos(\beta) - \cos(\alpha))}{r(1 - \cos(\alpha) \cos(\beta)) + a(\cos(\beta) - \cos(\alpha))} \right)$$

and

$$\left(r \frac{r - a \cos(\alpha)}{a - r \cos(\alpha)}, r \frac{a(1 + \cos(\alpha) \cos(\beta)) - r(\cos(\beta) + \cos(\alpha))}{a(\cos(\beta) + \cos(\alpha)) - r(1 + \cos(\alpha) \cos(\beta))} \right).$$

ACKNOWLEDGMENTS

Computations are assisted through Maple and GeoGebra softwares. Some simplification formulas are obtained on <https://www.dcode.fr>.

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