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Reconstruction from the Fourier transform on the ball via prolate spheroidal wave functions *

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Abstract

We give new formulas for finding a compactly supported function v on \mathbb{R}^d , $d \geq 1$, from its Fourier transform $\mathcal{F}v$ given within the ball B_r . For the one-dimensional case, these formulas are based on the theory of prolate spheroidal wave functions (PSWF's). In multidimensions, well-known results of the Radon transform theory reduce the problem to the one-dimensional case. Related results on stability and convergence rates are also given.

Keywords: ill-posed inverse problems, band-limited Fourier transform, prolate spheroidal wave functions, Radon transform, Hölder-logarithmic stability.

AMS subject classification: 42A38, 35R30, 49K40

1 Introduction

Following D. Slepian, H. Landau, and H. Pollak (see, for example, the survey paper [18]), we consider the compact integral operator \mathcal{F}_c on $\mathcal{L}^2([-1, 1])$ defined by

$$\mathcal{F}_c[f](x) := \int_{-1}^1 e^{icxy} f(y) dy, \quad (1.1)$$

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where f is a test function and the parameter $c > 0$ is the bandwidth. Let $\mathbb{N} := \{0, 1, \dots\}$. The eigenfunctions $(\psi_{j,c})_{j \in \mathbb{N}}$ of \mathcal{F}_c are *prolate spheroidal wave functions* (PSWFs). These functions are real-valued and form an orthonormal basis in $\mathcal{L}^2([-1, 1])$. Let $(\mu_{j,c})_{j \in \mathbb{N}}$ denote the corresponding eigenvalues. It is known that all these eigenvalues are simple and non-zero, so we can assume that $0 < |\mu_{j+1,c}| < |\mu_{j,c}|$ for all $j \in \mathbb{N}$.

The properties of $(\psi_{j,c})_{j \in \mathbb{N}}$ and $(\mu_{j,c})_{j \in \mathbb{N}}$ are recalled in Section 2.1 of this paper. In particular, we have that

$$\mathcal{F}_c[f](x) = \sum_{j \in \mathbb{N}} \mu_{j,c} \psi_{j,c}(x) \int_{-1}^1 \psi_{j,c}(y) f(y) dy, \quad (1.2)$$

and, for $g = \mathcal{F}_c[f]$,

$$\mathcal{F}_c^{-1}[g](y) = \sum_{j \in \mathbb{N}} \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^1 \psi_{j,c}(x) g(x) dx, \quad (1.3)$$

where \mathcal{F}_c^{-1} is the inverse operator, that is $\mathcal{F}_c^{-1}[\mathcal{F}_c[f]] \equiv f$ for all $f \in \mathcal{L}^2[-1, 1]$.

The operator \mathcal{F}_c appears naturally in the theory of the classical Fourier transform \mathcal{F} defined in the multidimensional case $d \geq 1$ by

$$\mathcal{F}[v](p) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ipq} v(q) dq, \quad p \in \mathbb{R}^d, \quad (1.4)$$

where v is a complex-valued test function on \mathbb{R}^d . To avoid any possible confusion with \mathcal{F}_c , we employ the simplified notation $\hat{v} := \mathcal{F}[v]$ throughout the paper. Let

$$B_\rho := \{q \in \mathbb{R}^d : |q| < \rho\}, \quad \text{for any } \rho > 0.$$

We consider the following inverse problem.

Problem 1.1. Let $d \geq 1$ and $r, \sigma > 0$. Find $v \in \mathcal{L}^2(\mathbb{R}^d)$ from \hat{v} given on the ball B_r (possibly with some noise), under a priori assumption that v is supported in B_σ .

Problem 1.1 is a classical problem of the Fourier analysis, inverse scattering, and image processing; see, for example, [1, 4–6, 10–12, 15] and references therein. In the present work, we suggest a new approach to Problem 1.1, proceeding from the singular value decomposition formulas (1.1), (1.2) and further results of the PSWF theory. Surprisingly, to our knowledge, the PSWF theory was omitted in the context of Problem 1.1 in the literature even though it is quite natural. In particular, in dimension $d = 1$, Problem 1.1 reduces to finding a function $f \in \mathcal{L}^2([-1, 1])$ from $\mathcal{F}_c[f]$ (possibly with some noise).

In multidimensions, in addition to the PSWF theory, we use inversion methods for the classical Radon transform \mathcal{R} ; see, for example [13, 16]. Recall that \mathcal{R} is defined by

$$\mathcal{R}[v](y, \theta) := \int_{q \in \mathbb{R}^d : q\theta = y} v(q) dq, \quad y \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}, \quad (1.5)$$

where v is a complex-valued test function on \mathbb{R}^d , $d \geq 1$. In the present work, for simplicity, we define the inverse Radon transform \mathcal{R}^{-1} via the projection theorem; see formula (2.13) for details.

Theorem 1.1. *Let $d \geq 1$, $r, \sigma > 0$ and $c = r\sigma$. Let $v \in \mathcal{L}^2(\mathbb{R}^d)$ and $\text{supp } v \subset B_\sigma$. Then, its Fourier transform \hat{v} restricted to B_r determines v via the following formulas:*

$$\begin{aligned} v(q) &= \mathcal{R}^{-1}[f_{r,\sigma}](\sigma^{-1}q), \quad q \in \mathbb{R}^d, \\ f_{r,\sigma}(y, \theta) &:= \begin{cases} \mathcal{F}_c^{-1}[g_{r,\theta}](y), & \text{if } y \in [-1, 1] \\ 0, & \text{otherwise,} \end{cases} \\ g_{r,\theta}(x) &:= \left(\frac{2\pi}{\sigma}\right)^d \hat{v}(rx\theta), \quad x \in [-1, 1], \theta \in \mathbb{S}^{d-1}, \end{aligned}$$

where \mathcal{F}_c^{-1} is defined by (1.3) and \mathcal{R}^{-1} is the inverse Radon transform.

Remark 1.2. For $d = 1$, the formulas of Theorem 1.1 reduce to

$$v(q) = \mathcal{F}_c^{-1}[g_r](\sigma^{-1}q), \quad g_r(x) := \frac{2\pi}{\sigma} \hat{v}(rx),$$

where $q \in (-\sigma, \sigma)$ and $x \in [-1, 1]$.

We prove Theorem 1.1 in Section 4.1.

Unfortunately, the reconstruction procedure given in Theorem 1.1 and Remark 1.2 is severely unstable. The reason is that the numbers $(\mu_{j,c})_{j \in \mathbb{N}}$ decay superexponentially as $j \rightarrow \infty$; see formulas (2.4) and (2.6). To overcome this difficulty, we approximate \mathcal{F}_c^{-1} by the operator $\mathcal{F}_{n,c}^{-1}$ defined by

$$\mathcal{F}_{n,c}^{-1}[w](y) := \sum_{j=0}^n \frac{1}{\mu_{j,c}} \psi_{j,c}(y) \int_{-1}^1 \psi_{j,c}(x) w(x) dx. \quad (1.6)$$

Note that (1.6) correctly defines the operator $\mathcal{F}_{n,c}^{-1}$ on $\mathcal{L}^2([-1, 1])$ for any $n \in \mathbb{N}$. Let

$$\pi_{n,c}[f] := \sum_{j=0}^n \hat{f}_{j,c} \psi_{j,c}, \quad \hat{f}_{j,c} := \int_{-1}^1 \psi_{j,c}(y) f(y) dy. \quad (1.7)$$

That is, $\pi_{n,c}[\cdot]$ is the orthogonal projection in $\mathcal{L}^2([-1, 1])$ onto the span of the first $n + 1$ functions $(\psi_{j,c})_{j \leq n}$.

Lemma 1.3. *Let $f, w \in \mathcal{L}^2([-1, 1])$ and $\|\mathcal{F}_c[f] - w\|_{\mathcal{L}^2} \leq \delta$ for some $\delta \geq 0$. Then, for any $n \in \mathbb{N}$,*

$$\|f - \mathcal{F}_{n,c}^{-1}[w]\|_{\mathcal{L}^2([-1,1])} \leq \frac{\delta}{|\mu_{n,c}|} + \|f - \pi_{n,c}[f]\|_{\mathcal{L}^2([-1,1])}. \quad (1.8)$$

Estimates of the type (1.8) are of general nature for operators admitting a singular value decomposition like (1.2). For completeness of the presentation, we prove Lemma 1.3 in Section 2.2. Combining Theorem 1.1, Remark 1.2, Lemma 1.3, inversion methods for the Radon transform \mathcal{R} , and known estimates of the PSWF theory for $|\mu_{n,c}|$ and $\|f - \pi_{n,c}[f]\|_{\mathcal{L}^2}$ (see Section 2.1) yields numerical methods for Problem 1.1. In this connection, in the present work we give a regularised version of the reconstruction procedure of Theorem 1.1; see Theorem 1.4, Theorem 3.1 and Corollary 3.3.

For $\alpha, \delta \in (0, 1)$, let

$$n^* = n^*(c, \alpha, \delta) = \left\lfloor 3 + \tau \frac{ec}{4} \right\rfloor, \quad (1.9)$$

where $\lfloor \cdot \rfloor$ denotes the floor function and $\tau = \tau(c, \alpha, \delta) \geq 1$ is the solution of the equation

$$\tau \log \tau = \frac{4}{ec} \alpha \log(\delta^{-1}). \quad (1.10)$$

Let

$$\begin{aligned} \mathcal{L}_r^2 &:= \{w \in \mathcal{L}^2(B_r) : \|w\|_r < \infty\}, \\ \|w\|_r &:= \left(\int_{B_r} p^{1-d} |w(p)|^2 dp \right)^{1/2}. \end{aligned} \quad (1.11)$$

Theorem 1.4. *Let the assumptions of Theorem 1.1 hold and $v \in \mathcal{H}^\nu(\mathbb{R}^d)$ for some $\nu \geq 0$ (and $\nu > 0$ for $d = 1$). Suppose that $w \in \mathcal{L}_r^2$ and $\|w - \hat{v}\|_r \leq \delta N$ for some $\delta \in (0, 1)$. Let $\alpha \in (0, 1)$ and n^* be defined by (1.9). Let*

$$\begin{aligned} v^\delta(q) &:= \mathcal{R}^{-1}[u_{r,\sigma}](\sigma^{-1}q), \quad q \in \mathbb{R}^d, \\ u_{r,\sigma}(y, \theta) &:= \begin{cases} \mathcal{F}_{n^*,c}^{-1}[w_{r,\theta}](y), & \text{if } y \in [-1, 1], \\ 0, & \text{otherwise,} \end{cases} \\ w_{r,\theta}(x) &:= \left(\frac{2\pi}{\sigma}\right)^d w(rx\theta), \quad x \in [-1, 1], \theta \in \mathbb{S}^{d-1}. \end{aligned}$$

Then, for any $\beta \in (0, 1 - \alpha)$ and any $\mu \in (0, \nu + \frac{d-1}{2})$,

$$\|v - v^\delta\|_{\mathcal{H}^{-(d-1)/2}(\mathbb{R}^d)} \leq \kappa_1 N \delta^\beta + \kappa_2 \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)} (\log \delta^{-1})^{-\mu}, \quad (1.12)$$

where $\kappa_1 = \kappa_1(c, d, r, \sigma, \alpha, \beta) > 0$ and $\kappa_2 = \kappa_2(c, d, r, \sigma, \alpha, \nu, \mu) > 0$.

Similarly to Remark 1.2, the statement of Theorem 1.4 simplifies significantly for the case $d = 1$; see Corollary 3.3. We prove Theorem 1.4 in Section 4.2.

The parameter N from Theorem 1.4 can be considered as an a priori upper bound for $\|\hat{v}\|_r$. Indeed, the assumption $\|w - \hat{v}\|_r \leq \delta\|\hat{v}\|_r$ is natural. If the noise level is such that $\|w - \hat{v}\|_r \geq \|\hat{v}\|_r$, then the given data w tells about v as little as the trivial function $w_0 \equiv 0$. An accurate reconstruction is hardly possible in this case, since it is equivalent to no data given at all.

The function v^δ in Theorem 1.4 is not compactly supported, in general; see also the related remark about v after Lemma 2.4. Nevertheless, only v^δ restricted to B_σ is of interest under the assumptions of Theorem 1.4.

Our stability estimate (1.12) is given in \mathcal{H}^s with $s \leq 0$. One can improve the regularity in such estimates using the apodized reconstruction $\phi * v^\delta$, where $*$ denotes the convolution operator and ϕ is an appropriate sufficiently regular non-negative compactly supported function with $\|\phi\|_{\mathcal{L}^1(\mathbb{R}^d)} = 1$; see, for example, [10, Section 6.1]. In particular, (1.12) implies estimates for $\phi * v - \phi * v^\delta$ in \mathcal{H}^t with $t \geq 0$.

Applying Theorem 1.4 with $v := v_1 - v_2$ and $w \equiv 0$, we get the following result.

Corollary 1.5. *Let the assumptions of Theorem 1.1 hold for $v := v_1 - v_2$. Let $v_1 - v_2 \in \mathcal{H}^\nu(\mathbb{R}^d)$ for some $\nu \geq 0$ (and $\nu > 0$ for $d = 1$). Suppose that $\|\hat{v}_1 - \hat{v}_2\|_r \leq \delta N$ for some $\delta \in (0, 1)$ and $N > 0$. Let $\alpha \in (0, 1)$. Then, for any $\beta \in (0, 1 - \alpha)$ and any $\mu \in (0, \nu + \frac{d-1}{2})$,*

$$\|v_1 - v_2\|_{\mathcal{H}^{-(d-1)/2}(\mathbb{R}^d)} \leq \kappa_1 N \delta^\beta + \kappa_2 \|v_1 - v_2\|_{\mathcal{H}^\nu(\mathbb{R}^d)} (\log \delta^{-1})^{-\mu}, \quad (1.13)$$

where $\kappa_1 = \kappa_1(c, d, r, \sigma, \alpha, \beta)$ and $\kappa_2 = \kappa_2(c, d, r, \sigma, \alpha, \nu, \mu)$ are the same as in (1.12).

The present work continues studies of [10, 11], where we approached Problem 1.1 via a Hölder-stable extrapolation of \hat{v} from B_r to a larger ball, using truncated series of Chebyshev polynomials. The reconstruction of the present work is essentially different; in particular, it does not use any extrapolation. However, the resulting stability estimates are analogous for both reconstructions. In particular, estimate (1.12) resembles [10, Theorem 3.1] in dimension $d = 1$ and resembles [11, Theorem 3.2] (with $s = -\frac{d-1}{2}$ and $\kappa = 1$) in dimension $d \geq 1$; estimate (1.13) resembles [10, Corollary 3.3] in dimension $d = 1$ and resembles [11, Corollary 3.4] (with $s = -\frac{d-1}{2}$ and $\kappa = 1$) in dimension $d \geq 1$. Note also that, in the domain of coefficient inverse problems, estimates of the form (1.12) and (1.13) are known as Hölder-logarithmic stability estimates; see [7–11] and references therein.

The main advantages of the present work in comparison with [10, 11] are the following:

- We allow the "noise" in Problem 1.1 to be from a larger space \mathcal{L}_r^2 defined by (1.11) in contrast with \mathcal{L}^∞ .
- We use the straightforward formulas (1.3), (1.6), (1.8) in place of the roundabout way that requires extrapolation of \hat{v} from B_r to a larger ball and leads to additional numerical issues.

On the other hand, the advantages of [10, 11] in comparison with the present work include: explicit expressions for quantities like κ_1 and κ_2 in (1.12); more advanced norms $\|\cdot\|$ for reconstruction errors like $v - v^\delta$ in (1.12), where $\|\cdot\| = \|\cdot\|_{\mathcal{L}^2(\mathbb{R}^d)}$ in [10] and $\|\cdot\| = \|\cdot\|_{\mathcal{H}^s(\mathbb{R}^d)}$ with any $s \in (-\infty, \nu)$ in [11]. The reason is purely due to the fact that the PSWFs theory is still less developed than the theory of Chebyshev polynomials and the classical Fourier transform theory. In connection with further developments in the PSWFs theory that would improve the results of the present work on Problem 1.1, see Remarks 2.1, 2.2, and 2.3 in Section 2.1.

Note also that the functions $(\psi_{j,c})_{j \in \mathbb{N}}$ for large j , yield a new example of exponential instability for Problem 1.1 in dimension $d = 1$. This instability behaviour follows from the properties of $\psi_{j,c}$ and $\mu_{j,c}$ recalled in Section 2.1 and the result formulated in Remark 2.2. However, known estimates for the derivatives of PSWFs do not allow yet to say that this example is more strong than the example constructed in [10, Theorem 5.2].

The aforementioned possible developments in the PSWFs theory and further development of the approach of the present work to Problem 1.1, including its numerical implementation, will be addressed in further articles.

The further structure of the paper is as follows. Some primary results are recalled in Section 2. In Section 3, we prove our estimates in dimension $d = 1$ modulo a technical lemma, namely, Lemma 3.2. In Section 4.2, we prove Theorem 1.1, Theorem 1.4 and Corollary 1.5 based on the results given in Sections 2 and 3. In Section 5, we prove Lemma 3.2.

2 Preliminaries

In this section, we recall some known results on PSWFs and on the Radon transform that we will use in the proofs of Theorems 1.1 and 1.4. In addition, we prove Lemma 1.3 and give a stability estimate for the inverse Radon transform; see Lemma 2.4.

2.1 Prolate spheroidal wave functions

In connection with the facts presented in this subsection we refer to [2, 3, 17–20] and references therein.

Originally, the prolate spheroidal wave functions $(\psi_{n,c})_{n \in \mathbb{N}}$ were discovered as the eigenfunctions of the following spectral problem:

$$\mathcal{L}_c \psi = \chi \psi, \quad \psi \in C^2([-1, 1]), \quad (2.1)$$

where χ is the spectral parameter and

$$\mathcal{L}_c[\psi] := -\frac{d}{dx} \left[(1-x^2) \frac{d\psi}{dx} \right] + c^2 x^2 \psi.$$

We also consider the operator \mathcal{Q}_c defined on $\mathcal{L}_2([-1, 1])$ by

$$\mathcal{Q}_c[f](x) := \frac{c}{2\pi} \mathcal{F}_c^* [\mathcal{F}_c[f]](x) = \int_{-1}^1 \frac{\sin c(x-y)}{\pi(x-y)} f(y) dy, \quad (2.2)$$

where \mathcal{F}_c^* is the conjugate operator to \mathcal{F}_c defined by (1.1). The prolate spheroidal wave functions $(\psi_{n,c})_{n \in \mathbb{N}}$ are eigenfunctions for problem (2.1) and for both operators \mathcal{F}_c and \mathcal{Q}_c .

Let $(\chi_{n,c})_{n \in \mathbb{N}}$ denote the eigenvalues of problem (2.1). It is known that $(\chi_{n,c})_{n \in \mathbb{N}}$ are real, positive, simple, that is, one can assume that

$$0 < \chi_{n,c} < \chi_{n+1,c}, \quad \text{for all } n \in \mathbb{N}.$$

In addition, the following estimates hold:

$$n(n+1) < \chi_{n,c} < n(n+1) + c^2. \quad (2.3)$$

If $\mu_{n,c}$ and $\lambda_{n,c}$ are the corresponding eigenvalues of \mathcal{F}_c and \mathcal{Q}_c , respectively, then

$$\mu_{n,c} = i^n \sqrt{\frac{2\pi}{c} \lambda_{n,c}} \quad \text{and} \quad 1 > \lambda_{n,c} > \lambda_{n+1,c} > 0. \quad (2.4)$$

Furthermore, each $\lambda_{n,c}$ is non-decreasing with respect to c . Using also [2, formula (6)], we find that

$$\left\lfloor \frac{2c}{\pi} \right\rfloor - 1 \leq \left| \{n \in \mathbb{N} : \lambda_{n,c} \geq 1/2\} \right| \leq \left\lceil \frac{2c}{\pi} \right\rceil + 1. \quad (2.5)$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and the ceiling functions, respectively, and $|\cdot|$ is the number of elements. We also employ the following estimate from [2, Corollary 3]: for $n \geq \max\{3, \frac{2c}{\pi}\}$,

$$A(n, c)^{-1} e^{-2\tilde{n}(\log \tilde{n} - \kappa)} \leq \lambda_{n,c} \leq A(n, c) e^{-2\tilde{n}(\log \tilde{n} - \kappa)}, \quad (2.6)$$

where $\nu_1 \geq 1$, $\nu_2, \nu_3 \geq 0$ are some fixed constants,

$$A(n, c) := \nu_1 n^{\nu_2} \left(\frac{c}{c+1} \right)^{-\nu_3} e^{(\pi c)^2/4n}.$$

and

$$\kappa := \log \left(\frac{ec}{4} \right), \quad \tilde{n} = \tilde{n}(n) := n + \frac{1}{2}. \quad (2.7)$$

Remark 2.1. Apparently, proceeding from the approach of [2], one can give explicit values for the constants ν_1, ν_2, ν_3 in the expression for $A(n, c)$.

We also recall from [19, formula (11)] that, for all $n \in \mathbb{N}$ and $c > 0$,

$$\max_{0 \leq j \leq n} \max_{|x| \leq 1} |\psi_{j,c}(x)| \leq 2\sqrt{n}. \quad (2.8)$$

Remark 2.2. Proceeding from (2.1), (2.3), and (2.8), one can show that, for any $m \in \mathbb{N}$,

$$\|\psi_{n,c}\|_{C^m[-1,1]} = O(n^{2m+1/2}) \quad \text{as } n \rightarrow \infty.$$

Next, we recall results on the spectral approximation by PSWFs in Sobolev-type spaces; see [20]. For a real $\nu \geq 0$, let

$$\tilde{\mathcal{H}}_c^\nu([-1, 1]) := \left\{ f \in \mathcal{L}^2([-1, 1]) : \|f\|_{\tilde{\mathcal{H}}_c^\nu} < \infty \right\}, \quad (2.9)$$

where

$$\|f\|_{\tilde{\mathcal{H}}_c^\nu([-1,1])} := \left(\sum_{n \in \mathbb{N}} (\chi_{n,c})^\nu |\hat{f}_{n,c}|^2 \right)^{1/2} \quad \hat{f}_{n,c} := \int_{-1}^1 \psi_{n,c}(y) f(y) dy.$$

Recall from (1.7) that

$$\pi_n[f] = \sum_{j=0}^n \hat{f}_{j,c} \psi_{j,c}(x), \quad n \in \mathbb{N}.$$

Note that $\pi_n[f] \rightarrow f$ as $n \rightarrow \infty$ since $(\psi_{j,c}(x))_{j \in \mathbb{N}}$ form an orthonormal basis in $\mathcal{L}^2([-1, 1])$. Furthermore, for any $0 \leq \mu \leq \nu$,

$$\|f - \pi_n[f]\|_{\tilde{\mathcal{H}}_c^\mu([-1,1])} \leq n^{\mu-\nu} \|f\|_{\tilde{\mathcal{H}}_c^\nu([-1,1])}. \quad (2.10)$$

The standard Sobolev space $\mathcal{H}^\nu[(-1, 1)]$ is embedded in $\tilde{\mathcal{H}}_c^\nu([-1, 1])$. In fact, we have that

$$\|f\|_{\tilde{\mathcal{H}}_c^\nu[(-1,1)]} \leq C(1 + c^2)^{\nu/2} \|f\|_{\mathcal{H}^\nu[(-1,1)]}, \quad (2.11)$$

where C is a constant independent of c and f assuming that $c \geq c_0 > 0$.

Remark 2.3. Proceeding from the results of [20], one can obtain an explicit estimate for the constant $C = C(c_0, \nu)$ in (2.11). Besides, one can establish an upper bound for $\|\varphi f\|_{\mathcal{H}^\nu[(-1,1)]}$ in terms of $\|f\|_{\tilde{\mathcal{H}}_c^\nu[(-1,1)]}$, for fixed $\nu > 0$, where φ is a smooth real-valued function appropriately vanishing at the ends of the interval $[-1, 1]$ and non-vanishing elsewhere.

2.2 Proof of Lemma 1.3

First, we observe that

$$\mathcal{F}_{n,c}^{-1}[\mathcal{F}_c[f]] = \pi_n[f].$$

Using also the linearity of $\mathcal{F}_{n,c}^{-1}$, we derive

$$f - \mathcal{F}_{n,c}^{-1}[w] = f - \pi_n[f] + \mathcal{F}_{n,c}^{-1}[\mathcal{F}_c[f]] - \mathcal{F}_{n,c}^{-1}[w] = f - \pi_n[f] + \mathcal{F}_{n,c}^{-1}[u],$$

where $u := \mathcal{F}_c[f] - w$. Therefore,

$$\|f - \mathcal{F}_{n,c}^{-1}[w]\|_{\mathcal{L}^2([-1,1])} \leq \|\mathcal{F}_{n,c}^{-1}[u]\|_{\mathcal{L}^2([-1,1])} + \|f - \pi_n[f]\|_{\mathcal{L}^2([-1,1])}.$$

Due to (2.4), we have that $|\mu_{j,c}| \geq |\mu_{n,c}|$ for all $j \leq n$. Using also the orthonormality of the basis $(\psi_{j,c})_{j \in \mathbb{N}}$ in $\mathcal{L}^2([-1, 1])$, we estimate

$$\begin{aligned} \|\mathcal{F}_{n,c}^{-1}[u]\|_{\mathcal{L}^2([-1,1])}^2 &= \left\| \sum_{j=0}^n \frac{1}{\mu_{j,c}} \psi_{j,c}(\cdot) \int_{-1}^1 \psi_{j,c}(x) u(x) dx \right\|_{\mathcal{L}^2([-1,1])}^2 \\ &= \sum_{j=0}^n \frac{1}{|\mu_{j,c}|^2} \left\| \psi_{j,c}(\cdot) \int_{-1}^1 \psi_{j,c}(x) u(x) dx \right\|_{\mathcal{L}^2([-1,1])}^2 \\ &\leq \frac{1}{|\mu_{n,c}|^2} \sum_{j=0}^n \left\| \psi_{j,c}(\cdot) \int_{-1}^1 \psi_{j,c}(x) u(x) dx \right\|_{\mathcal{L}^2([-1,1])}^2 \\ &\leq \frac{1}{|\mu_{n,c}|^2} \sum_{j=0}^{\infty} \left\| \psi_{j,c}(\cdot) \int_{-1}^1 \psi_{j,c}(x) u(x) dx \right\|_{\mathcal{L}^2([-1,1])}^2 = \left(\frac{\|u\|_{\mathcal{L}^2([-1,1])}}{|\mu_{n,c}|} \right)^2. \end{aligned}$$

Recalling that $\|u\|_{\mathcal{L}^2([-1,1])} \leq \delta$ (by assumptions) and combining the formulas above, we complete the proof.

2.3 Radon Transform

The Radon transform \mathcal{R} defined in (1.5) arises in various domains of pure and applied mathematics. Since Radon's work [16], this transform and its applications received significant attention and its properties are well studied; see, for example, [13] and references therein. In particular, the Radon transform $\mathcal{R}[v]$ is closely related to the Fourier transform \hat{v} (see (1.4)) via the following formula:

$$\hat{v}(s\theta) = \frac{1}{(2\pi)^d} \int_{-\infty}^{\infty} e^{ist} \mathcal{R}[v](t, \theta) dt, \quad s \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}. \quad (2.12)$$

In the theory of Radon transform, formula (2.12) is known as the projection theorem. Note that one can define the inverse transform \mathcal{R}^{-1} by combining (2.12) with inversion formulas for the Fourier transform:

$$\begin{aligned}\mathcal{R}^{-1}[u](q) &:= \frac{1}{(2\pi)^{d-1}} \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} e^{-is\theta q} \hat{u}(s, \theta) s^{d-1} ds d\theta, \quad q \in \mathbb{R}^d, \\ \hat{u}(s, \theta) &:= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ist} u(t, \theta) dt, \quad s \in \mathbb{R}, \theta \in \mathbb{S}^{d-1}.\end{aligned}\tag{2.13}$$

For other inversion formulas for \mathcal{R} ; see [16] and, for example, [13, Section II.2].

For real ν , let

$$\begin{aligned}\mathcal{H}^\nu(\mathbb{R}^d) &:= \{v : \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)} < \infty\}, \\ \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)} &:= \left(\int_{\mathbb{R}^d} (1+p^2)^\nu |\hat{v}(p)|^2 dp \right)^{1/2}, \\ \mathcal{H}^\nu(\mathbb{R} \times \mathbb{S}^{d-1}) &:= \{u : \|u\|_{\mathcal{H}^\nu(\mathbb{R} \times \mathbb{S}^{d-1})} < \infty\}, \\ \|u\|_{\mathcal{H}^\nu(\mathbb{R} \times \mathbb{S}^{d-1})} &:= \left(\int_{\mathbb{S}^{d-1}} \int_{-\infty}^{+\infty} (1+s^2)^\nu |\hat{u}(s, \theta)|^2 ds d\theta \right)^{1/2},\end{aligned}$$

where v, u are distributions on \mathbb{R}^d and $\mathbb{R} \times \mathbb{S}^{d-1}$, respectively. According to [13, Theorem 5.1], if $v \in \mathcal{H}^\nu(\mathbb{R}^d)$ and $\text{supp } v \subset B_1$ then

$$a(\nu, d) \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)} \leq \|\mathcal{R}[v]\|_{\mathcal{H}^{\nu+(d-1)/2}(\mathbb{R} \times \mathbb{S}^{d-1})} \leq b(\nu, d) \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)}.\tag{2.14}$$

In addition, one can recover explicit expressions for $a(\nu, d)$ and $b(\nu, d)$ from the proof of [13, Theorem 5.1]. We will also use the following result generalizing the left inequality in (2.14).

Lemma 2.4. *Let $u \in \mathcal{H}^{\nu+(d-1)/2}(\mathbb{R} \times \mathbb{S}^{d-1})$, $\text{supp } u \subseteq [-1, 1] \times \mathbb{S}^{d-1}$, and $u(s, \theta) = u(-s, -\theta)$ for all $(s, \theta) \in \mathbb{R} \times \mathbb{S}^{d-1}$. Then,*

$$a(\nu, d) \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)} \leq \|u\|_{\mathcal{H}^{\nu+(d-1)/2}(\mathbb{R} \times \mathbb{S}^{d-1})},$$

where $v := \mathcal{R}^{-1}[u]$ is defined by (2.13) and $a(\nu, d)$ is the same as in (2.14).

In fact, the proof of Lemma 2.4 is identical to the arguments of [13, Theorem 5.1] for the left inequality in (2.14). In addition, we use also that $u = \mathcal{R}[v]$. Note that v defined by (2.13) might not be compactly supported; see, for example, [14] for the asymptotic analysis of $\mathcal{R}^{-1}[u]$ at infinity.

3 Stability estimates in 1D

The main result of this section is the following theorem.

Theorem 3.1. *Let $f, w \in \mathcal{L}^2([-1, 1])$ and $\|\mathcal{F}_c[f] - w\|_{\mathcal{L}^2} \leq \delta$ for some $\delta \in (0, 1)$. Suppose that $f \in \mathcal{H}^\nu([-1, 1])$, $\nu > 0$. Then,*

$$\begin{aligned} \|f - \mathcal{F}_{n^*,c}^{-1}[w]\|_{\mathcal{L}^2([-1,1])} &\leq \gamma_1 c^{-\gamma_2} (1+c)^{\gamma_3} (1+\rho)^{\gamma_4} \exp\left(\frac{\pi^2 c \log(1+\rho)}{2e\rho}\right) \delta^{1-\alpha} \\ &\quad + C(1+c^2)^{\nu/2} \|f\|_{\mathcal{H}^\nu([-1,1])} \left(2 + \frac{ec}{4} \cdot \frac{\rho}{\log(1+\rho)}\right)^{-\nu}, \end{aligned} \quad (3.1)$$

where $\alpha \in (0, 1)$, $\rho = \frac{4}{ec} \alpha \log(\delta^{-1})$, $n^* = n^*(c, \alpha, \delta)$ is defined by (1.9), C is the constant from (2.11), and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are some positive constants independent of c, α, δ .

Theorem 3.1 follows directly by combining estimate (2.10) with $\mu = 0$, estimate (2.11), Lemma 1.3, and the following lemma.

Lemma 3.2. *Let $c, \alpha, \delta, \rho, n^*$ be the same as in Theorem 3.1. Then*

$$\frac{\delta}{|\mu_{n^*,c}|} \leq \gamma_1 c^{-\gamma_2} (1+c)^{\gamma_3} (1+\rho)^{\gamma_4} \exp\left(\frac{\pi^2 c \log(1+\rho)}{2e\rho}\right) \delta^{1-\alpha}. \quad (3.2)$$

We prove Lemma 3.2 in Section 5. The proof of Lemma 3.2 is based on two additional technical lemmas, namely, Lemma 5.1 and Lemma 5.2.

Theorem 3.1 implies the following corollary, which is equivalent to Theorem 1.4 in dimension $d = 1$. This corollary is also crucial for our considerations for $d \geq 2$ given in Section 4.2.

Corollary 3.3. *Let $f, w \in \mathcal{L}^2([-1, 1])$ and $\|\mathcal{F}_c[f] - w\|_{\mathcal{L}^2} \leq \delta M$ for some $\delta \in (0, 1)$ and $M > 0$. Suppose that $f \in \mathcal{H}^\nu([-1, 1])$, $\nu > 0$. Let $\alpha \in (0, 1)$ and n^* be defined by (1.9). Then, for any $\beta \in (0, 1 - \alpha)$ and any $\mu \in (0, \nu)$,*

$$\|f - \mathcal{F}_{n^*,c}^{-1}[w]\|_{\mathcal{L}^2([-1,1])} \leq C_1 M \delta^\beta + C_2 \|f\|_{\mathcal{H}^\nu([-1,1])} (\log \delta^{-1})^{-\mu},$$

where $C_1 = C_1(c, \alpha, \beta) > 0$ and $C_2 = C_2(c, \alpha, \nu, \mu) > 0$.

Proof. It is sufficient to prove Corollary 3.3 for the case $M = 1$. The case $M \neq 1$ is reduced to $M = 1$ by scaling $f \rightarrow \tilde{f} = f/M$ and $w \rightarrow \tilde{w} = w/M$. Therefore, it remains to show that, under the assumptions of Theorem 3.1, the following estimate holds for any $\beta \in (0, 1 - \alpha)$ and any $\mu \in (0, \nu)$:

$$\|f - \mathcal{F}_{n^*,c}^{-1}[w]\|_{\mathcal{L}^2([-1,1])} \leq C_1 \delta^\beta + C_2 \|f\|_{\mathcal{H}^\nu([-1,1])} (\log \delta^{-1})^{-\mu}, \quad (3.3)$$

where $C_1 = C_1(c, \alpha, \beta) > 0$ and $C_2 = C_2(c, \alpha, \nu, \mu) > 0$.

Under our assumptions, we have that:

$$\rho = \frac{4}{ec} \alpha \log(\delta^{-1}) > 0; \quad \frac{\pi^2 c \log(1+\rho)}{2e\rho} \leq \frac{\pi^2 c}{2e};$$

and, for some positive constants $m_1 = m_1(c, \alpha, \beta, \gamma_4)$ and $m_2 = m_2(c, \alpha, \nu, \mu)$,

$$(1 + \rho)^{\gamma_4} \delta^{1-\alpha} \leq m_1 \delta^\beta;$$

$$\left(2 + \frac{ec}{4} \cdot \frac{\rho}{\log(1+\rho)}\right)^{-\nu} \leq m_2 (\log \delta^{-1})^{-\mu}.$$

Applying these estimates in (3.1), we derive (3.3) with

$$C_1 = \gamma_1 c^{-\gamma_2} (1+c)^{\gamma_3} e^{\pi^2 c/(2e)} m_1, \quad C_2 = C(1+c^2)^{\nu/2} m_2.$$

This completes the proof of Corollary 3.3. □

4 Multidimensional reconstruction

In this section, we prove Theorems 1.1 and 1.4.

4.1 Proof of Theorem 1.1

Let $\mathcal{R}[v]$ be the Radon transform of v ; see formula (1.5). Since $\text{supp } v \subset B_\sigma$, we have that

$$\mathcal{R}[v](t, \theta) = 0 \text{ for } |t| > \sigma. \quad (4.1)$$

Therefore, we only need to integrate over $t \in [-\sigma, \sigma]$ in (2.12). Then, using the change of variables $s = rx$, $t = \sigma y$ and recalling $c = r\sigma$, we get that

$$g_{r,\theta}(x) = \left(\frac{2\pi}{\sigma}\right)^d \hat{v}(rx\theta) = \frac{1}{\sigma^{d-1}} \int_{-1}^1 e^{icxy} \mathcal{R}[v](\sigma y, \theta) dy, \quad x \in [-1, 1]. \quad (4.2)$$

Using (4.1), (4.2) and recalling the definitions of \mathcal{F}_c and $f_{r,\sigma}$, we obtain that

$$\mathcal{R}[v](\sigma y, \theta) = \sigma^{d-1} \mathcal{F}_c^{-1}[g_{r,\theta}](y) = \sigma^{d-1} f_{r,\sigma}(y, \theta). \quad (4.3)$$

Let

$$v_\sigma(q) := v(\sigma q), \quad q \in \mathbb{R}^d. \quad (4.4)$$

Using (1.5) and the change of variables $q = \sigma q'$, we find that

$$\mathcal{R}[v](\sigma y, \theta) = \int_{q \in \mathbb{R}^d : q\theta = \sigma y} v(q) dq = \sigma^{d-1} \int_{q' \in \mathbb{R}^d : q'\theta = y} v(\sigma q') dq' = \sigma^{d-1} \mathcal{R}[v_\sigma](y, \theta).$$

Thus, also using (4.3), we get

$$\mathcal{R}[v_\sigma] = f_{r, \sigma}. \quad (4.5)$$

Applying the inverse Radon transform and formula (4.4) completes the proof.

4.2 Proof of Theorem 1.4

We will repeatedly use the following bounds for the Sobolev norm with respect to the argument scaling.

Lemma 4.1. *Let $\mathbf{v} \in \mathcal{H}^\eta(\mathbb{R}^d)$ for some $\eta \in \mathbb{R}$. Then, for any $\sigma > 0$,*

$$\begin{aligned} \frac{\sigma^{\eta-d/2}}{(1+\sigma)^\eta} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)} &\leq \|\mathbf{v}_\sigma\|_{\mathcal{H}^\eta(\mathbb{R}^d)} \leq \frac{(1+\sigma)^\eta}{\sigma^{d/2}} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)}, & \text{for } \eta \geq 0, \\ \frac{(1+\sigma)^\eta}{\sigma^{d/2}} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)} &\leq \|\mathbf{v}_\sigma\|_{\mathcal{H}^\eta(\mathbb{R}^d)} \leq \frac{\sigma^{\eta-d/2}}{(1+\sigma)^\eta} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)}, & \text{for } \eta \leq 0, \end{aligned}$$

where \mathbf{v}_σ is defined by $\mathbf{v}_\sigma(q) := \mathbf{v}(\sigma q)$, $q \in \mathbb{R}^d$.

Proof. Recall that

$$\begin{aligned} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} (1+p^2)^\eta |\hat{\mathbf{v}}(p)|^2 dp, \\ \|\mathbf{v}_\sigma\|_{\mathcal{H}^\eta(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} (1+p^2)^\eta |\hat{\mathbf{v}}_\sigma(p)|^2 dp \\ &= \sigma^{-2d} \int_{\mathbb{R}^d} (1+p^2)^\eta |\hat{\mathbf{v}}(p/\sigma)|^2 dp \\ &= \sigma^{-d} \int_{\mathbb{R}^d} (1+(\sigma p')^2)^\eta |\hat{\mathbf{v}}(p')|^2 dp'. \end{aligned}$$

Bounding

$$\min\{1, \sigma^{2\eta}\} \leq \frac{(1+(\sigma p')^2)^\eta}{(1+(p')^2)^\eta} \leq \max\{1, \sigma^{2\eta}\},$$

we derive that

$$\min\{1, \sigma^{2\eta}\} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)}^2 \leq \sigma^d \|\mathbf{v}_\sigma\|_{\mathcal{H}^\eta(\mathbb{R}^d)}^2 \leq \max\{1, \sigma^{2\eta}\} \|\mathbf{v}\|_{\mathcal{H}^\eta(\mathbb{R}^d)}^2.$$

To complete the proof, it remains to observe that

$$\begin{aligned} \max\{1, \sigma^{2\eta}\} &\leq \begin{cases} (1 + \sigma)^{2\eta}, & \text{if } \eta \geq 0, \\ \frac{\sigma^{2\eta}}{(1 + \sigma)^{2\eta}}, & \text{if } \eta \leq 0, \end{cases} \\ \min\{1, \sigma^{2\eta}\} &\geq \begin{cases} \frac{\sigma^{2\eta}}{(1 + \sigma)^{2\eta}}, & \text{if } \eta \geq 0, \\ (1 + \sigma)^{2\eta}, & \text{if } \eta \leq 0. \end{cases} \end{aligned}$$

□

Now we are ready to prove Theorem 1.4. Let v_σ be defined by (4.4) and

$$v_\sigma^\delta(q) := v^\delta(\sigma q), \quad q \in \mathbb{R}^d.$$

Applying Lemma 4.1 with $\mathbf{v} = v - v^\delta$ and $\eta = -\frac{d-1}{2}$, we find that

$$\|v - v^\delta\|_{\mathcal{H}^{-(d-1)/2}(\mathbb{R}^d)} \leq (1 + \sigma)^{(d-1)/2} \sigma^{d/2} \|v_\sigma - v_\sigma^\delta\|_{\mathcal{H}^{-(d-1)/2}(\mathbb{R}^d)}. \quad (4.6)$$

Using the formulas for $f_{r,\sigma}$ and $u_{r,\sigma}$ of Theorems 1.1 and 1.4, we find that

$$v_\sigma - v_\sigma^\delta = \mathcal{R}^{-1}[f_{r,\sigma} - u_{r,\sigma}].$$

Note also that both $f_{r,\sigma}$ and $u_{r,\sigma}$ are supported in $[-1, 1] \times \mathbb{S}^{d-1}$. Applying Lemma 2.4 for $u = f_{r,\sigma} - u_{r,\sigma}$, we get that

$$\|v_\sigma - v_\sigma^\delta\|_{\mathcal{H}^{-(d-1)/2}(\mathbb{R}^d)} \leq \frac{1}{a} \|f_{r,\sigma} - u_{r,\sigma}\|_{\mathcal{L}^2(\mathbb{R} \times \mathbb{S}^{d-1})}, \quad (4.7)$$

where $a = a(-\frac{d-1}{2}, d)$ is the constant from (2.14).

Observe that

$$\|f_{r,\sigma} - u_{r,\sigma}\|_{\mathcal{L}^2(\mathbb{R} \times \mathbb{S}^{d-1})}^2 = \int_{\mathbb{S}^{d-1}} \|f_{r,\sigma}(\cdot, \theta) - u_{r,\sigma}(\cdot, \theta)\|_{\mathcal{L}^2([-1,1])}^2 d\theta. \quad (4.8)$$

Applying Corollary 3.3 with functions $f = f_{r,\sigma}(\cdot, \theta)$ and $w = w_{r,\theta}$, we obtain that, for any $\mu \in (0, \nu + \frac{d-1}{2})$ and almost all $\theta \in \mathbb{S}^{d-1}$,

$$\begin{aligned} \|f_{r,\sigma}(\cdot, \theta) - u_{r,\sigma}(\cdot, \theta)\|_{\mathcal{L}^2([-1,1])} &\leq C_1 M(\theta) \delta^\beta + C_2 H(\theta) (\log \delta^{-1})^{-\mu}, \\ M(\theta) &:= \frac{1}{\delta} \|g_{r,\theta} - w_{r,\theta}\|_{\mathcal{L}^2([-1,1])}, \end{aligned} \quad (4.9)$$

$$H(\theta) := \|f_{r,\sigma}(\cdot, \theta)\|_{\mathcal{H}^{\nu+(d-1)/2}([-1,1])},$$

where $f_{r,\sigma}$, $g_{r,\theta}$ and $w_{r,\theta}$ are defined in Theorems 1.1 and 1.4, C_1 and C_2 are the constants of Corollary 3.3 with $\nu + \frac{d-1}{2}$ in place of ν . Here, the assumption of Corollary 3.3 that

$$\|\mathcal{F}_c[f_{r,\sigma}(\cdot, \theta)] - w_{r,\theta}\|_{\mathcal{L}^2([-1,1])} \leq \delta M(\theta)$$

is fulfilled automatically, since $f_{r,\sigma}(\cdot, \theta) \equiv \mathcal{F}_c^{-1}[g_{r,\sigma}]$ on $[-1, 1]$ by definition.

In fact, the functions M, H belong to $\mathcal{L}^2(\mathbb{S}^{d-1})$; see formulas (4.11) and (4.12) below. Combining formulas (4.8), (4.9) and the Cauchy–Schwarz inequality

$$\int_{\mathbb{S}^{d-1}} H(\theta)M(\theta)d\theta \leq \|M\|_{\mathcal{L}^2(\mathbb{S}^{d-1})}\|H\|_{\mathcal{L}^2(\mathbb{S}^{d-1})},$$

we get that

$$\begin{aligned} \|f_{r,\sigma} - u_{r,\sigma}\|_{\mathcal{L}^2(\mathbb{R} \times \mathbb{S}^{d-1})}^2 &\leq \int_{\mathbb{S}^{d-1}} \left(C_1 M(\theta) \delta^\beta + C_2 H(\theta) (\log \delta^{-1})^{-\mu} \right)^2 d\theta \\ &\leq \left(C_1 \|M\|_{\mathcal{L}^2(\mathbb{S}^{d-1})} \delta^\beta + C_2 \|H\|_{\mathcal{L}^2(\mathbb{S}^{d-1})} (\log \delta^{-1})^{-\mu} \right)^2. \end{aligned} \quad (4.10)$$

Next, we estimate $\|M\|_{\mathcal{L}^2(\mathbb{S}^{d-1})}$ and $\|H\|_{\mathcal{L}^2(\mathbb{S}^{d-1})}$. Since $\|w - \hat{v}\|_r \leq \delta N$, we get

$$\begin{aligned} \|M\|_{\mathcal{L}^2(\mathbb{S}^{d-1})}^2 &= \int_{\mathbb{S}^{d-1}} \frac{1}{\delta^2} \|g_{r,\theta} - w_{r,\theta}\|_{\mathcal{L}^2([-1,1])}^2 d\theta \\ &= \frac{1}{r\delta^2} \left(\frac{2\pi}{\sigma} \right)^{2d} \int_{\mathbb{S}^{d-1}} \int_{-r}^r |w(s\theta) - \hat{v}(s\theta)|^2 ds d\theta. \\ &= \frac{2}{r\delta^2} \left(\frac{2\pi}{\sigma} \right)^{2d} \|w - \hat{v}\|_r^2 \leq \frac{2}{r} \left(\frac{2\pi}{\sigma} \right)^{2d} N^2. \end{aligned} \quad (4.11)$$

In addition, using (4.5), we get

$$\|H\|_{\mathcal{L}^2(\mathbb{S}^{d-1})} = \|f_{r,\sigma}\|_{\mathcal{H}^{\nu+(d-1)/2}(\mathbb{R} \times \mathbb{S}^{d-1})} = \|\mathcal{R}[v_\sigma]\|_{\mathcal{H}^{\nu+(d-1)/2}(\mathbb{R} \times \mathbb{S}^{d-1})}. \quad (4.12)$$

Using formula (4.12), the right inequality of (2.14), and applying Lemma 4.1 with $\mathbf{v} = v$ and $\eta = \nu$, we obtain that

$$\|H\|_{\mathcal{L}^2(\mathbb{S}^{d-1})} \leq b \|v_\sigma\|_{\mathcal{H}^\nu(\mathbb{R}^d)} \leq b \frac{(1+\sigma)^\nu}{\sigma^{d/2}} \|v\|_{\mathcal{H}^\nu(\mathbb{R}^d)}, \quad (4.13)$$

where $b = b(\nu, d)$ is the constant from (2.14).

Combining (4.6) – (4.13), we derive the required bound (1.12) with

$$\kappa_1 := \frac{\sqrt{2}(2\pi)^d (1+\sigma)^{(d-1)/2} C_1}{a\sigma^{\frac{d}{2}} \sqrt{r}}, \quad \kappa_2 := \frac{b}{a} (1+\sigma)^{\nu+(d-1)/2} C_2.$$

5 Proof of Lemma 3.2

To prove Lemma 3.2, we need two additional technical results given below.

Lemma 5.1. For any $\rho > 0$, the equation

$$\tau \log \tau = \rho \quad (5.1)$$

has the unique solution $\tau = \tau(\rho) > 1$. Furthermore,

$$1 \leq \frac{\rho}{\log(1+\rho)} \leq \tau(\rho) \leq 1 + \rho. \quad (5.2)$$

Proof. Observe that $u_1(\tau) = \tau \log \tau$ is a strictly increasing continuous function on $[1, +\infty)$, $u_1(1) = 0$, and $u_1(\tau) \rightarrow +\infty$ as $\tau \rightarrow +\infty$. Then, by the intermediate value theorem, equation (5.1) has the unique solution $\tau(\rho) \in (0, +\infty)$ for any $\rho > 0$.

Next, note that $u_2(\tau) = \tau - \tau \log \tau$ is a strictly decreasing function on $[1, +\infty)$ since its derivative $u_2'(\tau) = -\log \tau$ is negative for $\tau > 1$. Therefore,

$$\tau(\rho) - \rho = \tau(\rho) - \tau(\rho) \log \tau(\rho) \leq u_2(1) = 1.$$

Thus, we proved that $\tau(\rho) \leq 1 + \rho$. Then, we get $\log(\tau(\rho)) \leq \log(1 + \rho)$ which implies the other bound

$$\tau(\rho) = \frac{\rho}{\log(\tau(\rho))} \geq \frac{\rho}{\log(1+\rho)}.$$

The remaining inequality $\frac{\rho}{\log(1+\rho)} \geq 1$ is equivalent to $e^\theta - 1 \geq \theta$ with $\theta = \log(1 + \rho)$. \square

Lemma 5.2. Let $\alpha, \delta \in (0, 1)$ and τ be defined according to (5.1) with $\rho = \frac{4}{ec} \alpha \log(\delta^{-1})$. Then, for any $q \geq 0$, we have

$$e^{\eta(\log \eta - \kappa)} \leq \left(\frac{4\eta}{c}\right)^q \delta^{-\alpha},$$

where κ is defined according to (2.7) and $\eta = \eta(q, \alpha, \delta, c) := q + \tau \frac{ec}{4}$.

Proof. First, observe that

$$\begin{aligned} \eta(\log \eta - \kappa) &= (q + \tau \frac{ec}{4})(\log \eta - \kappa) \\ &= q(\log \eta - \kappa) + \tau \frac{ec}{4} (\log(\tau \frac{ec}{4}) - \log(\frac{ec}{4}) + \log \eta - \log(\tau \frac{ec}{4})) \\ &= q(\log \eta - \kappa) + \tau \frac{ec}{4} \log \tau + \tau \frac{ec}{4} (\log \eta - \log(\tau \frac{ec}{4})). \end{aligned}$$

By the definition of τ , we have that

$$\tau \frac{ec}{4} \log \tau = \alpha \log(\delta^{-1}).$$

Besides,

$$\tau \frac{ec}{4} (\log \eta - \log(\tau \frac{ec}{4})) = \tau \frac{ec}{4} \log \left(1 + \frac{q}{\tau \frac{ec}{4}}\right) \leq q.$$

Combining the formulas above and recalling the definition of κ , we derive that

$$\eta(\log \eta - \kappa) \leq q(\log \eta - \log(\frac{ec}{4})) + \alpha \log(\delta^{-1}) + q = q \log\left(\frac{4\eta}{c}\right) + \alpha \log(\delta^{-1}).$$

The required bound follows by exponentiating the both sides of the last formula. \square

Now, we are ready to prove Lemma 3.2. First, we combine formulas (2.4) and (2.6) to get

$$|\mu_{n^*,c}| \geq \sqrt{\frac{2\pi}{c A(n^*,c)}} e^{-\tilde{n}(\log \tilde{n} - \kappa)}, \quad (5.3)$$

where $\tilde{n} = n^* + \frac{1}{2}$. Note that (2.6) requires $n^* \geq \max\left\{3, \frac{2c}{\pi}\right\}$. The inequality $n^* \geq 3$ is immediate by the definition of n^* . In addition, using that $\tau > 1$ by Lemma 5.1, we can estimate

$$n^* \geq 2 + \tau \frac{ec}{4} \geq 2 + \frac{ec}{4} > \frac{ec}{4} > \frac{2c}{\pi}.$$

Thus, we justified (5.3).

Using the inequalities $1 \leq \tau \leq 1 + \rho$ from Lemma 5.1, we estimate

$$n^* \leq 3 + \tau \frac{ec}{4} \leq 3(c+1)\tau \leq 3(c+1)(1+\rho).$$

Using the inequality $\tau \geq \frac{\rho}{\log(1+\rho)}$ from Lemma 5.1, we also find that

$$e^{(\pi c)^2/4n^*} \leq e^{\pi^2 c/(e\tau)} \leq \exp\left(\frac{\pi^2 c \log(1+\rho)}{e\rho}\right).$$

Thus, we get that

$$\begin{aligned} A(n^*, c) &= \nu_1(n^*)^{\nu_2} \left(\frac{c}{c+1}\right)^{-\nu_3} e^{(\pi c)^2/4n^*} \\ &\leq \nu_1 3^{\nu_2} (c+1)^{\nu_2-\nu_3} c^{-\nu_3} (1+\rho)^{\nu_2} \exp\left(\frac{\pi^2 c \log(1+\rho)}{e\rho}\right). \end{aligned} \quad (5.4)$$

Similarly as before, using the inequalities $1 \leq \tau \leq 1 + \rho$ from Lemma 5.1, we estimate

$$\tilde{n} \leq 3.5 + \tau \frac{ec}{4} \leq 3.5(c+1)(1+\rho).$$

Then, using Lemma 5.2 with $q := \tilde{n} - \tau \frac{ec}{4}$ and observing that $0 \leq q \leq 3.5$, we find that

$$e^{\tilde{n}(\log \tilde{n} - \kappa)} \leq \left(\frac{4\tilde{n}}{c}\right)^q \delta^{-\alpha} \leq \left(\frac{14(c+1)}{c}\right)^{3.5} (1+\rho)^{3.5} \delta^{-\alpha}. \quad (5.5)$$

Substituting the bounds of (5.4) and of (5.5) into (5.3), we derive estimate (3.2) with

$$\gamma_1 = \sqrt{\frac{\nu_1 3^{\nu_2}}{2\pi}} 14^{3.5}, \quad \gamma_2 = \frac{\nu_3}{2} + 3, \quad \gamma_3 = \frac{\nu_2 - \nu_3}{2} + 3.5, \quad \gamma_4 = \frac{\nu_2}{2} + 3.5.$$

Note that if $\gamma_3 \leq 0$ then we can replace it with zero, since $(1+c)^{\gamma_3} \leq 1$ in this case. This completes the proof of Lemma 3.2.

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