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Abstract. In this article, we are interested in the Gevrey properties of the formal power series solution in time of the partial differential equations with a polynomial semilinearity and with analytic coefficients at the origin of $\mathbb{C}^{n+1}$. We prove in particular that the inhomogeneity of the equation and the formal solution are together s-Gevrey for any $s \geqslant s_c$, where s_c is a nonnegative rational number fully determined by the Newton polygon of the associated linear PDE. In the opposite case $s < s_c$, we show that the solution is generically s_c-Gevrey while the inhomogeneity is s-Gevrey, and we give an explicit example in which the solution is s'-Gevrey for no $s' < s_c$.

Contents

1. Introduction

In this article, we consider an inhomogeneous semilinear partial differential equation with a 1-dimensional time variable $t \in \mathbb{C}$ and a n-dimensional spatial variable $x = (x_1, ..., x_n) \in \mathbb{C}^n$ of the form

$$(1.1)\qquad \begin{cases} \partial_t^\kappa u - \displaystyle\sum_{i\in\mathcal{K}} \sum_{q\in Q_i} t^{v_{i,q}} a_{i,q}(t,x) \partial_t^i \partial_x^q u - P(u) = \widetilde{f}(t,x) \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j = 0, ..., \kappa - 1 \end{cases}$$

where

- $\kappa \geqslant 1$ is a positive integer;
- $\mathcal{K}$ is a nonempty subset of $\{0, ..., \kappa - 1\}$;
- Q_i is a nonempty finite subset of $\mathbb{N}^n$ for all $i \in \mathcal{K}$ ($\mathbb{N}$ denotes the set of the nonnegative integers);
- ∂_x^q denotes the derivative $\partial_{x_1}^{q_1} ... \partial_{x_n}^{q_n}$ while $q := (q_1, ..., q_n) \in \mathbb{N}^n$;
- $v_{i,q} \geqslant 0$ is a nonnegative integer for all $i \in \mathcal{K}$ and $q \in Q_i$;

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- the coefficients $a_{i,q}(t,x)$ are analytic on a polydisc $D_{\rho_0,\rho_1,...,\rho_n} := D_{\rho_0} \times D_{\rho_1} \times ... \times D_{\rho_n}$ centered at the origin of $\mathbb{C}^{n+1}$ (D_ρ denotes the disc with center $0 \in \mathbb{C}$ and radius $\rho > 0$) and satisfy $a_{i,q}(0,x) \not\equiv 0$ for all $i \in \mathcal{K}$ and $q \in Q_i$;
- $P(X) := \sum_{m=2}^{d} b_m(t,x)X^m$ is a polynomial of degree $d \geqslant 2$ with analytic coefficients on $D_{\rho_0,\rho_1,...,\rho_n}$;
- the inhomogeneity $\widetilde{f}(t,x)$ is a formal power series in t with analytic coefficients in $D_{\rho_1,...,\rho_n}$ (we denote by $\widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]$) which may be smooth, or not[1];
- the initial conditions $\varphi_j(x)$ are analytic on $D_{\rho_1,...,\rho_n}$ for all $j = 0, ..., \kappa - 1$.

Looking for a formal solution $\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]$, and writing any element $\widetilde{g}(t,x)$ of $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]$ on the form

$$\widetilde{g}(t,x) = \sum_{j\geqslant 0} g_{j,*}(x)\frac{t^j}{j!} \text{ with } g_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n}) \text{ for all } j,$$

we easily get that the coefficients $u_{j,*}(x)$ of $\widetilde{u}(t,x)$ are uniquely determined by the recurrence relations

$$\begin{aligned}u_{j+\kappa,*}(x) = f_{j,*}(x) + & \\ \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell=0}^{j-v_{i,q}} & \frac{j!}{\ell!(j-v_{i,q}-\ell)!}a_{i,q;\ell,*}(x)\partial_x^q u_{j-v_{i,q}-\ell+i,*}(x) + \\ & \sum_{m=2}^{d}\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}} \frac{j!}{\ell!\ell_1!...\ell_m!} b_{m;\ell,*}(x)u_{\ell_1,*}(x)...u_{\ell_m,*}(x)\end{aligned} \tag{1.2}$$

together with the initial conditions $u_{j,*}(x) = \varphi(x)$ for $j = 0, ..., \kappa - 1$. As usual, we use the classical convention that the first sum is zero as soon as $j - v_{i,q} < 0$.

The purpose of the paper is to answer to the following question:

"What relationship exists between the Gevrey order of the solution $\widetilde{u}(t,x)$ and the Gevrey order of the inhomogeneity $\widetilde{f}(t,x)$?"

Indeed, according to the algebraic structure of the s-Gevrey spaces $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ (see section 3.1 for the exact definition of theses spaces), it is classical one has

$$\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s.$$

But, what can we say about the converse?

In previous articles [24, 25], the author studied two particular cases of eq. (1.1): the inhomogeneous n-dimensional heat equation

$$\begin{cases}\partial_t u - a(x)\Delta_x u - b(x)u^m = \widetilde{f}(t,x)\\ u(0,x) = \varphi(x)\end{cases} \tag{1.3}$$

[1] We denote $\widetilde{f}$ with a tilde to emphasize the possible divergence of the series $\widetilde{f}$.

and the most general equation

$$\begin{cases} \partial_t^\kappa u - a(t,x)\partial_x^p u - b(t,x)u^m = \widetilde{f}(t,x), (t,x) \in \mathbb{C}^2 \\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j = 0, ..., \kappa - 1 \end{cases} \tag{1.4}$$

In this two cases, he proved that the Gevrey orders of $\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are closely related:

Proposition 1.1 ([24,25])**.** *Let s_c denote the nonnegative rational number equal to the inverse of the smallest positive slope of the Newton polygon at $t = 0$ of the associated linear part of eq. (1.3) (resp. eq. (1.4)) if any exists, and equal to 0 otherwise*[2]*. Then,*

1. *$\widetilde{u}(t,x)$ and $\widetilde{f}(t,x)$ are together s-Gevrey for any $s \geqslant s_c$;*
2. *$\widetilde{u}(t,x)$ is generically s_c-Gevrey while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$.*

Remark 1.2. When the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$, the hypotheses made on eqs. (1.3) and (1.4) do not allow in general to specify the exact Gevrey order of the solution $\widetilde{u}(t,x)$ as in the opposite case $s \geqslant s_c$ (Point 1). However, the second point of proposition 1.1 asserts that this order is always less or equal to s_c[3] and that this inequality is the *best* possible. Indeed, one can easily find cases for which the solution $\widetilde{u}(t,x)$ is exactly s_c-Gevrey (see [25, Prop. 3.2] and [24, Prop. 4.11] for more details).

In this paper, we propose to extend the result of proposition 1.1 to the very general eq. (1.1). Let us mention here that a similar problem has already been studied by H. Tahara in [36] in the case of *real* variables. However, the calculations we develop in this paper are based on a very different approach.

Let us also mention that other slightly different works have also been done for several years by many authors towards the convergence [15,17,33] and the Gevrey order [8,9,16,29–32,34,35] of the formal power series solutions of some singular nonlinear partial differential equations, and towards the summability [11,13,20] of the formal power series solution of some nonlinear partial differential equations. Furthermore, in [6,7], A. Lastra and S. Malek considered some parametric nonlinear partial differential equations; in [12], S. Malek investigated the Gevrey properties of some nonlinear integro-differential equations. Of course, given the technical and computational difficulties inherent in the nonlinearity, the known results are currently far fewer than in the linear case.

The organization of the paper is as follows. In section 2, we introduce and we describe the Newton polygon at $t = 0$ of the linear part of eq. (1.1). In section 3, we recall some definitions and basic properties about the Gevrey formal series which are needed in the sequel. Next, we state our main result (theorem 3.3) which displays the explicit relationship between the Gevrey order of the solution $\widetilde{u}(t,x)$ and the one of the inhomogeneity $\widetilde{f}(t,x)$. The proof of this result is detailed in section 4.

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[2] We have thereby $s_c = 1$ in the case of eq. (1.3); and, in the case of eq. (1.4), $s_c = p/\kappa - 1$ when $p > \kappa$, and $s_c = 0$ when $p \leqslant \kappa$. For the definition of the Newton polygon, we refer to section 2 below.

[3] This is obvious due to the filtration of the Gevrey spaces (see section 3.1) and the first point of proposition 1.1.

2. Newton polygon

Let L denote the linear part of eq. (1.1):

$$L = \partial_t^\kappa - \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} t^{v_{i,q}} a_{i,q}(t,x)\partial_t^i \partial_x^q.$$

As definition of the Newton polygon, we choose the definition of M. Miyake [14] (see also A. Yonemura [39] or S. Ouchi [19]) which is an analogue to the one given by J.-P. Ramis [23] for linear ordinary differential equations. Recall that, H. Tahara and H. Yamazawa use in [37] a slightly different one.

For any $(a,b)\in\mathbb{R}^2$, we denote by $C(a,b)$ the domain

$$C(a,b) = \{(x,y)\in\mathbb{R}^2; x \leqslant a \text{ and } y \geqslant b\}.$$

Definition 2.1. *The Newton polygon $N_t(L)$ of L at $t=0$* is defined as the convex hull of

$$C(\kappa,-\kappa) \cup \bigcup_{i\in\mathcal{K}} \bigcup_{q\in Q_i} C(\lambda(q)+i, v_{i,q}-i),$$

where $\lambda(q) = q_1 + \dots + q_n$ denotes the length of $q = (q_1,\dots,q_n)\in\mathbb{N}^n$.

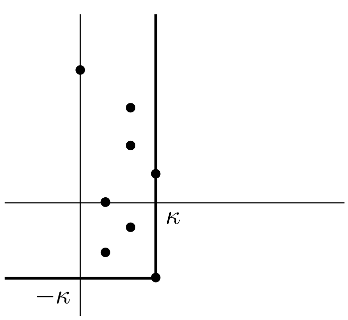
Proposition 2.2 below specifies the geometric structure of $N_t(L)$.

Proposition 2.2. *Let $\mathcal{S} = \{(i,q)$ such that $i\in\mathcal{K},\ q\in Q_i$ and $\lambda(q) > \kappa - i\}$ be.*

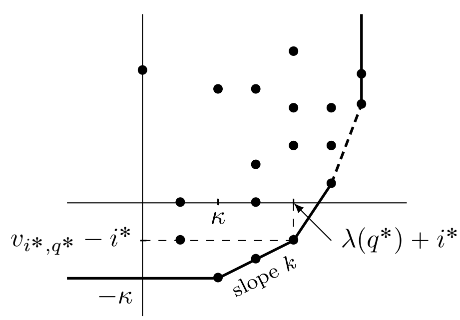
1. *Suppose $\mathcal{S}=\varnothing$. Then, $N_t(L) = C(\kappa,-\kappa)$. In particular, $N_t(L)$ has no side with a positive slope (see fig. 1a).*
2. *Suppose $\mathcal{S}\neq\varnothing$. Then, $N_t(L)$ has at least one side with a positive slope. Moreover, its smallest positive slope k is given by*

$$k = \min_{(i,q)\in\mathcal{S}} \left(\frac{v_{i,q}+\kappa-i}{\lambda(q)-\kappa+i}\right) = \frac{v_{i^*,q^*}+\kappa-i^*}{\lambda(q^*)-\kappa+i^*},$$

where $(i^*,q^*)\in\mathcal{S}$ stands for any convenient pair (see fig. 1b) which we assume from now on fixed once and for all.



(A) Case $\mathcal{S}=\varnothing$



(B) Case $\mathcal{S}\neq\varnothing$

Figure 1. The Newton polygon $N_t(L)$

Proof. The first point stems obvious from the fact that the condition $\mathcal{S}=\varnothing$ implies $C(\lambda(q)+i, v_{i,q}-i) \subset C(\kappa,-\kappa)$ for all $i\in\mathcal{K}$ and $q\in Q_i$. As for the second point, it suffices to remark, on one hand, that $C(\lambda(q)+i, v_{i,q}-i)\subset C(\kappa,-\kappa)$ for all pairs $(i,q)\notin\mathcal{S}$, and, on the other hand, that the segment with the two end points $(\kappa,-\kappa)$

and $(\lambda(q)+i, v_{i,q}-i)$ has a positive slope equal to $(v_{i,q}+\kappa-i)/(\lambda(q)-\kappa+i)$ for all pairs $(i,q)\in\mathcal{S}$. □

The following result is a direct consequence of the definition of the pair (i^*,q^*). It will be very useful to us in the sequel.

Corollary 2.3. *Suppose $\mathcal{S}\neq\varnothing$. Then, the inequality*

$$\frac{\lambda(q^*)+v_{i^*,q^*}}{v_{i^*,q^*}+\kappa-i^*}(v_{i,q}+\kappa-i)\geqslant\lambda(q)+v_{i,q}$$

holds for all $i\in\mathcal{K}$ and all $q\in Q_i$.

Proof. Due to the definition of the pair (i^*,q^*), we get

$$\frac{\lambda(q^*)-\kappa+i^*}{v_{i^*,q^*}+\kappa-i^*}\geqslant\frac{\lambda(q)-\kappa+i}{v_{i,q}+\kappa-i}>0$$

for all $(i,q)\in\mathcal{S}$, and next

$$\frac{\lambda(q^*)-\kappa+i^*}{v_{i^*,q^*}+\kappa-i^*}\geqslant\frac{\lambda(q)-\kappa+i}{v_{i,q}+\kappa-i}\tag{2.1}$$

for all $(i,q)\in\mathcal{K}\times Q_i$. We have indeed $\lambda(q)-\kappa+i\leqslant 0$ when $(i,q)\notin\mathcal{S}$. Corollary 2.3 follows by first adding "+1" to both sides of (2.1), and then by multiplying by the positive term $v_{i,q}+\kappa-i$. □

Let us now turn to the Gevrey properties of the solution $\widetilde{u}(t,x)$.

3. Gevrey properties of $\widetilde{u}(t,x)$

As we said in section 1, the purpose of this article is to make explicit the relationship between the Gevrey order of the solution $\widetilde{u}(t,x)$ and the Gevrey order of the inhomogeneity $\widetilde{f}(t,x)$.

Before stating our main result (see theorem 3.3 below), let us first recall for the convenience of the reader some definitions and basic properties about the Gevrey formal series in $\mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]$, which are needed in the sequel.

3.1. Gevrey formal series.

All along the article, we consider t as the variable and x as a parameter. Thereby, to define the notion of *Gevrey classes of formal power series in* $\mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]$, one extends the classical notion of *Gevrey classes of elements in* $\mathbb{C}[[t]]$ to families parametrized by x in requiring similar conditions, the estimates being however uniform with respect to x. Doing that, any formal power series of $\mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]$ can be seen as a formal power series in t with coefficients in a convenient Banach space defined as the space of functions that are holomorphic on a polydisc $\mathcal{D}_{\rho,\dots,\rho}$ $(0<\rho\leqslant\min\rho_\ell)$ and continuous up to its boundary, equipped with the usual supremum norm. For a general study of the formal power series with coefficients in a Banach space, we refer for instance to [1].

In the sequel, we endow $\mathbb{C}^n$ with the maximum norm: for $x=(x_1,\dots,x_n)\in\mathbb{C}^n$,

$$\|x\|=\max_{\ell\in\{1,\dots,n\}}|x_\ell|.$$

Definition 3.1. Let $s\geqslant 0$ be. A formal series

$$\widetilde{u}(t,x)=\sum_{j\geqslant 0}u_{j,*}(x)\frac{t^j}{j!}\in\mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]$$

is said to be *Gevrey of order s* (in short, *s-Gevrey*) if there exist three positive constants $0<\rho<\min\rho_\ell$, $C>0$ and $K>0$ such that the inequalities

$$\sup_{\|x\|\leqslant\rho}|u_{j,*}(x)|\leqslant CK^j\Gamma(1+(s+1)j)$$

hold for all $j\geqslant 0$.

In other words, definition 3.1 means that $\widetilde{u}(t,x)$ is s-Gevrey in t, uniformly in x on a neighborhood of $x=(0,...,0)\in\mathbb{C}^n$.

We denote by $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ the set of all the formal series in $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]$ which are s-Gevrey. Observe that the set $\mathbb{C}\{t,x\}$ of germs of analytic functions at the origin of $\mathbb{C}^{n+1}$ coincides with the union $\bigcup_{\rho_1>0,...,\rho_n>0}\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_0$; in particular, any element of $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_0$ is convergent and $\mathbb{C}\{t,x\}\cap\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]=\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_0$. Observe also that the sets $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ are filtered as follows:

$$\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_0\subset\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s\subset\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_{s'}\subset\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]$$

for all s and s' satisfying $0<s<s'<+\infty$.

Following proposition 3.2 specifies the algebraic structure of $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$.

Proposition 3.2. *Let $s\geqslant 0$. Then, the set $(\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s,\partial_t,\partial_{x_1},...,\partial_{x_n})$ is a $\mathbb{C}$-differential algebra.*

Proof. Since $(\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]],\partial_t,\partial_{x_1},...,\partial_{x_n})$ is a $\mathbb{C}$-differential algebra, it is sufficient to prove that $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ is stable under multiplication and derivations.

The proof of the stability under the multiplication and the derivation ∂_t is similar to the one already detailed in [26, Prop. 1] (see also [1, p. 64]) in the case $n=1$.

To prove the stability under the derivation ∂_{x_ℓ} with $\ell\in\{1,...,n\}$, we proceed as follows. Let $\widetilde{u}(t,x)\in\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ as in definition 3.1 and $\widetilde{w}(t,x)=\partial_{x_\ell}\widetilde{u}(t,x)$. For a given $0<\rho'<\rho$, the Cauchy Integral Formula gives us, for all $j\geqslant 0$ and all $\|x\|\leqslant\rho'$:

$$w_{j,*}(x)=\partial_{x_\ell}u_{j,*}(x)=\frac{1}{(2i\pi)^n}\int_{\gamma(x)}\frac{u_{j,*}(x')}{(x'_\ell-x_\ell)^2\displaystyle\prod_{\substack{k=1\\k\neq\ell}}^{n}(x'_k-x_k)}dx',$$

where $\gamma(x):=\{x'=(x'_1,...,x'_n)\in\mathbb{C}^n;|x'_k-x_k|=\rho-\rho'\text{ for all }k\in\{1,...,n\}\}$. Hence, the inequalities

$$\sup_{\|x\|\leqslant\rho'}|w_{j,*}(x)|\leqslant C'K^j\Gamma(1+(s+1)j)\quad\text{with } C'=\frac{C}{\rho-\rho'}\text{ for all }j\geqslant 0.$$

Indeed, the definition of the path $\gamma(x)$ implies $\|x'\|\leqslant\rho$. The proof is complete. □

Observe that the stability under the derivation ∂_{x_ℓ} would not be guaranteed without the condition *"there exist* $0<\rho<\min\rho_\ell$ *..."* in definition 3.1.

3.2. **Main result.** We are now able to state the result in view in this article.

Theorem 3.3. *Recall that* $\mathcal{S} = \{(i,q)$ *such that* $i \in \mathcal{K},\ q \in Q_i$ *and* $\lambda(q) > \kappa - i\}$. *Let* s_c *be the nonnegative rational number defined by*

$$s_c := \begin{cases} 0 & \text{if } \mathcal{S} = \varnothing \\ \dfrac{1}{k} = \dfrac{\lambda(q^*) - \kappa + i^*}{v_{i^*,q^*} + \kappa - i^*} & \text{if } \mathcal{S} \neq \varnothing \end{cases}$$

Then,

(1) $\widetilde{u}(t,x)$ *and* $\widetilde{f}(t,x)$ *are together* s*-Gevrey for any* $s \geqslant s_c$*;*
(2) $\widetilde{u}(t,x)$ *is generically* s_c*-Gevrey while* $\widetilde{f}(t,x)$ *is* s*-Gevrey with* $s < s_c$.

Definition 3.4. The number s_c defined in theorem 3.3 is called *the critical value of eq. (1.1)*.

Observe that theorem 3.3 coincides with proposition 1.1 in the case of eqs. (1.3) and (1.4). Besides, since no condition is made on the polynomial P except its coefficients are analytic at the origin of $\mathbb{C}^{n+1}$, theorem 3.3 applies as well to the linear case $P \equiv 0$, and, consequently, generalizes the results already obtained by the author in [26, 28].

Observe also that theorem 3.3 yields a result similar to the Maillet-Ramis theorem for the ordinary linear differential equations [22, 23] (see also [10, Thm. 4.2.7]).

Corollary 3.5. *Assume that the inhomogeneity* $\widetilde{f}(t,x)$ *is convergent. Then, the solution* $\widetilde{u}(t,x)$ *is either convergent or* $1/k$*-Gevrey, where* k *stands for the smallest positive slope of the Newton polygon* $N_t(L)$.

The proof of theorem 3.3 is detailed in section 4 below. The first point is the most technical and the most complicated. Its proof is based on the Nagumo norms, a technique of majorant series and a fixed point procedure (see section 4.1). As for the second point, it stems both from the first one and from proposition 4.11 that gives an explicit example for which $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$ while $\widetilde{f}(t,x)$ is s-Gevrey with $s < s_c$ (see section 4.2).

4. Proof of theorem 3.3

4.1. **Proof of the first point.** According to proposition 3.2, it is clear that

$$\widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]_s \Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})[[t]]_s.$$

Reciprocally, let us fix $s \geqslant s_c$ and let us suppose that the inhomogeneity $\widetilde{f}(t,x)$ is s-Gevrey. By assumption, its coefficients $f_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})$ satisfy the following condition (see definition 3.1): there exist three positive constants $0 < \rho < \min \rho_\ell$, $C > 0$ and $K > 0$ such that the inequalities

$$|f_{j,*}(x)| \leqslant CK^j \Gamma(1 + (s+1)j) \tag{4.1}$$

hold for all $j \geqslant 0$ and all $\|x\| \leqslant \rho$.

We must prove that the coefficients $u_{j,*}(x) \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})$ of $\widetilde{u}(t,x)$ satisfy similar inequalities. The approach we present below is analoguous to the ones already developed in [2, 26–28] in the framework of linear partial and integro-differential equations, and in [24, 25] in the case of eqs. (1.3) and (1.4). It is based on the Nagumo norms [3, 18, 38] and on a technique of majorant series. However, as we shall see, our calculations appear to be much more technical and complicated.

Furthermore, the nonlinear polynomial term $P(u)$ instead of the power term u^m used in eqs. (1.3) and (1.4) generates several new technical combinatorial situations.

Before starting the calculations, let us first recall for the convenience of the reader the definition of the Nagumo norms and some of their properties which are needed in the sequel.

4.1.1. *Nagumo norms.*

Definition 4.1. Let $f \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})$, $p \geqslant 0$ and $0 < r < \min \rho_\ell$ be. Then, the *Nagumo norm* $\|f\|_{p,r}$ *with indices* (p,r) *of* f is defined by

$$\|f\|_{p,r} := \sup_{\|x\| \leqslant r} |f(x) d_r(x)^p| ,$$

where $d_r(x)$ denotes the Euclidian distance $d_r(x) := r - \|x\|$.

Following proposition 4.2 gives us some properties of the Nagumo norms.

Proposition 4.2. *Let* $f, g \in \mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})$, $p, p' \geqslant 0$ *and* $0 < r < \min \rho_\ell$ *be. Then,*

(1) $\|\cdot\|_{p,r}$ *is a norm on* $\mathcal{O}(\mathcal{D}_{\rho_1,\dots,\rho_n})$.
(2) $|f(x)| \leqslant \|f\|_{p,r} d_r(x)^{-p}$ *for all* $\|x\| < r$.
(3) $\|f\|_{0,r} = \sup_{\|x\| \leqslant r} |f(x)|$ *is the usual sup-norm on the polydisc* $\mathcal{D}_{r,\dots,r}$.
(4) $\|fg\|_{p+p',r} \leqslant \|f\|_{p,r} \|g\|_{p',r}$.
(5) $\|\partial_{x_\ell} f\|_{p+1,r} \leqslant e(p+1) \|f\|_{p,r}$ *for all* $\ell \in \{1, \dots, n\}$.

Proof. Properties 1–4 are straightforward and are left to the reader. To prove Property 5, we proceed as follows. Let $\ell \in \{1, \dots, n\}$ be, $x \in \mathbb{C}^n$ such that $\|x\| < r$ and $0 < R < d_r(x)$. Using the Cauchy Integral Formula, we have

$$\partial_{x_\ell} f(x) = \frac{1}{(2i\pi)^n} \int_{\gamma(x)} \frac{f(x')}{(x'_\ell - x_\ell)^2 \prod\limits_{\substack{k=1 \\ k \neq \ell}}^{n} (x'_k - x_k)} dx',$$

where $\gamma(x) := \{x' = (x'_1, \dots, x'_n) \in \mathbb{C}^n ; |x'_k - x_k| = R \text{ for all } k \in \{1, \dots, n\}\}$. Since

$$x' \in \gamma(x) \Rightarrow \|x'\| < r,$$

we can apply Property 2 of proposition 4.2; hence, the inequalities

$$|\partial_{x_\ell} f(x)| \leqslant \frac{1}{R} \max_{x' \in \gamma(x)} |f(x')| \leqslant \frac{1}{R} \|f\|_{p,r} \max_{x' \in \gamma(x)} d_r(x')^{-p} = \frac{1}{R} \|f\|_{p,r} (d_r(x) - R)^{-p}.$$

Observe that the last equality stems from the relations

$$d_r(x') = r - \|x'\| = r - \|x + x' - x\| \geqslant d_r(x) - \|x' - x\| = d_r(x) - R > 0.$$

When $p = 0$, the choice $R = \dfrac{d_r(x)}{e}$ implies the inequality

$$|\partial_{x_\ell} f(x)| \leqslant e \|f\|_{0,r} d_r(x)^{-1};$$

hence, the inequality

$$|\partial_{x_\ell} f(x)| d_r(x) \leqslant e \|f\|_{0,r} . \tag{4.2}$$

When $p > 0$, the choice $R = \dfrac{d_r(x)}{p+1}$ and the relations

$$\left(1 - \frac{1}{p+1}\right)^{-p} = \left(1 + \frac{1}{p}\right)^{p} < e,$$

brings us to the inequalities

$$|\partial_{x_\ell} f(x)| \leqslant \|f\|_{p,r}\, d_r(x)^{-p-1}(p+1)\left(1 - \frac{1}{p+1}\right)^{-p} \leqslant e(p+1)\, \|f\|_{p,r}\, d_r(x)^{-p-1}$$

and then to the inequality

$$|\partial_{x_\ell} f(x)|\, d_r(x)^{p+1} \leqslant e(p+1)\, \|f\|_{p,r}\,. \tag{4.3}$$

Property 5 follows since inequalities (4.2) and (4.3) are still valid when $\|x\| = r$. This achieves the proof of proposition 4.2. ☐

Remark 4.3. Inequalities 4–5 of proposition 4.2 are the most important properties. Observe besides that the same index r occurs on their both sides, allowing thus to get estimates for the product fg in terms of f and g and for the derivatives $\partial_{x_\ell} f$ for any $\ell \in \{1, ..., n\}$ in terms of f without having to shrink the polydisc $\mathcal{D}_{r,...,r}$.

Let us now turn to the proof of the first point of theorem 3.3.

4.1.2. *Some inequalities.* From the recurrence relations (1.2), we first get the identities

$$\begin{aligned}\frac{u_{j+\kappa,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} &= \frac{f_{j,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} + \\ &\sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell=0}^{j-v_{i,q}} \frac{j!}{\ell!(j-v_{i,q}-\ell)!}\frac{a_{i,q;\ell,*}(x)\partial_x^q u_{j-v_{i,q}-\ell+i,*}(x)}{\Gamma(1+(s+1)(j+\kappa))} + \\ &\qquad \sum_{m=2}^{d}\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}} \frac{j!}{\ell!\ell_1!...\ell_m!}\frac{b_{m;\ell,*}(x)u_{\ell_1,*}(x)...u_{\ell_m,*}(x)}{\Gamma(1+(s+1)(j+\kappa))}\end{aligned} \tag{4.4}$$

for all $j \geqslant 0$, together with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0, ..., \kappa - 1$. As usual, we use the classical convention that the first sum is zero as soon as $j - v_{i,q} < 0$.

Let us now define the positive constant $\sigma_s = (s+1)(\kappa + v)$ with $v = \max v_{i,q}$. The following lemma yields various inequalities which will play a crucial role in the sequel of our proof.

Lemma 4.4. *The inequalities*

$$\sigma_s \geqslant (s+1)(\kappa + v_{i,q}) \geqslant (s+1)(\kappa - i + v_{i,q}) \geqslant \lambda(q) + v_{i,q}$$

hold for all $i \in \mathcal{K}$ and all $q \in Q_i$.

Proof. The first two inequalities are trivial. To prove the third, we proceed as follows.

Let us first assume $\mathcal{S} = \varnothing$. Since $s \geqslant s_c = 0$, we have

$$(s+1)(\kappa - i + v_{i,q}) \geqslant \kappa - i + v_{i,q},$$

and the result stems from the inequality $\lambda(q) \leqslant \kappa - i$.

Let us now assume $\mathcal{S} \neq \varnothing$. Then,

$$(s+1)(\kappa - i + v_{i,q}) \geqslant (s_c+1)(\kappa - i + v_{i,q}) = \frac{\lambda(q^*) + v_{i^*,q^*}}{v_{i^*,q^*} + \kappa - i^*}(\kappa - i + v_{i,q}),$$

and the inequality follows from corollary 2.3. □

Let us apply the Nagumo norms of indices $((j+\kappa)\sigma_s, \rho)$ to relations (4.4). From the first property of proposition 4.2, we obtain

$$\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell=0}^{j-v_{i,q}} \frac{j!}{\ell!(j-v_{i,q}-\ell)!}\frac{\left\|a_{i,q;\ell,*}\partial_x^q u_{j-v_{i,q}-\ell+i,*}\right\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{m=2}^{d}\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}} \frac{j!}{\ell!\ell_1!\ldots\ell_m!}\frac{\|b_{m;\ell,*}u_{\ell_1,*}\ldots u_{\ell_m,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}$$

for all $j \geqslant 0$. Next, using the fourth and fifth property of proposition 4.2, we derive the following inequalities for all j:

$$\frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} + \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell=0}^{j-v_{i,q}} A_{i,q,j,\ell,s}\frac{\|a_{i,q;\ell,*}\|_{(v_{i,q}+\kappa-i+\ell)\sigma_s-\lambda(q),\rho}}{\ell!}\frac{\left\|u_{j-v_{i,q}-\ell+i,*}\right\|_{(j-v_{i,q}-\ell+i)\sigma_s,\rho}}{\Gamma(1+(s+1)(j-v_{i,q}-\ell+i))} + \sum_{m=2}^{d}\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}} B_{j,\ell,\ell_1,\ldots\ell_m,s}\frac{\|b_{m;\ell,*}\|_{(\ell+\kappa)\sigma_s,\rho}\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}\ldots\|u_{\ell_m,*}\|_{\ell_m\sigma_s,\rho}}{\ell!\Gamma(1+(s+1)\ell_1)\ldots\Gamma(1+(s+1)\ell_m)}, \tag{4.5}$$

where the constants $A_{i,q,j,\ell,s}$ and $B_{j,\ell,\ell_1,\ldots\ell_m,s}$ are positive and defined by

$$A_{i,q,j,\ell,s} = \frac{j!e^{\lambda(q)}\left(\displaystyle\prod_{\ell'=0}^{\lambda(q)-1}((j-v_{i,q}-\ell+i)\sigma_s+\lambda(q)-\ell')\right)}{(j-v_{i,q}-\ell)!\Gamma(1+(s+1)(j+\kappa))}\times \Gamma(1+(s+1)(j-v_{i,q}-\ell+i))$$

and

$$B_{j,\ell,\ell_1,\ldots\ell_m,s} = \frac{j!}{\ell_1!\ldots\ell_m!}\frac{\Gamma(1+(s+1)\ell_1)\ldots\Gamma(1+(s+1)\ell_m)}{\Gamma(1+(s+1)(j+\kappa))}.$$

In the definition of the constants $A_{i,q,j,\ell,s}$, we use of course the classical convention that the product is 1 when $\lambda(q) = 0$

Observe that all the norms written in inequality (4.5), and especially the norms $\|a_{i,q;\ell,*}\|_{(v_{i,q}+\kappa-i+\ell)\sigma_s-\lambda(q),\rho}$ are well-defined. Indeed, the inequalities $\kappa - i \geqslant 1$ and $v_{i,q} \geqslant 0$, and lemma 4.4 imply for all i, q and ℓ:

$$(v_{i,q}+\kappa-i+\ell)\sigma_s - \lambda(q) \geqslant \sigma_s - \lambda(q) \geqslant \lambda(q) + v_{i,q} - \lambda(q) = v_{i,q} \geqslant 0.$$

The following proposition allows to bound the constants $A_{i,q,j,\ell,s}$ and $B_{j,\ell,\ell_1,\ldots\ell_m,s}$.

Proposition 4.5.

(1) *Let $i \in \mathcal{K}$, $q \in Q_i$, $j \geqslant v_{i,q}$ and $\ell \in \{0, ..., j - v_{i,q}\}$ be. Then,*

$$A_{i,q,j,\ell,s} \leqslant (e(\kappa + v))^{\lambda(q)}.$$

(2) *Let $m \in \{2, ..., d\}$, $j \geqslant 0$ and $\ell \in \{0, ..., j\}$ be. Then,*

$$B_{j,\ell,\ell_1,...\ell_m,s} \leqslant 1$$

for all $\ell_1, ..., \ell_m \in \mathbb{N}$ such that $\ell_1 + ... + \ell_m = j - \ell$.

The first point is straightforward from the two following technical lemmas 4.6 and 4.7 below. The second point is proved in [24, Prop. 4.8].

Lemma 4.6. *Let $i \in \mathcal{K}$, $q \in Q_i$, $j \geqslant v_{i,q}$ and $\ell \in \{0, ..., j - v_{i,q}\}$ be. Then,*

$$\frac{j!}{(j - v_{i,q} - \ell)!\Gamma(1 + (s + 1)(j + \kappa))} \leqslant \frac{1}{\Gamma(1 + (s + 1)(j + \kappa - \ell) - v_{i,q})}.$$

Proof. Lemma 4.6 is clear for $\ell + v_{i,q} = 0$. Let us now assume $\ell + v_{i,q} \geqslant 1$ and let us write the quotient $j!/(j - v_{i,q} - \ell)!$ on the form

$$\frac{j!}{(j - v_{i,q} - \ell)!} = \prod_{\ell'=0}^{\ell+v_{i,q}-1} (j - \ell'). \tag{4.6}$$

On the other hand, applying $\ell + v_{i,q}$ times the recurrence relation $\Gamma(1 + z) = z\Gamma(z)$ to $\Gamma(1 + (s + 1)(j + \kappa))$, we get

$$\Gamma(1+(s+1)(j+\kappa)) = \Gamma(1+(s+1)(j+\kappa)-\ell-v_{i,q}) \prod_{\ell'=0}^{\ell+v_{i,q}-1} ((s+1)(j+\kappa)-\ell'). \tag{4.7}$$

Combinating then (4.6) and (4.7), we obtain

$$\begin{aligned}\frac{j!}{(j - v_{i,q} - \ell)!\Gamma(1 + (s + 1)(j + \kappa))} &= \frac{\displaystyle\prod_{\ell'=0}^{\ell+v_{i,q}-1} \frac{j - \ell'}{(s + 1)(j + \kappa) - \ell'}}{\Gamma(1 + (s + 1)(j + \kappa) - \ell - v_{i,q})} \\ &\leqslant \frac{1}{\Gamma(1 + (s + 1)(j + \kappa) - \ell - v_{i,q})}\end{aligned}$$

and lemma 4.6 follows from the inequalities

$$\begin{aligned}1 + (s + 1)(j + \kappa) - \ell - v_{i,q} &\geqslant 1 + (s + 1)(j + \kappa - \ell) - v_{i,q} \\ &\geqslant 1 + (s + 1)(\kappa + v_{i,q}) - v_{i,q} \\ &\geqslant 1 + \kappa \\ &\geqslant 2\end{aligned}$$

and from the increase of the Gamma function on $[2, +\infty[$. □

Lemma 4.7. *Let $i \in \mathcal{K}$, $q \in Q_i$, $j \geqslant v_{i,q}$ and $\ell \in \{0, ..., j - v_{i,q}\}$ be. Then,*

$$\frac{\displaystyle\prod_{\ell'=0}^{\lambda(q)-1} ((j - v_{i,q} - \ell + i)\sigma_s + \lambda(q) - \ell')}{\Gamma(1 + (s + 1)(j + \kappa - \ell) - v_{i,q})} \leqslant \frac{(\kappa + v)^{\lambda(q)}}{\Gamma(1 + (s + 1)(j - v_{i,q} - \ell + i))}.$$

Proof. $\lhd$ Let us first assume $\ell = j - v_{i,q}$ and $i = 0$. We must prove the inequality

$$\frac{\prod_{\ell'=0}^{\lambda(q)-1} (p-\ell')}{\Gamma(1+(s+1)(\kappa+v_{i,q})-v_{i,q})} \leqslant (\kappa+v)^{\lambda(q)}. \tag{4.8}$$

Let us begin by observing that

$$\prod_{\ell'=0}^{\lambda(q)-1} (\lambda(q)-\ell') = (\lambda(q))! = \Gamma(1+\lambda(q))$$

for all $\lambda(q)$, including the case $\lambda(q) = 0$ since the product is 1 by convention.

On the other hand, in the case $\lambda(q) > 0$, lemma 4.4 implies the inequalities

$$1+(s+1)(\kappa+v_{i,q})-v_{i,q} \geqslant 1+\lambda(q) \geqslant 2;$$

hence,

$$\Gamma(1+(s+1)(\kappa+v_{i,q})-v_{i,q}) \geqslant \Gamma(1+\lambda(q))$$

since the Gamma function is increasing on $[2,+\infty[$. In the special case $\lambda(q) = 0$, we observe that the increase of the Gamma function applied to the inequalities

$$1+(s+1)(\kappa+v_{i,q})-v_{i,q} \geqslant 1+\kappa \geqslant 2$$

implies

$$\Gamma(1+(s+1)(\kappa+v_{i,q})-v_{i,q}) \geqslant \Gamma(2) = \Gamma(1) = \Gamma(1+\lambda(q)).$$

Consequently, the left hand-side of (4.8) is $\leqslant 1$ and lemma 4.7 follows then from the inequality $\kappa+v \geqslant 1$.

$\lhd$ Let us now assume $(\ell, i) \neq (j-v_{i,q}, 0)$. According to the definition of σ_s, we first have the identity

$$\prod_{\ell'=0}^{\lambda(q)-1} ((j-v_{i,q}-\ell+i)\sigma_s+\lambda(q)-\ell') = (\kappa+v)^{\lambda(q)} \times \prod_{\ell'=0}^{\lambda(q)-1} \left((s+1)(j-v_{i,q}-\ell+i)+\frac{\lambda(q)-\ell'}{\kappa+v}\right). \tag{4.9}$$

On the other hand, applying $\lambda(q)$ times the recurrence relation $\Gamma(1+z) = z\Gamma(z)$ to $\Gamma(1+(s+1)(j+\kappa-\ell)-v_{i,q})$, we besides have

$$\Gamma(1+(s+1)(j+\kappa-\ell)-v_{i,q}) = \Gamma(1+(s+1)(j+\kappa-\ell)-v_{i,q}-\lambda(q)) \prod_{\ell'=0}^{\lambda(q)-1} ((s+1)(j+\kappa-\ell)-v_{i,q}-\ell'). \tag{4.10}$$

Observe that this identity makes since lemma 4.4 implies

$$(s+1)(j+\kappa-\ell)-v_{i,q}-\lambda(q) \geqslant (s+1)(\kappa+v_{i,q})-v_{i,q}-\lambda(q) \geqslant 0.$$

Observe also that we have the inequality

$$(s+1)(j-v_{i,q}-\ell+i)+\frac{\lambda(q)-\ell'}{\kappa+v} \leqslant (s+1)(j+\kappa-\ell)-v_{i,q}-\ell'$$

for all $\ell' \in \{0, ..., \lambda(q)-1\}$ when $\lambda(q) > 0$. Indeed, according to lemma 4.4 and the inequality $\kappa + v \geqslant 1$, we have

$$\begin{aligned}(s{+}1)(j+\kappa-\ell) - v_{i,q} - \ell' - (s+1)(j-v_{i,q}-\ell+i) - \frac{\lambda(q)-\ell'}{\kappa+v} \\ = (s+1)(v_{i,q}+\kappa-i) - v_{i,q} - \ell' - \frac{\lambda(q)-\ell'}{\kappa+v} \geqslant (\lambda(q)-\ell')\left(1-\frac{1}{\kappa+v}\right) \geqslant 0.\end{aligned}$$

Consequently, identities (4.9) and (4.10) implies the inequality

$$\frac{\displaystyle\prod_{\ell'=0}^{\lambda(q)-1}((j-v_{i,q}-\ell+i)\sigma_s+\lambda(q)-\ell')}{\Gamma(1+(s+1)(j+\kappa-\ell)-v_{i,q})} \leqslant \frac{(\kappa+v)^{\lambda(q)}}{\Gamma(1+(s+1)(j+\kappa-\ell)-v_{i,q}-\lambda(q))}$$

for all $\lambda(q)$. Lemma 4.7 follows then from the relations

$$\begin{aligned}1+(s+1)(j+\kappa-\ell)-v_{i,q}-\lambda(q) &\geqslant 1+(s+1)(j+\kappa-\ell)-(s+1)(\kappa-i+v_{i,q}) \\ &= 1+(s+1)(j-v_{i,q}-\ell+i) \\ &\geqslant 2\end{aligned}$$

and from the increase of the Gamma function on $[2,+\infty[$. Observe that the first inequality stems again from lemma 4.4. Observe also that, without the condition $(\ell, i) \neq (j-v_{i,q}, 0)$, the second inequality is no longer valid.

This ends the proof of lemma 4.7. □

Let us now apply proposition 4.5 to inequalities (4.5). We get:

$$\begin{aligned}(4.11)\quad \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant g_{j,s} + \\ \sum_{i\in\mathcal{K}}\sum_{q\in Q_i}\sum_{\ell=0}^{j-v_{i,q}} \alpha_{i,q,\ell,s}\frac{\|u_{j-v_{i,q}-\ell+i,*}\|_{(j-v_{i,q}-\ell+i)\sigma_s,\rho}}{\Gamma(1+(s+1)(j-v_{i,q}-\ell+i))} + \\ \sum_{m=2}^{d}\sum_{\ell=0}^{j}\sum_{\substack{\ell_1+\ldots+\ell_m\\=j-\ell}} \beta_{m,\ell,s}\frac{\|u_{\ell_1,*}\|_{\ell_1\sigma_s,\rho}}{\Gamma(1+(s+1)\ell_1)}\cdots\frac{\|u_{\ell_m,*}\|_{\ell_m\sigma_s,\rho}}{\Gamma(1+(s+1)\ell_m)},\end{aligned}$$

for all $j \geqslant 0$, where the constants $g_{j,s}$, $\alpha_{i,q,\ell,s}$ and $\beta_{m,\ell,s}$ are positive and defined by

$$\begin{aligned}g_{j,s} &= \frac{\|f_{j,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))}, \\ \alpha_{i,q,\ell,s} &= (e(\kappa+v))^{\lambda(q)}\frac{\|a_{i,q;\ell,*}\|_{(v_{i,q}+\kappa-i+\ell)\sigma_s-\lambda(q),\rho}}{\ell!}, \\ \beta_{m,\ell,s} &= \frac{\|b_{m;\ell,*}\|_{(\ell+\kappa)\sigma_s,\rho}}{\ell!}.\end{aligned}$$

We shall now bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ for any $j \geqslant 0$. To do that, we shall proceed similarly as in [2, 24–28] by using a technique of majorant series. However, as we shall see, the calculations are much more complicated.

4.1.3. *A Majorant Series.* Let us consider the formal series $v(X) = \sum_{j\geqslant 0} v_j X^j$, where the coefficients v_j are determined for all $j \geqslant 0$ by the recurrence relations

$$v_{j+\kappa} = g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell=0}^{j-v_{i,q}+i} \alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i} + \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+...+\ell_d\\=j-\ell}} \beta_{\ell,s} v_{\ell_1}...v_{\ell_d} \tag{4.12}$$

with

$$\beta_{\ell,s} = \sum_{m=2}^{d} \beta_{m,\ell,s},$$

and together with the initial conditions

$$v_0 = 1 + \|\varphi_0\|_{0,\rho}\,, \text{ and, for } j = 1, ..., \kappa-1 \text{ (if } \kappa \geqslant 2\text{):}$$
$$v_j = \frac{\|\varphi_j\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} + \sum_{(i,q)\in V_j} \sum_{\ell=0}^{j-\kappa-v_{i,q}+i} \alpha_{i,q,\ell,s} v_{j-\kappa-v_{i,q}-\ell+i}$$

where $V_j = \{(i,q) \in \mathcal{K} \times Q_i \text{ such that } j-\kappa-v_{i,q}+i \geqslant 0\}$. Observe that the condition "$\kappa > i$ for all $i \in \mathcal{K}$" implies $j-\kappa-v_{i,q}+i < j$; hence, the initial conditions on the v_j's with $j = 1, ..., \kappa-1$ make sense. Observe also that the set V_j may be empty (this is particularly the case when $\mathcal{K} = \{0\}$, or when $v_{i,q} \geqslant i$ for all i and q).

Proposition 4.8. *The inequalities*

$$0 \leqslant \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)} \leqslant v_j \tag{4.13}$$

hold for all $j \geqslant 0$.

Proof. According to the initial conditions on the u_j's and on the v_j's, inequalities (4.13) hold for all $j = 0, ..., \kappa-1$. Let us now suppose that these inequalities are true for all $k \leqslant j-1+\kappa$ for a certain $j \geqslant 0$, and let us prove them for $j+\kappa$.

First of all, applying our hypotheses to relations (4.11), we get

$$\begin{aligned} 0 \leqslant \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} &\leqslant g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell=0}^{j-v_{i,q}} \alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i} + \\ &\qquad \sum_{m=2}^{d} \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}} \beta_{m,\ell,s} v_{\ell_1}...v_{\ell_m} \\ &\leqslant g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell=0}^{j-v_{i,q}+i} \alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i} + \\ &\qquad \sum_{m=2}^{d} \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+...+\ell_m\\=j-\ell}} \beta_{m,\ell,s} v_{\ell_1}...v_{\ell_m}, \end{aligned}$$

since all the terms $\alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i}$ are nonnegative.

Next, let us observe that, for all $\ell \in \{0, ..., j\}$ and all $m \in \{2, ..., d-1\}$ if $d \geqslant 3$, any tuple-m $(\ell_1, ..., \ell_m) \in \mathbb{N}^m$ such that $\ell_1 + ... + \ell_m = j-\ell$ can be seen as the tuple-d $(\ell_1, ..., \ell_m, \ell_{m+1}, ..., \ell_d) \in \mathbb{N}^d$, where $\ell_{m+1} = ... = \ell_d = 0$. Therefore, using the fact that $v_0 \geqslant 1$, we have

$$0 \leqslant v_{\ell_1} ... v_{\ell_m} \leqslant v_{\ell_1} ... v_{\ell_m} v_0^{d-m} = v_{\ell_1} ... v_{\ell_m} v_{\ell_{m+1}} ... v_{\ell_d},$$

and, consequently, the inequalities

$$0 \leqslant \sum_{\substack{\ell_1+...+\ell_m \\ =j-\ell}} v_{\ell_1} ... v_{\ell_m} \leqslant \sum_{\substack{\ell_1+...+\ell_m+ \\ 0+...+0=j-\ell}} v_{\ell_1} ... v_{\ell_d} \leqslant \sum_{\substack{\ell_1+...+\ell_d \\ =j-\ell}} v_{\ell_1} ... v_{\ell_d}$$

since all the terms are nonnegative.

Hence, the relations

$$\begin{aligned} 0 \leqslant \frac{\|u_{j+\kappa,*}\|_{(j+\kappa)\sigma_s,\rho}}{\Gamma(1+(s+1)(j+\kappa))} &\leqslant g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell=0}^{j-v_{i,q}+i} \alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i} + \\ &\qquad \sum_{m=2}^{d} \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+...+\ell_d \\ =j-\ell}} \beta_{m,\ell,s} v_{\ell_1} ... v_{\ell_d} \\ &= g_{j,s} + \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} \sum_{\ell=0}^{j-v_{i,q}+i} \alpha_{i,q,\ell,s} v_{j-v_{i,q}-\ell+i} + \\ &\qquad \sum_{\ell=0}^{j} \sum_{\substack{\ell_1+...+\ell_d \\ =j-\ell}} \beta_{\ell,s} v_{\ell_1} ... v_{\ell_d} \\ &= v_{j+\kappa}, \end{aligned}$$

which ends the proof of proposition 4.8. □

Following proposition 4.9 allows us to bound the v_j's.

Proposition 4.9. *The formal series $v(X)$ is convergent. In particular, there exist two positive constants $C', K' > 0$ such that $v_j \leqslant C'K'^j$ for all $j \geqslant 0$.*

Proof. It is sufficient to prove the convergence of $v(X)$.

First of all, let us observe that $v(X)$ is the unique formal power series in X solution of the functional equation

$$(1 - \alpha(X))v(X) = X^\kappa \beta(X)(v(X))^d + h(X), \tag{4.14}$$

where

$$\begin{aligned} \alpha(X) &= \sum_{i\in\mathcal{K}} \sum_{q\in Q_i} X^{\kappa-i+v_{i,q}} \alpha_{i,q}(X) \text{ with } \alpha_{i,q}(X) = \sum_{j\geqslant 0} \alpha_{i,q,j,s} X^j; \\ \beta(X) &= \sum_{j\geqslant 0} \beta_{j,s} X^j; \\ h(X) &= A_0 + A_1 X + ... + A_{\kappa-1} X^{\kappa-1} + X^\kappa \sum_{j\geqslant 0} g_{j,s} X^j \end{aligned}$$

with

$$A_0 = 1 + \|\varphi_0\|_{0,\rho}, \text{ and, for } j = 1, ..., \kappa-1 \text{ (if } \kappa \geqslant 2), \ A_j = \frac{\|\varphi_j\|_{j\sigma_s,\rho}}{\Gamma(1+(s+1)j)}.$$

Observe that, according to the analyticity of the functions $a_{i,q}(t,x)$ and $b_m(t,x)$ at the origin of $\mathbb{C}^{n+1}$, and the hypothesis on the coefficients $f_{j,*}(x)$ of $\widetilde{f}(t,x)$ (see inequality (4.1)), we have the inequalities

- $0 \leqslant \alpha_{i,q,j,s} \leqslant \dfrac{(e(\kappa+v))^{\lambda(q)} C_1 K_1^j j! \rho^{(v_{i,q}+\kappa-i+j)\sigma_s - \lambda(q)}}{j!} = C_1' K_1'^j$,
- $0 \leqslant \beta_{j,s} \leqslant (d-1) \dfrac{C_1 K_1^j j! \rho^{(j+\kappa)\sigma_s}}{j!} = C_1'' K_1''^j$, and
- $0 \leqslant g_j \leqslant \dfrac{C K^j \Gamma(1+(s+1)j) \rho^{(j+\kappa)\sigma_s}}{\Gamma(1+(s+1)(j+\kappa))} \leqslant C \rho^{\kappa\sigma_s} (K\rho^{\sigma_s})^j$

with convenient positive constants C_1, K_1, C_1', K_1', C_1'' and K_1''. Hence, the series $\alpha(X)$, $\beta(X)$ and $h(X)$ are all convergent with nonnegative coefficients. In the sequel, we denote by $r_\alpha > 0$ (resp. $r_\beta > 0$, $r_h > 0$) the radius of convergence of the series $\alpha(X)$ (resp. $\beta(X)$, $h(X)$). We also denote by $r_\alpha' > 0$ the radius of convergence of the series $1/(1-\alpha(X))$ (which is of course well-defined since $\alpha(0) = 0$).

The convergence of $v(X)$ being obvious when $\beta(X) \equiv 0$ (we have indeed $(1-\alpha(X))v(X) = h(X)$), we suppose in the sequel that $\beta(X) \not\equiv 0$. In particular, we have $\beta(X) > 0$ for all $X \in]0, r_\beta[$. Notice that we also have $h(X) \geqslant 1$ for all $X \in [0, r_h[$. To prove the convergence of the series $v(X)$, we proceed through a fixed point method as follows. Let us set

$$V(X) = \sum_{m \geqslant 0} V_m(X)$$

and let us choose the solution of eq. (4.14) given by the system

$$\begin{cases} (1-\alpha(X))V_0(X) = h(X) \\ (1-\alpha(X))V_{m+1}(X) = X^\kappa \beta(X) \displaystyle\sum_{\substack{\ell_1+\ldots+\ell_d \\ =m}} V_{\ell_1}(X) \ldots V_{\ell_d}(X) \quad \text{for } m \geqslant 0. \end{cases}$$

By induction on $m \geqslant 0$, we easily check that

$$V_m(X) = \frac{C_{m,d} X^{\kappa m} (\beta(X))^m (h(X))^{m(d-1)+1}}{(1-\alpha(X))^{md+1}}, \tag{4.15}$$

where the $C_{m,d}$'s are the positive constants recursively determined from $C_{0,d} := 1$ by the relations

$$C_{m+1,d} = \sum_{k_1+\ldots+k_d=m} C_{k_1,d} \ldots C_{k_d,d}.$$

Thereby, all the $V_m(X)$'s are analytic functions on the disc with center $0 \in \mathbb{C}$ and radius $\min(r_\alpha', r_\beta, r_h)$ at least. Moreover, identities (4.15) show us that $V_m(X)$ is of order $X^{\kappa m}$ for all $m \geqslant 0$. Consequently, the series $V(X)$ makes sense as a formal power series in X and we get $V(X) = v(X)$ by unicity.

We are left to prove the convergence of $V(X)$. To do that, let us choose $0 < r < \min(r_\alpha', r_\alpha, r_\beta, r_h)$. By definition, the constants $C_{m,d}$'s are the generalized Catalan numbers of order d and we have[4]

$$C_{m,d} = \frac{1}{(d-1)m+1}\binom{md}{m} \leqslant 2^{md}$$

for all $m \geqslant 0$ (see [4,5,21] for instance). On the other hand, the convergent series $\alpha(X)$, $\beta(X)$ and $h(X)$ define increasing functions on $[0,r]$, since theirs coefficients are nonnegative. Therefore, identities (4.15) imply the inequalities

$$|V_m(X)| \leqslant \frac{h(r)}{1-\alpha(r)}\left(\frac{2^d\beta(r)(h(r))^{d-1}}{(1-\alpha(r))^d}|X|^\kappa\right)^m$$

for all $m \geqslant 0$ and all $|X| \leqslant r$. Consequently, since $\beta(r) > 0$ and $h(r) > 0$ (see the remark just above), the series $V(X)$ is normally convergent on any disc with center $0 \in \mathbb{C}$ and radius

$$0 < r' < \min\left(r, \left(\frac{(1-\alpha(r))^d}{2^d\beta(r)(h(r))^{d-1}}\right)^{1/\kappa}\right).$$

This proves the analyticity of $V(X)$ at 0 and achieves then the proof of proposition 4.9. □

According to propositions 4.8 and 4.9, we can now bound the Nagumo norms $\|u_{j,*}\|_{j\sigma_s,\rho}$.

Corollary 4.10. *Let $C', K' > 0$ be as in proposition 4.9. Then, the inequalities*

$$\|u_{j,*}\|_{j\sigma_s,\rho} \leqslant C'K'^j\Gamma(1+(s+1)j)$$

hold for all $j \geqslant 0$.

We are now able to conclude the proof of theorem 3.3.

4.1.4. *Conclusion.* We must prove on the sup-norm of the $u_{j,*}(x)$ estimates similar to the ones on the norms $\|u_{j,*}\|_{j\sigma_s,\rho}$ (see corollary 4.10). To this end, we proceed by shrinking the closed polydisc $\|x\| \leqslant \rho$. Let $0 < \rho' < \rho$. Then, for all $j \geqslant 0$ and all $\|x\| \leqslant \rho'$, we have

$$|u_{j,*}(x)| = \left|u_{j,*}(x)d_\rho(x)^{j\sigma_s}\frac{1}{d_\rho(x)^{j\sigma_s}}\right| \leqslant \frac{|u_{j,*}(x)d_\rho(x)^{j\sigma_s}|}{(\rho-\rho')^{j\sigma_s}} \leqslant \frac{\|u_{j,*}\|_{j\sigma_s,\rho}}{(\rho-\rho')^{j\sigma_s}}$$

and, consequently,

$$\sup_{\|x\|\leqslant\rho'}|u_{j,*}(x)| \leqslant C'\left(\frac{K'}{(\rho-\rho')^{\sigma_s}}\right)^j\Gamma(1+(s+1)j)$$

by applying corollary 4.10. This ends the proof of the first point of theorem 3.3.

[4]These numbers were named in honor of the mathematician Eugène Charles Catalan (1814-1894). They appear in many probabilist, graphs and combinatorial problems. For example, they can be seen as the number of d-ary trees with m source-nodes, or as the number of ways of associating m applications of a given d-ary operation, or as the number of ways of subdividing a convex polygon into m disjoint $(d+1)$-gons by means of non-intersecting diagonals. They also appear in theoretical computers through the generalized Dyck words. See for instance [4] and the references inside.

4.2. **Proof of the second point.** Let us now fix $s < s_c$[5]. According to the filtration of the s-Gevrey spaces $\mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s$ (see section 3) and the first point of theorem 3.3, it is clear that we have the following implications:

$$\begin{aligned}\widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_s &\Rightarrow \widetilde{f}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_{s_c} \\ &\Rightarrow \widetilde{u}(t,x) \in \mathcal{O}(\mathcal{D}_{\rho_1,...,\rho_n})[[t]]_{s_c}.\end{aligned}$$

To conclude that we can not say better about the Gevrey order of $\widetilde{u}(t,x)$, that is $\widetilde{u}(t,x)$ is *generically* s_c-Gevrey, we need to find an example for which the solution $\widetilde{u}(t,x)$ of eq. (1.1) is s'-Gevrey for no $s' < s_c$. Proposition 4.11 below provides such an example.

Proposition 4.11. *Let us consider the equation*

$$(4.16)\quad \begin{cases}\partial_t^\kappa u - \displaystyle\sum_{i\in\mathcal{K}}\sum_{q\in Q_i} a_{i,q}t^{v_{i,q}}\partial_t^i\partial_x^q u - \sum_{m=2}^{d} b_m u^m = \widetilde{f}(t,x), \;\; a_{i,q} > 0, b_m \geqslant 0\\ \partial_t^j u(t,x)_{|t=0} = \varphi_j(x), j = 0,...,\kappa-1\end{cases}$$

where the initial condition $\varphi_{i^*}(x)$ is the analytic function defined by

$$\varphi_{i^*}(x) = \frac{1}{1 - x_1 - ... - x_n}$$

on the disc $\mathcal{D}_{1/n,...,1/n}$, and where the initial conditions $\varphi_j(x)$ for $j \neq i^*$ are analytic functions on $\mathcal{D}_{1/n,...,1/n}$ satisfying $\partial_x^\ell\varphi_j(0) \geqslant 0$ for all $\ell \in \mathbb{N}^n$. Suppose also that the inhomogeneity $\widetilde{f}(t,x)$ satisfies the following conditions:

- *$\widetilde{f}(t,x)$ is s-Gevrey;*
- *$\partial_x^\ell f_{j,*}(0) \geqslant 0$ for all $j \geqslant 0$ and all $\ell \in \mathbb{N}^n$.*

Then, the formal solution $\widetilde{u}(t,x)$ of eq. (4.16) is exactly s_c-Gevrey.

Proof. Due to the calculations above, it is sufficient to prove that $\widetilde{u}(t,x)$ is s'-Gevrey for no $s' < s_c$.

First of all, let us rewrite the general relations (1.2) by isolating the term in (i^*, q^*). We get

$$u_{j+\kappa,*}(x) = \frac{j!a_{i^*,q^*}}{(j - v_{i^*,q^*})!}\partial_x^{q^*} u_{j-v_{i^*,q^*}+i^*,*}(x) + R_j(x)$$

with

$$R_j(x) = f_{j,*}(x) + \sum_{\substack{(i,q)\in\mathcal{K}\times Q_i\\(i,q)\neq(i^*,q^*)}} \frac{j!a_{i,q}}{(j - v_{i,q})!}\partial_x^q u_{j-v_{i,q}+i,*}(x) + \sum_{m=2}^{d}\sum_{\ell_1+...+\ell_m=j}\frac{j!b_m}{\ell_1!...\ell_m!}u_{\ell_1,*}(x)...u_{\ell_m,*}(x)$$

for all $j \geqslant 0$, together with the initial conditions $u_{j,*}(x) = \varphi_j(x)$ for $j = 0,...,\kappa-1$. Using then our hypotheses on the coefficients $a_{i,q}$ and b_m, on the initial conditions

[5] Of course, this case only occurs when $\mathcal{S} \neq \varnothing$.

$\varphi_j(x)$, and on the inhomogeneity $\widetilde{f}(t,x)$, we easily check that, for all $j \geqslant 0$:

$$u_{j(v_{i^*,q^*}+\kappa-i^*)+i^*,*}(x) = \frac{a_{i^*,q^*}^j (j\lambda(q^*))!}{(1-x_1-\ldots-x_n)^{j\lambda(q^*)+1}} \prod_{\ell=1}^{j} \frac{(\ell v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!}{((\ell-1)v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!} + \mathrm{rem}_j(x)$$

with $\mathrm{rem}_j(0) \geqslant 0$. Hence, applying technical lemmas 4.12 and 4.13 below, the following inequalities:

$$u_{j(v_{i^*,q^*}+\kappa-i^*)+i^*,*}(0) \geqslant \left(\frac{a_{i^*,q^*}}{2^{\lambda(q^*)+v_{i^*,q^*}}}\right)^j (j(\lambda(q^*)+v_{i^*,q^*}))!. \tag{4.17}$$

Let us now suppose that $\widetilde{u}(t,x)$ is s'-Gevrey for some $s' < s_c$. Then, definition 3.1 and inequality (4.17) imply

$$1 \leqslant C\left(\frac{2^{\lambda(q^*)+v_{i^*,q^*}}K}{a_{i^*,q^*}}\right)^j \frac{\Gamma(1+i^*(s'+1)+j(s'+1)(v_{i^*,q^*}+\kappa-i^*))}{\Gamma(1+j(\lambda(q^*)+v_{i^*,q^*}))} \tag{4.18}$$

for all $j \geqslant 0$ and some convenient positive constants C and K independent of j. Proposition 4.11 follows since such inequalities are impossible: applying the Stirling's Formula, the right hand-side of (4.18) is equivalent to

$$C' j^{i^*(s'+1)} \left(\frac{K'}{j^\sigma}\right)^j, \ j \to +\infty \tag{4.19}$$

with

- $C' = C((s'+1)(v_{i^*,q^*}+\kappa-i^*))^{i^*(s'+1)} \sqrt{\dfrac{(s'+1)(v_{i^*,q^*}+\kappa-i^*)}{\lambda(q^*)+v_{i^*,q^*}}}$;
- $K' = K\dfrac{2^{\lambda(q^*)+v_{i^*,q^*}}}{a_{i^*,q^*}} \dfrac{((s'+1)(v_{i^*,q^*}+\kappa-i^*))^{(s'+1)(v_{i^*,q^*}+\kappa-i^*)}e^\sigma}{(\lambda(q^*)+v_{i^*,q^*})^{\lambda(q^*)+v_{i^*,q^*}}}$;
- $\sigma := \lambda(q^*)+v_{i^*,q^*}-(s'+1)(v_{i^*,q^*}+\kappa-i^*)$,

and (4.19) goes to 0 when j tends to infinity. Indeed, the condition $s' < s_c$ implies

$$\sigma > \lambda(q^*)+v_{i^*,q^*}-(s_c+1)(v_{i^*,q^*}+\kappa-i^*) = 0.$$

This ends the proof. □

Lemma 4.12. *Let $j \geqslant 1$ be. Then,*

$$\prod_{\ell=1}^{j} \frac{(\ell v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!}{((\ell-1)v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!} \geqslant (jv_{i^*,q^*})!. \tag{4.20}$$

Proof. Lemma 4.12 is clear for $j=1$. Let us now suppose that inequality (4.20) holds for a certain $j \geqslant 1$. Then,

$$\prod_{\ell=1}^{j+1} \frac{(\ell v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!}{((\ell-1)v_{i^*,q^*}+(\ell-1)(\kappa-i^*))!} \geqslant \frac{((j+1)v_{i^*,q^*}+j(\kappa-i^*))!}{(jv_{i^*,q^*}+j(\kappa-i^*))!}(jv_{i^*,q^*})!$$

and we conclude due to the inequality

$$\binom{(j+1)v_{i^*,q^*}+j(\kappa-i^*)}{j(\kappa-i^*)} \geqslant \binom{jv_{i^*,q^*}+j(\kappa-i^*)}{j(\kappa-i^*)}.$$

□

Lemma 4.13. *Let $j \geqslant 1$ be. Then,*

$$(j\lambda(q^*))!(jv_{i*,q*})! \geqslant \frac{(j(\lambda(q^*)+v_{i*,q*}))!}{2^{j(\lambda(q^*)+v_{i*,q*})}}.$$

Proof. Lemma 4.13 is straightforward from the inequality

$$\binom{j(\lambda(q^*)+v_{i*,q*})}{j\lambda(q^*)} \leqslant 2^{j(\lambda(q^*)+v_{i*,q*})}.$$

□

This ends the proof of the second point of theorem 3.3

References

[1] W. Balser. *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New-York, 2000.

[2] W. Balser and M. Loday-Richaud. Summability of solutions of the heat equation with inhomogeneous thermal conductivity in two variables. *Adv. Dyn. Syst. Appl.*, 4(2):159–177, 2009.

[3] M. Canalis-Durand, J.-P. Ramis, R. Schäfke, and Y. Sibuya. Gevrey solutions of singularly perturbed differential equations. *J. Reine Angew. Math.*, 518:95–129, 2000.

[4] P. Hilton and J. Pedersen. Catalan numbers, their generalization, and their uses. *Math. Intelligencer*, 13(2):64–75, 1991.

[5] D. A. Klarner. Correspondences between plane trees and binary sequences. *J. Combinatorial Theory*, 9:401–411, 1970.

[6] A. Lastra and S. Malek. On parametric Gevrey asymptotics for some nonlinear initial value Cauchy problems. *J. Differential Equations*, 259:5220–5270, 2015.

[7] A. Lastra and S. Malek. On parametric multisummable formal solutions to some nonlinear initial value Cauchy problems. *Adv. Differ. Equ.*, 2015:200, 2015.

[8] A. Lastra, S. Malek, and J. Sanz. On Gevrey solutions of threefold singular nonlinear partial differential equations. *J. Differential Equations*, 255:3205–3232, 2013.

[9] A. Lastra and H. Tahara. Maillet type theorem for nonlinear totally characteristic partial differential equations. *Math. Ann.*, https://doi.org/10.1007/s00208-019-01864-x, 2019.

[10] M. Loday-Richaud. *Divergent Series, Summability and Resurgence II. Simple and Multiple Summability*, volume 2154 of *Lecture Notes in Math.* Springer-Verlag, 2016.

[11] S. Malek. On the summability of formal solutions of nonlinear partial differential equations with shrinkings. *J. Dyn. Control Syst.*, 13(1):1–13, 2007.

[12] S. Malek. On Gevrey asymptotic for some nonlinear integro-differential equations. *J. Dyn. Control Syst.*, 16(3):377–406, 2010.

[13] S. Malek. On the summability of formal solutions for doubly singular nonlinear partial differential equations. *J. Dyn. Control Syst.*, 18(1):45–82, 2012.

[14] M. Miyake. Newton polygons and formal Gevrey indices in the Cauchy-Goursat-Fuchs type equations. *J. Math. Soc. Japan*, 43(2):305–330, 1991.

[15] M. Miyake and A. Shirai. Convergence of formal solutions of first order singular nonlinear partial differential equations in the complex domain. *Ann. Polon. Math.*, 74:215–228, 2000.

[16] M. Miyake and A. Shirai. Structure of formal solutions of nonlinear first order singular partial differential equations in complex domain. *Funkcial. Ekvac.*, 48:113–136, 2005.

[17] M. Miyake and A. Shirai. Two proofs for the convergence of formal solutions of singular first order nonlinear partial differential equations in complex domain. *Surikaiseki Kenkyujo Kokyuroku Bessatsu, Kyoto Unviversity*, B37:137–151, 2013.

[18] M. Nagumo. Über das Anfangswertproblem partieller Differentialgleichungen. *Jap. J. Math.*, 18:41–47, 1942.

[19] S. Ouchi. Multisummability of formal solutions of some linear partial differential equations. *J. Differential Equations*, 185(2):513–549, 2002.

[20] M. E. Pliś and B. Ziemian. Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables. *Ann. Polon. Math.*, 67(1):31–41, 1997.

[21] G. Pólya and G. Szegö. *Aufgaben und Lehrsätze aus der Analysis, Vol. I*, volume 125. Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.

[22] J.-P. Ramis. Dévissage Gevrey. *Astérisque, Soc. Math. France, Paris*, 59-60:173–204, 1978.

[23] J.-P. Ramis. Théorèmes d'indices Gevrey pour les équations différentielles ordinaires. *Mem. Amer. Math. Soc.*, 48:viii+95, 1984.

[24] P. Remy. Gevrey index theorem for some inhomogeneous semilinear partial differential equations with variable coefficients. *hal-02263353, version 1.*

[25] P. Remy. Gevrey index theorem for the inhomogeneous n-dimensional heat equation with a power-law nonlinearity and variable coefficients. *hal-02117418, version 1.*

[26] P. Remy. Gevrey order and summability of formal series solutions of some classes of inhomogeneous linear partial differential equations with variable coefficients. *J. Dyn. Control Syst.*, 22(4):693–711, 2016.

[27] P. Remy. Gevrey order and summability of formal series solutions of certain classes of inhomogeneous linear integro-differential equations with variable coefficients. *J. Dyn. Control Syst.*, 23(4):853–878, 2017.

[28] P. Remy. Gevrey properties and summability of formal power series solutions of some inhomogeneous linear Cauchy-Goursat problems. *J. Dyn. Control Syst.*, 26(1):69–108, 2020.

[29] A. Shirai. Maillet type theorem for nonlinear partial differential equations and Newton polygons. *J. Math. Soc. Japan*, 53:565–587, 2001.

[30] A. Shirai. Convergence of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type. *Funkcial. Ekvac.*, 45:187–208, 2002.

[31] A. Shirai. A Maillet type theorem for first order singular nonlinear partial differential equations. *Publ. RIMS. Kyoto Univ.*, 39:275–296, 2003.

[32] A. Shirai. Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type. *Surikaiseki Kenkyujo Kokyuroku, Kyoto University*, 1431:94–106, 2005.

[33] A. Shirai. Alternative proof for the convergence of formal solutions of singular first order nonlinear partial differential equations. *Journal of the School of Education, Sugiyama Jogakuen University*, 1:91–102, 2008.

[34] A. Shirai. Gevrey order of formal solutions of singular first order nonlinear partial differential equations of totally characteristic type. *Journal of the School of Education, Sugiyama Jogakuen University*, 6:159–172, 2013.

[35] A. Shirai. Maillet type theorem for singular first order nonlinear partial differential equations of totally characteristic type, part ii. *Opuscula Math.*, 35(5):689–712, 2015.

[36] H. Tahara. Gevrey regularity in time of solutions to nonlinear partial differential equations. *J. Math. Sci. Univ. Tokyo*, 18:67–137, 2011.

[37] H. Tahara and H. Yamazawa. Multisummability of formal solutions to the Cauchy problem for some linear partial differential equations. *J. Differential Equations*, 255(10):3592–3637, 2013.

[38] W. Walter. An elementary proof of the Cauchy-Kowalevsky theorem. *Amer. Math. Monthly*, 92(2):115–126, 1985.

[39] A. Yonemura. Newton polygons and formal Gevrey classes. *Publ. Res. Inst. Math. Sci.*, 26:197–204, 1990.

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