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## ► To cite this version:

Nicolas Auffray, Saad El Ouafa, Giuseppe Rosi, Boris Desmorat. Anisotropic Structure Of Two-Dimensional Linear Cosserat Elasticity. Mathematics and Mechanics of Complex Systems, 2022, 10.2140/memocs.2022.10.321 . hal-03287608v2

**HAL Id: hal-03287608**

**<https://hal.science/hal-03287608v2>**

Submitted on 3 Jan 2022

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# ANISOTROPIC STRUCTURE OF TWO-DIMENSIONAL LINEAR COSSERAT ELASTICITY

N. AUFRAY, S. EL OUAFA, G. ROSI, AND B. DESMORAT

**ABSTRACT.** In the present contribution the anisotropic structure of the two-dimensional linear Cosserat elasticity is investigated. The symmetry classes of this model are derived and detailed in a synthetic way. Particular attention is paid to specific features of Cosserat Elasticity which are the sensitivity to non-centrosymmetry as well as to chirality. These aspects are important for the application of this continuum theory to the mechanical modelling of lattices and metamaterials. In order to give a parameterisation to the Cosserat constitutive law, an explicit harmonic decomposition of its constitutive tensors is provided. Finally, using an algorithm introduced in a side paper, a minimal integrity basis, which is the minimal set of polynomial invariants generating the algebra of  $O(2)$ -invariant polynomials, is finally reported.

## 1. INTRODUCTION

The study and the design of architected materials often involves the use of an homogenised model that amounts to capture the main effects emerging from a specific mesostructure, without considering all its geometrical details [36, 66]. This approach is mainly interesting at two stages of the study of the architected material:

**Mesostructure design:** When designing a mesostructure, it can be interesting to forget the geometry and focus on the resulting properties by defining a homogenised substitution continuum. Once the overall properties determined, an actual mesostructure can be obtained by a deshomogenisation process [36]. This step can be achieved, for instance, by using topological optimisation as detailed, in [3, 20].

**Large scale simulation:** The numerical simulation of a large structure made of an architected material is challenging due to the complex mesh imposed by the geometry of constitutive material at mesoscale. This problem can be solved by using a multiscale approach for which, at the structure scale, the architected material is replaced by an equivalent homogeneous continuum [40, 41]. In the spirit of  $FE^2$  method, the local problem can be studied in a second step once the macroscopic fields have been determined. [32].

Despite this strategy being attractive from a theoretical point of view, its practical application imposes to first answer some key questions. The principal one concerns the choice of a specific continuum model for operating this substitution. When the different scales of the problem are well-separated, the problem is solved since, at least, the 1980s and the introduction of the classical theory of two-scale homogenisation [13]. In this case, the substitution medium is a classical Cauchy continuum and there is no specific mesostructural effect emerging at the macroscale. On the contrary, in the absence of scale separation, the presence of the mesostructure plays an important role in the actual behaviour of the material. The substitution continuum model is then no longer of Cauchy type, and its formulation must be generalised. In this framework, Mindlin, among others, made several important contributions the most famous of its being "*Elasticity with microstructure*" [44]. In this fundamental paper, he proposed to append to the displacement field  $\underline{u}$ , a second-order tensor field  $\underline{\chi}$  that intends to describe the micro-deformation of the matter, see also [22, 30]. This general setting can be restricted by not retaining the full second-order tensor  $\underline{\chi}$  as new Degrees Of Freedom (*DOF*), but only some of its parts. According to the considered choice different theories can be formulated (cf. [30, 34] for a complete classification). Restricting  $\underline{\chi}$  to its antisymmetric part amounts to take into account the rigid rotation of the mesostructure in the mechanical formulation. The resulting continuum is known as micropolar model, and is in agreement with the one proposed by the Cosserat brothers [21, 33]. In this work we will refer to this model as *Cosserat elasticity*.

With the recent interest in metamaterials with exotic dynamical properties, we have seen a vigorous revival of this century-old classical generalized continuum theory in the last few years. The chiral sensitivity of this theory has motivated numerous

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Date: November 10, 2021.

authors to study the non-standard dynamic coupling occurring in lattice materials within its framework [18, 19, 43]. Following the path opened by promising applications to elastic cloaking, other authors aiming to extend the tools of transformational optic to elasticity have developed the concept of polar materials, whose modelling strategy involved a Cosserat-type continuum [48]. Not restricted to elastic wave propagation, Cosserat effective description has interesting applications in multiple and diverse domains such, for instance: in civil engineering with block masonry modelling [52] or the modelling of granular materials [37], and also in biomechanics to model the behaviour of bones [51, 31]. **Another strategy to enrich the Cauchy continuum is to keep the same DOFs but consider the contribution of their higher-order gradients in the mechanical energy. The archetype of such a model is Mindlin's strain-gradient elasticity, a well-known generalised continuum that is of increasing interest for engineering applications [57, 23, 17]. As Cosserat elasticity, Strain-gradient elasticity is also sensitive to chirality and to the lack of centro-symmetry [6].**

When considering a generalized continuum theory, it is important to know the mechanical problem that this model can describe. When it comes to modelling lattice materials or metamaterials, the ability of a theory to describe anisotropic mechanical couplings is a crucial issue. According to their tensorial nature some theories can, or cannot, describe physical phenomena that may be of central importance for the situation of interest. This problem has been studied in the literature for the strain gradient elasticity model both in 2D [5, 6] and 3D [9, 7]. **The results of these studies reveal the interesting extended anisotropic modelling capabilities offered by this model: sensitivity to chirality and to the lack of centro symmetry, appearance of higher order anisotropy classes...** To the best of our knowledge, a complete classification has not been provided yet for Cosserat elasticity. **This result is important for comparing the Cosserat and Strain-Gradient elasticities and for rationalising their use according to the modelling context.** In [61] the problem of symmetry classes of a bidimensional Cosserat continuum is discussed and partial results are provided, while the complete classification is missing. The same result can be found in the references [43, 19] where specific situations are considered but without providing general results. For the complete three-dimensional situation, that will not be discussed in this paper, the main reference is [68]. The results presented in this last reference are themselves incomplete, we will come back to them in a future contribution.

A first objective of this paper is therefore to fill this gap by providing the complete symmetry classification of the linear Cosserat elasticity in two dimensions. The motivation of the bidimensional setting is related to the its strong interest for the overall description of lattice materials [43, 58, 19, 11], **especially for investigating their dynamical features.** The extension to full 3D will be considered in a future contribution. In addition, we were motivated by the observation that there is **some confusion** regarding the concepts of chirality and centrosymmetry. This confusion is maintained, it seems to us, by:

- (1) the fact that these concepts differ slightly in 2D and 3D. The inversion in 2D is a rotation by  $\pi$  ( $\det \mathbf{i} = 1$ ) while in 3D it is an improper transformation ( $\det \mathbf{i} = -1$ );
- (2) an ambiguous terminology for designating lattices. For examples the famous hexachiral pattern [53, 2], if chiral in 2D is not chiral in 3D. Another example is the so-called trichiral pattern which generates a tiling that is invariant with respect to a rotation of  $\frac{2\pi}{6}$  contrary to what its name suggests...

A focus on the consequences of the lack of centrosymmetry has been discussed in [42]. For the nonlinear case, these concepts have been studied in [29].

Another major question concerns the transition between a real mesostructure and its actual properties, especially when these are not standard. This is particularly interesting in the aim of synthesising architected materials with a tailored effect [35, 65]. The recent advances in topological optimisation could lead to major breakthrough in this field (see [14] in the case of wave propagation). Some recent studies considered the generalization of classical topological homogenisation algorithms to the Cosserat elasticity model [56, 15, 16]. In the perspective of extending approaches involving the use of the topological gradient [3, 49] having a fine parameterisation of the anisotropic constitutive law is valuable. For this purpose, we introduce the complete explicit harmonic decomposition of the three constitutive tensors of the model. Further, in order to allow invariant parameterisation of the Cosserat model, we propose for the first time a minimal  $O(2)$ -integrity basis. This set of invariant polynomials allows both to identify the constitutive law modulo its orientation, and to express any  $O(2)$ -invariant polynomial function. For a complete anisotropic law, this set is fairly large since comprising 325 elements, but its cardinality quickly decreases for higher symmetry classes.

**Organisation** This article is organised as follows. In Section 2, we introduce the notation that we will use throughout the text. Section 3 is dedicated to a brief introduction to linear Cosserat elasticity in two dimensions. We introduced there the constitutive tensors defining the model. In section 4, the notion of harmonic structure is introduced and applied to the constitutive tensors of Cosserat model. Using results from a side paper, this decomposition allows to determinate the symmetry classes of the theory. This section concludes with some comments on the physical content of this result, particularly

with respect to the modelling of chiral or non-centrosymmetric lattices. In section 5, the explicit harmonic decompositions of the constitutive tensors of the model are provided. From these elements, minimum integrity bases are produced. Firstly, the one related to the constitutive tensors considered separately will be considered and then, in a second step, the basis of the complete law is provided. The elements constituting this basis are reported in the form of tables.

The proofs of the various results presented in this paper are generally direct applications of theorems and methods that we have introduced in parallel papers. Therefore, for reasons of readability and to focus on the physical consequences of the results, we have decided not to detail the proofs but to cite the associated papers and theorems directly.

## 2. NOTATION

Throughout this paper, the two-dimensional Euclidean space  $\mathcal{E}^2$  is equipped with a rectangular Cartesian coordinate system with origin  $O$  and an orthonormal basis  $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$ . Upon the choice of a reference point  $O$  and a given basis  $\mathcal{B}$ ,  $\mathcal{E}^2$  is identified with the vector space  $\mathbb{R}^2$ . As a consequence, points in  $\mathcal{E}^2$  are designated by their vector positions with respect to  $O$ . In the following,  $\mathbf{x} = x_i \mathbf{e}_i$ , where Einstein summation convention is used, i.e. when an index appears twice in an expression it implies summation of this term over all the values of the index. Below are the specific notations and conventions used in this article.

*Groups:*

- $O(2)$ : the group of invertible transformations of  $\mathbb{R}^2$  satisfying  $\mathbf{g}^{-1} = \mathbf{g}^T$ , where  $\mathbf{g}^{-1}$  and  $\mathbf{g}^T$  stand for the inverse and the transpose of  $\mathbf{g}$ . This group is called the orthogonal group [59];
- $SO(2)$ : the subgroup of  $O(2)$  of elements with determinant 1, called the special orthogonal group;
- $\mathbb{1}$ : the trivial group solely containing the identity.

Let us denote by  $\mathbf{r}(\theta)$  the rotation by an angle  $\theta$  and by  $\boldsymbol{\pi}(\underline{n})$  the mirror across the line normal to  $\underline{n}$ :

$$\boldsymbol{\pi}(\underline{n}) := \mathbb{I} - 2\underline{n} \otimes \underline{n}, \quad \|\underline{n}\| = 1,$$

with  $\mathbb{I}$  the second order identity tensor. These transformations are represented in a matrix by

$$[\mathbf{r}(\theta)] = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}_{\mathcal{B}} \quad \text{with } 0 \leq \theta < 2\pi \quad \text{and} \quad [\boldsymbol{\pi}(\mathbf{e}_2)] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}.$$

As matrix groups,  $SO(2)$  is generated by  $\mathbf{r}(\theta)$ , while  $O(2)$  is generated by  $\mathbf{r}(\theta)$  and  $\boldsymbol{\pi}(\mathbf{e}_2)$ .

This collection has to be supplemented with the closed subgroups of  $O(2)$ . They are of two types:

- $\mathbb{Z}_k (k \geq 2)$  the cyclic group with  $k$  elements, generated by  $\mathbf{r}(2\pi/k)$ . For  $k = 1$ , we have  $\mathbb{Z}_k = \mathbb{1}$ , and the limit group is  $SO(2)$ ;
- $\mathbb{D}_k^{\underline{n}} (k \geq 2)$  the dihedral group with  $2k$  elements, generated by  $\mathbf{r}(2\pi/k)$  and  $\boldsymbol{\pi}(\underline{n})$ . For  $k = 1$ , the group  $\mathbb{D}_1^{\underline{n}}$  will be denoted by  $\mathbb{Z}_2^{\pi, \underline{n}}$ , and the limit group is  $O(2)$ .

It has to be noted that, up to conjugacy by an element of  $SO(2)$ , any  $\mathbb{D}_k^{\underline{n}}$  is conjugate to  $\mathbb{D}_k^{\mathbf{e}_2}$ , and will simply be denoted by  $\mathbb{D}_k$ . Accordingly,  $\mathbb{Z}_2^{\pi, \mathbf{e}_2}$  will simply be denoted  $\mathbb{Z}_2^{\pi}$ .

To conclude, let us introduce the Lie algebra  $\mathfrak{so}(2)$  which is the tangent space to  $SO(2)$  at  $\mathbb{I}$  and which is the vector space of asymmetric matrices on  $\mathbb{R}^2$ . As a matrix Lie algebra,  $\mathfrak{so}(2)$  is further equipped with the matrix commutator.

*Actions on tensors and the Levi-Civita pseudo-tensor.* We consider the action of  $O(2)$  on the space  $\mathbb{T}^n$  given by

$$\forall \mathbf{g} \in O(2), \quad \mathbf{T}^{\mathbf{g}} := \mathbf{g} \star^n \mathbf{T} = g_{i_1 j_1} g_{i_2 j_2} \dots g_{i_n j_n} T_{j_1 j_2 \dots j_n} \mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_n}.$$

This action is the tensorial action, sometimes also known as the Rayleigh product. When clear from the context, no mention will be made of the tensor order in the product, in this case the notation  $\star$  simplifies to  $\star$ .

In two dimensions, the Levi-Civita pseudo-tensor is defined by  $\epsilon_{ij} = \begin{cases} 1 & \text{if } ij = 12, \\ -1 & \text{if } ij = 21, \text{ and can be identified with } \mathbf{r}(-\frac{\pi}{2}). \\ 0 & \text{if } i = j. \end{cases}$

The Levi-Civita symbol is called a pseudo-tensor because for the elements  $\mathbf{g} \in O(2)$ ,  $\epsilon$  transforms as  $\mathbf{g} \star \epsilon = (\det \mathbf{g}) \epsilon$ .

*Tensor products:* Classical notations will be used to denote tensor products:

- $\otimes$  stands for the classical product, and  $\otimes^n$  indicates its  $n$ -th power;
- $S^2$  denotes its symmetrized version, and  $S^n$  its extension to product of  $n$  elements.

Tensor spaces:

- $\mathbb{T}^n$  is a space of  $n$ th-order tensors, possibly satisfying index symmetries;
- $\mathbb{K}^n$  is the space of  $n$ th-order *harmonic tensors* (i.e. totally symmetric and traceless tensors), with

$$\dim(\mathbb{K}^n) = \begin{cases} 2 & \text{if } n > 0, \\ 1 & \text{if } n \in \{0, -1\}. \end{cases}$$

Among them:

- $\mathbb{K}^0$  is the space of scalars;
- $\mathbb{K}^{-1}$  is the space of *pseudo-scalars* (i.e. scalars which change sign under improper transformations).

Tensors of order  $-1, 0, 1, 2, 3, 4$  are denoted by  $\beta, \alpha, \underline{\mathbf{y}}, \underline{\mathbf{a}}, \underline{\mathbf{A}}, \underline{\mathbf{B}}$  respectively. General tensors (i.e. with no mention of their order) are denoted using bold fonts, as for instance  $\mathbf{T}$ . With respect to  $\mathcal{B}$ , the components of  $\mathbf{T} \in \mathbb{T}^n$  are denoted as  $T_{i_1 \dots i_n}$ .

The simple, double, triple and fourth-order contractions are written  $\cdot, :, \vdots, ::$ , respectively. Generic  $k$ th-order contraction will be indicated by the notation  $\overset{(k)}{\cdot}$ . In components with respect to  $\mathcal{B}$ , for general tensors  $\mathbf{A}$  and  $\mathbf{B}$ , these notations correspond to

$$\begin{aligned} (\mathbf{A} \cdot \mathbf{B})_{i_1 \dots i_n} &= A_{i_1 \dots i_p j} B_{j i_{p+1} \dots i_n}, & (\mathbf{A} : \mathbf{B})_{i_1 \dots i_n} &= A_{i_1 \dots i_p j k} B_{j k i_{p+1} \dots i_n}, \\ (\mathbf{A} : \mathbf{B})_{i_1 \dots i_n} &= A_{i_1 \dots i_p j k l} B_{j k l i_{p+1} \dots i_n}, & (\mathbf{A} :: \mathbf{B})_{i_1 \dots i_n} &= A_{i_1 \dots i_p j k l m} B_{j k l m i_{p+1} \dots i_n}. \end{aligned}$$

With the chosen summation convention  $\cdot, :$  and  $::$  correspond to inner products between second, third and fourth order tensors, respectively:

$$\underset{\sim}{\mathbf{a}} : \underset{\sim}{\mathbf{b}} = a_{pq} b_{pq}, \quad \underset{\sim}{\mathbf{A}} : \underset{\sim}{\mathbf{B}} = A_{pqr} B_{pqr}, \quad \underset{\sim}{\mathbf{A}} : \underset{\sim}{\mathbf{B}} = A_{pqrs} B_{pqrs}.$$

When needed, index symmetries of both spaces and their elements are expressed as follows:  $(\dots)$  indicates invariance under permutations of the indices in parentheses and  $\underline{\dots}$  indicates symmetry with respect to permutations of the underlined blocks.

### 3. COSSERAT ELASTICITY

In this section, the fundamentals of the Cosserat model are summarised. For a more detailed presentation, the interested reader is advised to consult the following references [44, 28, 30].

**3.1. Degrees of Freedom (DOF).** A Cosserat continuum, also known as a micropolar continuum, is a medium in which material points are described as a collection of infinitesimal rigid bodies. Consider  $\Omega$  an open and bounded domain of  $\mathbb{R}^2$ , this domain represents the body in its reference configuration. With respect to this configuration, the position of each particle is given by its position vector  $\underline{\mathbf{X}}$  with respect to  $\mathbf{O}$ . The orientation of the rigid particle is defined by an orthonormal trihedron  $\{\underline{\mathbf{D}}_k\}$ ,  $k = \{1, 2\}$ . During the transformation of the body, the material point initially located in  $\underline{\mathbf{X}}$  moves to another point  $\underline{\mathbf{x}}$  of space and its associated trihedron turns (without deformation) into another orthonormal trihedron  $\{\underline{\mathbf{d}}_k\}$ . Within this picture, the *generalized displacement* of a material point is described as follows

$$\underline{\mathbf{u}} := \underline{\mathbf{x}} - \underline{\mathbf{X}}, \quad \underset{\sim}{\mathbf{R}} := \underline{\mathbf{d}}_k \otimes \underline{\mathbf{D}}_k,$$

in which  $\underline{\mathbf{u}}$  is the linear displacement of a particle initially located at  $\underline{\mathbf{X}}$ , and  $\underset{\sim}{\mathbf{R}} \in \text{SO}(2)$  is the rotation of the particle from its original orientation  $\{\underline{\mathbf{D}}_k\}$  to the new one  $\{\underline{\mathbf{d}}_k\}$ . In terms of fields, the transformation is parametrised by a vector field  $\underline{\mathbf{u}}$  and a second-order orthogonal tensor field both over  $\Omega$ . Hence the DOF of this mechanical theory are

$$\text{DOF} := \{\underline{\mathbf{u}}, \underset{\sim}{\mathbf{R}}\} \in \mathbb{R}^2 \times \text{SO}(2).$$

The tensor field  $\underset{\sim}{\mathbf{R}}$  is known as the micro-rotation field. For the following, let us denote by  $\theta$  the angle of rotation associated with  $\underset{\sim}{\mathbf{R}}$ , since we are in 2D, it is not necessary to specify the associated axis of rotation. More precisely,  $\underset{\sim}{\mathbf{R}}$  is parameterized as follows in terms of  $\theta$

$$\underset{\sim}{\mathbf{R}}(\theta) := \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Now consider the case where the translations and microrotations are point-wisely infinitesimally small:

$$\|\underline{\mathbf{u}}\| \ll 1, \quad \|\underline{\mathbf{u}} \otimes \underline{\nabla}\| \ll 1, \quad \|\theta\| \ll 1, \quad \|\theta \otimes \underline{\nabla}\| \ll 1.$$

with  $\|\cdot\|$  the norm associated to the dot product. Under this hypothesis the reference and the actual configurations can be associated,  $\underline{\mathbf{u}}$  is now the infinitesimal linear displacement vector while the infinitesimal rotation can be expressed as

$$\underset{\sim}{\mathbf{R}} \simeq \underset{\sim}{\mathbf{I}} + \underset{\sim}{\mathbf{w}}$$

with  $\underline{\mathbf{w}} \in \mathfrak{so}(2)$ . In the classical formulation of Cosserat elasticity, the second-order skew symmetric tensor is usually substituted with the pseudo-scalar  $\theta$ :

$$\underline{\mathbf{w}} = -\theta \underline{\underline{\epsilon}} \quad , \quad \theta = -\frac{1}{2} \underline{\underline{\epsilon}} : \underline{\mathbf{w}}.$$

where  $\underline{\underline{\epsilon}}$  is the 2D Levi-Civita pseudo-tensor. Hence the linearised DOF are

$$\text{DOF} := \{\underline{\mathbf{u}}, \underline{\mathbf{w}}\} \in \mathbb{R}^2 \times \mathfrak{so}(2).$$

**3.2. Strain measures.** The Cosserat elasticity is a first gradient theory, as such the set of primary state variables is [33]:

$$\mathcal{V} := \{\underline{\mathbf{u}}, \underline{\mathbf{u}} \otimes \underline{\nabla}, \theta, \underline{\nabla} \theta\}.$$

Two equivalent sets of strain measures can be found in the literature:

- *Classical formulation:* as introduced for instance in [30, 33] the following measures are used

$$\underline{\mathbf{e}} = \underline{\mathbf{u}} \otimes \underline{\nabla} - \theta \underline{\underline{\epsilon}} \quad ; \quad \underline{\mathbf{\kappa}} = \underline{\nabla} \theta, \quad (\underline{\mathbf{e}}, \underline{\mathbf{\kappa}}) \in \otimes^2 \mathbb{R}^2 \times \mathbb{R}^2$$

where  $\underline{\mathbf{e}}$  is the linear stretch tensor and  $\underline{\mathbf{\kappa}}$  the linear wryness tensor;

- *Micromorphic formulation:* using the classical strain  $\underline{\underline{\varepsilon}}$  and spin  $\underline{\underline{\omega}}$  tensors, such as  $\underline{\mathbf{u}} \otimes \underline{\nabla} = \underline{\underline{\varepsilon}} + \underline{\underline{\omega}}$ , the following measures can be used as well

$$\underline{\underline{\varepsilon}} \quad ; \quad \underline{\underline{\eta}} = \underline{\underline{\omega}} - \theta \underline{\underline{\epsilon}} \quad ; \quad \underline{\mathbf{\kappa}} = \underline{\nabla} \theta \quad (\underline{\underline{\varepsilon}}, \underline{\underline{\eta}}, \underline{\mathbf{\kappa}}) \in S^2(\mathbb{R}^2) \times \mathfrak{so}(2) \times \mathbb{R}^2$$

This last partition follows the one classically used in the micromorphic approach of the elasticity [44], from which Cosserat is a special case. This choice highlights the fact that the kinematic constraint  $\underline{\underline{\eta}} = \underline{\underline{0}}$  degenerates the Cosserat model into the Koiter one [60], also know as *constrained couple-stress*, which is a special case of a strain-gradient continuum.

While in what follows only the first formulation will be considered, the second will, in a way, reappear when studying the harmonic decomposition of the linear constitutive law.

**3.3. Constitutive law.** By duality we define the stress tensors:  $\underline{\mathbf{s}}$  the asymmetric stress tensor, and  $\underline{\mathbf{m}}$  the couple-stress tensor. Consequently,  $(\underline{\mathbf{s}}, \underline{\mathbf{m}}) \in \otimes^2 \mathbb{R}^2 \times \mathbb{R}^2$ , and the linear constitutive law between the primary and dual state tensors is expressed as:

$$\begin{pmatrix} \underline{\mathbf{s}} \\ \underline{\mathbf{m}} \end{pmatrix} = \begin{pmatrix} \underline{\underline{\mathbf{A}}} & \underline{\underline{\mathbf{B}}} \\ \underline{\underline{\mathbf{B}}}^T & \underline{\underline{\mathbf{d}}} \end{pmatrix} \begin{pmatrix} \underline{\mathbf{e}} \\ \underline{\mathbf{\kappa}} \end{pmatrix}.$$

In the above notation, the symmetry of the coupled constitutive law comes from the assumption that it derives from a potential energy. As the theory is linear, this potential is considered quadratic with respect to the deformation tensors. Let us define the following vector spaces to which the constitutive tensors belong<sup>1</sup>:

$$\underline{\underline{\mathbf{A}}} \in \text{Cos} := \{\underline{\underline{\mathbf{T}}} \in S^2(\otimes^2 \mathbb{R}^2) | \underline{\underline{\mathbf{T}}}_{ij} = \underline{\underline{\mathbf{T}}}_{kl}\}, \quad \underline{\underline{\mathbf{B}}} \in \text{Cou} := \{\underline{\underline{\mathbf{T}}} \in \otimes^3 \mathbb{R}^2\} \quad \underline{\underline{\mathbf{d}}} \in \text{Rot} := \{\underline{\underline{\mathbf{T}}} \in S^2(\mathbb{R}^2)\}.$$

From them, we define the complete vector space of Cosserat elasticity law:

$$(1) \quad \mathcal{C}os = \text{Cos} \oplus \text{Cou} \oplus \text{Rot}.$$

As such, linear Cosserat elastic continuum is defined by a triplet  $\mathcal{C} := (\underline{\underline{\mathbf{A}}}, \underline{\underline{\mathbf{B}}}, \underline{\underline{\mathbf{d}}}) \in \mathcal{C}os$ . Even in a two-dimensional framework, the full anisotropic law is quite a complex object. A better understanding of its anisotropic properties can be obtained by breaking down each of its constitutive tensors into elementary parts, called harmonic tensors. To the authors' knowledge, this study has not yet been carried out neither for three-dimensional Cosserat elasticity, nor for its two-dimensional version. The associated results are important since the harmonic decomposition of the constitutive tensors encodes:

- (1) the symmetry classes of the tensors taken independently, as well as of the complete law [8];
- (2) the structure of the integrity basis of each tensor taken independently, as well as of the complete law [63, 25].

The determination of the symmetry classes only requires the determination of the harmonic structure of the constitutive tensors. This point will hence be considered first in the next section. Considering the integrity basis, it is better to further have determined an explicit harmonic decomposition of the constitutive tensors. This aspect will be examined in a second step.

<sup>1</sup>To avoid any misunderstanding, these results are specific to the 2D context, in a full 3D setting, all the constitutive tensors would be of order 4.

## 4. HARMONIC STRUCTURE OF CONSTITUTIVE TENSORS AND SYMMETRY CLASSIFICATION

**4.1. Harmonic structure.** When a material is rotated<sup>2</sup>, its physical nature is not affected but, with respect to a fixed reference, tensors associated to the description of its physical properties are transformed. Since constitutive tensors are usually of order greater than 2, the way they transform is not simple and their different parts transform differently: some components are left fixed while others *turn* at different speeds. The different mechanisms of transformation of a tensor with respect to an orthogonal transformation are revealed by its harmonic structure. It is known that tensor spaces in  $\mathbb{R}^2$  are isomorphic to a direct sum of harmonic tensor spaces  $\mathbb{K}^p$  [38, 8]. Such a decomposition is interesting since the  $O(2)$ -action on  $\mathbb{K}^p$  is elementary and given by  $\rho_p$  [8], with for  $p \geq 1$ :

$$(2) \quad [\rho_p(\mathbf{r}(\theta))] := \begin{pmatrix} \cos p\theta & -\sin p\theta \\ \sin p\theta & \cos p\theta \end{pmatrix}_{\mathcal{B}}, \quad [\rho_p(\boldsymbol{\pi}(\mathbf{e}_2))] := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_{\mathcal{B}}.$$

The  $O(2)$ -action on  $\mathbb{K}^0$  is the identity while the  $O(2)$ -action on  $\mathbb{K}^{-1}$  is given by the determinant of the transformation:

$$(3) \quad \rho_0(\mathbf{g}) := 1, \quad \rho_{-1}(\mathbf{g}) := \det \mathbf{g}.$$

The harmonic structure of a tensor space can be determined without making heavy computations by using the *Clebsch-Gordan formulas*. These formulas indicate how the tensor product of two irreducible spaces decomposes into a direct sum of irreducible spaces. Note that this formula only indicates the structure of the resulting vector space and does not provide an explicit construction of the decomposition. For the determination of the harmonic structure, we use the following result, the proof of which is found in [8].

**Lemma 4.1.** *For every integers  $p > 0$  and  $q > 0$ , we have the following isotypic decompositions, where the meaningless products are indicated by  $\times$ :*

$\otimes$	$\mathbb{K}^q$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^p$	$\begin{cases} \mathbb{K}^{p+q} \oplus \mathbb{K}^{ p-q }, & p \neq q \\ \mathbb{K}^{2p} \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1}, & p = q \end{cases}$	$\mathbb{K}^p$	$\mathbb{K}^p$
$\mathbb{K}^0$	$\mathbb{K}^q$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^{-1}$	$\mathbb{K}^q$	$\mathbb{K}^{-1}$	$\mathbb{K}^0$

$\otimes^s$	$\mathbb{K}^p$	$\mathbb{K}^0$	$\mathbb{K}^{-1}$
$\mathbb{K}^p$	$\mathbb{K}^{2p} \oplus \mathbb{K}^0$	$\times$	$\times$
$\mathbb{K}^0$	$\times$	$\mathbb{K}^0$	$\times$
$\mathbb{K}^{-1}$	$\times$	$\times$	$\mathbb{K}^0$

The previous lemma can be applied to the different tensor spaces involved in Cosserat elasticity. The first step is to determine the irreducible decomposition of the DOF used in the model, in the two dimensional context we have

$$(\underline{\mathbf{u}}, \boldsymbol{\theta}) \in \mathbb{K}^1 \times \mathbb{K}^{-1}.$$

From their definitions, the harmonic structure of the state tensors are easily determined, and from them the harmonic structure of the constitutive tensors. For 2D Cosserat elasticity these results are provided in the following tables:

State tensor space	Harmonic structure
$\underline{\mathbf{e}}, \underline{\mathbf{s}} \in \otimes^2 \mathbb{R}^2$	$\mathbb{K}^2 \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1}$
$\underline{\mathbf{m}}, \underline{\mathbf{k}} \in \mathbb{R}^2$	$\mathbb{K}^1$

Constitutive tensor space	Harmonic structure
Cos	$\mathbb{K}^4 \oplus 2\mathbb{K}^2 \oplus 3\mathbb{K}^0 \oplus \mathbb{K}^{-1}$
Cou	$\mathbb{K}^3 \oplus 3\mathbb{K}^1$
Rot	$\mathbb{K}^2 \oplus \mathbb{K}^0$

From the knowledge of the harmonic structure, the next step is to construct an explicit isomorphism that realises the decomposition. It should be noted that as soon as the harmonic structure involves multiple spaces of the same order, the isomorphism is not uniquely defined<sup>3</sup> [38] and therefore choices have to be made. As can be seen in the previous table, this is the case for Cos and Cou. However, the determination of the symmetry classes is independent of the explicit harmonic decomposition considered. Let us first examine this question.

**4.2. Symmetry classes.** In this subsection, we provide the complete set of symmetry classes that can be modelled by bidimensional Cosserat elasticity.

<sup>2</sup>Here *rotated* is understood in the broad sense of a full orthogonal transformation.

<sup>3</sup>This is indeed the same situation as for an eigenspace associated with an eigenvalue of multiplicity greater than one.



4.2.1. *O(2)-action.* Before giving the result we shall give a precise definition of what a symmetry class is. To this end, consider a triplet  $\mathcal{C} := (\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{d}) \in \text{Cos}$ . The action of  $O(2)$  on this triplet is given by

$$\mathbf{g} \star \mathcal{C} = (\mathbf{g} \star \underset{\sim}{A}, \mathbf{g} \star \underset{\sim}{B}, \mathbf{g} \star \underset{\sim}{d}),$$

The group of symmetry of a tensor  $\mathbf{T}$  is defined as

$$G_{\mathbf{T}} = \{\mathbf{g} \in O(2), \mathbf{g} \star \mathbf{T} = \mathbf{T}\},$$

The symmetry group of  $\text{Cos}$  is defined as:

$$G_{\mathcal{C}} = G_{\underset{\sim}{A}} \cap G_{\underset{\sim}{B}} \cap G_{\underset{\sim}{d}},$$

and its symmetry class is given by

$$(4) \quad [G_{\mathcal{C}}] = \{G \subseteq O(2) | G = \mathbf{g} G_{\mathcal{C}} \mathbf{g}^T, \mathbf{g} \in O(2)\}.$$

This introduces a fundamental distinction between symmetry groups and symmetry classes. In a symmetry group, the orientations of the symmetry elements are specified, whereas this notion is removed when speaking of a symmetry class. Roughly speaking, a symmetry class is a *type* of symmetry.

4.2.2. *Material invariance and tensorial invariance.* The previous definitions only concerned the invariance properties of tensors. In practice, these tensors represent the effective physical properties of a material. For the sake of clarity, let us assume that those tensors reflect the effective mechanical behaviour of a RVE (Representative Volume Element). Let us denote by  $G_{\mathcal{M}}$  the symmetry group of the RVE. The link between the symmetries of matter (material symmetries) and the symmetries of tensors (physical symmetries) is given by the Curie-Neumann principle [67]:

$$G_{\mathcal{M}} \subset G_{\mathcal{C}},$$

which means that any operation that leaves the RVE invariant will leave the resulting physical behaviour invariant. However, this is just an inclusion and the effective behaviour may be more symmetrical than the underlying microstructure. A well-known example of this is the fact that hexagonal material symmetries induce isotropic Cauchy elasticity in 2D (or transverse isotropy in 3D).

4.2.3. *Classification.* Knowing the harmonic structure of the constitutive tensors, and using the clips operators defined in [8], the symmetry classes of each constitutive tensor space can be obtained:

**Theorem 4.2.** *The spaces Cos, Cou and Rot are respectively partitioned into 6, 4 and 2 symmetry classes:*

$$\begin{aligned} \mathfrak{J}(\text{Cos}) &= \{[\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_4], [\mathbb{D}_4], [\text{SO}(2)], [\text{O}(2)]\}. \\ \mathfrak{J}(\text{Cou}) &= \{[\mathbf{1}], [\mathbb{Z}_2^s], [\mathbb{D}_3], [\text{O}(2)]\}. \\ \mathfrak{J}(\text{Rot}) &= \{[\mathbb{D}_2], [\text{O}(2)]\} \end{aligned}$$

The proof of Theorem (4.2) is obtained as a direct application of [8, Theorem 2.4]. For practical applications, the matrix forms of the different constitutive tensors are detailed, symmetry class by symmetry class, in Appendix A. By combining these results we obtain the set of symmetry classes of the complete elasticity of Cosserat:

**Theorem 4.3.** *The space Cos is partitioned into 10 symmetry classes:*

$$\mathfrak{J}(\text{Cos}) = \{[\mathbf{1}], [\mathbb{Z}_2^s], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\text{SO}(2)], [\text{O}(2)]\}.$$

The global form of the constitutive law can be detailed for each symmetry class, the constitutive laws have the following synthetic form:

$$\begin{aligned} \mathcal{L}_{\mathbf{1}} &= \begin{pmatrix} \mathbf{A}_{\mathbb{Z}_2^s} & \mathbf{K}_{\mathbf{1}} \\ \mathbf{K}_{\mathbf{1}}^T & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{Z}_2} = \begin{pmatrix} \mathbf{A}_{\mathbb{Z}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{Z}_3} = \begin{pmatrix} \mathbf{A}_{\text{SO}(2)} & \mathbf{K}_{\mathbb{D}_3} \\ \mathbf{K}_{\mathbb{D}_3}^T & \mathbf{H}_{\text{O}(2)} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{Z}_4} = \begin{pmatrix} \mathbf{A}_{\mathbb{Z}_4} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\text{O}(2)} \end{pmatrix}, \quad \mathcal{L}_{\text{SO}(2)} = \begin{pmatrix} \mathbf{A}_{\text{SO}(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\text{O}(2)} \end{pmatrix}; \\ \mathcal{L}_{\mathbb{Z}_2^s} &= \begin{pmatrix} \mathbf{A}_{\mathbb{D}_2} & \mathbf{K}_{\mathbb{Z}_2^s} \\ \mathbf{K}_{\mathbb{Z}_2^s}^T & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{D}_2} = \begin{pmatrix} \mathbf{A}_{\mathbb{D}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{D}_3} = \begin{pmatrix} \mathbf{A}_{\text{O}(2)} & \mathbf{K}_{\mathbb{D}_3} \\ \mathbf{K}_{\mathbb{D}_3}^T & \mathbf{H}_{\text{O}(2)} \end{pmatrix}, \quad \mathcal{L}_{\mathbb{D}_4} = \begin{pmatrix} \mathbf{A}_{\mathbb{D}_4} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\text{O}(2)} \end{pmatrix}, \quad \mathcal{L}_{\text{O}(2)} = \begin{pmatrix} \mathbf{A}_{\text{O}(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\text{O}(2)} \end{pmatrix}. \end{aligned}$$

Using trace formula [8], the following table, which gives the number of independent constitutive coefficients for each symmetry class, can be constructed:



$[G_C]$	$[1]$	$[Z_2^\pi]$	$[Z_2]$	$[D_2]$	$[Z_3]$	$[D_3]$	$[Z_4]$	$[D_4]$	$[SO(2)]$	$[O(2)]$
$[G_M]$	$[1]$	$[Z_2^\pi]$	$[Z_2]$	$[D_2]$	$[Z_3]$	$[D_3]$	$[Z_4]$	$[D_4]$	$[Z_{k>4}]$	$[D_{k>4}]$
$\#_{\text{indep}}(\underline{d})$	(2)	(2)	<b>2</b>	(2)	(1)	(1)	(1)	(1)	(1)	<b>1</b>
$\#_{\text{indep}}(\underline{\mathcal{B}})$	<b>8</b>	<b>4</b>	(0)	(0)	(1)	<b>1</b>	(0)	(0)	(0)	<b>0</b>
$\#_{\text{indep}}(\underline{A})$	(10)	(10)	<b>10</b>	<b>6</b>	(4)	(3)	<b>6</b>	<b>4</b>	<b>4</b>	<b>3</b>
$\#_{\text{indep}}(\underline{C})$	20	16	12	8	6	5	7	5	5	4

In the table above, when the symmetry class of a constitutive tensor coincides with that of  $\underline{C}$ , the dimension is noted in bold. On the contrary, when the symmetry classes differ, the dimension is indicated in parenthesis. For instance, the class  $[1]$  is a symmetry class for  $\underline{C}$  but not for  $\underline{A}$ , whose symmetry class is  $[Z_2^\pi]$ , as such the number of parameters is indicated in parenthesis.

**4.3. Chirality and non-centrosymmetry.** If from a mathematical point of view the previous classification is sufficient, it can nevertheless be made more precise by explicitly introducing the notions of chirality and centrosymmetry. These concepts are very important for the intended physical applications and are of growing interest in the metamaterials community. Moreover these concepts are sometimes confused. Non-centrosymmetry and chirality are two different invariance properties.

**Definition 4.4.** A subgroup of  $O(2)$  will be said to be:

- *centrosymmetric* (denoted by **i**) if it contains the inversion  $\mathbf{r}(\pi)$ , and *non-centrosymmetric* ( **$\bar{i}$** ) otherwise;
- *chiral* (noted **c**) if it only contains rotations, and *achiral* (noted  **$\bar{c}$** ) otherwise (i.e. if it contains reflections).

$O(2)$ -closed subgroups can be divided into four subsets according to the nature of their generators. The different situations are reported in the following table:

	<b>i</b>	<b><math>\bar{i}</math></b>
<b><math>\bar{c}</math></b>	$D_{2k}$	$D_{2k+1}$
<b>c</b>	$Z_{2k}$	$Z_{2k+1}$

Their implication on constitutive tensors are distinct since, centrosymmetry implies the vanishing of odd-order tensors, while there is no such consequence for chirality which can be modelled by tensors of any order. Basically, the lack of centrosymmetry will result in a coupling between the state tensors associated with displacement and microrotation. For example, in a non-centrosymmetric medium, uniaxial stress will induce both strain and curvature. Other related effects are discussed in [42]. For its part, the absence of mirror symmetry, i.e. chirality, induces a coupling between shear and strain as can be seen in the matrix representations (cf. [Appendix A](#)).

The following observations can be made concerning the symmetry classes of Cosserat elasticity:

- Cosserat elasticity is sensitive to chirality, we can observe for example the appearance of hemitropy (ie.  **$SO(2)$ -invariance**) in the list of possible symmetry classes. This chiral sensitivity can be read directly from the harmonic structure of  $\text{Cos}$  since a  $\mathbb{K}^{-1}$  space appears there;
- the complete Cosserat's law, through the coupling tensor  $\underline{\mathcal{B}}$  is sensitive to the centrosymmetric nature of the medium.

**4.4. Comparison with two dimensional linear Strain-Gradient elasticity.** The linear strain-gradient elasticity model [45, 46] is obtained by extending the set of state tensors by including the strain-gradient tensor  $\underline{\eta} := \underline{\varepsilon} \otimes \underline{\nabla}$  and its dual quantity, the hyperstress tensor  $\underline{\tau}$ . This construction corresponds to the Mindlin's *Type II* formulation [46]. Those tensors are elements of  $(\otimes^2 \mathbb{R}^2) \otimes \mathbb{R}^2$ . The constitutive equations of the model define the stress tensor  $\underline{\sigma}$  and the hyperstress tensor  $\underline{\tau}$  as linear functions of the strain tensor  $\underline{\varepsilon}$  and the strain-gradient tensor  $\underline{\eta}$ . This coupled constitutive law requires tensors belonging to the following spaces:

$$\underline{C} \in \text{Ela}_4 \simeq \mathcal{L}^s(\otimes^2 \mathbb{R}^2, \otimes^2 \mathbb{R}^2), \quad \underline{M} \in \text{Ela}_5 \simeq \mathcal{L}((\otimes^2 \mathbb{R}^2) \otimes \mathbb{R}^2, \otimes^2 \mathbb{R}^2), \quad \underline{A} \in \text{Ela}_6 \simeq \mathcal{L}^s((\otimes^2 \mathbb{R}^2) \otimes \mathbb{R}^2, (\otimes^2 \mathbb{R}^2) \otimes \mathbb{R}^2).$$

To be more specific, the constitutive equations read:

$$(5) \quad \begin{cases} \underset{\sim}{\sigma} &= \underset{\approx}{C} : \underset{\sim}{\varepsilon} + \underset{\approx}{M} : \underset{\approx}{\eta} \\ \underset{\sim}{\tau} &= \underset{\approx}{M}^\top : \underset{\sim}{\varepsilon} + \underset{\approx}{A} : \underset{\approx}{\eta} \end{cases}$$

where

- $\underset{\approx}{C} \in \text{Ela}_4 := \{\underset{\approx}{T} \in \overset{4}{\otimes} \mathbb{R}^2 \mid \underset{\approx}{T} \in \mathbb{T}_{(ij)(kl)}\}$  is the fourth-order elasticity tensor;
- $\underset{\approx}{M} \in \text{Ela}_5 := \{\underset{\approx}{T} \in \overset{5}{\otimes} \mathbb{R}^2 \mid \underset{\approx}{T} \in \mathbb{T}_{(ij)(kl)m}\}$  is the fifth-order elasticity tensor;
- $\underset{\approx}{A} \in \text{Ela}_6 := \{\underset{\approx}{T} \in \overset{6}{\otimes} \mathbb{R}^2 \mid \underset{\approx}{T} \in \mathbb{T}_{(ij)k(lm)n}\}$  is the sixth-order elasticity tensor.

Let us define  $\mathcal{S}\text{grd}$  the tensor space of the strain-gradient constitutive tensors as

$$(6) \quad \mathcal{S}\text{grd} = \text{Ela}_4 \oplus \text{Ela}_5 \oplus \text{Ela}_6.$$

A strain-gradient elastic law is defined by a triplet  $\mathcal{E} := (\underset{\approx}{C}, \underset{\approx}{M}, \underset{\approx}{A}) \in \mathcal{S}\text{grd}$ . The complete symmetry classification of two dimensional linear Strain-Gradient elasticity has been established in [6]. In this reference we obtained the following result

**Theorem 4.5.** *The space  $\mathcal{S}\text{ge}$  is partitioned into 14 symmetry classes:*

$$\mathcal{I}(\mathcal{S}\text{ge}) = \{[\mathbf{1}], [\mathbb{Z}_2^\pi], [\mathbb{Z}_2], [\mathbb{D}_2], [\mathbb{Z}_3], [\mathbb{D}_3], [\mathbb{Z}_4], [\mathbb{D}_4], [\mathbb{Z}_5], [\mathbb{D}_5], [\mathbb{Z}_6], [\mathbb{D}_6], [\text{SO}(2)], [\text{O}(2)]\}.$$

As a result we obtain 14 non equivalent symmetry classes, which are reported together with their number of independent components in the following table:

$[\mathbb{G}_C]$	$[\mathbf{1}]$	$[\mathbb{Z}_2^\pi]$	$[\mathbb{Z}_2]$	$[\mathbb{D}_2]$	$[\mathbb{Z}_3]$	$[\mathbb{D}_3]$	$[\mathbb{Z}_4]$	$[\mathbb{D}_4]$	$[\mathbb{Z}_5]$	$[\mathbb{D}_5]$	$[\mathbb{Z}_6]$	$[\mathbb{D}_6]$	$[\text{SO}(2)]$	$[\text{O}(2)]$
$[\mathbb{G}_M]$	$[\mathbf{1}]$	$[\mathbb{Z}_2^\pi]$	$[\mathbb{Z}_2]$	$[\mathbb{D}_2]$	$[\mathbb{Z}_3]$	$[\mathbb{D}_3]$	$[\mathbb{Z}_4]$	$[\mathbb{D}_4]$	$[\mathbb{Z}_5]$	$[\mathbb{D}_5]$	$[\mathbb{Z}_6]$	$[\mathbb{D}_6]$	$[\mathbb{Z}_{k>6}]$	$[\mathbb{D}_{k>6}]$
$\#_{\text{indep}}(\underset{\approx}{A})$	(21)	(21)	<b>21</b>	<b>12</b>	(7)	(5)	<b>9</b>	<b>6</b>	(5)	(4)	<b>7</b>	<b>5</b>	<b>5</b>	<b>4</b>
$\#_{\text{indep}}(\underset{\approx}{M})$	<b>18</b>	<b>9</b>	(0)	(0)	<b>6</b>	<b>3</b>	(0)	(0)	(1)	1	(0)	(0)	(0)	<b>0</b>
$\#_{\text{indep}}(\underset{\approx}{E})$	(6)	(6)	<b>6</b>	<b>4</b>	(2)	(2)	(4)	<b>4</b>	(2)	(2)	(2)	(2)	(2)	<b>2</b>
$\#_{\text{indep}}(\mathcal{E})$	45	27	36	16	15	10	13	9	9	7	9	7	7	6

TABLE 1. The names, the sets of subgroups  $[\mathbb{G}_\mathcal{E}]$  and the numbers of independent components  $\#_{\text{indep}}(\mathcal{E})$  for the 14 symmetry classes of  $\mathcal{E}$ .

When comparing Cosserat elasticity with Strain-gradient elasticity, the following observations can be made

- **Sensitivity to chirality and non centro symmetry:** both models are able to describe this feature absent from Cauchy elasticity. But this description involves fewer coefficients using the Cosserat model rather than the full Strain-gradient model. Another option would be to consider the constrained couple stress model [60] (also called Koiter elasticity) which is a sub-model of the strain gradient elasticity corresponding to a constrained Cosserat model. This model is expected to be the least expensive to take into account the sensitivity to chirality and non centro-symmetry;
- **Maximum anisotropic class before isotropy:** the maximal anisotropic class before isotropy is  $[\mathbb{D}_4]$  with Cosserat and  $[\mathbb{D}_6]$  with Strain-gradient elasticity. Thus, the propagation of hexagonal elastic waves as described in [54, 55] cannot be modelled in the framework of Cosserat elasticity;
- **Acoustical and optical branches:** optical branches associated to rotational mode of vibration can only be described within Cosserat elasticity. Midlin strain gradient elasticity is unable to describe optical branches.

## 5. EXPLICIT HARMONIC DECOMPOSITION AND INTEGRITY BASIS

In this section an explicit harmonic decomposition for the tensors of the constitutive law will be determined. From this parameterisation, a complete set of  $\text{O}(2)$ -polynomial invariants, constituting a minimal integrity basis, will be provided. Before starting, it should be noted that a harmonic decomposition for tensors in  $\mathbb{C}\text{os}$  can be found in the work of Vanucci et al. [62]. As not provided in a closed form, this decomposition does not correspond to our needs. Moreover, due to the non-uniqueness of the decomposition, its mechanical content is not clear. That is why in the next subsection another construction will be proposed.

**5.1. Clebsch-Gordan harmonic decomposition.** In the literature, there exist different methods for proceeding the explicit harmonic decomposition of a tensor, we refer to [4] for an overview of these different approaches. Here we will follow the method introduced in the former reference and named *Clebsch-Gordan approach*. The main idea of the approach, and the main difference from all other methods, is first to decompose not the constitutive tensors but the state tensors on which they act. This first step induces a *block structure* on the constitutive tensors. These *elementary blocks* are generally not harmonic, but their harmonic structures are very simple to decompose into irreducible tensors. The combination of these different steps leads to an explicit harmonic decomposition of the constitutive tensors. What is gained over other methods is that we know the explicit role of each harmonic tensor presents in the decomposition of the constitutive operator. This point will be detailed below.

**Step 1:** To compute the harmonic decomposition of the Cosserat's constitutive tensors, the first step is to write the harmonic decomposition of  $\otimes^2 \mathbb{R}^2$ .

**Proposition 5.1.** *There exists an  $O(2)$ -equivariant isomorphism  $\varphi$  between  $\otimes^2 \mathbb{R}^2$  and  $\mathbb{K}^2 \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1}$  such that*

$$\underline{\mathbb{T}} = \underline{\mathbb{d}} + \underline{\mathbb{s}} + \underline{\mathbb{w}} = \underline{\mathbb{d}} + \underline{\Phi}^{\{2,0\}} \alpha + \underline{\Phi}^{\{2,-1\}} \beta = \varphi(\underline{\mathbb{d}}, \alpha, \beta)$$

with  $(\underline{\mathbb{d}}, \alpha, \beta) \in \mathbb{K}^2 \times \mathbb{K}^0 \times \mathbb{K}^{-1}$  and  $\underline{\Phi}^{\{2,0\}}$  and  $\underline{\Phi}^{\{2,-1\}}$  are such that

$$\underline{\mathbb{s}} = \underline{\Phi}^{\{2,0\}} \alpha \quad \text{with} \quad \underline{\Phi}^{\{2,0\}} := \frac{1}{2} \underline{\mathbb{I}}, \quad \text{and} \quad \underline{\mathbb{w}} = \underline{\Phi}^{\{2,-1\}} \beta \quad \text{with} \quad \underline{\Phi}^{\{2,-1\}} := \frac{1}{2} \underline{\epsilon}.$$

Conversely, we have

$$\alpha = 2 \underline{\Phi}^{\{2,0\}} : \underline{\mathbb{T}} = \text{tr}(\underline{\mathbb{T}}), \quad \text{and} \quad \beta = 2 \underline{\Phi}^{\{2,-1\}} : \underline{\mathbb{T}} = \underline{\epsilon} : \underline{\mathbb{T}}.$$

In the former proposition the notation  $\underline{\Phi}^{\{i,j\}}$  indicates an embedding of  $\mathbb{K}^j$  into  $\mathbb{T}^i$ , the embedding space will be denoted  $\mathbb{H}^{i,j} \subset \mathbb{T}^i$ . From this elementary result, we can introduce the following family of tensors:

$$(7) \quad \underline{\mathbb{P}}^{(2)} := \frac{1}{2} (\underline{\mathbb{I}}_{\approx 2}^{(4)} + \underline{\mathbb{I}}_{\approx 3}^{(4)} - \underline{\mathbb{I}}_{\approx 1}^{(4)}), \quad \underline{\mathbb{P}}^{(0)} = \frac{1}{2} \underline{\mathbb{I}} \otimes \underline{\mathbb{I}} = \frac{1}{2} \underline{\mathbb{I}}_{\approx 1}^{(4)}, \quad \underline{\mathbb{P}}^{(-1)} = \frac{1}{2} \underline{\epsilon} \otimes \underline{\epsilon}.$$

with the 3 elementary isotropic tensors:

$$(8) \quad \left( \underline{\mathbb{I}}_{\approx 1}^{(4)} \right)_{ijkl} = \delta_{ij} \delta_{kl}, \quad \left( \underline{\mathbb{I}}_{\approx 2}^{(4)} \right)_{ijkl} = \delta_{ik} \delta_{jl}, \quad \left( \underline{\mathbb{I}}_{\approx 3}^{(4)} \right)_{ijkl} = \delta_{il} \delta_{jk}.$$

The following table<sup>4</sup> can be constructed showing that  $(\underline{\mathbb{P}}^{(2)}, \underline{\mathbb{P}}^{(0)}, \underline{\mathbb{P}}^{(-1)})$  is a set of orthogonal projectors:

:	$\underline{\mathbb{P}}^{(2)}$	$\underline{\mathbb{P}}^{(0)}$	$\underline{\mathbb{P}}^{(-1)}$
$\underline{\mathbb{P}}^{(2)}$	$\underline{\mathbb{P}}^{(2)}$	$\underline{\mathbb{0}}$	$\underline{\mathbb{0}}$
$\underline{\mathbb{P}}^{(0)}$	$\underline{\mathbb{0}}$	$\underline{\mathbb{P}}^{(0)}$	$\underline{\mathbb{0}}$
$\underline{\mathbb{P}}^{(-1)}$	$\underline{\mathbb{0}}$	$\underline{\mathbb{0}}$	$\underline{\mathbb{P}}^{(-1)}$

To be more explicit, the projector  $\underline{\mathbb{P}}^{(j)}$  is a projection from  $\mathbb{T}^2$  onto  $\mathbb{H}^{2,j}$ . In the present context  $(\underline{\mathbb{P}}^{(2)}, \underline{\mathbb{P}}^{(0)}, \underline{\mathbb{P}}^{(-1)})$  are, respectively, the deviatoric, spheric and hemitropic projectors.

*Remark 5.2.* This decomposition implies the following partition of the state tensors

$$(\underline{\mathbb{e}}, \underline{\kappa}) \in (\otimes^2 \mathbb{R}^2) \oplus \mathbb{R}^2 \rightarrow (\underline{\mathbb{e}}^{(2)}, \underline{\mathbb{e}}^{(0)}, \underline{\mathbb{e}}^{(-1)}, \underline{\kappa}) \in \mathbb{K}^2 \oplus \mathbb{K}^0 \oplus \mathbb{K}^{-1} \oplus \mathbb{K}^1$$

in which  $\underline{\mathbb{e}}^{(j)} = \underline{\mathbb{P}}^{(j)} : \underline{\mathbb{e}}$  belongs to  $\mathbb{H}^{2,j}$ . It can be observed that  $\underline{\varepsilon} \simeq (\underline{\mathbb{e}}^{(2)}, \underline{\mathbb{e}}^{(0)})$  and  $\underline{\eta} \simeq \underline{\mathbb{e}}^{(-1)}$ . In other words, the harmonic decomposition of the classical state tensors coincides with the *micromorphic* type partition proposed by Mindlin.

<sup>4</sup>In this table, the double scalar products : between the elements of the considered family are calculated. The result of such a product is a fourth-order tensor, as it should be.

The harmonic decomposition of the state tensors implies the following block decomposition<sup>5</sup> for the constitutive operator:

$$\begin{pmatrix} \mathbb{A} \\ \mathbb{B} \\ \mathbb{C} \end{pmatrix} = \begin{pmatrix} [\mathbb{K}^2 \otimes^s \mathbb{K}^2] & [\mathbb{K}^2 \otimes \mathbb{K}^0] & [\mathbb{K}^2 \otimes \mathbb{K}^{-1}] \\ & [\mathbb{K}^0 \otimes^s \mathbb{K}^0] & [\mathbb{K}^0 \otimes \mathbb{K}^{-1}] \\ & & [\mathbb{K}^{-1} \otimes \mathbb{K}^{-1}] \end{pmatrix} \begin{pmatrix} [\mathbb{K}^2 \otimes \mathbb{K}^1] \\ [\mathbb{K}^0 \otimes \mathbb{K}^1] \\ [\mathbb{K}^{-1} \otimes \mathbb{K}^1] \\ [\mathbb{K}^1 \otimes^s \mathbb{K}^1] \end{pmatrix} = \begin{pmatrix} [\mathbb{K}^4 \oplus \mathbb{K}^0] & [\mathbb{K}^2] & [\mathbb{K}^2] \\ & [\mathbb{K}^0] & [\mathbb{K}^{-1}] \\ & & [\mathbb{K}^0] \end{pmatrix} \begin{pmatrix} [\mathbb{K}^3 \oplus \mathbb{K}^1] \\ [\mathbb{K}^1] \\ [\mathbb{K}^1] \\ [\mathbb{K}^2 \oplus \mathbb{K}^0] \end{pmatrix}$$

The blue blocks are associated to the Cosserat asymmetric elasticity, while the red ones are associated to the micro-rotation elasticity. The violet blocks are responsible for the coupling between those two mechanisms. In other terms,

$$\begin{pmatrix} \mathbb{s}^{(2)} \\ \mathbb{s}^{(0)} \\ \mathbb{s}^{(-1)} \\ \underline{\mathbb{m}} \end{pmatrix} = \begin{pmatrix} [\mathbb{K}^4 \oplus \mathbb{K}^0] & [\mathbb{K}^2] & [\mathbb{K}^2] \\ & [\mathbb{K}^0] & [\mathbb{K}^{-1}] \\ & & [\mathbb{K}^0] \end{pmatrix} \begin{pmatrix} [\mathbb{K}^3 \oplus \mathbb{K}^1] \\ [\mathbb{K}^1] \\ [\mathbb{K}^1] \\ [\mathbb{K}^2 \oplus \mathbb{K}^0] \end{pmatrix} \begin{pmatrix} \mathbb{e}^{(2)} \\ \mathbb{e}^{(0)} \\ \mathbb{e}^{(-1)} \\ \underline{\kappa} \end{pmatrix}.$$

From this result, and applying the procedure detailed in [4], Clesbch-Gordan Harmonic Decompositions are obtained for the constitutive tensors of Cos.

**Proposition 5.3.** *The tensor  $\mathbb{A} \in \text{Cos}$  admits the uniquely defined Clebsch-Gordan Harmonic Decomposition associated to the family of projectors  $(\mathbb{P}^{(2)}, \mathbb{P}^{(0)}, \mathbb{P}^{(-1)})$ :*

$$\begin{aligned} \mathbb{A} &= \mathbb{H}^{(2)} + \frac{\alpha^{(2)}}{2} \mathbb{P}^{(2)} + 2 \left( \mathbb{h}^{(2,0)} \otimes \mathbb{P}^{\{2,0\}} + \mathbb{P}^{\{2,0\}} \otimes \mathbb{h}^{(2,0)} \right) + 2 \left( \mathbb{h}^{(2,-1)} \otimes \mathbb{P}^{\{-1,2\}} + \mathbb{P}^{\{-1,2\}} \otimes \mathbb{h}^{(2,-1)} \right) \\ &\quad + 2\alpha^{(0)} \mathbb{P}^{(0)} + 2\alpha^{(-1)} \mathbb{P}^{(-1)} + 4\beta^{(0,-1)} \left( \mathbb{P}^{\{2,0\}} \otimes \mathbb{P}^{\{-1,2\}} + \mathbb{P}^{\{-1,2\}} \otimes \mathbb{P}^{\{2,0\}} \right) \end{aligned}$$

in which  $(\mathbb{H}^{(2)}, \mathbb{h}^{(2,0)}, \mathbb{h}^{(2,-1)}, \alpha^{(2)}, \alpha^{(0)}, \alpha^{(-1)}, \beta^{(0,-1)})$  are elements of  $\mathbb{K}^4 \times (\mathbb{K}^2)^2 \times (\mathbb{K}^0)^3 \times \mathbb{K}^{-1}$  defined from  $\mathbb{A}$  as follows:

$\mathbb{K}^{-1}$	$\mathbb{K}^0$	$\mathbb{K}^2$	$\mathbb{K}^4$
$\beta^{(0,-1)} = \mathbb{P}^{\{2,0\}} : \mathbb{A} : \mathbb{P}^{\{2,-1\}}$	$\alpha^{(-1)} = \frac{1}{2} \mathbb{P}^{(-1)} :: \mathbb{A}$ $\alpha^{(0)} = \frac{1}{2} \mathbb{P}^{(0)} :: \mathbb{A}$ $\alpha^{(2)} = \mathbb{A}^{(2,2)} :: \mathbb{P}^{(2)}$	$\mathbb{h}^{(2,-1)} = \mathbb{P}^{(2)} : \mathbb{A} : \mathbb{P}^{\{2,-1\}}$ $\mathbb{h}^{(2,0)} = \mathbb{P}^{(2)} : \mathbb{A} : \mathbb{P}^{\{2,0\}}$	$\mathbb{H}^{(2)} = \mathbb{A}^{(2,2)} - \frac{\alpha^{(2)}}{2} \mathbb{P}^{(2)}$

where  $\mathbb{A}^{(2,2)} = \mathbb{P}^{(2)} : \mathbb{A} : \mathbb{P}^{(2)}$ .

In the above harmonic decomposition, the notation  $\mathbf{H}^{(i,j)}$  indicates a harmonic stiffness tensors that will take a harmonic strain tensor of type  $\mathbb{K}^j$  to produce a stress tensor of type  $\mathbb{K}^i$ . For instance, the deviatoric tensor  $\mathbb{h}^{(2,0)}$  is responsible for generating a deviatoric stress from a spheric strain. When  $j = i$  the notation  $\mathbf{H}^{(i,j)}$  is simplified to  $\mathbf{H}^{(i)}$ . A schematic location of the different terms of the harmonic decomposition is provided in the following matrix-type representation<sup>6</sup>:

$$\begin{bmatrix} [\mathbb{K}^4 \oplus \mathbb{K}^0] & [\mathbb{K}^2] & [\mathbb{K}^2] \\ & [\mathbb{K}^0] & [\mathbb{K}^{-1}] \\ & & [\mathbb{K}^0] \end{bmatrix} = \begin{bmatrix} (\mathbb{H}^{(2)}, \alpha^{(2)}) & \mathbb{h}^{(2,0)} & \mathbb{h}^{(2,-1)} \\ & \alpha^{(0)} & \beta^{(0,-1)} \\ & & \alpha^{(-1)} \end{bmatrix}$$

*Remark 5.4.* In the isotropic case we directly obtain the following parameterisation

$$\mathbb{A} = \frac{\alpha^{(2)}}{2} \mathbb{P}^{(2)} + 2\alpha^{(0)} \mathbb{P}^{(0)} + 2\alpha^{(-1)} \mathbb{P}^{(-1)}$$

which corresponds to the extension of the classical (K, G) parameterisation of the isotropic elasticity<sup>7</sup>. This parameterisation differs from the one generally used in the literature [30] which reads

$$\mathbb{A} = \lambda \mathbf{I}_{\approx 1}^{(4)} + (\mu + \kappa) \mathbf{I}_{\approx 2}^{(4)} + \mu \mathbf{I}_{\approx 3}^{(4)}$$

As an extension of the (K, G) parameterisation, an interest of the proposed system is that the eigenvalues of the matrix representation are directly  $(\frac{\alpha^{(2)}}{2}, 2\alpha^{(0)}, 2\alpha^{(-1)})$ .

<sup>5</sup>For the sake of brevity, only the upper right part of the symmetric matrix structure is detailed.

<sup>6</sup>Here the representation is essentially formal, but it should be noted that by using a suitable change of basis, this matrix representation can be obtained from the conventional one [24, 1].

<sup>7</sup>The relation between our convention and the (K, G) is  $\alpha^{(0)} = K$  and  $\alpha^{(2)} = \frac{G}{4}$ .

**Proposition 5.5.** *The tensor  $\mathcal{B} \in \mathbb{C}\text{ou}$  admits the uniquely defined Clebsch-Gordan Harmonic Decomposition associated to the family of projectors  $(\mathcal{P}^{(2)}, \mathcal{P}^{(0)}, \mathcal{P}^{(-1)})$ :*

$$\mathcal{B} = \mathcal{H}^{(2,1)} + \mathcal{P}^{(2)} \cdot \mathcal{V}^{(2,1)} + 2\mathcal{P}^{(0)} \otimes \mathcal{V}^{(0,1)} + 2\mathcal{P}^{(-1)} \otimes \mathcal{V}^{(-1,1)}$$

in which  $(\mathcal{H}^{(2,1)}, \mathcal{V}^{(2,1)}, \mathcal{V}^{(0,1)}, \mathcal{V}^{(-1,1)})$  are elements of  $\mathbb{K}^3 \times (\mathbb{K}^1)^3$ . Those elements are defined from  $\mathcal{B}$  as follows:

$\mathbb{K}^1$	$\mathbb{K}^3$
$\mathcal{V}^{(2,1)} = 2\mathcal{P}^{(2)} : \mathcal{B} : \mathcal{P}^{(2,0)}$	$\mathcal{H}^{(2,1)} = \mathcal{P}^{(2)} : \mathcal{B} - \mathcal{P}^{(2)} \cdot \mathcal{V}^{(2,1)}$
$\mathcal{V}^{(0,1)} = \mathcal{P}^{(0)} : \mathcal{B}$	
$\mathcal{V}^{(-1,1)} = \mathcal{P}^{(-1)} : \mathcal{B}$	

The schematic location of the terms of the harmonic decomposition is provided in the following matrix-type representation:

$$\begin{bmatrix} [\mathbb{K}^3 \oplus \mathbb{K}^1] \\ [\mathbb{K}^1] \\ [\mathbb{K}^1] \end{bmatrix} = \begin{bmatrix} (\mathcal{H}^{(2,1)}, \mathcal{V}^{(2,1)}) \\ \mathcal{V}^{(0,1)} \\ \mathcal{V}^{(-1,1)} \end{bmatrix}$$

Even if trivial, these results have to be completed by the harmonic decomposition of  $\mathcal{d}$ :

**Proposition 5.6.** *The tensor  $\mathcal{d} \in \mathbb{R}\text{ot}$  admits the uniquely defined Clebsch-Gordan Harmonic Decomposition*

$$\mathcal{d} = \mathcal{h}^{(1)} + \mathcal{P}^{(2,0)} \alpha^{(1)}$$

in which  $(\mathcal{h}^{(1)}, \alpha^{(1)})$  are elements of  $\mathbb{K}^2 \times \mathbb{K}^0$ .

At the end, we have the following harmonic parameterisation for the two-dimensional linear Cosserat elasticity. There exists a  $O(2)$ -equivariant isomorphism  $\varphi$  between  $\mathcal{C}\text{os}$  and  $\mathbb{K}^{-1} \oplus 4\mathbb{K}^0 \oplus 3\mathbb{K}^1 \oplus 3\mathbb{K}^2 \oplus \mathbb{K}^3 \oplus \mathbb{K}^4$  such that

$$\mathcal{C} = (\mathcal{A}, \mathcal{B}, \mathcal{d}) = \varphi(\beta^{(0,-1)}, \alpha^{(-1)}, \alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \mathcal{V}^{(-1,1)}, \mathcal{V}^{(0,1)}, \mathcal{V}^{(2,1)}, \mathcal{h}^{(1)}, \mathcal{h}^{(2,0)}, \mathcal{h}^{(2,-1)}, \mathcal{H}^{(2,1)}, \mathcal{H}^{(2)})$$

**5.2. Application to the parameterisation of effective behaviour of lattice materials.** In the literature devoted to lattice materials, numerous publications investigate the properties of so-called chiral lattices. We will briefly discuss a family of lattices comprising a trichiral<sup>8</sup> and a hexachiral lattice. The family considered also contains their achiral versions, which are the trigonal and hexagonal lattices.

<sup>8</sup>It should be noted that the pattern usually referred to as trichiral in the literature generates a  $\mathbb{Z}_6$ -invariant lattice. The trichiral cell we consider here generates a true  $\mathbb{Z}_3$ -invariant material.

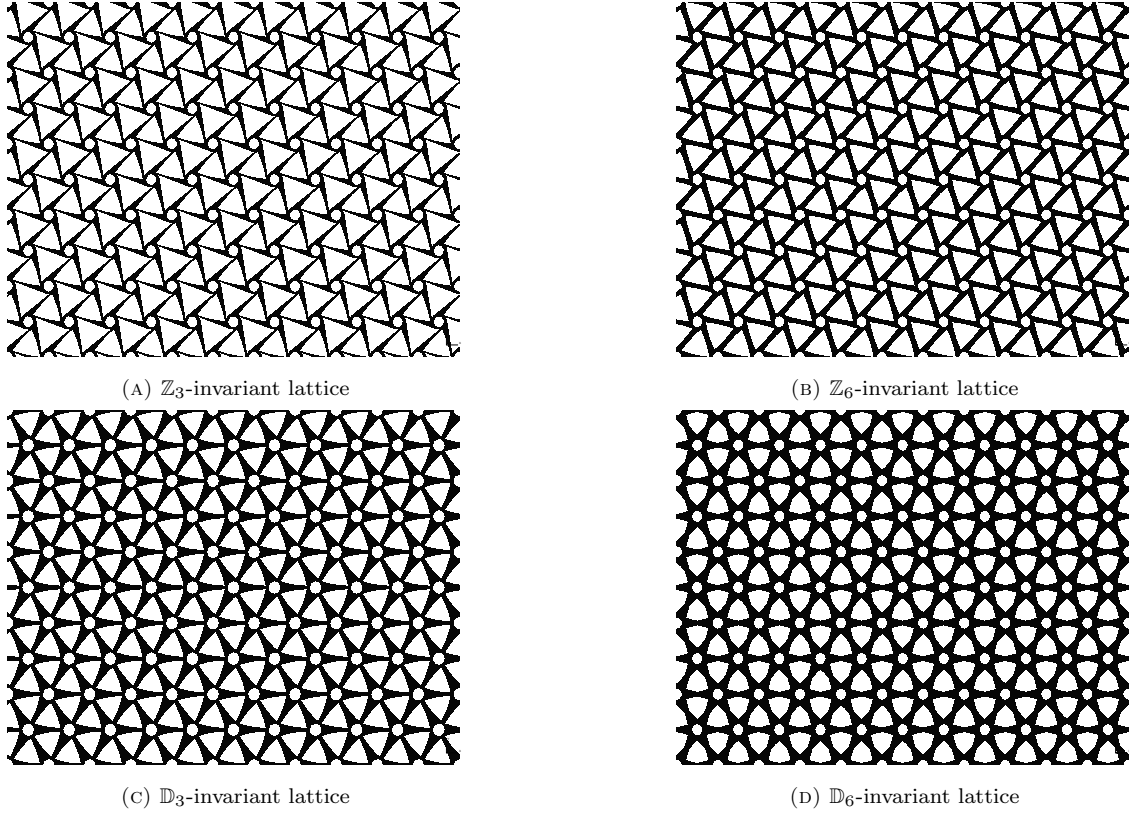




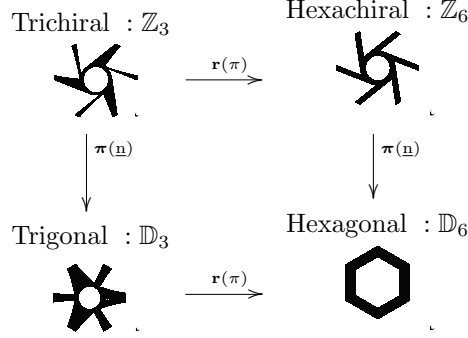


FIGURE 1. Lattices with different invariance properties

Some properties of these lattices, together with references concerning their homogenised properties, are reported in the table below:

Pattern	Name	Point group	Wallpaper group	Homogenisation
	hexagonal	$\mathbb{D}_6$	p6m	<ul style="list-style-type: none"> <li>• Cauchy : [26];</li> <li>• Strain-gradient : [55],</li> <li>• Cosserat : [50]</li> </ul>
	hexachiral	$\mathbb{Z}_6$	p6	<ul style="list-style-type: none"> <li>• Cauchy : [26];</li> <li>• Strain-gradient : [54];</li> <li>• Cosserat : [47, 58, 10, 27, 43, 39]</li> </ul>
	trigonal	$\mathbb{D}_3$	p31m/p31m	<ul style="list-style-type: none"> <li>• Cosserat : [12]</li> </ul>
	trichiral	$\mathbb{Z}_3$	p3	cf. footnote 8

Members of this family are of the same type but differ with respect to their centrosymmetric or chiral properties as indicated in the following diagram.

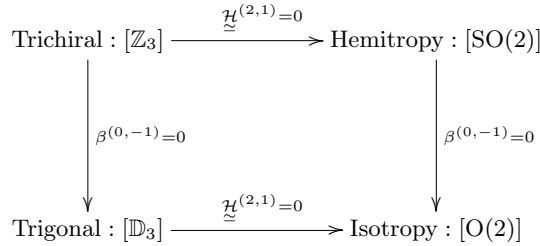


This family is interesting because for Cauchy elasticity, all its members produce an effective isotropic behaviour, but when considering Cosserat elasticity, they produce different anisotropic ones.

In the view of the Cosserat continuum, the mechanical behaviour of these lattices can be considered as degenerated cases of the symmetry class  $[\mathbb{Z}_3]$  :

$$\begin{cases} \underset{\sim}{A} = \frac{\alpha^{(2)}}{2} \underset{\sim}{P}^{(2)} + 2\alpha^{(0)} \underset{\sim}{P}^{(0)} + 2\alpha^{(-1)} \underset{\sim}{P}^{(-1)} + 4\beta^{(0,-1)} \left( \underset{\sim}{\Phi}^{\{2,0\}} \otimes \underset{\sim}{\Phi}^{\{-1,2\}} + \underset{\sim}{\Phi}^{\{2,-1\}} \otimes \underset{\sim}{\Phi}^{\{2,0\}} \right) \\ \underset{\sim}{B} = \underset{\sim}{\mathcal{H}}^{(2,1)} \\ \underset{\sim}{d} = \gamma \underset{\sim}{I} \end{cases}$$

The different degeneracies are shown in the following diagram:



**5.3. Integrity basis.** In this last subsection a complete set of  $O(2)$ -invariant polynomials of Cosserat dimensional elasticity is provided. Such a set, which is usually known as an *integrity basis* [64, 63, 25], is interesting for applications since their elements:

- **Separate the  $O(2)$ -orbits.** The invariants of the integrity basis take the same value if evaluated on two sets of constitutive tensors that just differ up to an isometry, and take different values if not;
- **Generate the algebra of  $O(2)$ -invariant polynomials.** Any  $O(2)$ -polynomial function can be written as a polynomial in the elements of the integrity basis.

Strictly speaking, the nature of this basis (number and order of elements) is independent of a specific explicit harmonic decomposition. However, the knowledge of such a decomposition gives a physical content to its elements. Concerning the Cosserat elasticity, the state of the art is as follows:

- the minimal integrity basis for  $\mathbf{Inv}(\mathbb{R}ot, O(2))$  is known and constituted of two quantities  $\alpha^{(1)}$  and  $\underset{\sim}{h}^{(1)} : \underset{\sim}{h}^{(1)}$ ;
- the minimum integrity bases for  $\mathbb{C}ou$  and  $\mathbb{C}os$  is a non-trivial result. They have been determined using an algorithm introduced in [25], we refer to this contribution for the details of the method. We have the following results:

**Theorem 5.7.** *A minimal integrity basis for  $\mathbf{Inv}(\mathbb{C}ou, O(2))$  consists in the 17 invariants provided in table 2.*

**Theorem 5.8.** *A minimal integrity basis for  $\mathbf{Inv}(\mathbb{C}os, O(2))$  consists in the 15 invariants provided in table 2.*

The definition of invariants of  $\underset{\sim}{A} \in \mathbb{C}os$  in table 2 involves a non-standard tensorial operation called the generalized cross product [25]. For two totally symmetric tensors  $\mathbf{S}_1 \in \mathbb{S}^{n_1}$  and  $\mathbf{S}_2 \in \mathbb{S}^{n_2}$  the generalized cross product is defined as

$$(9) \quad (\mathbf{S}_1 \times \mathbf{S}_2) := -(\mathbf{S}_1 \cdot \underset{\sim}{\epsilon} \cdot \mathbf{S}_2)^s \in \mathbb{S}^{n_1+n_2-2}(\mathbb{R}^2).$$



TABLE 2. Isotropic invariants of  $\text{Cou}$  (left) with  $(v_1, v_2, v_3, \mathcal{H}) := (\underline{v}^{(-1)}, \underline{v}^{(0)}, \underline{v}^{(2)}, \underline{\mathcal{H}}^{(2,1)})$  and  $\text{Cos}$  (right) with  $(\xi, \alpha_1, \alpha_2, \alpha_3, h_1, h_2, H) := (\beta^{(0,-1)}, \alpha^{(-1)}, \alpha^{(0)}, \alpha^{(2)}, \underline{h}^{(2,0)}, \underline{h}^{(2,-1)}, \underline{H}^{(2)})$

#	degree	Formula	#	degree	Formula
1	2	$v_1 \cdot v_1$	1	1	$\alpha_1$
2	2	$v_2 \cdot v_2$	2	1	$\alpha_2$
3	2	$v_3 \cdot v_3$	3	1	$\alpha_3$
4	2	$\mathcal{H} : \mathcal{H}$	4	2	$\xi^2$
5	2	$v_1 \cdot v_2$	5	2	$h_1 : h_1$
6	2	$v_1 \cdot v_3$	6	2	$h_2 : h_2$
7	2	$v_2 \cdot v_3$	7	2	$H :: H$
8	4	$\mathcal{H} : (v_1 \otimes v_1 \otimes v_1)$	8	2	$h_1 : h_2$
9	4	$\mathcal{H} : (v_2 \otimes v_2 \otimes v_2)$	9	3	$h_1 : H : h_1$
10	4	$\mathcal{H} : (v_3 \otimes v_3 \otimes v_3)$	10	3	$h_2 : H : h_2$
11	4	$\mathcal{H} : (v_1 \otimes v_2 \otimes v_2)$	11	3	$h_1 : H : h_2$
12	4	$\mathcal{H} : (v_2 \otimes v_3 \otimes v_3)$	12	3	$\xi(h_1 : (\underline{I} \times h_2))$
13	4	$\mathcal{H} : (v_2 \otimes v_2 \otimes v_3)$	13	4	$\xi(h_2 : H : (\underline{I} \times h_2))$
14	4	$\mathcal{H} : (v_1 \otimes v_1 \otimes v_3)$	14	4	$\xi(h_1 : H : (\underline{I} \times h_1))$
15	4	$\mathcal{H} : (v_1 \otimes v_1 \otimes v_2)$	15	4	$\xi(h_2 : H : (\underline{I} \times h_1))$
16	4	$\mathcal{H} : (v_1 \otimes v_3 \otimes v_3)$			
17	4	$\mathcal{H} : (v_1 \otimes v_2 \otimes v_3)$			

It remains to determine the integrity basis for the full law. The harmonic decomposition of the constitutive tensors  $(\underline{A}, \underline{B}, \underline{d})$ , with respect to  $O(2)$  writes as

$$\mathbb{K}^4 \oplus \mathbb{K}^3 \oplus 3\mathbb{K}^2 \oplus 3\mathbb{K}^1 \oplus 4\mathbb{K}^0 \oplus \mathbb{K}^{-1}.$$

Using the algorithm detailed in [25], the following result is obtained:

**Theorem 5.9.** *A minimal integrity basis for the  $O(2)$ -invariant algebra for the triplet  $(\underline{A}, \underline{B}, \underline{d}) \in \text{Cos}$  consists in the 325 invariants of Tables 3, 4 and 5.*

*Remark 5.10.* Although this is a fairly large set of invariants, it should be borne in mind that this set is valid for tensors that may be completely anisotropic. If integrity bases are considered for tensors that are, at least, say orthotropic, this number of basic invariants will decrease drastically.

## 6. CONCLUSION

In this paper, different aspects of the 2D Cosserat anisotropic linear elasticity continuum have been investigated. Using results developed in previous contributions [8, 4, 25] we obtained the following original results:

- the symmetry classes of the model and the way to combined elementary matrices in each case.
- the harmonic decompositions of the three constitutive tensors of the model;
- the integrity basis of the full coupled law.

If, concerning the symmetry classes, the obtained result was partially known, the two other results are, up to authors' knowledge, new. **In addition, we compared the anisotropic modelling capability of two-dimensional linear Cosserat elasticity with that of strain-gradient elasticity, which is another well-known generalised continuum. These models seem closed and sometimes the reason for choosing one model over another is a matter of personal taste. We believe that our results can provide some clues to rationalise this choice.** We believe that this different results may find interesting applications for the optimal design of architected materials, synthesis problems for granular materials etc. It should be noted that these different results can be extended to the 3D context. The study of this extension will be the subject of a future contribution.

## ACKNOWLEDGEMENT

The authors acknowledge the support of the French Agence Nationale de la Recherche (ANR), under grant ANR-19-CE08-0005 (project Max-Oasis).

## APPENDIX A. MATRIX FORM

### A.1. Tensor $\underline{\hat{A}}$ .

$$\left[ \underline{\hat{A}} \right]_{\mathbb{Z}_2} = \begin{pmatrix} A_{1111} & A_{1122} & A_{1112} & A_{1121} \\ & A_{2222} & A_{2212} & A_{2221} \\ & & A_{1212} & A_{1221} \\ & & & A_{2121} \end{pmatrix}, \quad \left[ \underline{\hat{A}} \right]_{\mathbb{D}_2} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 \\ & A_{2222} & 0 & 0 \\ & & A_{1212} & A_{1221} \\ & & & A_{2121} \end{pmatrix}$$



## A.4.2. Achiral subgroups.

$$\mathcal{L}_{\mathbb{Z}_2^\pi} = \begin{pmatrix} \mathbf{A}_{\mathbb{D}_2} & \mathbf{K}_{\mathbb{Z}_2^\pi}^T \\ \mathbf{K}_{\mathbb{Z}_2^\pi}^T & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 & 0 & B_{112} \\ & A_{2222} & 0 & 0 & 0 & B_{222} \\ & & A_{1212} & A_{1221} & B_{121} & 0 \\ & & & A_{2121} & B_{211} & 0 \\ & & & & d_{11} & 0 \\ & & & & & d_{22} \end{pmatrix}$$

$$\mathcal{L}_{\mathbb{D}_2} = \begin{pmatrix} \mathbf{A}_{\mathbb{D}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbb{D}_2} \end{pmatrix} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 & 0 & 0 \\ & A_{2222} & 0 & 0 & 0 & 0 \\ & & A_{1212} & A_{1221} & 0 & 0 \\ & & & A_{2121} & 0 & 0 \\ & & & & d_{11} & 0 \\ & & & & & d_{22} \end{pmatrix}$$

$$\mathcal{L}_{\mathbb{D}_3} = \begin{pmatrix} \mathbf{A}_{\mathbb{O}(2)} & \mathbf{K}_{\mathbb{D}_3}^T \\ \mathbf{K}_{\mathbb{D}_3}^T & \mathbf{H}_{\mathbb{O}(2)} \end{pmatrix} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 & 0 & B_{112} \\ & A_{1111} & 0 & 0 & 0 & -B_{112} \\ & & A_{1111} - A_{1122} - A_{1221} & A_{1221} & B_{112} & 0 \\ & & & A_{1111} - A_{1122} - A_{1221} & B_{112} & 0 \\ & & & & d_{11} & 0 \\ & & & & & d_{11} \end{pmatrix}$$

$$\mathcal{L}_{\mathbb{D}_4} = \begin{pmatrix} \mathbf{A}_{\mathbb{D}_4} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbb{O}(2)} \end{pmatrix} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 & 0 & 0 \\ & A_{1111} & 0 & 0 & 0 & 0 \\ & & A_{1212} & A_{1221} & 0 & 0 \\ & & & A_{1212} & 0 & 0 \\ & & & & d_{11} & 0 \\ & & & & & d_{11} \end{pmatrix}$$

$$\mathcal{L}_{\mathbb{O}(2)} = \begin{pmatrix} \mathbf{A}_{\mathbb{O}(2)} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{\mathbb{O}(2)} \end{pmatrix} = \begin{pmatrix} A_{1111} & A_{1122} & 0 & 0 & 0 & 0 & 0 & 0 \\ & A_{1111} & 0 & 0 & 0 & 0 & 0 & 0 \\ & & A_{1111} - A_{1122} - A_{1221} & A_{1221} & 0 & 0 & 0 & 0 \\ & & & A_{1111} - A_{1122} - A_{1221} & 0 & 0 & 0 & 0 \\ & & & & d_{11} & 0 & 0 & 0 \\ & & & & & d_{11} & 0 & 0 \\ & & & & & & d_{11} & 0 \\ & & & & & & & d_{11} \end{pmatrix}$$

## APPENDIX B. FULL TABLES

The complete 325 invariants of  $\mathbb{O}(2)$ -invariant algebra for the triplet  $(\mathbf{A}, \mathbf{B}, \mathbf{d}) \in \text{Cos}$  are provided in the tables below. The definition of some of these invariants require the use of special tensorial product denoted by  $*$  and called the *harmonic product*. Consider  $\mathbf{H}_1 \in \mathbb{H}^{n_1}$  and  $\mathbf{H}_2 \in \mathbb{H}^{n_2}$ , the harmonic product  $\mathbf{H}_1 * \mathbf{H}_2$  is the projection of the classical tensor product on  $\mathbb{H}^{n_1+n_2}$ , this product can be computed as follows

$$(\mathbf{H}_1 * \mathbf{H}_2) = \mathbf{H}_1 \odot \mathbf{H}_2 - \frac{1}{2}(\mathbf{I} \otimes (\mathbf{H}_1 \cdot \mathbf{H}_2))^s$$

In this expression  $(\cdot)^s$  denotes the complete symmetrisation of a tensor.

TABLE 3. Isotropic invariants of  $\mathcal{C}os = \mathcal{C}os \oplus \mathcal{C}ou \oplus \mathcal{R}ot$  with

$$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathcal{H}) := (\underline{\mathbf{v}}^{(-1)}, \underline{\mathbf{v}}^{(0)}, \underline{\mathbf{v}}^{(2)}, \underline{\mathcal{H}}^{(2,1)})$$

$$(\xi, \alpha_1, \alpha_2, \alpha_3, h_1, h_2, H) := (\beta^{(0,-1)}, \alpha^{(-1)}, \alpha^{(0)}, \alpha^{(2)}, h^{(2,0)}, h^{(2,-1)}, H^{(2)})$$

$$(\alpha_4, h_3) := (\alpha^{(1)}, h^{(1)})$$

#	degree	Formula
1	1	$\alpha_1$
2	1	$\alpha_2$
3	1	$\alpha_3$
4	1	$\alpha_4$
5	2	$\mathbf{v}_1 \cdot \mathbf{v}_1$
6	2	$\mathbf{v}_2 \cdot \mathbf{v}_2$
7	2	$\mathbf{v}_3 \cdot \mathbf{v}_3$
8	2	$h_1 : h_1$
9	2	$h_2 : h_2$
10	2	$h_3 : h_3$
11	2	$\mathcal{H} : \mathcal{H}$
12	2	$H :: H$
13	2	$\mathbf{v}_1 \cdot \mathbf{v}_2$
14	2	$\mathbf{v}_1 \cdot \mathbf{v}_3$
15	2	$\mathbf{v}_2 \cdot \mathbf{v}_3$
16	2	$h_1 : h_2$
17	2	$h_1 : h_3$
18	2	$h_2 : h_3$
19	3	$h_1 : (\mathbf{v}_1 \otimes \mathbf{v}_1)$
20	3	$h_1 : (\mathbf{v}_2 \otimes \mathbf{v}_2)$
21	3	$h_1 : (\mathbf{v}_3 \otimes \mathbf{v}_3)$
22	3	$h_2 : (\mathbf{v}_1 \otimes \mathbf{v}_1)$
23	3	$h_2 : (\mathbf{v}_2 \otimes \mathbf{v}_2)$
24	3	$h_2 : (\mathbf{v}_3 \otimes \mathbf{v}_3)$
25	3	$h_3 : (\mathbf{v}_1 \otimes \mathbf{v}_1)$
26	3	$h_3 : (\mathbf{v}_2 \otimes \mathbf{v}_2)$
27	3	$h_3 : (\mathbf{v}_3 \otimes \mathbf{v}_3)$
28	3	$h_1 : (\mathbf{v}_1 \otimes \mathbf{v}_2)$
29	3	$h_1 : (\mathbf{v}_1 \otimes \mathbf{v}_3)$
30	3	$h_1 : (\mathbf{v}_2 \otimes \mathbf{v}_3)$
31	3	$h_2 : (\mathbf{v}_1 \otimes \mathbf{v}_2)$
32	3	$h_2 : (\mathbf{v}_1 \otimes \mathbf{v}_3)$
33	3	$h_2 : (\mathbf{v}_2 \otimes \mathbf{v}_3)$
34	3	$h_3 : (\mathbf{v}_1 \otimes \mathbf{v}_2)$
35	3	$h_3 : (\mathbf{v}_1 \otimes \mathbf{v}_3)$
36	3	$h_3 : (\mathbf{v}_2 \otimes \mathbf{v}_3)$
37	3	$\mathcal{H} : (\mathbf{v}_1 \otimes h_1)$
38	3	$\mathcal{H} : (\mathbf{v}_1 \otimes h_2)$
39	3	$\mathcal{H} : (\mathbf{v}_1 \otimes h_3)$
40	3	$\mathcal{H} : (\mathbf{v}_2 \otimes h_1)$
41	3	$\mathcal{H} : (\mathbf{v}_2 \otimes h_2)$
42	3	$\mathcal{H} : (\mathbf{v}_2 \otimes h_3)$
43	3	$\mathcal{H} : (\mathbf{v}_3 \otimes h_1)$
44	3	$\mathcal{H} : (\mathbf{v}_3 \otimes h_2)$
45	3	$\mathcal{H} : (\mathbf{v}_3 \otimes h_3)$
46	3	$H :: (h_1 \otimes h_1)$
47	3	$H :: (h_2 \otimes h_2)$
48	3	$H :: (h_3 \otimes h_3)$
49	3	$H :: (h_1 \otimes h_2)$
50	3	$H :: (h_1 \otimes h_3)$
51	3	$H :: (h_2 \otimes h_3)$
52	3	$H :: (\mathbf{v}_1 \otimes \mathcal{H})$
53	3	$H :: (\mathbf{v}_2 \otimes \mathcal{H})$
54	3	$H :: (\mathbf{v}_3 \otimes \mathcal{H})$
55	4	$\mathcal{H} : (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
56	4	$\mathcal{H} : (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
57	4	$\mathcal{H} : (\mathbf{v}_3 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
58	4	$\mathcal{H} : (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
59	4	$\mathcal{H} : (\mathbf{v}_1 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
60	4	$\mathcal{H} : (\mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
61	4	$\mathcal{H} : (\mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
62	4	$\mathcal{H} : (\mathbf{v}_3 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
63	4	$\mathcal{H} : (\mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
64	4	$\mathcal{H} : (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
65	4	$(h_1 * h_1) :: (\mathbf{v}_1 \otimes \mathcal{H})$
66	4	$(h_2 * h_2) :: (\mathbf{v}_1 \otimes \mathcal{H})$
67	4	$(h_3 * h_3) :: (\mathbf{v}_1 \otimes \mathcal{H})$
68	4	$(h_1 * h_1) :: (\mathbf{v}_2 \otimes \mathcal{H})$
69	4	$(h_2 * h_2) :: (\mathbf{v}_2 \otimes \mathcal{H})$
70	4	$(h_3 * h_3) :: (\mathbf{v}_2 \otimes \mathcal{H})$
71	4	$(h_1 * h_1) :: (\mathbf{v}_3 \otimes \mathcal{H})$
72	4	$(h_2 * h_2) :: (\mathbf{v}_3 \otimes \mathcal{H})$
73	4	$(h_3 * h_3) :: (\mathbf{v}_3 \otimes \mathcal{H})$
74	4	$(h_1 * h_2) :: (\mathbf{v}_1 \otimes \mathcal{H})$
75	4	$(h_1 * h_3) :: (\mathbf{v}_1 \otimes \mathcal{H})$
76	4	$(h_2 * h_3) :: (\mathbf{v}_1 \otimes \mathcal{H})$
77	4	$(h_1 * h_2) :: (\mathbf{v}_2 \otimes \mathcal{H})$
78	4	$(h_1 * h_3) :: (\mathbf{v}_2 \otimes \mathcal{H})$
79	4	$(h_2 * h_3) :: (\mathbf{v}_2 \otimes \mathcal{H})$
80	4	$(h_1 * h_2) :: (\mathbf{v}_3 \otimes \mathcal{H})$
81	4	$(h_1 * h_3) :: (\mathbf{v}_3 \otimes \mathcal{H})$
82	4	$(h_2 * h_3) :: (\mathbf{v}_3 \otimes \mathcal{H})$

#	degree	Formula
83	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes h_1)$
84	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes h_1)$
85	4	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_3 \otimes h_1)$
86	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes h_2)$
87	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes h_2)$
88	4	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_3 \otimes h_2)$
89	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes h_3)$
90	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes h_3)$
91	4	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_3 \otimes h_3)$
92	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes h_1)$
93	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_3 \otimes h_1)$
94	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_3 \otimes h_1)$
95	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes h_2)$
96	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_3 \otimes h_2)$
97	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_3 \otimes h_2)$
98	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes h_3)$
99	4	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_3 \otimes h_3)$
100	4	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_3 \otimes h_3)$
101	4	$(\mathcal{H} * h_1) :: (\mathbf{H} \otimes \mathbf{v}_1)$
102	4	$(\mathcal{H} * h_1) :: (\mathbf{H} \otimes \mathbf{v}_2)$
103	4	$(\mathcal{H} * h_1) :: (\mathbf{H} \otimes \mathbf{v}_3)$
104	4	$(\mathcal{H} * h_2) :: (\mathbf{H} \otimes \mathbf{v}_1)$
105	4	$(\mathcal{H} * h_2) :: (\mathbf{H} \otimes \mathbf{v}_2)$
106	4	$(\mathcal{H} * h_2) :: (\mathbf{H} \otimes \mathbf{v}_3)$
107	4	$(\mathcal{H} * h_3) :: (\mathbf{H} \otimes \mathbf{v}_1)$
108	4	$(\mathcal{H} * h_3) :: (\mathbf{H} \otimes \mathbf{v}_2)$
109	4	$(\mathcal{H} * h_3) :: (\mathbf{H} \otimes \mathbf{v}_3)$
110	4	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes h_1)$
111	4	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes h_2)$
112	4	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes h_3)$
113	5	$(\mathcal{H} * \mathcal{H}) :: (h_1 \otimes h_1 \otimes h_1)$
114	5	$(\mathcal{H} * \mathcal{H}) :: (h_2 \otimes h_2 \otimes h_2)$
115	5	$(\mathcal{H} * \mathcal{H}) :: (h_3 \otimes h_3 \otimes h_3)$
116	5	$(\mathcal{H} * \mathcal{H}) :: (h_1 \otimes h_2 \otimes h_2)$
117	5	$(\mathcal{H} * \mathcal{H}) :: (h_1 \otimes h_3 \otimes h_3)$
118	5	$(\mathcal{H} * \mathcal{H}) :: (h_2 \otimes h_1 \otimes h_1)$
119	5	$(\mathcal{H} * \mathcal{H}) :: (h_2 \otimes h_3 \otimes h_3)$
120	5	$(\mathcal{H} * \mathcal{H}) :: (h_3 \otimes h_1 \otimes h_1)$
121	5	$(\mathcal{H} * \mathcal{H}) :: (h_3 \otimes h_2 \otimes h_2)$
122	5	$(\mathcal{H} * \mathcal{H}) :: (h_1 \otimes h_2 \otimes h_3)$
123	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
124	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
125	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
126	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_1 \otimes \mathbf{v}_2)$
127	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_1 \otimes \mathbf{v}_3)$
128	5	$(\mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
129	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
130	5	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
131	5	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
132	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
133	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
134	5	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
135	5	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_3 \otimes \mathbf{v}_3)$
136	5	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_1 \otimes \mathbf{v}_1)$
137	5	$H :: (\mathbf{v}_3 \otimes \mathbf{v}_2 \otimes \mathbf{v}_2)$
138	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes \mathbf{v}_2)$
139	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes \mathbf{v}_3)$
140	5	$H :: (\mathbf{v}_2 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
141	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
142	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
143	5	$H :: (\mathbf{v}_1 \otimes \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \mathbf{v}_3)$
144	5	$(\mathbf{H} * \mathbf{H}) :: (\mathcal{H} \otimes \mathcal{H} \otimes h_1)$
145	5	$(\mathbf{H} * \mathbf{H}) :: (\mathcal{H} \otimes \mathcal{H} \otimes h_2)$
146	5	$(\mathbf{H} * \mathbf{H}) :: (\mathcal{H} \otimes \mathcal{H} \otimes h_3)$
147	6	$(\mathcal{H} * \mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{H} \otimes \mathbf{v}_1)$
148	6	$(\mathcal{H} * \mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{H} \otimes \mathbf{v}_2)$
149	6	$(\mathcal{H} * \mathcal{H} * \mathcal{H}) :: (\mathbf{H} \otimes \mathbf{H} \otimes \mathbf{v}_3)$
150	7	$(\mathbf{H} * \mathbf{H} * \mathbf{H}) :: (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})$



TABLE 5. Product of hemitropic invariants of  $\text{Cos} = \text{Cos} \oplus \text{Cou} \oplus \text{Rot}$  with  $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathcal{H}) := (\underline{\mathbf{v}}^{(-1)}, \underline{\mathbf{v}}^{(0)}, \underline{\mathbf{v}}^{(2)}, \underline{\mathcal{H}}^{(2,1)})$   
 $(\xi, \alpha_1, \alpha_2, \alpha_3, \mathbf{h}_1, \mathbf{h}_2, \mathbf{H}) := (\beta^{(0,-1)}, \alpha^{(-1)}, \alpha^{(0)}, \alpha^{(2)}, \mathbf{h}^{(2,0)}, \mathbf{h}^{(2,-1)}, \mathbf{H}^{(2)})$   
 $(\alpha_4, \mathbf{h}_3) := (\alpha^{(1)}, \mathbf{h}^{(1)})$

#	degree	Formula
1	4	$(\mathbf{v}_1 \times \mathbf{v}_2) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
2	4	$(\mathbf{v}_1 \times \mathbf{v}_2) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
3	4	$(\mathbf{v}_1 \times \mathbf{v}_2) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$
4	4	$(\mathbf{v}_1 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
5	4	$(\mathbf{v}_1 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
6	4	$(\mathbf{v}_1 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$
7	4	$(\mathbf{v}_2 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
8	4	$(\mathbf{v}_2 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
9	4	$(\mathbf{v}_2 \times \mathbf{v}_3) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$
10	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_1)) (\mathbf{v}_1 \times \mathbf{v}_2)$
11	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_1)) (\mathbf{v}_1 \times \mathbf{v}_3)$
12	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_1)) (\mathbf{v}_2 \times \mathbf{v}_3)$
13	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_2)) (\mathbf{v}_1 \times \mathbf{v}_2)$
14	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_2)) (\mathbf{v}_1 \times \mathbf{v}_3)$
15	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_2)) (\mathbf{v}_2 \times \mathbf{v}_3)$
16	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_3 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_2)$
17	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_3 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_3)$
18	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_3 \otimes \mathbf{h}_3)) (\mathbf{v}_2 \times \mathbf{v}_3)$
19	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_2)) (\mathbf{v}_1 \times \mathbf{v}_2)$
20	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_2)) (\mathbf{v}_1 \times \mathbf{v}_3)$
21	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_2)) (\mathbf{v}_2 \times \mathbf{v}_3)$
22	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_2)$
23	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_3)$
24	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_1 \otimes \mathbf{h}_3)) (\mathbf{v}_2 \times \mathbf{v}_3)$
25	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_2)$
26	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_3)) (\mathbf{v}_1 \times \mathbf{v}_3)$
27	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathbf{h}_2 \otimes \mathbf{h}_3)) (\mathbf{v}_2 \times \mathbf{v}_3)$
28	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_1)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
29	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_1)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
30	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_1)) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$
31	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_2)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
32	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_2)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
33	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_2)) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$
34	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_3)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_2)$
35	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_3)) ((\mathbf{I} \times \mathbf{h}_1) : \mathbf{h}_3)$
36	5	$((\mathbf{I} \times \mathbf{H}) :: (\mathcal{H} \otimes \mathbf{v}_3)) ((\mathbf{I} \times \mathbf{h}_2) : \mathbf{h}_3)$

## REFERENCES

- [1] H. Abdoul-Anziz, N. Auffray, and B. Desmorat. Symmetry classes and matrix representations of the 2d flexoelectric law. *Symmetry*, 12(4):674, 2020.
- [2] A. Alderson, K. Alderson, D. Attard, K. Evans, R. Gatt, J. Grima, W. Miller, N. Ravirala, C. Smith, and K. Zied. Elastic constants of 3-, 4-and 6-connected chiral and anti-chiral honeycombs subject to uniaxial in-plane loading. *Composites Science and Technology*, 70(7):1042–1048, 2010.
- [3] S. Amstutz, S. Giusti, A. Novotny, and E. de Souza Neto. Topological derivative for multi-scale linear elasticity models applied to the synthesis of microstructures. *International Journal for Numerical Methods in Engineering*, 84(6):733–756, 2010.
- [4] N. Auffray, H. Abdoul-Anziz, and B. Desmorat. Explicit harmonic structure of bidimensional linear strain-gradient elasticity. *European Journal of Mechanics-A/Solids*, 87:104202, 2021.
- [5] N. Auffray, R. Bouchet, and Y. Bréchet. Derivation of anisotropic matrix for bi-dimensional strain-gradient elasticity behavior. *International Journal of Solids and Structures*, 46(2):440–454, 2009.
- [6] N. Auffray, J. Dirrenberger, and G. Rosi. A complete description of bi-dimensional anisotropic strain-gradient elasticity. *International Journal of Solids and Structures*, 69-70:195–206, 2015.
- [7] N. Auffray, Q.C. He, and H. Le Quang. Complete symmetry classification and compact matrix representations for 3d strain gradient elasticity. *International Journal of Solids and Structures*, 159:197–210, 2019.
- [8] N. Auffray, B. Koley, and M. Olive. Handbook of bi-dimensional tensors: Part I: Harmonic decomposition and symmetry classes. *Mathematics and Mechanics of Solids*, 22(9):1847–1865, 2017.
- [9] N. Auffray, H. Le Quang, and Q.C. He. Matrix representations for 3D strain-gradient elasticity. *Journal of the Mechanics and Physics of Solids*, 61(5):1202–1223, 2013.
- [10] A. Bacigalupo and L. Gambarotta. Homogenization of periodic hexa-and tetrachiral cellular solids. *Composite Structures*, 116:461–476, 2014.
- [11] A. Bacigalupo and L. Gambarotta. Simplified modelling of chiral lattice materials with local resonators. *International Journal of Solids and Structures*, 83:126–141, 2016.
- [12] A. Bacigalupo and L. Gambarotta. Wave propagation in non-centrosymmetric beam-lattices with lumped masses: discrete and micropolar modeling. *International Journal of Solids and Structures*, 118:128–145, 2017.
- [13] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 374. American Mathematical Soc., 1978.
- [14] M. Bonnet, R. Cornaggia, and B. Guzina. Microstructural topological sensitivities of the second-order macroscopic model for waves in periodic media. *SIAM Journal on Applied Mathematics*, 78(4):2057–2082, 2018.

- [15] M. Bruggi, V. Zega, and A. Corigliano. Synthesis of auxetic structures using optimization of compliant mechanisms and a micropolar material model. *Structural and multidisciplinary optimization*, 55(1):1–12, 2017.
- [16] W. Chen and X. Huang. Topological design of 3d chiral metamaterials based on couple-stress homogenization. *Journal of the Mechanics and Physics of Solids*, 131:372–386, 2019.
- [17] X. Chen, J. Yvonnet, H. S. Park, and S. Yao. Enhanced converse flexoelectricity in piezoelectric composites by coupling topology optimization with homogenization. *Journal of Applied Physics*, 129(24):245104, 2021.
- [18] Y. Chen, T. Frenzel, S. Guenneau, M. Kadic, and M. Wegener. Mapping acoustical activity in 3d chiral mechanical metamaterials onto micropolar continuum elasticity. *Journal of the Mechanics and Physics of Solids*, 137:103877, 2020.
- [19] Y. Chen, X.N. Liu, G.K. Hu, and Q.P. Sun. Micropolar continuum modelling of bi-dimensional tetrachiral lattices. *Proc. R. Soc. A*, 470(2165):20130734, 2014.
- [20] R. Cornaggia and C. Bellis. Tuning effective dynamical properties of periodic media by fft-accelerated topological optimization. *International Journal for Numerical Methods in Engineering*, 121(14):3178–3205, 2020.
- [21] E. Cosserat and F. Cosserat. *Théorie des corps déformables*. A. Hermann et fils, 1909.
- [22] E. Cosserat and F. Cosserat. *Théorie des corps déformables*. 1909.
- [23] F. dell’Isola, P. Seppacher, M. Spagnuolo, E. Barchiesi, F. Hild, T. Lekszycki, I. Giorgio, L. Placidi, U. Andreaus, M. Cuomo, et al. Advances in pantographic structures: design, manufacturing, models, experiments and image analyses. *Continuum Mechanics and Thermodynamics*, 31(4):1231–1282, 2019.
- [24] B. Desmorat and N. Auffray. Space of 2D elastic materials: a geometric journey. *Continuum Mechanics and Thermodynamics*, 31(4):1205–1229, 2019.
- [25] B. Desmorat, M. Olive, N. Auffray, R. Desmorat, and B. Kolev. Computation of minimal covariants bases for 2D coupled constitutive laws. *arXiv preprint arXiv:2007.01576*, 2020.
- [26] J. Dirrenberger. *Propriétés effectives de matériaux architecturés*. PhD thesis, Ecole Nationale Supérieure des Mines de Paris, 2012.
- [27] K. El Nady, F. Dos Reis, and J.-F. Ganghoffer. Computation of the homogenized nonlinear elastic response of 2d and 3d auxetic structures based on micropolar continuum models. *Composite Structures*, 170:271–290, 2017.
- [28] V. Eremeyev, L. Lebedev, and H. Altenbach. *Foundations of micropolar mechanics*. Springer Science & Business Media, 2012.
- [29] V. Eremeyev and W. Pietraszkiewicz. Material symmetry group and constitutive equations of micropolar anisotropic elastic solids. *Mathematics and Mechanics of Solids*, 21(2):210–221, 2016.
- [30] A. Eringen. *Microcontinuum field theories: I. Foundations and solids*. Springer Science & Business Media, 2012.
- [31] J. Fatemi, P. Onck, G. Poort, and F. Van Keulen. Cosserat moduli of anisotropic cancellous bone: a micromechanical analysis. In *Journal de Physique IV (Proceedings)*, volume 105, pages 273–280. EDP sciences, 2003.
- [32] F. Feyel. A multilevel finite element method (fe2) to describe the response of highly non-linear structures using generalized continua. *Computer Methods in applied Mechanics and engineering*, 192(28-30):3233–3244, 2003.
- [33] S. Forest. Mechanics of cosserat media—an introduction. *Ecole des Mines de Paris, Paris*, pages 1–20, 2005.
- [34] S. Forest and R. Sievert. Nonlinear microstrain theories. *International Journal of Solids and Structures*, 43:7224–7245, 2006.
- [35] T. Frenzel, M. Kadic, and M. Wegener. Three-dimensional mechanical metamaterials with a twist. *Science*, 358(6366):1072–1074, 2017.
- [36] P. Geoffroy-Donders. *Homogenization method for topology optimization of structures built with lattice materials*. PhD thesis, Ecole Polytechnique, 2018.
- [37] I. Giorgio, F. dell’Isola, and A. Misra. Chirality in 2D Cosserat media related to stretch-micro-rotation coupling with links to granular micromechanics. *International Journal of Solids and Structures*, 202:28–38, 2020.
- [38] M. Golubitsky, I. Stewart, and D. Schaeffer. *Singularities and Groups in Bifurcation Theory: Volume II*, volume 69. Springer Science & Business Media, 1989.
- [39] H. Joumaa and M. Ostoj-Starzewski. Stress and couple-stress invariance in non-centrosymmetric micropolar planar elasticity. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2134):2896–2911, 2011.
- [40] P. Kanoute, D. Boso, J. Chaboche, and B. Schrefler. Multiscale methods for composites: a review. *Archives of Computational Methods in Engineering*, 16(1):31–75, 2009.
- [41] S. Khakalo and J. Niranen. Anisotropic strain gradient thermoelasticity for cellular structures: Plate models, homogenization and isogeometric analysis. *Journal of the Mechanics and Physics of Solids*, 134:103728, 2020.
- [42] R. Lakes and L. Benedict. Noncentrosymmetry in micropolar elasticity. *International Journal of Engineering Science*, 20(10):1161–1167, 1982.
- [43] X.N. Liu, G.L. Huang, and G.K. Hu. Chiral effect in plane isotropic micropolar elasticity and its application to chiral lattices. *Journal of the Mechanics and Physics of Solids*, 60(11):1907–1921, 2012.
- [44] R.D. Mindlin. Micro-structure in linear elasticity. *Archive for Rational Mechanics and Analysis*, 16(1), 1964.
- [45] R.D. Mindlin. Second gradient of strain and surface-tension in linear elasticity. *International Journal of Solids and Structures*, 1(4):417–438, 1965.
- [46] R.D. Mindlin and N.N. Eshel. On first strain-gradient theories in linear elasticity. *International Journal of Solids and Structures*, 4(1):109–124, 1968.
- [47] D. Mousanezhad, B. Haghpahan, R. Ghosh, A. Hamouda, H. Nayeb-Hashemi, and A. Vaziri. Elastic properties of chiral, anti-chiral, and hierarchical honeycombs: a simple energy-based approach. *Theoretical and Applied Mechanics Letters*, 6(2):81–96, 2016.
- [48] H. Nassar, Y.Y. Chen, and G.L. Huang. Polar metamaterials: a new outlook on resonance for cloaking applications. *Physical review letters*, 124(8):084301, 2020.
- [49] A. Novotny and J. Sokolowski. *Topological derivatives in shape optimization*. Springer Science & Business Media, 2012.
- [50] F. Ongaro. Estimation of the effective properties of two-dimensional cellular materials: a review. *Theoretical and Applied Mechanics Letters*, 8(4):209–230, 2018.
- [51] H.C. Park and R.S. Lakes. Cosserat micromechanics of human bone: strain redistribution by a hydration sensitive constituent. *Journal of biomechanics*, 19(5):385–397, 1986.
- [52] A. Pau and P. Trovalusci. Block masonry as equivalent micropolar continua: the role of relative rotations. *Acta Mechanica*, 223(7):1455–1471, 2012.
- [53] D. Prall and R. Lakes. Properties of a chiral honeycomb with a poisson’s ratio of—1. *International Journal of Mechanical Sciences*, 39(3):305–314, 1997.
- [54] G. Rosi and N. Auffray. Anisotropic and dispersive wave propagation within strain-gradient framework. *Wave Motion*, 63:120–134, 2016.
- [55] G. Rosi and N. Auffray. Continuum modelling of frequency dependent acoustic beam focussing and steering in hexagonal lattices. *European Journal of Mechanics-A/Solids*, 77:103803, 2019.
- [56] M. Rovati and D. Veber. Optimal topologies for micropolar solids. *Structural and Multidisciplinary Optimization*, 33(1):47–59, 2007.
- [57] S. Sahmani and M. Aghdam. Size-dependent axial instability of microtubules surrounded by cytoplasm of a living cell based on nonlocal strain gradient elasticity theory. *Journal of Theoretical Biology*, 422:59–71, 2017.



- [58] A. Spadoni and M. Ruzzene. Elasto-static micropolar behavior of a chiral auxetic lattice. *Journal of the Mechanics and Physics of Solids*, 60(1):156–171, 2012.
- [59] J. Stillwell. *Naive lie theory*. Springer Science & Business Media, 2008.
- [60] R. A. Toupin. Elastic materials with couple-stresses. *Archive for Rational Mechanics and Analysis*, 11:385–414, 1962.
- [61] P. Trovalusci and R. Masiani. Material symmetries of micropolar continua equivalent to lattices. *International Journal of Solids and Structures*, 36(14):2091–2108, 1999.
- [62] P. Vannucci and G. Verchery. Anisotropy of plane complex elastic bodies. *International Journal of Solids and Structures*, 47(9):1154–1166, 2010.
- [63] M. Vianello. An integrity basis for plane elasticity tensors. *Archives of Mechanics*, 49(1):197–208, 1997.
- [64] H. Weyl. *The classical groups: their invariants and representations*, volume 45. Princeton University Press, 1946.
- [65] M. Yildizdag, C. Tran, E. Barchiesi, M. Spagnuolo, F. dell’Isola, and F. Hild. A multi-disciplinary approach for mechanical metamaterial synthesis: a hierarchical modular multiscale cellular structure paradigm. In *State of the art and future trends in material modeling*, pages 485–505. Springer, 2019.
- [66] J. Yvonnet. *Computational homogenization of heterogeneous materials with finite elements*. Springer, 2019.
- [67] Q.-S. Zheng and J.-P. Boehler. The description, classification, and reality of material and physical symmetries. *Acta Mechanica*, 102(1-4):73–89, 1994.
- [68] Q-S Zheng and A. Spencer. On the canonical representations for kronecker powers of orthogonal tensors with application to material symmetry problems. *International Journal of Engineering Science*, 31(4):617–635, 1993.