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# BROWNIAN SHEET AND TIME INVERSION FROM $G$ -ORBIT TO $L(G)$ -ORBIT

MANON DEFOSSEUX

ABSTRACT. We have proved in a previous paper that a space-time Brownian motion conditioned to remain in a Weyl chamber associated to an affine Kac–Moody Lie algebra is distributed as the radial part process of a Brownian sheet on the compact real form of the underlying finite dimensional Lie algebra, the radial part being defined considering the coadjoint action of a loop group on the dual of a centrally extended loop algebra. We present here a very brief proof of this result based on a time inversion argument and on elementary stochastic differential calculus.

## 1. INTRODUCTION

We propose here a short proof of the main result of [5]. Let us briefly recall this result. For this we need to consider a connected simply connected simple compact Lie group  $G$  and its Lie algebra  $\mathfrak{g}$  equipped with an invariant scalar product for the adjoint action of  $G$  on  $\mathfrak{g}$ . One considers a standard Brownian sheet  $\{x_{s,t}, s \in [0, 1], t \geq 0\}$  with values in  $\mathfrak{g}$  and for each  $t > 0$ , the process  $\{Y_{s,t}, s \in [0, 1]\}$  starting from the identity element of  $G$  and satisfying the stochastic differential equation (in  $s$ )

$$t dY_{s,t} = Y_{s,t} \circ dx_{s,t},$$

where  $\circ$  stands for the Stratonovitch integral. This is a  $G$ -valued process. The adjoint orbits in  $G$  are in correspondence with an alcove which is a fundamental domain for the action on a Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$  of the extended Weyl group associated the roots of  $G$ . We have proved in [5] that if for any  $t > 0$  one denotes by  $\mathcal{O}(Y_{1,t})$  the element in the alcove corresponding to the orbit of  $Y_{1,t}$  then the random process<sup>1</sup>

$$\{(t, t\mathcal{O}(Y_{1,t})) : t \geq 0\}$$

is a space-time brownian motion in  $\mathbb{R} \times \mathfrak{t}$  conditioned in Doob's sense to remain in a Weyl chamber which occurs in the framework of affine Kac–Moody algebras [8]. The proof of [5] rests on a Kirillov–Frenkel character formula [7] from which follows an intertwining relation between the transition probability semi-group of the Brownian sheet and the one of the conditioned process. Then a Rogers and Pitman's criteria [11] can be applied, which provides the result. The conditioned process obtained when  $G = \mathrm{SU}(2)$  plays a crucial role in [2] where a Pitman type theorem is proved for a real Brownian motion in the unit interval. Time inversion is a key ingredient to get the Pitman type theorem in this case. In the present communication a new proof of the main result of [5] is proposed, which rests on such a time inversion firstly and secondly on an elementary but nice property of the Brownian sheet on  $\mathfrak{g}$  and its wrapping on  $G$ .

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<sup>1</sup>with the convention that  $Y_{1,0}$  is the identity element of  $G$

The results presented are valuable for themselves rather than for their proofs which are rudimentary. We present them in section 2 before giving the precise definitions of the objects that they involve. The rest of the communication is organized as follows. In section 3 we recall the general framework of [5]. In particular we describe the coadjoint orbits of the loop group  $L(G)$  in the dual of the centrally extended loop algebra  $L(\mathfrak{g})$  and the Weyl chamber associated to such an infinite dimensional Lie algebra which is an affine Kac–Moody algebra. In section 4 we define the radial process associated to the Brownian sheet on  $\mathfrak{g}$  and recall the main theorem of [5]. In section 5 we define two Doob conditioned processes living respectively in an alcove or in an affine Weyl chamber, and prove that the two processes are equal up to a time inversion. Finally in section 6 we propose a brief proof of the main result of [5].

## 2. STATEMENT OF THE RESULTS

Let us fix  $\gamma$  in a fixed alcove associated to  $G$  and consider  $\{X_{s,t}^\gamma : s \in [0, 1], t \geq 0\}$  a random sheet with values in  $G$ , such that for any  $t \geq 0$ ,

$$\begin{cases} X_{s,t}^\gamma = X_{s,t} \circ d(x_{s,t} + \gamma s) \\ X_{0,t}^\gamma = e, \end{cases}$$

where  $e$  is the identity element of  $G$ . Then one has the three following statements, the second one being an immediate consequence of the first, and the last one being deduced from the second by a time inversion argument.

### Statement 1 :

The random process  $\{X_{1,t}^\gamma : t \geq 0\}$  is a standard Brownian motion on  $G$  starting from  $\exp(\gamma)$ .

### Statement 2 :

The random process  $\{\mathcal{O}(X_{1,t}^\gamma) : t \geq 0\}$  is a standard Brownian motion starting from  $\gamma$  conditioned to remain in the alcove.

### Statement 3 :

The radial part process  $\{\text{rad}(t\Lambda_0 + \int_0^1 (\cdot | d(x_{s,t} + \gamma st))) : t \geq 0\}$  is a space-time Brownian motion with drift  $\gamma$  conditioned to remain in an affine Weyl chamber.

## 3. LOOP GROUP AND ITS ORBITS

In this part we fix succinctly the general framework of the results. One can find more details in [5] and references therein for instance. Let  $G$  be a connected simply connected simple compact Lie group and  $\mathfrak{g}$  its Lie algebra equipped with a Lie bracket denoted by  $[\cdot, \cdot]_{\mathfrak{g}}$ . We choose a maximal torus  $T$  in  $G$  and denote by  $\mathfrak{t}$  its Lie algebra. By compactness we suppose without loss of generality that  $G$  is a matrix Lie group. We denote by  $\text{Ad}$  the adjoint action of  $G$  on itself or on its Lie algebra  $\mathfrak{g}$  which is equipped with an  $\text{Ad}(G)$ -invariant scalar product  $(\cdot | \cdot)$ . We consider the real vector space  $L(\mathfrak{g})$  of smooth loops defined on the unit circle  $S^1$  with values in  $\mathfrak{g}$ ,  $S^1$  being identified with  $[0, 1]$ . We equip  $L(\mathfrak{g})$  with an

$\text{Ad}(G)$ -invariant scalar product also denoted by  $(\cdot|\cdot)$  letting

$$(\eta|\xi) = \int_0^1 (\eta(s)|\xi(s)) ds, \quad \eta, \xi \in L(\mathfrak{g}).$$

Equipped with the Lie bracket  $[\cdot, \cdot]_{\mathfrak{g}}$  pointwise defined,  $L(\mathfrak{g})$  is a Lie algebra. We consider its central extension

$$\tilde{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{R}c,$$

equipped with a Lie bracket  $[\cdot, \cdot]$  defined by

$$(1) \quad [\xi + \lambda c, \eta + \mu c] = [\xi, \eta]_{\mathfrak{g}} + (\xi'|\eta)c,$$

for  $\xi, \eta \in L(\mathfrak{g})$ ,  $\lambda, \mu \in \mathbb{R}$ . We consider the fundamental weight  $\Lambda_0$  in  $\tilde{L}(\mathfrak{g})^*$  defined by

$$\Lambda_0(L(\mathfrak{g})) = 0, \quad \Lambda_0(c) = 1,$$

the set of smooth loops  $L(G)$  with values on  $G$  and its coadjoint action  $\text{Ad}^*$  on  $L(\mathfrak{g})^* \oplus \mathbb{R}\Lambda_0$ . It is defined by

$$(2) \quad \text{Ad}^*(\gamma)(\phi + \tau\Lambda_0) = [\gamma \cdot \phi - \tau(\gamma' \gamma^{-1}|\cdot)] + \tau\Lambda_0,$$

for  $\gamma \in L(G)$ ,  $\phi \in L(\mathfrak{g})^*$ ,  $\tau \in \mathbb{R}$ , where  $(\gamma \cdot \phi)(\cdot) = \phi(\gamma^{-1} \cdot \gamma)$ . We notice that the coordinate along  $\Lambda_0$  of a linear form in  $\tilde{L}(\mathfrak{g})^*$ , which is called the level of the linear form, is not affected by the coadjoint action.

**Coadjoint  $L(G)$ -orbit.** For  $\zeta$  in  $\tilde{L}(\mathfrak{g})^*$  we denote by  $\tilde{\mathcal{O}}_{\zeta}$  the coadjoint orbit  $\text{Ad}^*(L(G))\{\zeta\}$  in  $\tilde{L}(\mathfrak{g})^*$ . We have recalled in [5] that roughly speaking for  $\xi$  in  $\tilde{\mathcal{O}}_{\zeta}$ , provided that  $\zeta$  has a positive level, we find  $\gamma \in L(G)$  such that  $\xi = \text{Ad}^*(\gamma)(\zeta)$  solving a differential equation. Actually if  $\xi$  is written

$$(3) \quad \xi = \tau\Lambda_0 + \int_0^1 (\cdot|\dot{x}_s) ds,$$

with  $\dot{x} \in L(\mathfrak{g})$  and  $\tau > 0$ , then  $\xi$  is in  $\tilde{\mathcal{O}}_{\tau\Lambda_0 + (a|\cdot)}$  for  $a \in \mathfrak{t}$  if and only if the  $G$ -valued function  $\{X_s : s \in [0, 1]\}$  starting from the identity element of  $G$  and satisfying the differential equation

$$\tau dX = X dx,$$

satisfies  $X_1 \in \text{Ad}(G)\{\exp(a/\tau)\}$  (see [12] for details or [5] in which the results of [12] are recalled). Thus coadjoint orbits in the subspace of linear forms in  $\tau\Lambda_0 + L(\mathfrak{g})^*$  written like in (3) are in one-to-one correspondence with the adjoint  $G$ -orbits in  $G$ . In order to parametrize these orbits, it is more convenient to consider the real roots of  $G$  rather than the infinitesimal ones. One can find for instance in chapters 5 and 7 of [3] definitions and properties recalled in the next two paragraphs.

**Real roots.** We consider the complexified Lie algebra  $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}$  of  $\mathfrak{g}$  that we denote by  $\mathfrak{g}_{\mathbb{C}}$ . The set of real roots is

$$\Phi = \{\alpha \in \mathfrak{t}^* : \exists X \in \mathfrak{g}_{\mathbb{C}} \setminus \{0\}, \forall H \in \mathfrak{t}, [H, X] = 2i\pi\alpha(H)X\}.$$

Suppose that  $\mathfrak{g}$  is of rank  $n$  and choose a set of simple real roots

$$\Pi = \{\alpha_k, k \in \{1, \dots, n\}\}.$$

We denote by  $\Phi_+$  the set of positive real roots. The half sum of positive real roots is denoted by  $\rho$ . Letting for  $\alpha \in \Pi$ ,

$$\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} : \forall H \in \mathfrak{t}, [H, X] = 2i\pi\alpha(H)X\},$$

the coroot  $\alpha^{\vee}$  of  $\alpha \in \Phi$  is defined as the only vector of  $\mathfrak{t}$  in  $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{-\alpha}]$  such that  $\alpha(\alpha^{\vee}) = 2$ . For  $\alpha \in \Pi$ , one defines two transformations on  $\mathfrak{t}$ , the reflection  $s_{\alpha^{\vee}}$  and the translation  $t_{\alpha^{\vee}}$ , letting for  $x \in \mathfrak{t}$

$$s_{\alpha^{\vee}}(x) = x - \alpha(x)\alpha^{\vee} \quad \text{and} \quad t_{\alpha^{\vee}}(x) = x + \alpha^{\vee}.$$

One considers the Weyl group  $W^{\vee}$  and the group  $\Gamma^{\vee}$  respectively generated by reflections  $s_{\alpha^{\vee}}$  and translations  $t_{\alpha^{\vee}}$ , for  $\alpha \in \Pi$ , and the extended Weyl group  $\Omega$  generated by  $W^{\vee}$  and  $\Gamma^{\vee}$ . Actually  $\Omega$  is the semi-direct product  $W^{\vee} \ltimes \Gamma^{\vee}$ . A fundamental domain for its action on  $\mathfrak{t}$  is

$$A = \{x \in \mathfrak{t} : \forall \alpha \in \Phi_+, 0 \leq \alpha(x) \leq 1\}.$$

**Adjoint  $G$ -orbit.** The group  $G$  being simply connected, the conjugaison classes  $G/\text{Ad}(G)$  is in correspondence with the fundamental domain  $A$ . Actually for every  $u \in G$ , there exists a unique element  $x \in A$  such that  $u \in \text{Ad}(G)\{\exp(x)\}$ . For  $\tau \in \mathbb{R}_+$ , one defines the alcove  $A_{\tau}$  of level  $\tau$  by

$$A_{\tau} = \{x \in \mathfrak{t} : \forall \alpha \in \Phi_+, 0 \leq \alpha(x) \leq \tau\},$$

i.e.  $A_{\tau} = \tau A$ . In particular  $A_1 = A$ .

**Alcoves and coadjoint  $L(G)$ -orbit.** For a positive real number  $\tau$  and a linear form  $\xi \in \tilde{L}(\mathfrak{g})^*$  written as in (3) there is a unique element in  $a \in A_{\tau}$  such that

$$X_1 \in \text{Ad}(G)\{\exp(a/\tau)\}$$

where  $X = \{X_s : s \in [0, 1]\}$  starts from the identity element  $e$  of  $G$  and satisfies

$$\tau dX = X dx.$$

Discussion above ensures that the pair  $(\tau, a)$  determines the orbit of  $\xi$ . Thus coadjoint orbits in the subspace of linear forms in  $\mathbb{R}_+^* \Lambda_0 + L(\mathfrak{g})^*$  written like in (3) are in one-to-one correspondence with

$$\{(\tau, a) \in \mathbb{R}_+^* \times \mathfrak{t} : a \in A_{\tau}\}.$$

This last domain can be identified (if we add it  $(0, 0)$ ) with a Weyl Chamber associated to an affine Kac–Moody algebra as it is explained in the following paragraph.

**Affine Weyl chamber.** From now on the scalar product on  $\mathfrak{g}$  is normalized such that  $(\theta|\theta) = 2$ . We denote by  $\theta$  the highest real root and we let  $\alpha_0^\vee = c - \theta^\vee$ . We consider

$$\widehat{\mathfrak{h}} = \text{Vect}_{\mathbb{C}}\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee, d\} \text{ and } \widehat{\mathfrak{h}}^* = \text{Vect}_{\mathbb{C}}\{\alpha_0, \alpha_1, \dots, \alpha_n, \Lambda_0\},$$

where  $\alpha_0 = \delta - \theta$  and for  $i \in \{0, \dots, n\}$

$$\alpha_i(d) = \delta_{i0}, \quad \delta(\alpha_i^\vee) = 0, \quad \Lambda_0(\alpha_i^\vee) = \delta_{i0}, \quad \Lambda_0(d) = 0.$$

We let

$$\widehat{\Pi} = \{\alpha_i : i \in \{0, \dots, n\}\} \text{ and } \widehat{\Pi}^\vee = \{\alpha_i^\vee : i \in \{0, \dots, n\}\}.$$

Then  $(\widehat{\mathfrak{h}}, \widehat{\Pi}, \widehat{\Pi}^\vee)$  is a realization of a generalized Cartan matrix of affine type. These objects are studied in details in [8]. The following definitions mainly come from chapters 1 and 6. We consider the restriction of  $(\cdot|\cdot)$  to  $\mathfrak{t}$  and extend it to  $\widehat{\mathfrak{h}}$  by  $\mathbb{C}$ -linearity and by letting

$$(\mathbb{C}c + \mathbb{C}d|\mathfrak{t}) = 0, \quad (c|c) = (d|d) = 0, \quad (c|d) = 1.$$

Then the linear isomorphism

$$\begin{aligned} \nu : \widehat{\mathfrak{h}} &\rightarrow \widehat{\mathfrak{h}}^* \\ h &\mapsto (h|\cdot) \end{aligned}$$

identifies  $\widehat{\mathfrak{h}}$  and  $\widehat{\mathfrak{h}}^*$ . We still denote  $(\cdot|\cdot)$  the induced bilinear form on  $\widehat{\mathfrak{h}}^*$ . We record that

$$(\delta|\alpha_i) = 0, \quad i = 0, \dots, n, \quad (\delta|\delta) = 0, \quad (\delta|\Lambda_0) = 1.$$

Due to the normalization we have  $\nu(\theta^\vee) = \theta$  and  $(\theta^\vee|\theta^\vee) = 2$ . We define the affine Weyl group  $\widehat{W}$  as the subgroup of  $\text{GL}(\widehat{\mathfrak{h}}^*)$  generated by fundamental reflections  $s_\alpha, \alpha \in \widehat{\Pi}$ , defined by

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha, \quad \beta \in \widehat{\mathfrak{h}}^*.$$

The bilinear form  $(\cdot|\cdot)$  is  $\widehat{W}$ -invariant. The affine Weyl group  $\widehat{W}$  is equal to the semi-direct product  $W \times \Gamma$ , where  $W$  is the Weyl group of  $G$  generated by  $s_{\alpha_i}, i \in \{1, \dots, n\}$ , and  $\Gamma$  the group of translations  $t_\alpha, \alpha \in \nu(Q^\vee)$ , defined by

$$(4) \quad t_\alpha(\lambda) = \lambda + \lambda(c)\alpha - [(\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\lambda(c)]\delta, \quad \lambda \in \widehat{\mathfrak{h}}^*.$$

Identification of  $\widehat{\mathfrak{h}}$  and  $\widehat{\mathfrak{h}}^*$  via  $\nu$  allows to define an action of  $\widehat{W}$  on  $\widehat{\mathfrak{h}}$ . One lets  $wx = \nu^{-1}w\nu x$ , for  $w \in \widehat{W}, x \in \widehat{\mathfrak{h}}$ . Then the action of  $\widehat{W}$  on  $\Lambda_0 \oplus \mathfrak{t}^* \oplus \mathbb{R}\delta/\mathbb{R}\delta$  or  $d \oplus \mathfrak{t} \oplus \mathbb{R}c/\mathbb{R}c$  is identified to the one of  $\Omega$  on  $\mathfrak{t}$ . Moreover a fundamental domain for the action of  $\widehat{W}$  on the quotient space  $(\mathbb{R}_+\Lambda_0 + \mathfrak{t}^* + \mathbb{R}\delta)/\mathbb{R}\delta$  is

$$\{\lambda \in \mathbb{R}\Lambda_0 \oplus \mathfrak{t}^* : \lambda(\alpha^\vee) \geq 0, \alpha \in \widehat{\Pi}\},$$

and for  $\tau \geq 0, \tau\Lambda_0 + \phi_a$ , with  $\phi_a = (a|\cdot)$ , is in this fundamental domain if and only if  $a \in A_\tau$ . Then we consider the following domain which is identified with the fundamental affine Weyl chamber viewed in the quotient space

$$C_W = \{(\tau, x) \in \mathbb{R}_+ \times \mathfrak{t} : x \in A_\tau\}.$$

4. COADJOINT  $L(G)$ -ORBIT AND BROWNIAN MOTION

When  $\{x_s : s \in [0, 1]\}$  is a continuous semi-martingale with values in  $\mathfrak{g}$ , then for  $\tau > 0$  the stochastic differential equation

$$(5) \quad \tau dX = X \circ dx,$$

where  $\circ$  stands for the Stratonovitch integral, has a unique solution starting from  $e$ . Such a solution is a  $G$ -valued process, that we denote by  $\epsilon(\tau, x)$  [9],[10]. This is the Stratonovitch stochastic exponential of  $\frac{x}{\tau}$ . The previous discussion leads naturally to the following definition.

**Definition 4.1.** For  $\tau \in \mathbb{R}_+^*$ , and  $x = \{x_s : s \in [0, 1]\}$  a  $\mathfrak{g}$ -valued continuous semi-martingale, we defines the radial part of  $\tau\Lambda_0 + \int_0^1 (\cdot | dx_s)$  that we denote by  $rad(\tau\Lambda_0 + \int_0^1 (\cdot | dx_s))$  by<sup>2</sup>

$$rad(\tau\Lambda_0 + \int_0^1 (\cdot | dx_s)) = (\tau, a),$$

where  $a$  is the unique element in  $A_\tau$  such that

$$\epsilon(\tau, x)_1 \in Ad(G)\{\exp(a/\tau)\}.$$

We have proved in [5] the following theorem, where the conditioned space-time Brownian motion is the one defined in section 5.2. This is this theorem for which we propose a new proof.

**Theorem 4.2.** If  $\{x_{s,t} : s \in [0, 1], t \geq 0\}$  is a Brownian sheet with values in  $\mathfrak{g}$  such that for any  $a, b \in \mathfrak{g}$ ,  $s_1, s_2 \in [0, 1]$ ,  $t_1, t_2 \in \mathbb{R}_+^*$ ,

$$\mathbb{E}((a|x_{s_1,t_1})(b|x_{s_2,t_2})) = \min(s_1, s_2) \min(t_1, t_2)(a|b),$$

then

$$\{rad(t\Lambda_0 + \int_0^1 (\cdot | dx_{s,t})) : t \geq 0\}$$

is a space-time Brownian motion in  $\mathbb{R} \times \mathfrak{t}$  conditioned to remain in the affine Weyl chamber  $C_W$ .

## 5. CONDITIONED BROWNIAN MOTIONS

In whole the communication, when we write  $f_t(x) \propto g_t(x)$  for  $f_t(x), g_t(x) \in \mathbb{C}$ , we mean that  $f_t(x)$  and  $g_t(x)$  are equal up to a multiplicative constant independent of the parameters  $t$  and  $x$ .

**5.1. A Brownian motion conditioned to remain in an alcove.** There is a common way to construct a Brownian motion conditioned in Doob sense to remain in an alcove, which is to consider at each time the  $Ad(G)$ -orbit of a brownian motion in  $G$ . The brownian motion on  $G$  is left Levy process. Its transition probability densities  $(\mathbf{p}_s)_{s \geq 0}$  with respect to the Haar measure on  $G$  can be expanded as a sum of characters of highest-weight complex representations of  $G$ . These representations are in correspondence with

$$P_+ = \{\lambda \in \mathfrak{t}^* : \lambda(\alpha_i^\vee) \in \mathbb{N}, i \in \{0, \dots, n\}\}.$$

<sup>2</sup>We do not specify in which space lives this distribution. We use this notation here just to keep track of the fact that when  $x$  is a Brownian motion the Wiener measure provides a natural measure on a coadjoint orbit in the original work of I. B. Frenkel.

One has for  $s \geq 0$ ,  $\mathbf{u}, \mathbf{v} \in G$ ,

$$\mathbf{p}_s(\mathbf{u}, \mathbf{v}) = \mathbf{p}_s(e, \mathbf{u}^{-1}\mathbf{v}) = \sum_{\lambda \in P_+} \text{ch}_\lambda(e) \text{ch}_\lambda(\mathbf{u}^{-1}\mathbf{v}) e^{-\frac{s(2\pi)^2}{2}(\|\lambda+\rho\|^2 - \|\rho\|^2)},$$

where  $\text{ch}_\lambda$  is the character of the irreducible representation of highest weight  $\lambda$  (see for instance [6]). By the Weyl character formula one has<sup>3</sup> for  $h \in \mathfrak{t}$

$$(6) \quad \text{ch}_\lambda(e^h) = \frac{\sum_{w \in W} \det(w) e^{2i\pi \langle w(\lambda+\rho), h \rangle}}{\sum_{w \in W} \det(w) e^{2i\pi \langle w(\rho), h \rangle}}.$$

We let

$$\pi(h) = \prod_{\alpha \in \Phi_+} \sin \pi \alpha(h),$$

which is the denominator in (6). Such a process starting from  $\mathbf{u} \in G$  can be obtained considering a standard Brownian motion  $\{x_s : s \geq 0\}$  with values in  $\mathfrak{g}$ , and the solution  $\{X_s : s \geq 0\}$  of the stochastic differential equation

$$dX = X \circ dx$$

with initial condition  $X_0 = \mathbf{u}$ . Then  $\{X_s : s \geq 0\}$  is a standard Brownian motion on  $G$  starting from  $\mathbf{u}$ . If  $\mathbf{u} = \exp(\gamma)$  with  $\gamma \in A$  then the process  $\{r_s^\gamma : s \geq 0\}$  such that for any  $s \geq 0$ ,  $r_s^\gamma$  is the unique element in  $A$  such that

$$X_s \in \text{Ad}(G)\{\exp(r_s^\gamma)\},$$

is a Markov process starting from  $\gamma$  with transition probability densities  $(q_t)_{t \geq 0}$  with respect to the Haar measure on  $G$  given by

$$(7) \quad q_t(x, y) \propto \pi(y)^2 \sum_{\lambda \in P_+} \text{ch}_\lambda(e^{-x}) \text{ch}_\lambda(e^y) e^{-\frac{t(2\pi)^2}{2}(\|\rho+\lambda\|^2 - \|\rho\|^2)},$$

for  $t \geq 0$ ,  $x, y \in A$ . This is obtained integrating over an  $\text{Ad}(G)$ -orbit (see (4.3.3) in [7] for instance) and using the Weyl integration formula. This Markov process is actually a Brownian motion killed on the boundary of  $A$  conditioned never to die. In fact if we denote by  $(u_t)_{t \geq 0}$  the transition densities of the standard Brownian motion on  $\mathfrak{t}$  killed on the boundary of  $A$ , a reflection principle gives that for  $t > 0$ ,  $x, y \in A$ ,

$$(8) \quad u_t(x, y) = \sum_{w \in \Omega} \det(w) p_t(x, w(y)),$$

where  $p_t$  is the standard heat kernel on  $\mathfrak{t}$  and  $\det(w)$  is the determinant of the linear part of  $w$ . A Poisson summation formula (see [1] for general results, and [7] or [5] for this particular case) then shows that

$$(9) \quad q_t(x, y) \propto \frac{\pi(y)}{\pi(x)} e^{2\pi^2(\rho|\rho)t} u_t(x, y),$$

which is the transition probability of the killed Brownian motion conditioned in the sense of Doob to remain in  $A$ .

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<sup>3</sup>The presence of a factor  $2i\pi$  is due to the fact that we have considered the real roots rather than the infinitesimal ones.



**5.2. A space-time Brownian motion conditioned to remain in an affine Weyl chamber.** We define a space-time Brownian motion conditioned to remain in an affine Weyl chamber as it has been defined in [5] and also in [2] when  $G = \text{SU}(2)$ . It is defined as an  $h$ -process, with the help of an anti-invariant classical theta function. For  $\tau \in \mathbb{R}_+^*$ ,  $b \in \mathfrak{t}$ ,  $a \in A_\tau$ , we define  $\widehat{\psi}_b(\tau, a)$  by

$$(10) \quad \widehat{\psi}_b(\tau, a) = \frac{1}{\pi(\widehat{b})} \sum_{w \in \widehat{W}} \det(w) e^{\langle w(\tau\Lambda_0 + \phi_a), d+b \rangle}.$$

From now on we fix  $\gamma \in A$ . One considers a standard Brownian motion  $\{b_t : t \geq 0\}$  with values in  $\mathfrak{t}$ , the space-time Brownian motion  $\{B_t^\gamma = (t, b_t + \gamma t) : t \geq 0\}$ , and the stopping time  $T = \inf\{t \geq 0 : B_t^\gamma \notin C_W\}$ . One defines a function  $\Psi_\gamma$  on  $C_W$  by

$$(11) \quad \Psi_\gamma : (t, x) \in C_W \rightarrow e^{-(\gamma|x)} \psi_\gamma(t, x).$$

Identity (8) and decomposition  $\widehat{W} = W \times \Gamma$  implies that

$$(12) \quad \Psi_\gamma(t, x) \pi(\gamma) \propto t^{-n/2} u_{\frac{1}{t}}(\gamma, x/t) e^{\frac{t}{2} \|\gamma - x/t\|^2}$$

**Proposition 5.1.** *The function  $\Psi_\gamma$  is a constant sign harmonic function for the killed process  $\{B_{t \wedge T}^\gamma : t \geq 0\}$ , vanishing on the boundary of  $C_W$ .*

*Proof.* The fact that  $\Psi_\gamma$  is harmonic and satisfies the boundary conditions is clear from (11). It is non negative by (12).  $\square$

**Definition 5.2.** *We define  $\{A_t^\gamma = (t, a_t^\gamma) : t \geq 0\}$  as the killed process  $\{B_{t \wedge T}^\gamma : t \geq 0\}$  starting from  $(0, 0)$  conditioned in Doob's sense not to die, via the harmonic function  $\Psi_\gamma$ .*

More explicitly, if we let for  $t \geq 0$ ,  $K_t^\gamma = B_{t \wedge T}^\gamma$ , and  $K_t^\gamma = (t, k_t^\gamma)$ , then  $\{A_t^\gamma = (t, a_t^\gamma) : t \geq 0\}$  is a Markov process starting from  $(0, 0)$  such that for  $r, t > 0$ , the probability density of  $a_{t+r}^\gamma$  given that  $a_r^\gamma = x$ , with  $x \in A_r$ , is

$$(13) \quad s_t^\gamma((r, x), (r+t, y)) = \frac{\Psi_\gamma(r+t, y)}{\Psi_\gamma(r, x)} w_t^\gamma((r, x), (r+t, y)), \quad (r+t, y) \in C_W,$$

where  $w_t^\gamma((r, x), (r+t, \cdot))$  is the probability density of  $k_{r+t}^\gamma$  given that  $k_t^\gamma = x$ , and the probability density of  $a_t^\gamma$  is given by

$$(14) \quad s_t^\gamma((0, 0), (t, y)) = C_t \Psi_\gamma(t, y) \pi\left(\frac{y}{t}\right) e^{-\frac{1}{2t} \|y - \gamma t\|^2}, \quad y \in A_t,$$

where  $C_t$  is a normalizing constant depending on  $t$ .

**5.3. The two conditioned processes and time inversion.** Actually the two Doob transformations previously defined are equal up to a time inversion. We prove this property as it is done in [2] for the Brownian motion in the unit interval. The following lemma is immediately deduced from (12) and (14).

**Lemma 5.3.** *For  $t > 0$ ,  $x \in A$ , one has*

$$s_{1/t}^\gamma((0, 0), (1/t, x/t)) = q_t(\gamma, x).$$

**Lemma 5.4.** *For  $0 < r \leq t$ ,  $x \in A_r$ ,  $y \in A_t$*

$$e^{-\frac{1}{2t} \|y\|^2} u_{\frac{1}{r} - \frac{1}{t}}(y/t, x/r) = e^{-\frac{1}{2r} \|x\|^2} w_{t-r}^0((r, x), (t, y)).$$

*Proof.* Using expression (8) and the time inversion invariance property for the standard heat kernel on  $\mathfrak{t}$ , one obtains that

$$e^{-\frac{1}{2t}\|y\|^2} u_{\frac{1}{r}-\frac{1}{t}}(y/t, x/r) = e^{-\frac{1}{2r}\|x\|^2} \sum_{w \in \Omega} e^{-\frac{1}{2t}(\|y\|^2 - \|tw(y/t)\|^2)} p_{t-r}(x, tw(y/t)).$$

The sum on the right-hand side of the identity is exactly  $w_{t-r}^0((r, x), (t, y))$  according to lemma 6.3 of [4], which achieves the proof.  $\square$

In the following proposition  $\{r_t^\gamma : t \geq 0\}$  is the conditioned process defined in section 5.1 and  $\{a_t^\gamma : t \geq 0\}$  is the one defined in section 5.2.

**Proposition 5.5.** *One has in distribution*

$$\{ta_{1/t}^\gamma : t \geq 0\} \stackrel{d}{=} \{r_t^\gamma : t \geq 0\}.$$

*Proof.* It follows immediately from the two previous lemmas and identity (12).  $\square$

## 6. A NEW PROOF OF THEOREM 4.2

For every  $t > 0$  one considers the diffusion process  $\{Y_{s,t}^\gamma : s \in [0, 1]\}$  starting from the identity element  $e$  of  $G$  satisfying the EDS (in  $s$ )

$$tdY_{s,t}^\gamma = Y_{s,t}^\gamma \circ d(x_{s,t} + \gamma st).$$

For  $u \in G$  one denotes by  $\mathcal{O}(u)$  the unique element in  $A$  such that

$$u \in \text{Ad}(G)\{\exp(\mathcal{O}(u))\}.$$

We have proved in [5] that the random process  $\{(t, t\mathcal{O}(Y_{1,t}^0)) : t \geq 0\}$  is distributed as  $\{A_t^0 : t \geq 0\}$ . As  $Y^\gamma$  satisfies

$$dY_{s,t}^\gamma = Y_{s,t}^\gamma \circ d\left(\frac{1}{t}x_{s,t} + \gamma s\right),$$

and  $\{\frac{1}{t}x_{s,t} : s, t > 0\} \stackrel{d}{=} \{x_{s,1/t} : s, t > 0\}$ , one could deduce from [5], with the help of a Kirillov-Frenkel character formula from [7] and a Cameron-Martin theorem, that the result remains true for any  $\gamma \in A$ .

We propose here a brief proof of the theorem, which is valid for every  $\gamma$ . For every  $t \geq 0$ , one considers the diffusion process  $\{X_{s,t}^\gamma : s \in [0, 1]\}$  starting from  $e \in G$  satisfying the stochastic differential equation (in  $s$ )

$$(15) \quad dX_{s,t}^\gamma = X_{s,t}^\gamma \circ d(x_{s,t} + \gamma s).$$

**Proposition 6.1.**

- (1) For  $t, t' \geq 0$ , the random process  $\{X_{s,t+t'}^\gamma (X_{s,t}^\gamma)^{-1} : s \in [0, 1]\}$  has the same law as  $\{X_{s,t'}^0 : s \in [0, 1]\}$ .
- (2) For  $t, t' \geq 0$ , the random process  $\{X_{s,t+t'}^\gamma (X_{s,t}^\gamma)^{-1} : s \in [0, 1]\}$  is independent of  $\{X_{s,r}^\gamma : s \in [0, 1], r \leq t\}$ .
- (3) The random process  $\{X_{1,t}^\gamma : t \geq 0\}$  is a standard Brownian motion in  $G$  starting from  $\exp(\gamma)$ .

*Proof.* For the first point, we let  $Z_s = X_{s,t+t'}^\gamma (X_{s,t}^\gamma)^{-1}$ ,  $s \in [0, 1]$ . The process  $\{(X_{s,t}^\gamma)^{-1} : s \in [0, 1]\}$  satisfies the EDS (in  $s$ )

$$d(X_{s,t}^\gamma)^{-1} = -d(x_{s,t} + \gamma s) \circ (X_{s,t}^\gamma)^{-1}$$

from which we immediately deduce that  $Z$  satisfies

$$dZ_s = Z_s \circ X_{s,t}^\gamma d(x_{s,t+t'} - x_{s,t})(X_{s,t}^\gamma)^{-1}.$$

As  $\{\int_0^s X_{r,t}^\gamma d(x_{r,t+t'} - x_{r,t})(X_{r,t}^\gamma)^{-1} : s \in [0, 1]\}$  has the same law as  $\{x_{s,t'} : s \in [0, 1]\}$ , and is independent of  $\{x_{s,r} : s \in [0, 1], r \leq t\}$ , one gets the first two points, which imply in particular that  $\{X_{1,t}^\gamma : t \geq 0\}$  is a right Levy process. The  $\text{Ad}(G)$ -invariance of the increments law implies that it is also a left Levy process. As for any  $t > 0$ ,  $X_{1,t}^0$  and  $X_{t,1}^0$  are equal in distribution, the third point follows. □

Proposition 6.1 has the two following corollaries, the second one being deduced from the first by proposition 5.5.

**Corollary 6.2.** *The random process  $\{\mathcal{O}(X_{1,t}^\gamma) : t \geq 0\}$  is a standard Brownian motion starting from  $\gamma$  killed on the boundary of  $A$  conditioned in Doob's sense to remain in  $A$ .*

**Corollary 6.3.** *The random process  $\{(t, t\mathcal{O}(X_{1,1/t}^\gamma)) : t \geq 0\}$  has the same distribution as the conditioned process  $\{A_t^\gamma : t \geq 0\}$ .*

As the two processes  $\{x_{s,1/t} : s, t > 0\}$  and  $\{\frac{1}{t}x_{s,t} : s, t > 0\}$  are equal in distribution, Theorem 4.2 follows from corollary 6.3 with  $\gamma = 0$ . For any  $\gamma \in A$ , one has under the same hypothesis as in the theorem the following one.

**Theorem 6.4.** *The radial part process*

$$\left\{ \text{rad}\left(t\Lambda_0 + \int_0^1 (\cdot |d(x_{s,t} + \gamma st))\right) : t \geq 0 \right\}$$

*is distributed as the Doob conditioned process  $\{A_t^\gamma : t \geq 0\}$ .*

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