

Some thoughts on Emil Post’s thoughts on “computation”



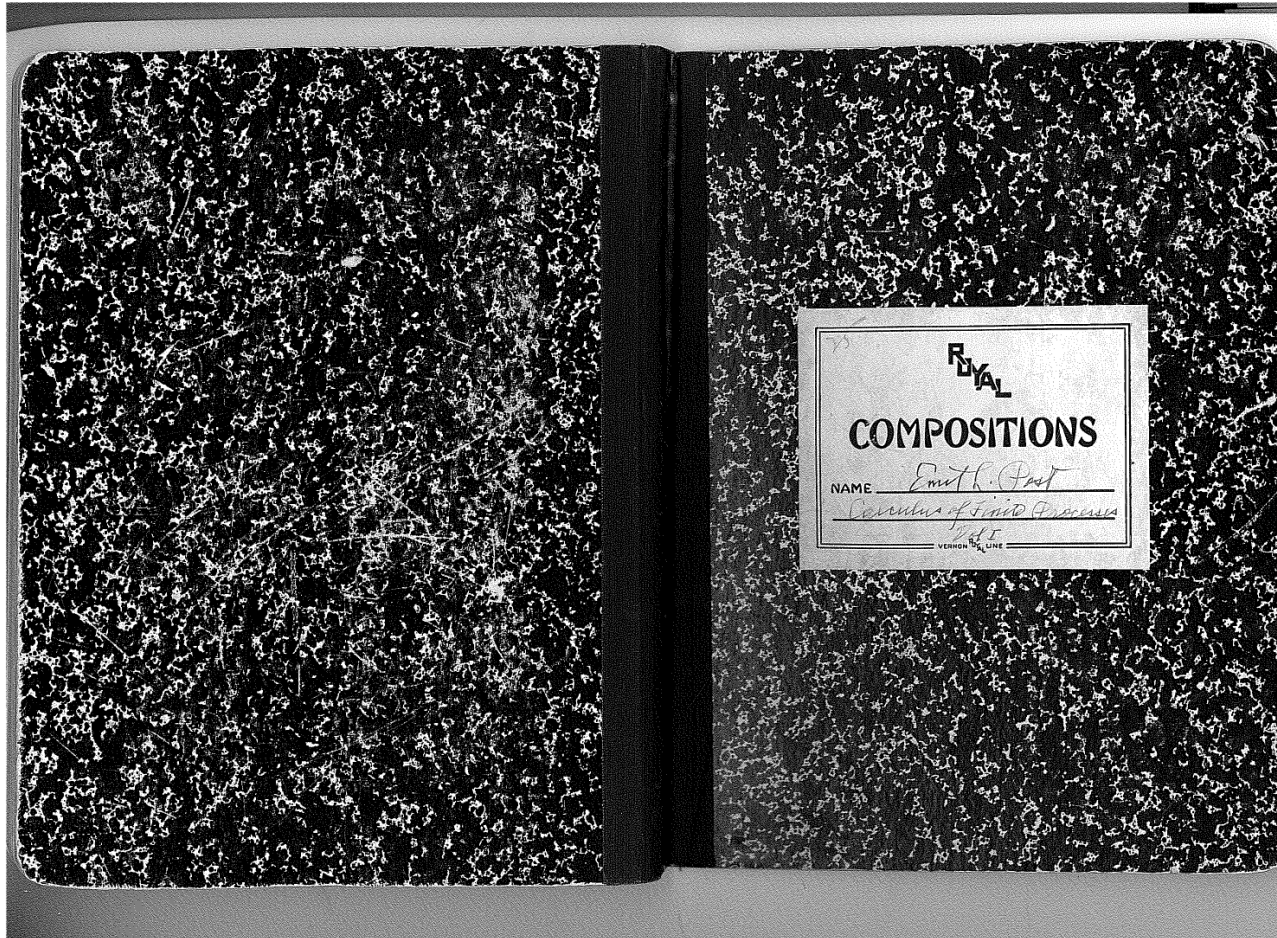
Liesbeth De Mol

CNRS - UMR 8163 Savoirs, textes, Langage

Université de Lille, France

liesbeth.de-mol@univ-lille.fr

Some thoughts on Emil Post’s thoughts on “computation” finite (combinatory) processes



Part 0. A context to look at Post’s work: the Church-Turing thesis (and Gödel)

Intellectual context – the Church-Turing thesis (in a nutshell)

⇒ **What was it about?**

	Vague notion	Formal device
Church (1936):	eff. calculability	λ -def. & gen. rec. functions
Turing (1936):	computability	Turing machines

⇒ **Why?** *“The very day on which the undecidability does not obtain any more, mathematics as we now understand it would cease to exist; it would be replaced by an absolutely mechanical prescription, by means of which anyone could decide the provability or unprovability of any given sentence.”* (Von Neumann, 1927)

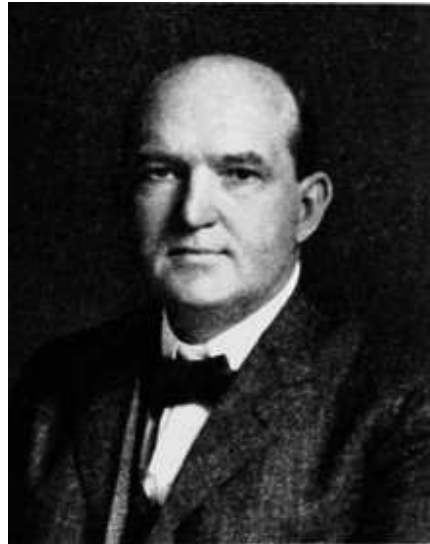
⇒ **Significance of Turing:** *“[I]t was Turing alone who [...] gave the first convincing formal definition of a computable function”* (Soare, 2007)

- **Turing’s main question:** “The real question at issue is: What are the possible processes which can be carried out in computing a number?” (Turing, 1936) – top-down

⇔ **Church’s ‘approach’?**: Thesis only *after* a thorough analysis of λ -calculus and recursive functions (bottom-up)

Part I. Post’s earlier work on finite combinatory iterations – the account of an anticipation

Put on the track of symbolic logic...



Columbia University
in the City of New York
DEPARTMENT OF MATHEMATICS

June 25, 1941

Dear Doctor Post:

It was exceedingly kind of you to write me such a beautiful and generous letter. Please accept my heartiest congratulations on the appearance of your brilliant memoir on "The Two-Valued Iterative Systems of Mathematical Logic". I had no suspicion that I had ever said or done anything that could by any possibility be regarded as in any sense the *fons et origo* of your profound researches in mathematical logic. If I was gazing in the right direction, as you credit me with doing, my vision had not penetrated very deeply or very far.

I have been spending some hours looking into this volume of yours and am fascinated by it. The ease and confidence with which you move in this sea of subtleties, intangibilities and tenuous abstractions is very remarkable. I doubt if any one else could have done this piece of work, in every way so distinguished.

Sincerely and gratefully yours,

C. J. Keyser

“I had no suspicion that I had ever said or done anything that could by any possibility be regarded as in any sense the fons et origo of your profound researches in mathematical logic. If I was gazing in the right direction, as you credit me with doing, my vision had not penetrated very deeply or very far” (June 25, 194, Keyser)

Put on the track of symbolic logic...



⇒ Russel and Whitehead’s **Principia Mathematica**: “*general theory of logic [...] to furnish a basis for all mathematics*” (Post, June 4 1920)

⇒ **C.I. Lewis** “*mathematics without meaning*”: “[W]hatever the mathematician has in his mind when he develops a system, what he does is to set down certain marks and proceed to manipulate them” (Lewis 1918)

...to “crack” the whole of mathematics...

INTRODUCTION TO A GENERAL THEORY OF ELEMENTARY PROPOSITIONS.

BY EMIL L. POST.

INTRODUCTION.

In the general theory of logic built up by Whitehead and Russell* to furnish a basis for all mathematics there is a certain subtheory† which is unique in its simplicity and precision; and though all other portions of the work have their roots in this subtheory, it itself is completely independent of them. Whereas the complete theory requires for the enunciation of its propositions real and apparent variables, which represent both individuals and propositional functions of different kinds, and as a result necessitates the introduction of the cumbersome theory of types, this subtheory uses only real variables, and these real variables represent but one kind of entity which the authors have chosen to call elementary propositions. The most general statements are formed by merely combining these variables by means of the two primitive propositional functions of propositions Negation and Disjunction; and the entire theory is concerned with the process of asserting those combinations which it regards as true propositions, employing for this purpose a few general rules which tell how to assert new combinations from old, and a certain number of primitive assertions from which to begin.

⇒ **Purpose?** Research in foundations: “[T]o obtain theorems about all [mathematical] assertions” using symbolic logic – proof consistency, completeness and decidability of propositional logic

Focus on the **finiteness problem**: “[...] to find a method, applicable to all the enunciations of the system [of mathematics], for determining in a finite number of steps whether a given enunciation of the system can or cannot be derived from the postulates of the system by means of its logic” (Post, June 4 1920)

“Since Principia was intended to formalize all of existing mathematics, Post was proposing no less than to find a single algorithm for all of mathematics.” (Davis, 1994)

...to “crack” the whole of mathematics...

Note on a Fundamental Problem in Postulate Theory.

In passing over from a study of particular theorems in a mathematical system to the study of the system as a whole, it is important to realize that such a system is not completely defined until not merely its own distinctive postulates are stated, but the logic to be used is explicitly specified. This logic serves two purposes. On the one hand it is in terms of its language in addition to the undefined terms of the system that the enunciations of the system are stated; on the other it is the sole instrument for deducing the

⇒ **Approach?** Development of a “general form of symbolic logic” as an “instrument of generalization” characterized by the “method of combinatory iteration” which “eschews all interpretation” (Post, June 4 1920):

[T]he method of combinatory iteration **completely neglects [...] meaning**, and considers the entire system purely from the symbolic standpoint as one in which both the enunciations and assertions are groups of symbols or symbol-complexes [...] and where these symbol assertions are obtained by starting with certain initial assertions and repeatedly applying certain rules for obtaining new symbol-assertions from old.

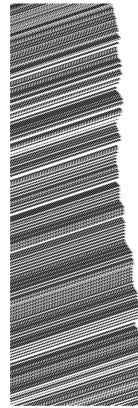
⇒ The idea of a “the canonical form of the [finiteness] problem in the method of combinatory iteration” (Post, June 4 1920)

...and finding the cracks

⇒ Sequence of reductions using method of combinatory iteration from PM to canonical forms A and B to both generalize and simplify

BUT: “The effect of these reductions is to **do away with** most of the difficulty there is in the general problem which is due to the **outward complexity** of its statements, and so to present **the real difficulties** of the problem unencumbered. But they do not appear to mitigate these real difficulties in the slightest. In fact in trying to bridge the gap between the particular problems which have been solved and these canonical forms, certain auxiliary problems were met with [...] and which resisted all attempts at solving them. Indeed **their difficulty was such that it seemed to be a vain illusion to even hope for a solution of the general problem.**” (Post, June 4 1920)

...and finding the cracks...



“More recently, however, one of the first of these auxiliary problems met with in the method of combinatory iteration has shown **signs of yielding**” (Post, June 4 1920)

⇒ **The problem of “tag”** – “It may be noticed that it is par-excellence a problem in **combinatory iteration**” (Post, June 4 1920)

⇒ “the solution of the general problem of ‘tag’ appeared **hopeless**, and with it our entire program of the solution of finiteness problems.” (Post 1965)

...and putting them center stage of a new reversed program.

⇒ To show that there is a base form for finite symbolic reasoning which is both simple and fundamentally limited

canonical form C (Post production systems):

$g_{11}P_{i_1^1}g_{12}P_{i_2^1} \dots g_{1m_1}P_{i_{m_1}^1}g_{1(m_1+1)}$	$abaP_1abP_2b$	$abaaaaabababbabaaaaab$
$g_{21}P_{i_1^2}g_{22}P_{i_2^2} \dots g_{2m_2}P_{i_{m_2}^2}g_{2(m_2+1)}$	$\underline{a}P_3\underline{bb}P_4\underline{b}$	$\underline{abaaaaabababbababbabaaaaab}$
.....
$g_{k1}P_{i_1^k}g_{k2}P_{i_2^k} \dots g_{km_k}P_{i_{m_k}^k}g_{k(m_k+1)}$	produce	produce
$g_1P_{i_1}g_2P_{i_2} \dots g_mP_{i_m}g_{(m+1)}$	$abaP_3b$	$ababaaaaababab$

.... and **Normal form:**

g_iP_i	1101 <i>P_i</i>	1101 11011101000000
	<i>produces</i>	
$P_i g_{i'}$	<i>P_i</i> 001	110111010000000001

... towards Post’s thesis I....

⇒ Apparent simplicity does not imply ‘real’ simplicity – Proof of “*the most beautiful theorem in mathematics*” (Minsky, 1961)

⇒ Idea that the whole PM can be reduced to normal form

[F]or if the meager formal apparatus of our final normal systems can wipe out all of the additional vastly greater complexities of canonical form [...], the more complicated machinery of [the canonical form] should clearly be able to handle formulations correspondingly more complicated than itself. (Post 1965)

(⇒ from this realization, “*the difficulty of “tag” is no longer surprising*”)

⇒ **Post’s thesis I** “*In view of the generality of the system of Principia Mathematica, and its seeming inability to lead to any other generated sets of sequences on a given set of letters than those given by the normal form, we are led to the following generalization*” (Post 1965) – any set of sequences that can be “generated” by some finite process can also be “generated” by the “primitive” normal form

...and a (partial) anticipation

⇒ *the finiteness problem for the class of all normal systems is unsolvable (Post 1965)*

(⇒ *“We shall refer to the system K as the complete normal system because, in a way, it contains all normal systems [...] The “complete normal system” would thus correspond to Turing’s “universal computing machine.”*) (Post 1965)

⇒ *A complete symbolic logic is impossible (Post 1965)*

Part II. Some (!) reflections on thesis I

...towards thesis II

Problem with thesis I: not intuitive immediately: *“it is clear that every generated set in your sense is lambda-enumerable (recursively enumerable), I can see no way of proving the converse of this [...] it seems to me possible that **the notion of a generated set is less general.**”* (Church to Post, June 26, 1936)

Post’s analysis: *“[for the thesis to obtain its full generality] an analysis should be made of all the possible ways **the human mind** can set up finite processes to generate sequences.”* (Post 1965) (\sim Turing’s *“What are the possible processes which can be carried out in computing a number?”*)

Post’s thesis II, 1936: Solvability in the intuitive sense coincides with solvability by Formulation 1 \sim standard version of Turing machine (cfr Post 1947)

...as a working hypothesis or natural law...

A working hypothesis “*Its purpose is not only to present a system of a certain logical potency but also, [...] of psychological fidelity*” (Post 1936)

[T]o mask this identification under a definition hides the fact that a fundamental discovery in the **limitations of the mathematicizing power of *Homo Sapiens*** has been made and blinds us to the need of its continual verification. (Post 1936)

Anchored in: “I consider mathematics as a product of the human mind, not as absolute” (Post 1965) – **absolutely unsolvable combinatory problems**

Post’s new programme – Towards a natural law In search of wider and wider formulations and to prove that all these are logically reducible to the original formulation 1

Re-Connecting to the Church-Turing thesis

⇒ Post’s 1936 formulation 1 is based on an analysis similar to Turing’s (but lost)

⇒ insistence on hypothetical character (unlike Turing and especially Church)

⇒ “Where in these formulations [Church and Turing] the informal basic idea is that of effective calculability, our own is that of a generated set. This **derives from the idea of a symbolic logic rather than that of an algorithm**, and may be described by saying that each member of the set is **at some time generated** [Produced, created – in practice, written down.] by the continued application of a given method” (Post 1965)

A kind of reversal (of history)...

“They were borrowed, sometimes under the label of ”Markov productions,” for use in computer systems programming (languages for compiling compilers). In about the mid- 1960’s, they were introduced into cognitive science at Carnegie Mellon University, some of their early uses being in the General Problem Solver [...] and in Tom Williams’ thesis [...] on a general game-playing program [...]. They also found early use as languages for formal grammars of natural language. Among production systems widely used in cognitive simulation are OPS5 [...], Prolog [...], and Act-R” (Simon, n.d.)

*“We are particularly interested in your “Productions” [...] which appeal to us as possibly **the most natural approach** to the theory of computability from the standpoint of one interested in syntactical machines” (Patterson 1952)*

⇒ Rather than recasting Post’s (early) systems in terms of computation and, especially, Turing machines, we could/should perhaps recast the latter as more specific instances of finite combinatory processes (or, at least, consider them equal).

Thank you

Generalization I: Systems in canonical form A

	Propositional Logic	Canonical form A
I.	<p>If p is an elementary proposition than so is $\sim p$</p> <p>If p and q are elementary propositions than so is $p \vee q$</p>	<p>If p_1, \dots, p_{m_1} are elementary propositions than so is $f_1(p_1, \dots, p_{m_1})$</p> <p style="text-align: center;">\vdots</p> <p>If p_1, \dots, p_{m_μ} are elementary propositions than so is $f_\mu(p_1, \dots, p_{m_\mu})$</p>
II.	<p>The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $\sim q$, or $(q \vee r)$^a</p>	<p>The assertion of a function involving a variable p produces the assertion of any function found from the given one by substituting for p any other variable q, or $f_1(q_1, \dots, q_{m_1})$, or $f_\mu(q_1, \dots, q_{m_\mu})$</p>
III.	<p>$\vdash P$</p> <p>$\vdash \sim P \vee Q$</p>	<p>$\vdash g_{11}(P_1, \dots, P_{k_1}) \dots \vdash g_{r_1}(P_1, \dots, P_{r_r})$</p> <p style="text-align: center;">\vdots</p> <p>$\vdash g_{1r_1}(P_1, \dots, P_{r_1}) \dots g_{rr_r}(P_1, \dots, P_{r_r})$</p>

^aThis corresponds to substitution Continued on next page

Table 1 – continued from previous page		
	Propositional Logic	Canonical form A
	produce $\vdash Q$	produce $\vdash g_1(P_1, \dots, P_{k_1}) \dots \vdash g_r(P_1, \dots, P_{r_r})$
IV.	Postulates: $\vdash \sim (p \vee p) \vee p$ $\vdash \sim (p \vee (q \vee r)) \vee (q \vee (p \vee r))$ $\vdash \sim q \vee (p \vee q)$ $\vdash \sim (\sim q \vee r) \vee (\sim (p \vee q) \vee (p \vee r))$ $\vdash (p \vee q) \vee (q \vee p)$	Postulates: $\vdash h_1(p_1, p_2, \dots, p_{l_1})$ $\vdash h_2(p_1, p_2, \dots, p_{l_2})$ \dots \dots $\vdash h_\lambda(p_1, p_2, \dots, p_{l_\lambda})$

(Tag systems)

Let T_{Post} be defined by $\Sigma = \{0, 1\}, v = 3, 1 \rightarrow 1101, 0 \rightarrow 00$

$A_0 = 10111011101000000 \Rightarrow$ Primitive assertion

~~101~~110111010000001101

~~110~~1110100000011011101

~~111~~01000000110111011101

~~010~~0000011011101110100

~~000~~001101110111010000

~~001~~10111011101000000 \Rightarrow Periodicity!
 A_0

...finite processes as human activity.

⇒ Why this insistence of Post on thesis as a hypothesis?

“I consider mathematics as a product of the human mind, not as absolute”
(Post 1965)

[T]he finitary character of symbolic logic follows from the fact that it is
“*essentially a human enterprise, and that when this is departed from, it is*
then incumbent on such a writer to add a qualifying “non-finitary”. (Post
1965)

⇒ “The writer cannot overemphasize the fundamental importance to mathematics of the existence of **absolutely unsolvable** combinatory problems. True, with a specific criterion of solvability under consideration, say recursiveness [...], the unsolvability in question [...] becomes merely unsolvability by a given set of instruments. **[The] fundamental new thing is that for the combinatory problems the given set of instruments is in effect the only humanly possible set.**” (Post 1965)