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# Why spinors do not form a vector space

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**Abstract.** Spinors  $\psi_j$  of  $SU(2)$  represent group elements, i.e. three-dimensional rotations  $R_j$ , because they are shorthands of  $SU(2)$  representation matrices  $\mathbf{R}_j$  obtained by taking their first columns. We explain that making linear combinations of spinors of  $SU(2)$  is feasible algebraically but geometrically meaningless. E.g.  $\psi_1 + \psi_2 \in \mathbb{C}^2$  would have to correspond to  $\mathbf{R}_1 + \mathbf{R}_2$ . But in  $SO(3)$ ,  $R_1 + R_2$  would be a function that transforms  $\mathbf{r} \in \mathbb{R}^3$  into  $R_1(\mathbf{r}) + R_2(\mathbf{r})$ , while in  $SU(2)$  it would correspond to a different function that is up to a multiplication constant equal to  $S : \mathbf{r} \rightarrow S(\mathbf{r}) = R_1(\mathbf{r}) + R_2(\mathbf{r}) + 2[\cos(\varphi_1/2)\cos(\varphi_2/2) - \mathbf{s}_1 \cdot \mathbf{s}_2]\mathbf{r} + 2\sin(\varphi_1/2)\sin(\varphi_2/2)[(\mathbf{s}_2 \cdot \mathbf{r})\mathbf{s}_1 + (\mathbf{s}_1 \cdot \mathbf{r})\mathbf{s}_2] + 2\sin(\varphi_1/2)\cos(\varphi_2/2)(\mathbf{s}_1 \wedge \mathbf{r}) + 2\cos(\varphi_1/2)\sin(\varphi_2/2)(\mathbf{s}_2 \wedge \mathbf{r})$ . The multiplication constant must be chosen such that  $|\mathbf{r}|$  is preserved. Such extensions by linear combinations for spinors from a manifold to an embedding vector space are therefore mindless algebra with a spurious, conceptually impenetrable geometrical counterpart. This should not surprise anybody because the group axioms only define products of group elements, not linear combinations of them. Redefining spinors as vectors of a Hilbert space is therefore a purely formal, would-be scholar generalization.

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## 1 Introduction

We have explained based on Eq. 4 on p. 7 of [1] that a spinor of  $SU(2)$  is the first column of a rotation matrix of  $SU(2)$ , and as such can be considered as a shorthand notation for a rotation matrix. It therefore also represents a rotation around the origin of  $\mathbb{R}^3$ . The group  $SO(3)$  of rotations around the origin of  $\mathbb{R}^3$  forms a three-dimensional manifold, because a rotation is defined by three independent real parameters, e.g. by the Euler angles  $(\alpha, \beta, \gamma)$  or by  $(\mathbf{s}, \varphi)$ , where  $\varphi$  is the rotation angle around the rotation axis defined by the unit vector  $\mathbf{s}$ . A linear combination of two rotations is not defined by the group axioms, only the product of two group elements is defined. Therefore, also the linear combination of two  $SU(2)$  matrices or of two spinors is geometrically not defined. But these spinors belong to  $\mathbb{C}^2$  which is a two-dimensional complex vector space, equivalent to the four-dimensional real vector space  $\mathbb{R}^4$ . This leads to the speculation that by definition every element of  $\mathbb{C}^2$  could be called a spinor and that  $\forall(c_1, c_2) \in \mathbb{C}^2$ ,  $c_1\psi_1 + c_2\psi_2$ , would be defined in the vector space  $\mathbb{C}^2$  [2, 3, 4]. These operations are indeed algebraically defined, but this is not the end of the story. The problem is to give the algebraic result a geometrical meaning. When a curved manifold  $M$  is embedded in a vector space  $V$ , the geometrical meaning one would have to attribute to the points of the set  $V \setminus M$  may not make sense. This is a well-known mathematical fact (see below), which is overlooked in such speculations.

It is actually even overlooked in the mathematical definitions one can find in most texts, in the style: “In the mathematical field of representation theory, group representations describe abstract groups in terms of bijective linear transformations (i.e. automorphisms) of vector spaces.” Perhaps, this definition should be tweaked. We would then need also the discussion in Section 2.5.2 of [1] to justify certain calculations of group representation theory where linear combinations are nevertheless used, e.g. the calculation of Casimir operators. In fact, on p.7 of his monograph [5], Sagan states clearly that the linear combinations are *purely formal*. Instead of a group ring [6], Sagan calls the set of these purely formal linear combinations a  $G$ -module, and the corresponding calculus the group algebra. The lack of geometrical meaning is something a mathematician should not be satisfied with, and even less a physicist because he applies the formalism to the real world.

The mathematical fact that spinors do not form a vector space flies in the face of what one can read elsewhere (see e.g. [2], which ignores the fact that the calculations are purely formal) and of the very existence of a monograph written by Dirac with the title “Spinors in Hilbert space”. Accordingly, in the eyes of a physicist, our remark seems to have all odds against it. Some physicists therefore immediately declare it anathema: The statement that spinors do

not form a vector space *very obviously has to be wrong*. They even refuse to read [1] where this mathematical fact is proved [7,8]. This should not refrain us from stoically pointing out the truth. Traditional quantum mechanics (QM) is teeming with such misinterpretations of the mathematics due to the way physicists use these mathematics as a blackbox according to the leitmotiv: “*Shut up and calculate!*” There is no pride to be drawn from such an expeditious slogan. In Section 4 we will provide a whole laundry list of such flawed interpretations.

Throughout this document the notation  $F(A, B)$  will be used for the functions whose definition domain is the set  $A$  and which take values in the set  $B$ . By  $L(A, B)$  we will design the linear mappings from the  $A$  to the set  $B$ .

## 2 The points of the vector space which do not belong to the embedded manifold

Column matrices representing group elements are not proverbial rare birds. We encounter them in every regular representation of a finite group. Let us consider e.g. the permutation group  $S_n$ . We can label arbitrarily each of the  $n!$  permutations  $p_j \in S_n$  with a number  $j \in [1, n!] \cap \mathbb{N}$ . The order the group elements acquire this way has no importance. Any order will do. The regular representation is then given by  $n! \times n!$  representation matrices and each group element  $p_j$  is also represented by a  $n! \times 1$  matrix  $\mathbf{p}^{(j)}$ , whose entries are  $[\mathbf{p}^{(j)}]_k = \delta_{kj}$ . That is, all entries take the value 0 except the one on line  $j$ , which takes the value 1. The square matrix representing a group element  $p_j$  just represents  $p_j$  by the group automorphism:  $T_{p_j} : q \in S_n \rightarrow T_{p_j}(q) = p_j \circ q \in S_n$ . We could call the  $n! \times 1$  column matrices  $\mathbf{p}^{(j)}$  “column vectors” and they would span a “vector space”  $(V, \mathbb{K})$  over some number field  $\mathbb{K}$  that could be  $\mathbb{R}$  or  $\mathbb{C}$ . The group  $S_n$  would then be a finite discrete subset (of  $n!$  points) of the vector space  $(V, \mathbb{K}) = (\mathbb{K}^{n!}, \mathbb{K})$ . The  $n!$  column vectors constitute an orthonormal basis for  $(V, \mathbb{K})$ . But this is just shallow nonsense, because in the representation the sum of two such  $n! \times 1$  column matrices  $\mathbf{p}$  and  $\mathbf{q}$  would by isomorphism correspond to:

$$\begin{pmatrix} 1 & 2 & \cdots & k & \cdots & n \\ p(1) & p(2) & \cdots & p(k) & \cdots & p(n) \end{pmatrix} + \begin{pmatrix} 1 & 2 & \cdots & k & \cdots & n \\ q(1) & q(2) & \cdots & q(k) & \cdots & q(n) \end{pmatrix}. \quad (1)$$

Good luck to anyone who wants to make sense of this. It is very obvious that this operation is just not defined, and the same applies for any other linear combination  $\sum_{j=1}^{n!} c_j \mathbf{p}^{(j)}$ , with  $c_j \in \mathbb{K}, \forall j \in [1, n!] \cap \mathbb{N}$ , that does not belong to  $S_n$ . All points of  $\mathbb{K}^{n!} \setminus S_n$  are *a priori* meaningless. This example illustrates the pitfalls of heedlessly carrying out algebraic calculations without bothering what they mean, as implied by the motto: “*Shut up and calculate!*”.

It is for the same reason that in general relativity the curved space-time manifold should not be considered as embedded in a vector space, but described *intrinsically*. The points you would have to add to obtain an extension in the form of a vector space wherein the curved space-time manifold could be embedded do not exist physically. That vector space would have to be  $\mathbb{R}^5$ , just like the two-dimensional surface of a sphere is embedded in  $\mathbb{R}^3$ . The points of the extension to  $\mathbb{R}^5$  that do not belong to space-time would just be physically meaningless. It is in order to avoid such utter nonsense and to describe accordingly space-time *intrinsically* that we need a whole artillery of concepts from differential geometry like manifolds, Riemann and Ricci tensors, curvilinear coordinates, covariant derivatives and parallel transport.

Within the representation  $\text{SO}(3) \subset L(\mathbb{R}^3, \mathbb{R}^3)$  it is possible to attribute a meaning to the sum of two rotations  $R_1 + R_2$  because we can figure out the result of the action of  $R_1 + R_2$  on a general vector  $\mathbf{r} \in \mathbb{R}^3$ . In fact  $R_1(\mathbf{r}) + R_2(\mathbf{r})$  is a sum of vectors, and we know what this means, such that it can be used to define  $R_1 + R_2$ . But the resulting definition does not introduce a very enlightening geometrical concept. It does not correspond to a Gestalt. A similar approach for the sum of two  $\text{SU}(2)$  matrices  $\mathbf{R}_1, \mathbf{R}_2$  and a general spinor  $\psi$  yields  $\mathbf{R}_1\psi + \mathbf{R}_2\psi$  which is a sum of spinors. This makes it impossible to figure out the geometrical meaning of  $\mathbf{R}_1 + \mathbf{R}_2$  for  $\mathbf{R}_j \in \text{SU}(2)$  without knowing the geometrical meaning of sums of spinors.<sup>1</sup> But these spinors  $\psi_j$  are the first columns of the matrices  $\mathbf{R}_j$ , such that to figure out their meaning we must know the meaning of  $\mathbf{R}_1 + \mathbf{R}_2$ . If we wanted to figure out what the sum  $\psi_1 + \psi_2$  of two spinors means, we could also try to calculate  $\mathbf{R}\psi_1 + \mathbf{R}\psi_2$ , but this is again a sum of two spinors. Whatever we try to make sense of such sums, we end up running in circles. In summary, we cannot figure out what the sum of two spinors means because we cannot figure out what the sum of two rotation matrices means, and we cannot figure out what the sum of two rotation matrices means because we cannot figure out what the sum of two spinors means. What we did for  $\text{SO}(3)$  can this way not be transposed to  $\text{SU}(2)$ . We cannot solve the problem in  $\text{SU}(2)$  with the method we used in  $\text{SO}(3)$ , because the column vectors the rotation matrices are operating on do not correspond to vectors but to group elements, which form a manifold rather than a vector space. To make sense of sums of rotation matrices or spinors in  $\text{SU}(2)$  we must therefore try again to figure out how they operate on vectors  $\mathbf{r} \in \mathbb{R}^3$ , but the problem is that the calculations to be performed are now no longer linear but “quadratic”, in the sense to be developed below, starting from Eq. 7.

<sup>1</sup> In reality  $\mathbf{R}_1, \mathbf{R}_2$  must here be considered as elements of  $F(\text{SU}(2), \text{SU}(2))$ , more precisely of the group of group automorphisms, which is isomorphic to  $\text{SU}(2)$ .

Consider now the complex vector space  $\mathbb{C}^2$  and a general point  $(\zeta_1, \zeta_2) \in \mathbb{C}^2$  of it. Until something is done about it, such a point has no obvious physical meaning. A first step consists in pointing out that the spinors  $\psi = [\zeta_1, \zeta_2]^\top$  of  $SU(2)$ , which (as explained on p.7 of [1]) can be given the meaning of rotations around the origin in  $\mathbb{R}^3$ , belong to the set  $\mathcal{C} = \{(\zeta_1, \zeta_2) \mid \zeta_1^* \zeta_1 + \zeta_2^* \zeta_2 = 1\} \subset \mathbb{C}^2$ , such that  $SU(2) \subset \mathcal{C}$ . These spinors are isometries, i.e. special elements of the vector space  $L(\mathbb{R}^3, \mathbb{R}^3)$ , which conserve the metric. They are geometrical/physical operators. Each rotation corresponds to two spinors which are identical up to a factor  $\pm 1$ . Each rotation defines therefore two points of the curved manifold  $\mathcal{C}$ , but the converse is also true: each such pair of points of  $\mathcal{C}$  corresponds to a rotation, i.e. a pair of spinors of  $SU(2)$ , which is a double covering of  $SO(3)$  due to the very existence of these two possible factors  $\pm 1$ . The simplest way to prove this is to identify  $(\zeta_1, \zeta_2)$  with the expression for a spinor that corresponds to the rotation  $R(\alpha, \beta, \gamma)$  where  $(\alpha, \beta, \gamma)$  are its Euler angles, as e.g. given by the first column of:

$$\mathbf{R}(\alpha, \beta, \gamma) = \begin{bmatrix} e^{-i(\alpha+\gamma)/2} \cos \frac{\beta}{2} & -ie^{-i(\alpha-\gamma)/2} \sin \frac{\beta}{2} \\ -ie^{i(\alpha-\gamma)/2} \sin \frac{\beta}{2} & e^{i(\alpha+\gamma)/2} \cos \frac{\beta}{2} \end{bmatrix}, \quad (2)$$

in Eq. 1.2.29 of [2]. The definition of the Euler angles used here is defined in Fig. 1.5 of [2]. Therefore we have also  $\mathcal{C} \subset SU(2)$ , such that  $\mathcal{C} \equiv SU(2)$ . We can consider the manifold  $\mathcal{C}$  as embedded in  $\mathbb{C}^2$ . In terms of real numbers, the rotation group, represented by  $\mathcal{C}$ , is then a three-dimensional manifold embedded in the four-dimensional vector space  $\mathbb{C}^2 \equiv \mathbb{R}^4$ . This is analogous to four-dimensional space-time embedded in  $\mathbb{R}^5$  as explained above, such that analogous caveats must prevail.

It is of course algebraically feasible to calculate linear combinations  $c_1\psi_1 + c_2\psi_2$ , where  $(c_1, c_2) \in \mathbb{C}^2$ , or to consider elements of  $\mathbb{C}^2 \setminus \mathcal{C}$  but this is purely formal and *a priori* devoid of any geometrical meaning in terms of some element of  $L(\mathbb{R}^3, \mathbb{R}^3)$ , which is the natural embedding for the rotation group  $SO(3) \subset L(\mathbb{R}^3, \mathbb{R}^3)$  [1]. What other kind of embedding of  $SO(3)$  could we else imagine to give  $(\zeta_1, \zeta_2) \in \mathbb{C}^2 \setminus \mathcal{C}$  meaning? This argument is similar to the one for  $S_n$  above. However, this time the group is a continuous Lie group and therefore no longer a discrete finite set but a differentiable manifold.

### 3 Attempt to attribute a meaning to the sum of two spinors of $SU(2)$ within $L(\mathbb{R}^3, \mathbb{R}^3)$

We can further illustrate that the meaning we would have to attribute to a sum of two spinors of  $SU(2)$  in  $L(\mathbb{R}^3, \mathbb{R}^3)$  is spurious, by making the following calculation on vectors anticipated above. Let us consider two rotations  $R_j(\mathbf{s}_j, \varphi_j)$ ,  $j \in \{1, 2\}$ , and a vector  $\mathbf{r} \in \mathbb{R}^3$ . Here  $\varphi_j$  are the rotation angles around the rotation axes  $\ell_j$  defined by the unit vectors  $\mathbf{s}_j \parallel \ell_j$ . The  $SU(2)$  representation matrices of  $R_j(\mathbf{s}_j, \varphi_j)$  are given by  $\mathbf{R}_j = \cos(\varphi/2)\mathbb{1} - i\sin(\varphi/2)[\mathbf{s}_j \cdot \boldsymbol{\sigma}]$ . The corresponding spinors  $\psi_j$  are obtained by taking the first columns of  $\mathbf{R}_j$ . The vector  $\mathbf{r}$  is represented by  $[\mathbf{r} \cdot \boldsymbol{\sigma}]$ .

We know that for a matrix  $\mathbf{R}$  of  $SU(2)$  its inverse is given by:  $\mathbf{R}^{-1} = \mathbf{R}^\dagger$ . In fact, the inverse of a general  $2 \times 2$  matrix:

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{with: } D = \det(\mathbf{M}) = ad - bc, \quad (3)$$

exists when  $D \neq 0$  and is then obtained by first calculating the matrix of the minors, subsequently transposing it and finally dividing it by its determinant. This yields:

$$\mathbf{M}^{-1} = \frac{1}{D} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \quad (4)$$

Applying this to a matrix of the form:

$$\mathbf{S} = \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}, \quad \text{with } (u, v) \in \mathbb{C}^2, \quad \det(\mathbf{S}) = uu^* + vv^* \in \mathbb{R}, \quad (5)$$

one obtains:

$$\mathbf{S}^{-1} = \frac{1}{uu^* + vv^*} \begin{bmatrix} u^* & v^* \\ -v & u \end{bmatrix} = \frac{1}{uu^* + vv^*} \begin{bmatrix} u & -v^* \\ v & u^* \end{bmatrix}^\dagger. \quad (6)$$

For matrices  $\mathbf{R} \in SU(2)$ , we have  $uu^* + vv^* = 1$ , such that then  $\mathbf{R}^{-1} = \mathbf{R}^\dagger$ . For a sum  $\mathbf{R}_1 + \mathbf{R}_2$  of two  $SU(2)$  matrices,  $\det(\mathbf{R}_1 + \mathbf{R}_2) \in \mathbb{R}$  will not be equal to 1. But to calculate the inverse of  $\mathbf{R}_1 + \mathbf{R}_2$  it suffices to calculate  $[\mathbf{R}_1 + \mathbf{R}_2]^\dagger$  and to divide the result by  $\det[\mathbf{R}_1 + \mathbf{R}_2]$ , provided  $D = \det[\mathbf{R}_1 + \mathbf{R}_2] \neq 0$ . Otherwise the inverse will not be defined, which will happen if and only if  $u = v = 0$ , i.e.  $\mathbf{R}_2 = -\mathbf{R}_1$ .

Let us calculate the effect of the sum  $\mathbf{R}_1 + \mathbf{R}_2$  on  $[\mathbf{r} \cdot \boldsymbol{\sigma}]$ . The original idea behind the speculation is that all elements of  $\mathbb{C}^2$  should be meaningful spinors. They therefore should be the first columns of  $2 \times 2$  matrices which work on vectors  $\mathbf{r} \in \mathbb{R}^3$  in the same way as the rotation matrices of  $\text{SU}(2)$ . Hence  $\psi_1 + \psi_2$  corresponds to  $\mathbf{R}_1 + \mathbf{R}_2$  which works on  $[\mathbf{r} \cdot \boldsymbol{\sigma}]$  according to:

$$[\mathbf{R}_1 + \mathbf{R}_2][\mathbf{r} \cdot \boldsymbol{\sigma}][\mathbf{R}_1 + \mathbf{R}_2]^{-1} = \frac{1}{D}[\mathbf{R}_1 + \mathbf{R}_2][\mathbf{r} \cdot \boldsymbol{\sigma}][\mathbf{R}_1 + \mathbf{R}_2]^\dagger \quad (7)$$

Note that for  $\mathbf{R} \in \text{SU}(2)$ , it is the rule  $[\mathbf{r} \cdot \boldsymbol{\sigma}] \rightarrow \mathbf{R}[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}^{-1}$  which is primal, while the rule  $[\mathbf{r} \cdot \boldsymbol{\sigma}] \rightarrow \mathbf{R}[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}^\dagger$  is derived and relies on the specificity  $\mathbf{R}^{-1} = \mathbf{R}^\dagger$  which is no longer valid for  $\mathbf{R}_1 + \mathbf{R}_2$ . Thus, the result must really be calculated according to Eq. 7, not according to Eq. 8 below. Let us now first calculate the result of the transformation:

$$[\mathbf{R}_1 + \mathbf{R}_2][\mathbf{r} \cdot \boldsymbol{\sigma}][\mathbf{R}_1 + \mathbf{R}_2]^\dagger = \underbrace{\mathbf{R}_1[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}_1^\dagger}_{R_1(\mathbf{r})} + \underbrace{\mathbf{R}_2[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}_2^\dagger}_{R_2(\mathbf{r})} + \underbrace{\mathbf{R}_1[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}_2^\dagger}_{T_3} + \underbrace{\mathbf{R}_2[\mathbf{r} \cdot \boldsymbol{\sigma}]\mathbf{R}_1^\dagger}_{T_4}. \quad (8)$$

As already mentioned, this action is no longer linear but “quadratic” or rank-2.

The term  $T_4$  can be obtained from the term  $T_3$  by carrying out the substitution  $(1, 2)|(2, 1)$  on all indices. It suffices therefore to calculate  $T_3$ . Using the algebraic identity  $[\mathbf{a} \cdot \boldsymbol{\sigma}][\mathbf{b} \cdot \boldsymbol{\sigma}] = (\mathbf{a} \cdot \mathbf{b})\mathbb{1} + \imath[(\mathbf{a} \wedge \mathbf{b}) \cdot \boldsymbol{\sigma}]$ , this yields:

$$\begin{aligned} T_3 &= \{ \cos(\varphi_1/2)\mathbb{1} - \imath \sin(\varphi_1/2)[\mathbf{s}_1 \cdot \boldsymbol{\sigma}] \} [\mathbf{r} \cdot \boldsymbol{\sigma}] \{ \cos(\varphi_2/2)\mathbb{1} + \imath \sin(\varphi_2/2)[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] \} = \\ &\{ \cos(\varphi_1/2)[\mathbf{r} \cdot \boldsymbol{\sigma}] - \imath \sin(\varphi_1/2)(\mathbf{s}_1 \cdot \mathbf{r})\mathbb{1} + \sin(\varphi_1/2)[(\mathbf{s}_1 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] \} \{ \cos(\varphi_2/2)\mathbb{1} + \imath \sin(\varphi_2/2)[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] \}. \end{aligned} \quad (9)$$

This equation contains a scalar and a vector term. The scalar term of  $T_3$  is:

$$- \imath \sin(\varphi_1/2) \cos(\varphi_2/2)(\mathbf{s}_1 \cdot \mathbf{r}) + \imath \sin(\varphi_2/2) \cos(\varphi_1/2)(\mathbf{s}_2 \cdot \mathbf{r}) + \imath \sin(\varphi_1/2) \sin(\varphi_2/2)(\mathbf{s}_2 \cdot (\mathbf{s}_1 \wedge \mathbf{r})). \quad (10)$$

The substitution  $(1, 2)|(2, 1)$  in the indices yields for the corresponding scalar term within  $T_4$ :

$$- \imath \sin(\varphi_2/2) \cos(\varphi_1/2)(\mathbf{s}_2 \cdot \mathbf{r}) + \imath \sin(\varphi_1/2) \cos(\varphi_2/2)(\mathbf{s}_1 \cdot \mathbf{r}) + \imath \sin(\varphi_2/2) \sin(\varphi_1/2)(\mathbf{s}_1 \cdot (\mathbf{s}_2 \wedge \mathbf{r})). \quad (11)$$

The sum of these two scalar terms is zero. In fact, the mixed products can be written under the form of determinants, and these two determinants are obtained one from another by exchanging two lines. Because the scalar terms vanish,  $T_3 + T_4$  is a true vector. The vector part of term  $T_3$  is:

$$\begin{aligned} &\cos(\varphi_1/2) \cos(\varphi_2/2)[\mathbf{r} \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \sin(\varphi_2/2)(\mathbf{s}_1 \cdot \mathbf{r})[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \cos(\varphi_2/2)[(\mathbf{s}_1 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] \\ &- \cos(\varphi_1/2) \sin(\varphi_2/2)[(\mathbf{r} \wedge \mathbf{s}_2) \cdot \boldsymbol{\sigma}] - \sin(\varphi_1/2) \sin(\varphi_2/2)[((\mathbf{s}_1 \wedge \mathbf{r}) \wedge \mathbf{s}_2) \cdot \boldsymbol{\sigma}]. \end{aligned} \quad (12)$$

This can be rewritten as:

$$\begin{aligned} &\cos(\varphi_1/2) \cos(\varphi_2/2)[\mathbf{r} \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \sin(\varphi_2/2)(\mathbf{s}_1 \cdot \mathbf{r})[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \cos(\varphi_2/2)[(\mathbf{s}_1 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] \\ &+ \cos(\varphi_1/2) \sin(\varphi_2/2)[(\mathbf{s}_2 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \sin(\varphi_2/2)[(\mathbf{s}_2 \wedge (\mathbf{s}_1 \wedge \mathbf{r})) \cdot \boldsymbol{\sigma}]. \end{aligned} \quad (13)$$

Now we can use the identity  $\mathbf{a} \wedge (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$  to rewrite this as:

$$\begin{aligned} &\cos(\varphi_1/2) \cos(\varphi_2/2)[\mathbf{r} \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \sin(\varphi_2/2)(\mathbf{s}_1 \cdot \mathbf{r})[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \cos(\varphi_2/2)[(\mathbf{s}_1 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] \\ &+ \cos(\varphi_1/2) \sin(\varphi_2/2)[(\mathbf{s}_2 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] + \sin(\varphi_1/2) \sin(\varphi_2/2)((\mathbf{s}_2 \cdot \mathbf{r})[\mathbf{s}_1 \cdot \boldsymbol{\sigma}] - (\mathbf{s}_2 \cdot \mathbf{s}_1)[\mathbf{r} \cdot \boldsymbol{\sigma}]). \end{aligned} \quad (14)$$

With the substitution  $(1, 2)|(2, 1)$  in the indices we obtain the corresponding term in  $T_4$ :

$$\begin{aligned} &\cos(\varphi_2/2) \cos(\varphi_1/2)[\mathbf{r} \cdot \boldsymbol{\sigma}] + \sin(\varphi_2/2) \sin(\varphi_1/2)(\mathbf{s}_2 \cdot \mathbf{r})[\mathbf{s}_1 \cdot \boldsymbol{\sigma}] + \sin(\varphi_2/2) \cos(\varphi_1/2)[(\mathbf{s}_2 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] \\ &+ \cos(\varphi_2/2) \sin(\varphi_1/2)[(\mathbf{s}_1 \wedge \mathbf{r}) \cdot \boldsymbol{\sigma}] + \sin(\varphi_2/2) \sin(\varphi_1/2)((\mathbf{s}_1 \cdot \mathbf{r})[\mathbf{s}_2 \cdot \boldsymbol{\sigma}] - (\mathbf{s}_1 \cdot \mathbf{s}_2)[\mathbf{r} \cdot \boldsymbol{\sigma}]). \end{aligned} \quad (15)$$

Summing the vector terms in  $T_3$  and  $T_4$  yields:

$$\begin{aligned} & 2 [\cos(\varphi_1/2) \cos(\varphi_2/2) - \mathbf{s}_1 \cdot \mathbf{s}_2] \mathbf{r} + 2 \sin(\varphi_1/2) \sin(\varphi_2/2) [(\mathbf{s}_2 \cdot \mathbf{r}) \mathbf{s}_1 + (\mathbf{s}_1 \cdot \mathbf{r}) \mathbf{s}_2] \\ & + 2 \sin(\varphi_1/2) \cos(\varphi_2/2) (\mathbf{s}_1 \wedge \mathbf{r}) + 2 \cos(\varphi_1/2) \sin(\varphi_2/2) (\mathbf{s}_2 \wedge \mathbf{r}). \end{aligned} \quad (16)$$

Therefore the transformation in Eq. 7 corresponds to the function  $S \in L(\mathbb{R}^3, \mathbb{R}^3) : \mathbf{r} \in \mathbb{R}^3 \rightarrow S(\mathbf{r})$  given by:

$$\begin{aligned} S(\mathbf{r}) = & R_1(\mathbf{r}) + R_2(\mathbf{r}) + 2 [\cos(\varphi_1/2) \cos(\varphi_2/2) - \mathbf{s}_1 \cdot \mathbf{s}_2] \mathbf{r} + 2 \sin(\varphi_1/2) \sin(\varphi_2/2) [(\mathbf{s}_2 \cdot \mathbf{r}) \mathbf{s}_1 + (\mathbf{s}_1 \cdot \mathbf{r}) \mathbf{s}_2] \\ & + 2 \sin(\varphi_1/2) \cos(\varphi_2/2) (\mathbf{s}_1 \wedge \mathbf{r}) + 2 \cos(\varphi_1/2) \sin(\varphi_2/2) (\mathbf{s}_2 \wedge \mathbf{r}). \end{aligned} \quad (17)$$

Let us now treat the division by  $D = \det[\mathbf{R}_1 + \mathbf{R}_2] \in \mathbb{R}$ , when  $D \neq 0$ . In this case  $D > 0$ , such that  $\sqrt{D} \in \mathbb{R}$ . The determinant of the matrix  $\mathbf{U} = \frac{1}{\sqrt{D}}[\mathbf{R}_1 + \mathbf{R}_2]$  is equal to 1. The matrix  $\mathbf{U}$  is therefore unitarian and:

$$[\mathbf{R}_1 + \mathbf{R}_2][\mathbf{r} \cdot \boldsymbol{\sigma}][\mathbf{R}_1 + \mathbf{R}_2]^{-1} = \sqrt{D} \mathbf{U}[\mathbf{r} \cdot \boldsymbol{\sigma}] \mathbf{U}^{-1} \frac{1}{\sqrt{D}} = \mathbf{U}[\mathbf{r} \cdot \boldsymbol{\sigma}] \mathbf{U}^{-1}. \quad (18)$$

The matrix  $\mathbf{U}$  is a rotation which preserves the length of  $\mathbf{r}$ . Therefore the operation  $[\mathbf{R}_1 + \mathbf{R}_2][\mathbf{r} \cdot \boldsymbol{\sigma}][\mathbf{R}_1 + \mathbf{R}_2]^{-1}$  also preserves the length. This means that this operation is nothing else than the rotation which transforms  $\mathbf{r}$  into  $|\mathbf{r}| S(\mathbf{r})/|S(\mathbf{r})|$ , provided  $S(\mathbf{r}) \neq \mathbf{0}$ . This would then be the transformation that corresponds to  $\psi = \psi_1 + \psi_2$ . The sum  $\psi = \psi_1 + \psi_2$  has this way been given a geometrical definition but it is completely abstruse. It is an elaboration of the calculation in Section 2.5.1 starting on p. 12 of [1], where we also introduced  $(\psi_1 + \psi_2)/|\psi_1 + \psi_2|$  provided  $\psi_1 + \psi_2 \neq 0$ . But now we have extended the scope to the action on vectors. We were wondering how we could justify introducing such a renormalization of  $\psi_1 + \psi_2$ . The present calculation gives the answer. It is a corollary of the agenda to call all elements of  $\mathbb{C}^2$  spinors, based on the assumption that spinors would constitute a vector space, whereby it turns out that the operator that corresponds to the sum  $\psi_1 + \psi_2$  of two spinors has the same action on a vector as these renormalized sums  $(\psi_1 + \psi_2)/|\psi_1 + \psi_2|$ .

The first two terms in  $S(\mathbf{r})$  correspond to  $R_1(\mathbf{r})$  and  $R_2(\mathbf{r})$ . Their sum corresponds to what we expect on the basis of the representation  $\text{SO}(3)$ . But in  $\text{SU}(2)$  there are extra terms and the sum of these terms is in general not identical to zero, as is easily checked. The result in  $\text{SU}(2)$  will in general be different from the result in  $\text{SO}(3)$ , be it only because the result in  $\text{SO}(3)$  does not preserve the length of the vector  $\mathbf{r}$  while the result in  $\text{SU}(2)$  does. But the vectors will in general also not be parallel. The finding that the value we must attribute to the sum of two group elements would depend on the choice of the representation gives a first inkling of the fact that the algebraic procedure of summing spinors or group elements does not necessarily define a meaningful geometrical result. We can compare it to a definition that would depend on the choice of a reference frame. The elaboration of  $|\psi_1 + \psi_2|^2 = |\psi_1|^2 + |\psi_2|^2 + \psi_1 \psi_2^\dagger + \psi_2 \psi_1^\dagger$  contains also two extra terms  $\psi_1 \psi_2^\dagger, \psi_2 \psi_1^\dagger$  which lead to conceptual problems in the double-slit experiment for electrons [9], raising also questions about the procedure of summing spinors.

Furthermore, the extravagant result  $|\mathbf{r}| S(\mathbf{r})/|S(\mathbf{r})|$  is admittedly providing  $\psi_1 + \psi_2$  with a geometrical definition, but one that is just not suited for any use in physics because its meaning is unfathomable. The vector  $S(\mathbf{r})$  is as useful as a point of  $\mathbb{R}^5$  in general relativity. It all just looks like inscrutable nonsense, let alone what we would obtain for  $\psi = c_1 \psi_1 + c_2 \psi_2$ . And when  $\mathbf{R}_2 = -\mathbf{R}_1$  the result is definitely meaningless. Nevertheless, sums of spinors are used in QM with useful results, such that this requires an investigation. It is therefore explained in [1] that linear combinations of spinors correspond to sets, which gives a reasonable meaning to the (incoherent) sums of spinor wave functions [9], whereas coherent summing cannot be justified [9].

## 4 Appendix - An anthology of wrong interpretations of the mathematics in traditional QM

*“When the wise man points to the Moon, the fool inspects the finger.” (Confucius).*

The following is a non-exhaustive list of errors in the traditional theory of QM. Most of them are treated in [1, 10]. We have always tried to signal these errors with some restraint out of respect for the founding fathers of QM and for the feelings of the readers. But peer review is infested with bullying psychopaths and a few of them wrote contemptuous sham peer reviews on a paper I submitted to Symmetry. They were endorsed by all editors, of which only Alessandro Sergi was not anonymous. The following list of errors in traditional QM that went unnoticed for all those years indicates that, if only they had empathy and cared, such gatekeepers could perhaps be a bit less self-righteous

and a bit more humble:

- ☐ Dirac's delta function does not exist.
- ☐ The algebraic expressions  $\mathbf{a} \cdot \boldsymbol{\sigma}$  (and  $\mathbf{a} \cdot \boldsymbol{\gamma}$ ) are the representations of vectors  $\mathbf{a}$ , not some coupling of  $\mathbf{a}$  with  $\boldsymbol{\sigma}$  where  $\frac{\hbar}{2}\boldsymbol{\sigma}$  would be the spin. The quantity  $\boldsymbol{\sigma}$  does not contain the spin, but just represents the triad of basis vectors  $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ . This erroneous interpretation of a mere notation as a true scalar product is used in the treatment of the anomalous Zeeman effect and the definition of the helicity of the neutrino. The correct expression for spin axis is  $\mathbf{s} \cdot \boldsymbol{\sigma}$ .
- ☐ Similarly  $c\boldsymbol{\alpha}$  in the Dirac equation represents a triad of vectors, rather than the three components of a "velocity operator"  $c\boldsymbol{\alpha}$  in terms of a so-called *Zitterbewegung* (which does not exist).
- ☐ The Dirac equation cannot be used to define neutrinos as fermions that would travel at the speed of light, because its derivation starts from describing a fermion as a spinning particle in its rest frame.
- ☐ The velocity of the electron in the Dirac equation with phase  $(Et - \mathbf{p} \cdot \mathbf{r})/\hbar$  is already  $v < c$  such that it is not necessary to introduce wave packets based on the observation that the phase velocity of the wave function is  $c^2/v > c$ .
- ☐ There is no collapse of the wave function.
- ☐ The Dirac equation cannot be applied to an electron with a varying direction of its spin axis. It can thus not be used to describe precession.
- ☐ The operators in QM are defined by the group theory, not by a procedure of trial and error based on guessing.
- ☐ The particle-duality is a wrong concept. Single electrons are particles, an electron wave corresponds to a statistical ensemble of electrons.
- ☐ Schrödinger's cat does not correspond to a cat that is both dead and alive but to a statistical ensemble of cats, half of which are dead and half of which are alive.
- ☐ Negative energies in the Dirac equation do not correspond to antiparticles. The change of sign just corresponds to a change of sense in the spinning motion. When an electron and a positron annihilate it yields two gamma rays of 511 keV, rather than a zero total energy.
- ☐ The derivation of the expression for the Thomas precession by the Dirac theory is wrong, because the wave function is a mixed state of opposite spinning motions.
- ☐ The direction of spin is not quantized.
- ☐ The traditional "derivation" of the Dirac equation is carried out within the exterior algebra while the correct derivation, which permits to understand what is going on behind the scenes, is made within the algebra of the group elements.
- ☐ The linearity of the Schrödinger, Pauli and Dirac equations does not imply that the wave functions can be added.
- ☐ The Heisenberg uncertainty principle is just a mathematical consequence of the way we define the wave functions.
- ☐ The probabilities that intervene in the Bell inequalities are conditional probabilities defined with respect to different contexts, i.e. different experimental set-ups with different polarizer settings (see [11] and the many references therein).

## References

1. G. Coddens, *Symmetry*, **13**, 659 (2021).
2. J. Hladik, J.M. Cole, in *Spinors in Physics*, (Springer, New York, NY, USA, 2012).
3. Anonymous on <https://physics.stackexchange.com/questions/636734/do-spinors-form-a-vector-space>.  
"Honestly the only way I can see this making some sense is if he uses some non-standard definition of spinors. In the conventional story spinors are by definition elements of a representation space of the spin group. In that case they form a vector space by definition".
4. Anonymous on <https://physics.stackexchange.com/questions/636734/do-spinors-form-a-vector-space>.  
"I think that he might be thinking of the orbit that you get when you choose a fixed starting spinor  $\psi_0$  and consider the set you get by acting on it by all possible SU(2) matrices. This subset of spinors (the orbit of  $\psi_0$ ) is not a vector space, but that does not mean that the set of all spinors is not a vector space."
5. Sagan, B.E. in *The Symmetric Group. Representations, Combinatorial Algorithms, and Symmetric Functions*, Springer Graduate Texts in Mathematics, Volume **203**, 2nd ed.; Springer: New York, NY, USA, 2001.
6. B.L. van der Waerden, in *Group Theory and Quantum Mechanics*, Springer, Berlin-Heidelberg (1974), translated from *Die gruppentheoretische Methode in der Quantenmechanik*, Springer, Berlin (1932).
7. Anonymous referee report *Symmetry*: "Every scientific paper should be, in my view, self-contained. I am sorry but I cannot accept the argument that for reading a given paper I will have to read another one as an introduction."
8. Anonymous on <https://physics.stackexchange.com/questions/636734/do-spinors-form-a-vector-space>.  
"I am loath to spend time on what may be a mathematical statement at variance to conventional definitions."
9. G. Coddens, <https://hal-cea.archives-ouvertes.fr/cea-01383609v6>.
10. G. Coddens, *Symmetry* **13**,134 (2021).
11. M. Kupczynski, *Frontiers in Physics*, **8**, 273 (2020).