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AN ALGORITHM FOR LINEARIZING THE COLLATZ CONVERGENCE

✉ Alexander Rahn, ✉ Max Henkel, ✉ Sourangshu Ghosh, ✉ Eldar Sultanow,
and ✉ Idriss J. Aberkane

ABSTRACT. The Collatz dynamic is known to generate a complex quiver of sequences over natural numbers which inflation propensity remains so unpredictable it could be used to generate reliable proof of work algorithms for the cryptocurrency industry; it has so far resisted every attempt at linearizing its behavior. Here we establish an *ad hoc* equivalent of modular arithmetic for Collatz sequences, based on five arithmetic rules we prove apply on the entire Collatz dynamical system and which iteration exactly define the full basin of attraction leading to any odd number. We further simulate these rules to gain insight on their quiver geometry and computational properties, and observe they allow to linearize the proof of convergence of the full rows of the binary tree over odd numbers in their natural order, a result which, along with the full description of the basin of any odd number, has never been achieved before. We then provide two theoretical programs to explain why the five rules allow to linearize Collatz convergence, one in ZFC and one in Peano arithmetic.

1. INTRODUCTION

In 1937 Lothar Collatz established a conjecture known as the $3n + 1$ problem, also known otherwise as Kakutani's problem, Syracuse algorithm, Hasse's algorithm, Thwaites conjecture, and Ulam's problem. The collatz problem involves the iterative sequence defined as (see OEIS [1] for the definition of the Collatz Map):

$$(1) \quad a_n = \begin{cases} a_{n-1}/2, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1} + 1, & \text{if } a_{n-1} \text{ is odd} \end{cases}$$

Conway [2] conjectured that given any initial term a_0 , the sequence always terminates at 1. He also showed that there is no non-trivial cycle with a length lesser than 400

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with Lagarias [3] later increasing this lower bound to 275000. Conway [2], Kurtz and Simon [4] proved that the generalization of the Collatz problem is undecidable. The Collatz conjecture was first verified up to $5.6 \cdot 10^{13}$ by Leavens et.al. [5], 10^{15} by Vardi [6], Oliveira [7] further extended the results to $5.48 \cdot 10^{18}$ and as of 2020 it had been verified beyond 2^{68} . The Collatz problem was stated differently by Terras [8] [9], to essentially compress the division by 2 [10] [11]:

$$(2) \quad a_n = \begin{cases} a_{n-1}/2, & \text{if } a_{n-1} \text{ is even} \\ 3a_{n-1}+1/2, & \text{if } a_{n-1} \text{ is odd} \end{cases}$$

Researchers have also tried to model the Problem in various ways. Wolfram [12] modelled it as an 8-register machine. Cloney et al. [10], Bruschi [13] modeled it as a quasi-cellular automaton, Zeleny [14] modeled it as an 6-color one-dimensional quasi-cellular automaton. There have been some more breakthroughs in this topic; for example Machado [15] provided an interesting clustering perspective of the Collatz Conjecture.

The dynamical system generated by the $3n + 1$ problem is known to create complex quivers over \mathbb{N} , one of the most picturesque being the so-called "Collatz Feather" or "Collatz Seaweed", a name popularized by Clojure programmer Oliver Caldwell in 2017. The inflation propensity of Collatz orbits remains so unpredictable it can form the core of a reliable proof-of-work algorithm for Blockchain solutions [16], with groundbreaking applications to the field of number-theoretical cryptography as such algorithms are unrelated to primes yet, being based on the class of congruential graphs, still allow for a wide diversity of practical variants. If Bocart thus demonstrated that graph-theoretical approaches to the $3n+1$ problem can be very fertile to applied mathematics, the authors have also endeavored to demonstrate its pure number-theoretical interest prior to this work [17], [18], [19], [20].

Our methodology consists of using the complete binary tree and the complete ternary tree¹ over $2\mathbb{N} + 1$ as a general coordinate system for each node of the Feather. We owe this strategy to earlier discussions with Feferman [21] on his investigations on the continuum hypothesis, as it is known the complete binary tree over natural numbers is one way of generating real numbers. The last author's discussion with Feferman argued morphisms, sections and origamis of n-ary trees over \mathbb{N} could be a promising strategy to define objects of intermediate cardinalities between \aleph_n and \aleph_{n+1} , in a manner inspired from Conway's construction of the surreal numbers [22], which itself began by investigating the branching factor of the game of Go.

¹We define the complete binary tree over odd numbers as $2\mathbb{N} + 1$ endowed with the following two linear applications $\{\cdot 2 - 1; \cdot 2 + 1\}$. The complete ternary tree over odd numbers is defined as $2\mathbb{N} + 1$ endowed with operations $\{\cdot 3 - 2; \cdot 3; \cdot 3 + 2\}$

2. RELATED RESEARCH

2.1. Goodstein Sequences and Hydra Games. The idea of attacking the Collatz conjecture from the angle of logic and set theory is not new. Hydra Games were first introduced by Kirby and Paris [23] and Arzarello [24] provided a rather wide outline of how their consideration could, in fact, lead to a set-theoretical solution of the Collatz conjecture. The convergence of Goodstein Sequences indeed, which form the core of Kirby and Paris' demonstration that no Hydra Game can be lost, cannot be proven in Peano Arithmetic alone. Their founding element however, which is the base- k hereditary representation of a number n , can be defined without the axiom of Choice.

Definition 1. *Let us write any given number n as a sum of powers of a base k . Let us further write the exponents themselves as sums of powers of a base k , this process will continue until we reach 1 in the exponent. This representation is denoted as the base- k hereditary representation of n .*

The Goodstein sequence is generated by repeatedly increasing or "bumping" the base k to $k+1$ and then subtract by 1. Mathematically this can be defined by the recursive sequence $G_0(n) = n$ and $G_k(n) = B[k+1](G_{k-1}(n)) - 1$. Here the operator $B[b](n)$ takes base- k hereditary representation of n and then replaces the base with $k+1$. An example of this as given by Klein [24] starts with 266

$$\begin{aligned} u_0 &= 2^{2^{2+1}} + 2^{2+1} + 2^1 = 266 \\ u_1 &= 3^{3^{3+1}} + 3^{3+1} + 3^1 - 1 = 3^{3^{3+1}} + 3^{3+1} + 2 \approx 10^{38} \\ u_2 &= 4^{4^{4+1}} + 4^{4+1} + 2 - 1 \approx 10^{616} \\ u_3 &= 5^{5^{5+1}} + 5^{5+1} \approx 10^{10921} \end{aligned}$$

Goodstein [25] proved that this sequence shall always terminate at 0. Kirby and Paris [23] showed that this theorem cannot be proven by Peano arithmetic alone. The idea of a Hydra game is similar to the Goodstein sequence, the name "Hydra" comes from Greek mythology describing the battle between Hercules and the Hydra of Lerna, with any of its multiple heads growing two more each time it is cut. In this game a tree is representing the Hydra, and the game consists of cutting a branch of the tree (or one of the multiple "heads") turn by turn. The Hydra then grows according to a set of rules, a finite number of new heads in response to the cutting. Kirby and Paris [23] showed that Hydra will get killed by Hercules regardless of the strategy used to cut its heads. They also proved that similar to Goodstein Sequences this property cannot be proven by Peano arithmetic alone. They proved this by demonstrating that if the well-ordering hypothesis for integers (i.e., within Peano arithmetic) could be used to

demonstrate the convergence, then the theorem regarding Goodstein sequences could be reduced to the famous result of Gentzen [26] named "Gentzen's consistency proof", meaning that from solving the Hydra Game From one may prove the consistency of Peano arithmetic, which cannot be achieved within Peano arithmetic, as known from the Gödel incompleteness theorem [27]. Cichon [28] and Hodgson [29] discussed a similar sequence than that of Goodstein, which is called a "weak Goodstein Sequence", also used in the Arzarello argument. In this sequence one also writes a number as a sum of powers of a base k , but the exponents are not further written in this way. An example of this is given by Arzarello [24] starts with 266 as earlier

$$\begin{aligned} u_0 &= 2^8 + 2^3 + 2^1 = 266 \\ u_1 &= 3^8 + 3^3 + 3^1 - 1 = 3^8 + 3^3 + 2 = 6590 \\ u_2 &= 4^8 + 4^3 + 2 - 1 = 4^8 + 4^3 + 1 = 65601 \\ u_3 &= 5^8 + 5^3 + 1 - 1 = 390750 \end{aligned}$$

Cichon [28] further proved the convergence of all weak Goodstein sequences by showing that one can assign the m -tuple of the coefficients of the decomposition in base $n + 2$ to each term u_n of any such sequence and then demonstrate that the m -tuples are well-ordered in a purely decreasing lexicographic way. In contrast to the Goodstein sequences, the convergence of weak Goodstein sequence *can* be proven by Peano Arithmetic. The above mentioned results of Cichon [28] and Kirby and Paris [23] were alternatively proved by Caicedo [30] using proof theoretic results of Lob-Wainer's fast growing hierarchy of functions. Other excellent literature discussing the independence of Goodstein Sequence and Peano Arithmetic Axioms are done by Kaplan [31] and Miller [32] in their thesis. Kaplan has further demonstrated a method for finding "non-standard models" of Peano Arithmetic that satisfy Goodstein's theorem using indicator theory, but a more significant contribution is that of Stepień and Stepień 2017 [33] with their groundbreaking approach to the demonstration of the consistency of arithmetic.

Barina [34] introduces a new algorithmic approach for a computational convergence verification of the Collatz problem. For this he used a single-threaded CPU implementation verifying $4.2 \cdot 2^{109}$ 128-bit numbers per second on Intel Xeon Gold 5218 CPU computer. His parallel OpenCL implementation reached a speed of $2.2 \cdot 10^{11}$ 128-bit numbers per second on NVIDIA GeForce RTX 2080. In conformity with the approach by Koch et al. [19] he used the advantage that is given by the binary representation of integers.

Kleinnijenhuis et al. [35] applied Hilbert's paradox of the Grand Hotel to the Collatz problem and used Wolfram Mathematica for computations that involve very large

numbers. These large numbers (and beyond) have also been simulated by Christian Koch, for this see the Collatz Python Library hosted by his GitHub repository [36].

2.2. L-systems and analogies with statistical physics. The fundamental concept of our approach is to identify inevitable collisions within the phase space of the Collatz dynamical system. To that end, we need to precisely define a coordinate system of this phase space, which will be the complete binary tree over $2\mathbb{N} + 1$ but also to describe the non-ergodicity of the system, which will be captured by the distribution of the intersections of the binary and ternary trees in Section 9. The most important contribution of this paper to the solving of the Collatz conjecture is the identification and demonstration of the five fundamental laws that characterize the basin of attraction of any odd number, which we can recursively apply to define an infinite L-system developing within the complete binary tree, and the characterization of some of their most essential emerging properties, in particular their comparative branching factor. From there, the objective is to demonstrate that the L-system starting from number 1 cannot fail to finitely collide with the L-system starting from any other number, a methodology that may rightly evoke ergodic theory and statistical physics. Indeed, demonstrating on the one side that the Collatz dynamical system tends to compress trajectories to certain bottlenecks of its phase space, and using this element of proof to further demonstrate that finite collisions between any two pairs of trajectories is therefore inevitable, is a proof program we borrowed from statistical physics. However, if the already existing representations of the "Collatz Feather" do already exhibit obvious bottlenecks and phase space confinements, the most essential contribution to their further understanding lies in establishing an *ad-hoc* coordinate system to characterize and demonstrate the nature of these confinements precisely.

3. CONTRIBUTIONS TO THE STATE OF THE ART

In acknowledgement of the intellectual influence of the study of quantum non ergodicity to the study of discrete dynamical systems (for a more precise example, see [37]) we have meant to not reduce this article to its mathematical proofs, but rather to accompany them with novel 3D visualizations of the Collatz phase space, along with specific empirical measurements of its behavior. As we explained in the previous section, both the mathematics proofs and 3D visualizations are based on the *ad-hoc* algebraic foundations, and in particular the coordinate system consisting of studying the intersections of both the binary and ternary trees over odd numbers, that we have established to gain further insight on the chaoticity of the Collatz feather. In Figure 1 we outline the fundamental contributions we intend to make. Green charts indicate results obtained from a two-dimensional coordinate system, purple charts indicate those obtained from a 3D-analysis of the feather. The Blue chart indicates a result obtained from both.

Fundamentally, our most essential theorems consist of the five rules that are exactly defining the basin of attraction of any odd number in the Collatz dynamical system. However, the emerging properties of those five rules are hard to predict and can be counter-intuitive. They require equally novel developments in mathematical visualization and beyond, a few novel concepts as well. This interplay between conceptual and visual progress is the reason we have here endeavored to develop many figures and many frameworks, both in 2D and in 3D, and from graph theory to cellular automata, transfinite set theory, space-filling L-systems and caustics. Though not intended *ab initio*, these many approaches practically complement each other in achieving what we believe is one of the finest understandings of the fundamental chaoticity of Collatz orbits ever achieved.

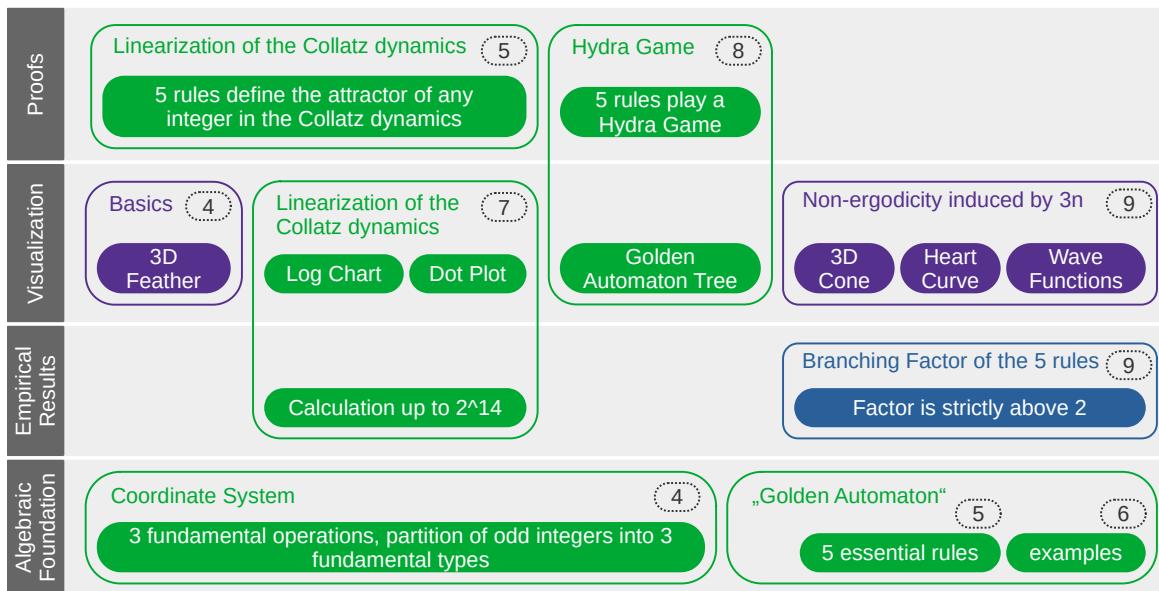


FIGURE 1. Structure of the original contributions of this paper.

4. BINARY AND TERNARY TREES AS A NOVEL COORDINATE SYSTEM FOR THE COLLATZ BASINS OF ATTRACTION

Remark 1. For all intent and purpose we will define $\text{Syr}(x)$ or the "Syracuse action" as "**the next odd number in the forward Collatz orbit of x** ". Whenever two numbers a and b have a common number in their orbit, we will also note $a \equiv b$, a

relation that is self-evidently transitive:

$$(a \equiv b) \wedge (b \equiv c) \Rightarrow a \equiv c$$

The choice of symbol " \equiv " is a deliberate one to acknowledge a kinship between our method and modular arithmetic.

Definition 2. Actions G , V and S For any natural number a are specified as follows:

- (1) $G(a) := 2a - 1$
- (2) $S(a) := 2a + 1$. The **rank** of a is its number of consecutive end digits 1 in base 2.
- (3) $V(a) := 4a + 1 = G \circ S(a)$

Definition 3. Type A, B and C

- (1) a number a is of type A if its base 3 representation ends with digit 2
- (2) a number b is of type B if its base 3 representation ends with digit 0
- (3) a number c is of type C if its base 3 representation ends with digit 1

To remember which is which one need only remember the order of ABC: $a+1$ is dividable by 3, and so is $c-1$, thus A is on the left of B and C is on the right.

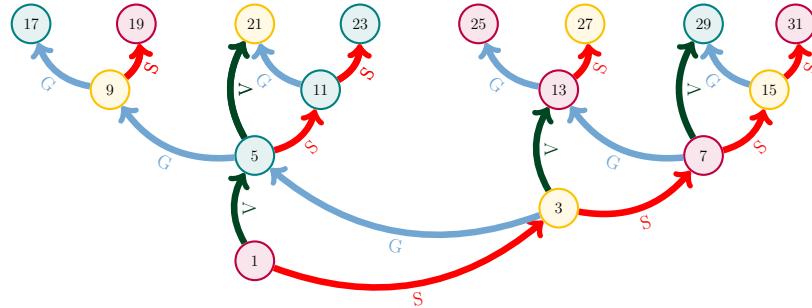


FIGURE 2. Quiver connecting all odd numbers from 1 to 31 with the arrows of actions S, V and G . The set $2\mathbb{N} + 1$ is thus endowed with three unary operations without a general inverse that are non commutative with $G \circ S = V$. Whenever we will mention the inverse of these operations, it will be assuming they exist on \mathbb{N} . Type A numbers are circled in teal, B in gold and C in purple.

We intend to use the quiver of Figure 3 as a general coordinate system for each node of the Collatz Feather. Paramount to our investigation will be the comparative analysis of the branching factor of the feather compared to that of the binary tree.



FIGURE 3. The Collatz feather rendered in Blender, this time using the same ternary typology as defined in 3.

The Collatz feather depicted by Figure 3 visualises the Collatz-sequences of the numbers from 1 to 15000 in 3D. Each branch is generated from the complete sequence of each number. For an even number, the branch is rotated around one axis, for an odd number the branch rotates in the other direction. The colors are set by Type of the number which was mentioned in Definition 3. Teal ones represent type A, golden branches represent type B numbers and purple ones represent type C numbers.

The rotation of the branches is based on a point that is moved by a vector and then rotated around a predefined axis. Two points, pre and cur which contain the previous and the current point in the form $[x, y, z]$, are handed over to the rotation function which executes the rotation of the current point. By rotating in different directions for even and odd numbers, the individual branches are generated, which together form the feather-like construct shown in Figure 3.

```

1  def oddRotation(pre, cur, rot, counter):
2      vec = [cur[0]-pre[0], cur[1]-pre[1], cur[2]-pre[2]]
3      rotation_degrees = -rot/30
4      rotation_radians = np.radians(rotation_degrees)
5
6      axis = [1,0,1]
7
8      rotation_axis = np.array(axis)
9
10     rotation_vector = rotation_radians * rotation_axis
11     rotation = R.from_rotvec(rotation_vector)
12     rotated_vec = rotation.apply(vec)
13     x = pre[0] + rotated_vec[0]
14     y = pre[1] + rotated_vec[1]
15     z = pre[2] + rotated_vec[2]
16
17     return [x,y,z]

```

LISTING 1. Code for rotating the branches of the Feather in Figure 3, see [38]

Although the Collatz feather has been often represented in the literature and in popular mathematics circle, its fundamental geometry remain poorly understood. In the next section, we define the five fundamental rules that define the complete basin of attraction of any point of the feather.

5. FIVE ESSENTIAL RULES OF COLLATZ DYNAMICS

Theorem 1. *The following arithmetic rules apply anywhere over the system $2\mathbb{N} + 1$ endowed with the Collatz dynamic. Their iteration ad infinitum from any odd number precisely define the entirety of the basin of attraction leading to it.*

- **Rule One:** $\forall x \text{ odd}, V(x) \equiv (x)$
- **Rule Two:** $\forall x, k \text{ odd}, S^k V(x) \equiv S^{k+1} V(x)$ and $\forall x, k \text{ even}, S^k V(x) \equiv S^{k+1} V(x)$
- **Rule Three:**

$$\forall \{n; y\} \in \mathbb{N}^2, \forall x \text{ odd non } B, 3^n x \equiv y \Rightarrow \bigwedge_{i=1}^n (V(4^i 3^{n-i} x)) \wedge S(V(4^i 3^{n-i} x)) \equiv y$$
- **Rule Four:**

$$\forall \{n; y\} \in \mathbb{N}^2, \forall x \text{ odd non } B, S(3^n x) \equiv y \Rightarrow \bigwedge_{i=1}^n (S(4^i 3^{n-i} x) \wedge S^2(4^i 3^{n-i} x)) \equiv y$$

- **Rule Five**

$$\begin{aligned} & \forall n \in \mathbb{N}, \forall y \in \mathbb{N}, \forall x \text{ odd non } B \text{ where } 3^n x \text{ is of rank 1, } a \equiv y, a = G(3^n x) \\ & \Rightarrow \bigwedge_{i=0}^n (S^i(G(3^{n-i} x)) \wedge S^{i+1}(G(3^{n-i} x))) \equiv y \end{aligned}$$

In the following we will prove these five rules.

Remark 2. In reference to [Figure 1](#) we will call "**vertical even**" a number that can be written $V(e)$ where e is even, and "**vertical odd**" if it can be written $V(o)$ where o is odd. For example, 9 is the first vertical even number and 5 is the first vertical odd.

5.1. Proving Rule One. If a is written $4b + 1$ then $3a + 1 = 12b + 4 = 4(3b + 1)$ therefore $a \equiv b$.

5.2. Proving Rule Two.

Lemma 1. Let a be a number of rank 1 thus with an odd number p so that $a = G(p)$ then $Syr(S(a)) = G(3 \cdot p)$. Let a be a number of rank n so that $S^{-(n-1)}(a) = G(p)$ then $Syr^{n-1}(a) = G(3^{n-1} \cdot p)$

Proof. If $a = 2p - 1$, p is odd, then it follows:

$$\begin{aligned} S(a) &= 4p - 1 \\ \frac{3 \cdot S(a) + 1}{2} &= \frac{12p - 2}{2} = 6p - 1 = G(3 \cdot p) \\ Syr(S(a)) &= G(3 \cdot p) \end{aligned}$$

Let's generalize to the n . If $Syr(S(a))$ can be written $G(3 \cdot p)$ it is also of rank 1, whereas $S(a)$ was of rank 2, therefore, the Syracuse action has made it lose one rank. All we have to prove now is that $Syr(S^2(a)) = S(Syr(S(a)))$ under those conditions:

$$\frac{3 \cdot (S^2(a)) + 1}{2} = 6a + 5$$

$$S(Syr(S(a))) = S(3a + 2) = 6a + 5 = Syr(S^2(a))$$

If a is of rank $n > 1$, $Syr(a)$ is of rank $n - 1$, and $Syr(S(a)) = S(Syr(a))$ \square

Remark 3. The $3n + 1$ action over an odd number, since it necessarily yields an even one, is *in fine* equivalent to adding 1 to it, then the half of the result, then -1 . How many times one can add an half to an odd number $+1$ directly depends on its base 2 representation, and in particular its number of consecutive end digits 1. Let us take Mersenne numbers for example, which are defined as $2^n - 1$. One can transform them consecutively in this way a number of time equal to their rank-1, indeed, 31, which is written 11111 in base 2 is of rank 5, because $32 = 2^5$ so if one repeats the action

"add to the number+1 the half of itself" this will yield an even result **exactly four consecutive times**. Thus, any strictly ascending Collatz orbit concerns only numbers a of rank $n > 1$, and is defined by

$$(a + 1) \cdot \left(\frac{3}{2}\right)^{n-1} - 1$$

Lemma 2. *Let a be an odd number of rank 1 that is vertical even, then $3a$ is of rank 2 or more, and $9a$ is vertical even. Let a be an odd number of rank 1 that is vertical odd, then $3a$ is of rank 2 or more, and $9a$ is vertical odd.*

Proof. If a is vertical even it can be written $8k + 1 \forall k : 3a = 24k + 3$ and this number admits an S^{-1} that is $12k + 1$, which is an odd number, therefore $3a$ is at least of rank 2.

Moreover, $9a = 72k + 9$ and this number admits a V^{-1} that is $18k + 2$, an even number. Now if a is vertical odd, it can be written $8k + 5$ and $\forall k : 3a = 24k + 15$ and $9a = 72k + 45$. It follows that $3a$ admits an S^{-1} and $9a$ admits a V^{-1} , respectively $12k + 7$ and $18k + 11$ and they are both odd. \square

Lemma 3. *Let a be a number that is vertical even, then $(a) \equiv S(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any even k . Let a be a number that is vertical odd, then $S(a) \equiv S^2(a)$ and $S^k(a) \equiv S^{k+1}(a)$ for any odd k .*

Proof. If a is vertical even then it can be written as $G(p)$ where p is necessarily vertical (odd or even). We proved that $3p$ is then of rank 2 or more and also that we have $Syr(S(a)) = G(p)$ so it is necessarily vertical odd (since $3d$ is of rank 2 or more) so $Syr(a) = V^{-1}(Syr(S(a)))$ and therefore $a \equiv S(a)$. This behavior we can now generalize to the n , because if a is vertical even with $a = G(p)$, then the lemmas we used also provide that $Syr^n(S^n(a)) = G(3^n \cdot p)$ and therefore $Syr^n(S^n(a))$ will be vertical even for any even n because $3^n \cdot p$ will be vertical something (even or odd, depending on p only) for any even n .

Now if a is vertical odd it can be written $G(p)$ and p is necessarily of rank 2 or more because $G \circ S = V$. Thus $3p$ is vertical (even or odd) and therefore $Syr(S(a)) = G(3p)$ is vertical even. \square

Remark 4. Observe that in the process of proving **Rule Two** we also demonstrated that any number of rank 2 or more is finitely turned into a rank 1 number of type A by the Collatz dynamic, and that any number x of rank 2 or more so that $x \equiv S(x)$ under **Rule Two** is finitely mapped to a type A number that is vertical even, therefore **proving the convergence of such numbers is enough to prove the Collatz Conjecture.**

5.3. Proving Rules Three and Four.

Lemma 4. Let a be a vertical even number with $a = G^{n+2}(S(b))$ where n and b are odd, then $a \equiv 3^{\frac{n+1}{2}}(b)$. Let a be a vertical even number with $a = G^{m+2}(S(b))$ where m is even (zero included) and b is odd, then $a \equiv S(3^{\frac{m}{2}}(b))$

Proof. If $a = G^{n+2}(S(b))$ by definition $a = 2^{n+3}b + 1$. Then $3 \cdot a + 1 = 3(2^{n+3}b + 1) + 1 = 2^{n+3} \cdot (3b) + 4$. As this expression can be divided by 2 no more than twice, we have $Syr(a) = 2^{n+1}3b + 1 = G^n(S(3b))$.

Note that if $n = 1$ then $V^{-1}(Syr(a)) = V^{-1}(2^2 \cdot (3a) + 1) = 2^2 \cdot \frac{1}{4} \cdot (3b) = 3b$ which is of course an odd number. Therefore $Syr(a)$ is vertical odd and $V^{-1}(Syr(a)) = 3b$ thus we have proven that $a \equiv 3b$.

If $n = 0$ then $a = 2^3 \cdot b + 1$ so $3(a + 1) = 2^3 \cdot 3b + 4$ therefore $Syr(a) = S(3b)$ and thus $a \equiv S(3b)$. From this we can generalise the progression of numbers that can be written $G^n(x)$ where x is of rank 2 or more. Let b be any odd number:

- All "Variety S" numbers above b are written $V(b \cdot 2^{2k-1})$ or $S(b \cdot 2^{2k}) = 2^{2k+1} \cdot b + 1$ and
- all "Variety V" numbers above b are written $V(b \cdot 4^k)$ or equivalently $S(b \cdot 2^{2k+1}) = 4^{k+1} \cdot b + 1$.

Any number g that can be written $G^n(V(x))$ with x odd and $n > 0$ may thus be finitely reduced under the Collatz dynamic to a number that can be written either $S(3^m x)$ or $V(3^m x)$ by the repeated following transformation:

$$(g - 1) \cdot \left(\frac{3}{4}\right)^k + 1$$

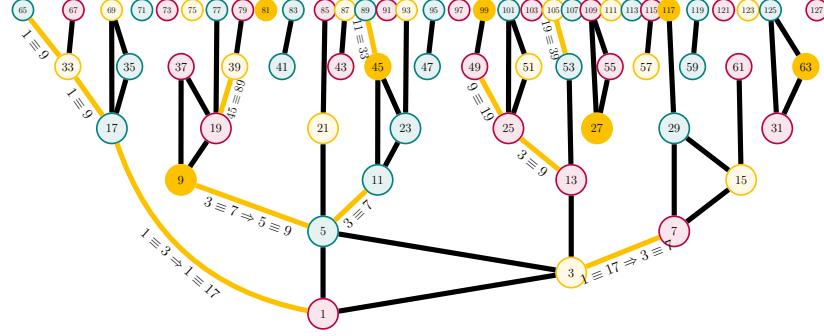
Therefore we have indeed that,

- for Variety S numbers: $2^{2k+1} \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 2b \cdot 3^k + 1 = S(b \cdot 3^k)$, which proves **Rule Four**.
- for Variety V numbers: $4 \cdot 4^k \cdot b \cdot \left(\frac{3}{4}\right)^k + 1 = 4b \cdot 3^k + 1 = V(b \cdot 3^k)$ which proves **Rule Three** because **Rule One** already provides that $V(b \cdot 3^k) \equiv b \cdot 3^k$.

□

5.4. Proving Rule Five. Any type A number of rank 1 can be written $a = G(b)$ where b is of type B. In proving **Rule Two** we showed that any number of rank $n > 1$ is finitely mapped by the Collatz dynamics to $G(3^{n-1} \cdot G^{-1}(S^{-(n-1)}(a)))$, which combined with **Rule Two** itself gives **Rule Five**.

Figure 4 shows a few applications of **Rules Three, Four and Five** plotted in gold. **Rules One and Two** are plotted in black. Whenever a number is connected to 1 by a finite path of black and/or gold edges it is proven to converge to 1.

FIGURE 4. Application of the five rules starting from $1 \equiv 3 \equiv 5$.

6. THE GOLDEN AUTOMATON

Definition 4. On $\{2N + 1; G, S\}$ where **Rules One and Two** are considered pre-computed (the black edges on Figure 4) the systematic computation of **Rules Three, Four and Five** from number 1 onward is called the "**Golden Automaton**".

6.1. **"Golden Arithmetic".** Our purpose is to develop an *ad hoc* multi-unary algebra that could found a congruence arithmetic specifically made to prove the Collatz conjecture, and which we intend as an epistemological extension of modular arithmetic, hence our use of the symbol \equiv in this article rather than the usual \sim which is seen more frequently in the Collatz-related literature. This "**Golden arithmetic**" involves words taken in the alphabet $\{G; S; V; 3\}$, which we will call in their order of application, just like in turtle graphics. For example $VGS3$ means $3 \cdot S \circ G \circ V$

Rules 3, 4 and 5 may now be reformulated as such, without loss of generality as long as **Rules One and Two** are still assumed:

- **Rule Three:** Let b be of type B, then $b \equiv VGS3^{-1}$ from b . We will call this action $R_b(x) = 16\frac{x}{3} + 1$
- **Rule Four:** Let c be of type C, then $c \equiv GS3^{-1}$ from c . We will call this action $R_c(x) = \frac{4x-1}{3}$
- **Rule Five:** Let a be of type A, then $a \equiv G3^{-1}$ from a . We will call this action $R_a(x) = \frac{2x-1}{3}$

As **Rules One and Two** ensure that the quiver generated by the Golden Automaton is branching, with each type B number that is vertical even providing both a new A and a new B to keep applying respectively rules 5 and 3, we may follow only the pathway of type A numbers to define a single non-branching series of arrows, forming a single infinite branch of the quiver. Starting with number 15, this leads straight to 31 and 27, solving a great deal of other numbers on the way:

$15 \equiv 81$	Rule 3
$81 \equiv 1025$	First type A reached by Rule 3
$1025 \equiv 303$	Rule 5
$303 \equiv 607$	Rule 2
$607 \equiv 809$	Rule 4
$809 \equiv 159$	Rule 5
$159 \equiv 319$	Rule 2
$319 \equiv 425$	Rule 4
$425 \equiv 283$	Rule 5
$283 \equiv 377$	Rule 4
$377 \equiv 111$	Rule 5
$111 \equiv 593$	Rule 3
$593 \equiv 175$	Rule 5
$175 \equiv 233$	Rule 4
$233 \equiv 103$	Rule 5
$103 \equiv 137$	Rule 4
$137 \equiv 91$	Rule 5
$91 \equiv 161$	Rule 4
$161 \equiv 31$	Rule 5
$31 \equiv 41$	Rule 4
$41 \equiv 27$	Rule 5

Again, it is in no way a problem, but rather a powerful property of the Golden Automaton that this particular quiver branch already cover 19 steps (and actually more) because each of them is branching into other solutions.

We may follow another interesting sequence to show that in the same way that Mersenne number 15 finitely solves Mersenne number 31, Mersenne number 7 solves Mersenne number 127, this time we will follow a B branch up to $Syr^6(127)$ which we know can be written $G(3^6)$ because 127 is the Mersenne of rank 7.

By Rule 4 we have the first equivalence $7 \equiv 9$ and $9 \equiv 25 \equiv 49$.

So by Rule 2 we also have $25 \equiv 51$.

Rule 3 gives $51 \equiv 273$ and again $273 \equiv 1457 = G(729) \equiv 127$.

The cases of 15 proving the convergence of 31 and 27 and of 7 proving the one of 127 naturally leads us to the following conjecture:

Conjecture 1. Suppose all odd numbers up to 2^n are proven to converge to 1 under the Collatz dynamic, then the Golden Automaton finitely proves the convergence of those up to 2^{n+1}

And indeed we already have that the Golden Automaton starting with 1 proves 3 by **Rule One**, then 3 proves all numbers from 5 to 15 which in turn prove all numbers from 33 to 127. In the next subsection we render larger quivers generated by the Golden Automaton to provide a better understanding of their geometry and fundamental properties, and to demonstrate why it is so, and more generally, why it can be proven they can reach any number in $2\mathbb{N} + 1$ in ZFC.

7. THE GOLDEN AUTOMATON LINEARIZES COLLATZ CONVERGENCE

In this cellular automaton, each cell is identified by a unique odd number and can only adopt three states:

- **Black**, meaning the odd number is not (yet) proven to converge under the iterated Collatz transformation or equivalently that it is only equivalent to another black number
- **Gold**, meaning the odd number is proven to converge and the consequences of its convergence have not yet been computed, *ie.* it can have an *offspring*
- **Blue**, meaning the number is proven to converge and the consequences of its convergence have been computed *ie.* its *offspring* has already been turned gold

In this *ad hoc* yet simpler game of life each gold cell yields and *offspring* then turns blue, and whenever a cell is blue or gold its odd number is proven to converge. Starting with one cell colored in gold at the positions 1, it applies the following algorithm to each gold cell in the natural order of odd numbers:

- (1) **Rule 1:** if a cell on x is gold color cell on $V(x)$ gold
- (2) **Rule 2:** if a cell on x is gold, color cell on $S(x)$ gold depending on the precise conditions of rule 2
- (3) If a cell on a of type A is gold, then color that on $R_a(x)$ in gold
- (4) If a cell on c of type C is gold, then color that on $R_c(x)$ in gold
- (5) After applying the previous rules on a gold cell, turn it blue

Note that applying R_b on a type B number being equivalent to **Rule 1** then R_c the algorithm needs not implement a defined R_b

Whenever a complete series of odd numbers between $2^n + 1$ and $2^{n+1} - 1$ has been colored in gold, it ticks it and returns what we will call its computational "expense", namely all the numbers colored blue and gold that are higher than $2^{n+1} - 1$, thus giving a clear measurement of the algorithmic time it takes the Golden Automaton to prove the convergence of each complete level of the binary tree over $2\mathbb{N} + 1$. We then plot the evolution of this expense on a linear and a logarithmic scale.

Figure 5 is illustrating of the game we defined for the case $n = 6$. On the middle image, row $\{5;7\}$ has been solved , with an "expense" of 8 numbers also solved above

it. On the right image, row $\{9; 11; 13; 15\}$ with an expense of 6. As number 1 is the neutral element of operation R_c we leave it in gold during all the simulation.

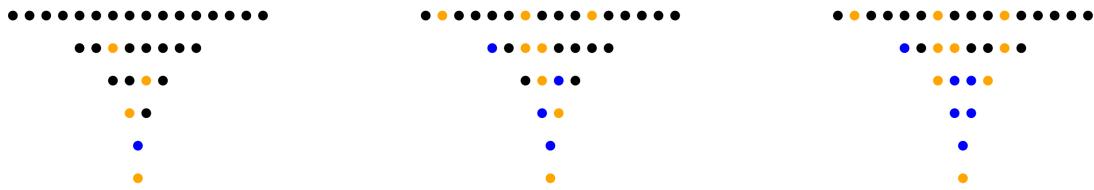


FIGURE 5. The five rules completing row by row.

In Figure 6 and Figure 7 we scaled up the simulation to $n = 12$.

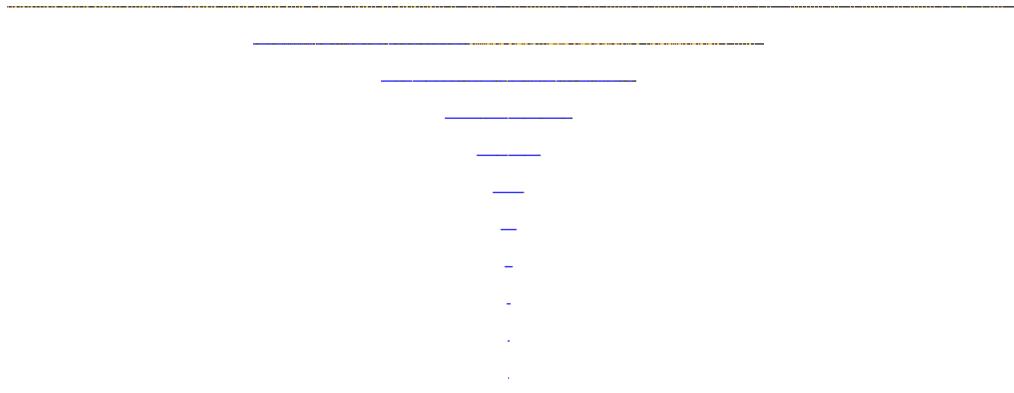


FIGURE 6. case 12, state of the tree when seventh row has just been completed.

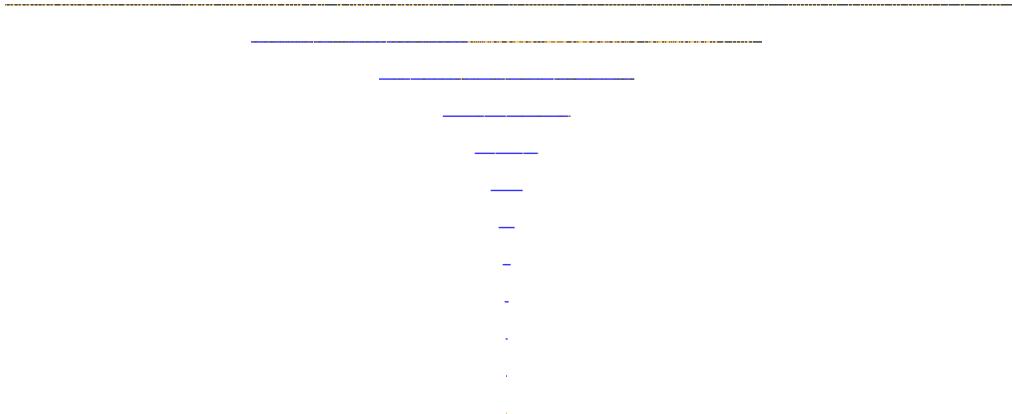


FIGURE 7. case 12, state of the tree when eighth row has just been completed.

To facilitate the observation of each row of the binary tree being covered by the Golden Automaton we zoom into each of them individually(Figure 8-11):



FIGURE 8. State of row 8 (129 to 255) when it has just been finished.



FIGURE 9. State of Row 9 (257 to 511) when Row 8 has just been finished.

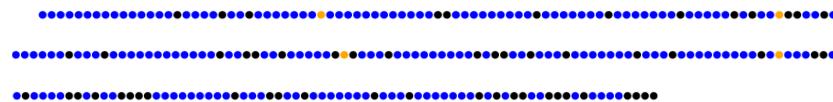


FIGURE 10. State of Row 10 (513 to 1023) when row 8 has just been finished

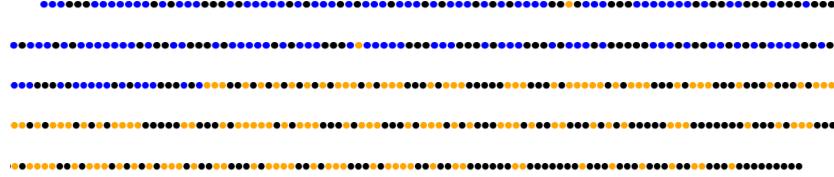


FIGURE 11. State of row 11 (going from 1025 to 2047: each line has about 100 dots)

The charts shown in Figure 12 plot the amount of extra numbers proven to converge above any row n of the binary tree when the Golden Automaton has just finished proving its entire convergence. The chart on the right plots the result against a logarithmic scale, with progressions $2 \cdot 5^n$ (orange line), 3^n (green) and $3 \cdot 5^n$ (red) in comparison. The line-like shape indicates the linear behavior of the Golden Automaton, solving the rows of the binary tree in their natural order, and solving about 3^n additional odd numbers above any row 2^n it has just solved.

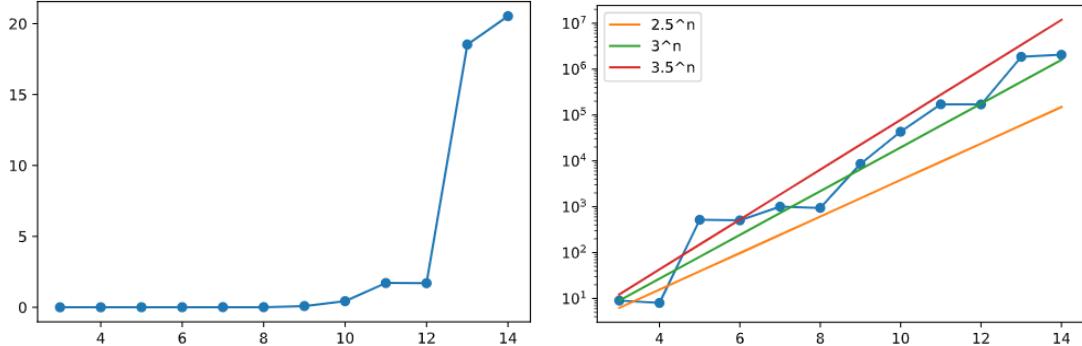


FIGURE 12. Amount of extra numbers proven to converge in a binary tree above row n and its logarithmic graph.

We further investigated the behavior of the Golden Automaton when mapped on the ternary tree over odd numbers, that is, the set of odd numbers endowed with operations $\{\cdot 3; \cdot 3 + 2; \cdot 3 - 2\}$. The automaton was still demonstrating the entire rows 3^n one after another, this time with about 5^n extra numbers solved above each row. These graphs are shown in Figure 13.

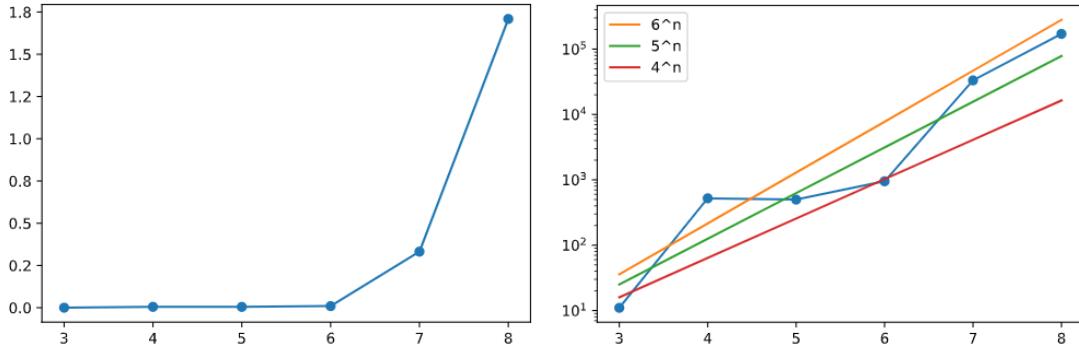


FIGURE 13. Amount of extra numbers proven to converge in a ternary tree above row n and its logarithmic graph.

From there we can thus provide two strategies to finalise a proof of the Collatz conjecture. The first would be to demonstrate the Golden Automaton defines a game that is strictly simpler than a Hydra Game over the graph of all unsolved numbers up to any arbitrary odd integer. The second would be to demonstrate the comparative branching factor of the Golden Automaton, as it is diagonal to the binary tree, is strictly above 2, and that thus the population of solved dots can only finitely take over the population of unsolved ones, or, put in another way, that the basin of attraction of any supposedly diverging odd number grows too fast not to collide the basin of number 1.

8. THE GOLDEN AUTOMATON AS A HYDRA GAME

As we mentioned in 2.1 the idea of attacking the Collatz conjecture from the angle of transfinite arithmetic and in particular the model of the Hydra Game is not new, as Arzarello and others considered it in 2015 [24]. Both Goodstein sequences and Collatz sequences iterate base changes, but the Collatz sequences do so in a much less diverging manner, involving only bases 2, 3 and 4, with each critical step of their trajectory obeying the following rules:

- (1) if a number is written $x \underbrace{1 \dots 1}_n$ in base 2, then it is finitely mapped to the result of operation G on the number written $y \underbrace{1 \dots 1}_n$ in base 3 with $y = (x-1)/2$. Note that this is the one and only way an orbit can grow in the Collatz dynamics.

- (2) if a number is written $z \underbrace{2 \dots 2}_n 1$ in base 4 then it is immediately mapped to a number written $x \underbrace{1 \dots 1}_n$ in base 2
- (3) if a number is written $s \underbrace{0 \dots 0}_{2n+1} 1$ in base 2 then it is equivalent to the result of operation S on $r \underbrace{0 \dots 0}_n$ in base 3 with r the base 3 representation of s
- (4) if a number is written $v \underbrace{0 \dots 0}_{2n} 1$ in base 2 then it is equivalent to $w \underbrace{0 \dots 0}_n$ in base 3 with w the base 3 representation of v

The purpose of this subsection is to identify provable fundamental properties of the Golden Automaton by computationally scaling it up on the full binary tree over $2N + 1$. To streamline its algorithmic scaling, we use the simplified rules we defined in the previous subsection, again, without loss of generality. Our precise purpose is to pave the way for a formal demonstration that proving the convergence of odd numbers up to n is always isomorphic to a Hydra Game. In Figure 14-17 we color all the elements of $24N - 7$ as for example $\{17, 41, 65, \dots\}$ in red to as we demonstrate in the next section they precisely from the "heads" in the Hydra Game.

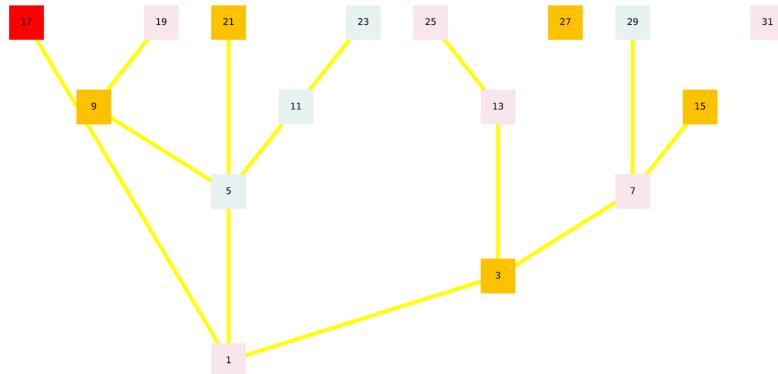


FIGURE 14. Golden Automaton confined to numbers smaller than 32.

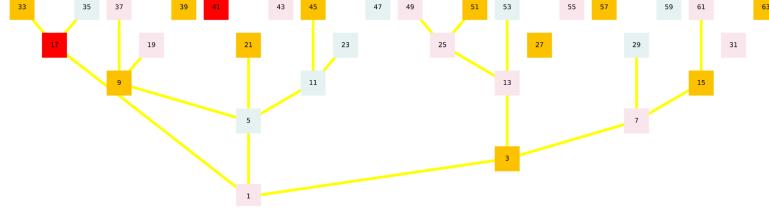


FIGURE 15. Golden Automaton confined to numbers smaller than 64.

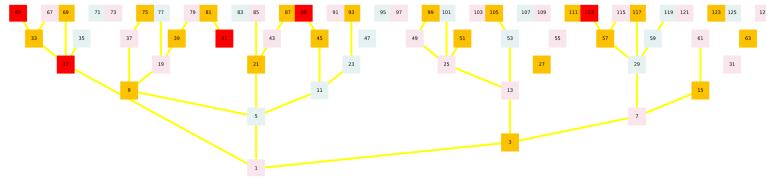


FIGURE 16. Golden Automaton confined to numbers smaller than 128.



FIGURE 17. Golden Automaton confined to numbers smaller than 256.

Theorem 2. *In ZFC, the Collatz conjecture is true.*

Definition 5. A **hydra** is a rooted tree with arbitrary many and arbitrary long finite branches. Leaf nodes are called **heads**. A head is **short** if the immediate parent of the head is the root; **long** if it is neither short nor the root. The object of the **Hydra game** is to cut down the hydra to its root. At each step, one can cut off one of the heads, after which the hydra grows new heads according to the following rules:

- if the head was long, grow n copies of the subtree of its parent node minus the cut head, rooted in the grandparent node.
- if the head was short, grow nothing

Lemma 5. *The Golden Automaton reaching any natural number is a Hydra Game over a finite subtree of the complete binary tree over $24\mathbb{N} - 7$.*

Proof. The essential questions to answer in demonstrating either a homomorphism between a Hydra game and the Golden Automaton reaching any odd number, or that the Golden Automaton is playing **at worst** a Hydra Game are:

- What are the Hydra's heads?
- How do they grow?
- Does the Golden Automaton cut them according to the rules (at worst)?

Definition 6. A type A number that is vertical even is called an A_g . The set of A_g numbers is $24\mathbb{N} - 7$. Type B numbers that verify $b \equiv S(b)$ and type C numbers that verify $c \equiv S(c)$ under **Rule Two** are called **Bups** and **Cups** respectively.

8.1. What are the Hydra's heads? A_g numbers are the heads of the Hydra. They are 12 points apart on $2\mathbb{N} + 1$ (24 in nominal value, e.g. 17 to 41) and any Bup or Cup of $\text{rank} > 1$ they represent under **Rule Five** is smaller than them since action R_a is strictly decreasing so up to the n^{th} A_g there are $2n$ (Bups + Cups) of rank 2 or more and half of them are equivalent to these A_g (e.g. between 17 and 41 Bup 27 is equivalent to A_g 41, which is equivalent to Cup 31 by **Rule Four**)

8.2. How do they grow? Between any two consecutive A_g in $2\mathbb{N} + 1$ there are

- 8 non-A numbers
- 1 of them at most is mapped to the second A_g
- 3 at most are "ups" (Bup or Cup) of rank 2 or more

Besides, we also have anywhere that:

- Let b be of type B, there are $\frac{2b}{3}$ numbers of type A_g that are smaller than $V^2(b)$
- Let c be of type C, there are $\frac{S(c)}{3}$ numbers of type A_g that are smaller than $V^2(c)$
- Let $3c$ be a type B where c is of type C, there are $\frac{S(c)}{3}$ numbers of type A_g up to $R_b(3c)$ included
- Let $3a$ be a type B where a is of type A, there are $\frac{G(a)}{3}$ numbers of type A_g smaller than $R_b(3a)$

Which is defining the growth of the heads. Indeed, any supposedly diverging A_g is forming a Hydra, as we have proven $24\mathbb{N} - 7$ contains an image of all undecided Collatz numbers and that any non-decreasing trajectory identifies a subtree within this set.

8.3. Does the Golden Automaton play a Hydra game? It could be demonstrated that the Golden Automaton is playing an even simpler game as it is branching and thus cutting heads several at a time and in particular cutting some long heads without them doubling² but as this is needless for the final proof we can now simply demonstrate

²In fact, the reason the Golden Automaton dominates $24\mathbb{N} - 7$ so fast is that it is playing a much simpler game one could call "Hecatonchire v. Hydra game" namely a Hydra game where Herakles' number of arms is also multiplying at each step

that even under the worst possible assumptions it follows at least the rules of a regular Hydra game.

The computing of $15 \equiv \dots \equiv 27$ that we detailed in **Subsection 3.1** is one case of the playing of a Hydra Game by the Golden Automaton; we underlined each use of **Rule 5** specifically so the reader can now report to it more easily, because each time this rule is used, a head (A_g) has just been cut.

The demonstration that 27 and 31 converge is the cutting of heads 41 and 161 respectively. This single branch of the Automaton having first cut head 17, reaches to the head 1025 via B-typed numbers 15 and 81. It is therefore playing a Hydra with $\frac{1025+7}{24} = 43$ heads of which one (17) is already cut at this point and of which at least 8 are rooted (so cutting them does not multiply any number of heads). This process being independent of the targeted number, we now have that the reaching of any number by the Golden Automaton is at least equivalent to the playing of a Hydra with n heads of which $0 < m < n$ are rooted. Even without demonstrating more precise limit theorems for each factors n and m (which could still be a fascinating endeavor) the road is now open for a final resolution of the Collatz conjecture. \square

From there indeed, we know with Goodstein [25] and Kirby and Paris [23] that assuming ϵ_0 is well-ordered, no Hydra game can be lost. Since we have that the reaching of any number n is a Hydra Game for the Golden Automaton, we have that the Golden Automaton cannot fail to finitely reach any natural number.

9. THE GOLDEN AUTOMATON AS A WINNING CELLULAR GAME

Beyond graph theory, we want to outline here a different strategy towards a resolution of the Collatz conjecture in Peano arithmetic, by studying the Golden Automaton as a cellular game invading the phase space defined by the complete binary tree over odd numbers. This is therefore essentially a 3D representation of the dynamic we studied in section 7, designed to specifically display potential collisions between the basins of attraction of number 1 and any supposedly diverging other number. We thus employ the same game, that is, a zero-player game which is significantly simpler than John Conway's Game of Life and played on the complete binary tree $\{2\mathbb{N} + 1; G, S\}$, except that we now allow it to start from any point rather than 1, and study its development within the basin of 1. The purpose of this approach is both to identify possibly provable patterns in the way any sub-basin would be embedded in the 1-basin, but also to simply observe whether or not the five rules, for any point, finitely spawn a population of points between any starting number x and $2^n x$ that is bigger than 2^n , which would imply that finite collisions between any two basins are inevitable.

In Figure 20 an orthogonal view of the Golden Automaton starting from 1 (in blue) merged with one starting from 1457 (in green), which is the first A_g in the trajectory

of 127. We input the A_g rather than 127 itself to specifically study the impact of divergence on the form of the basin.

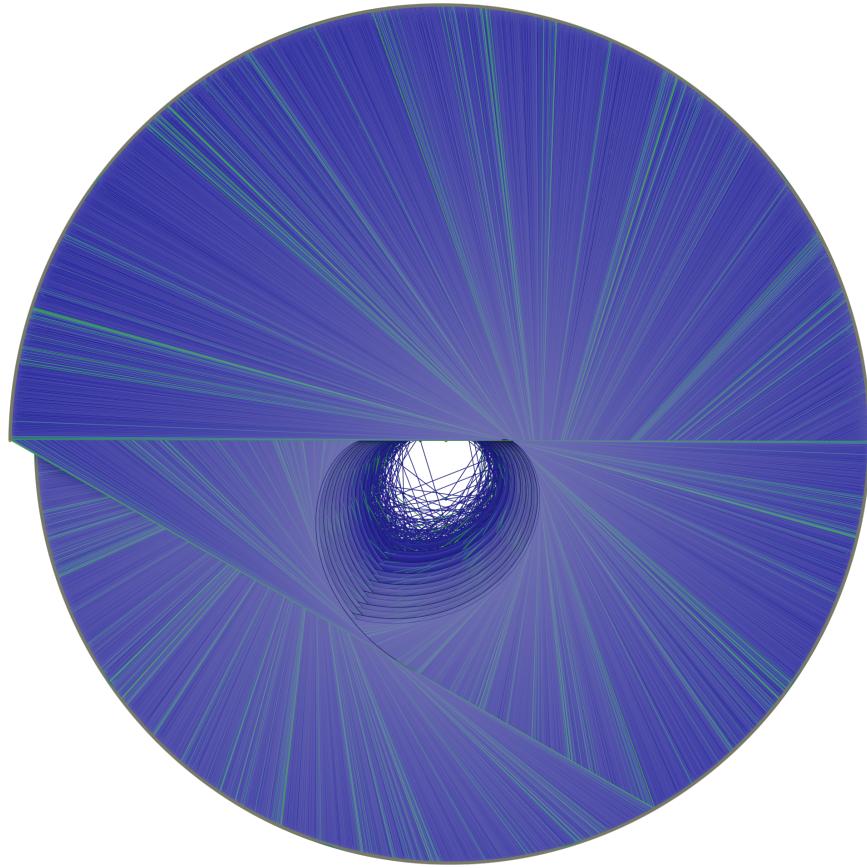


FIGURE 18. Orthogonal view of the Golden Automaton starting from 1 (in blue). All its intersections with the automaton starting independently from 1457 are shown in green. As expected from our 2D works in section 7 the Golden automaton starting from 1 covers all numbers.

Figure 19 visualizes the Golden Automaton starting from 1 (blue) merged with the one starting from 1457 (green). Mersenne numbers like 511, on which six iterations of the Collatz sequence yields 1457, are particularly interesting to study with the Golden Automaton in that they form the benchmark of Collatz divergence. Indeed, at any height 2^n of the binary tree over odd numbers, no number can monotonously increase

more than a Mersenne. As the Golden Automaton only accounts for the backward orbits of any number, precisely, defining the basin of attraction of any number, the interest of studying its expansion on the numbers generated by the monotonous increase of a Mersenne number specifically consists in observing, and further, measuring, **to which extent the basin of attraction of any number will increase if we assume its divergence.**

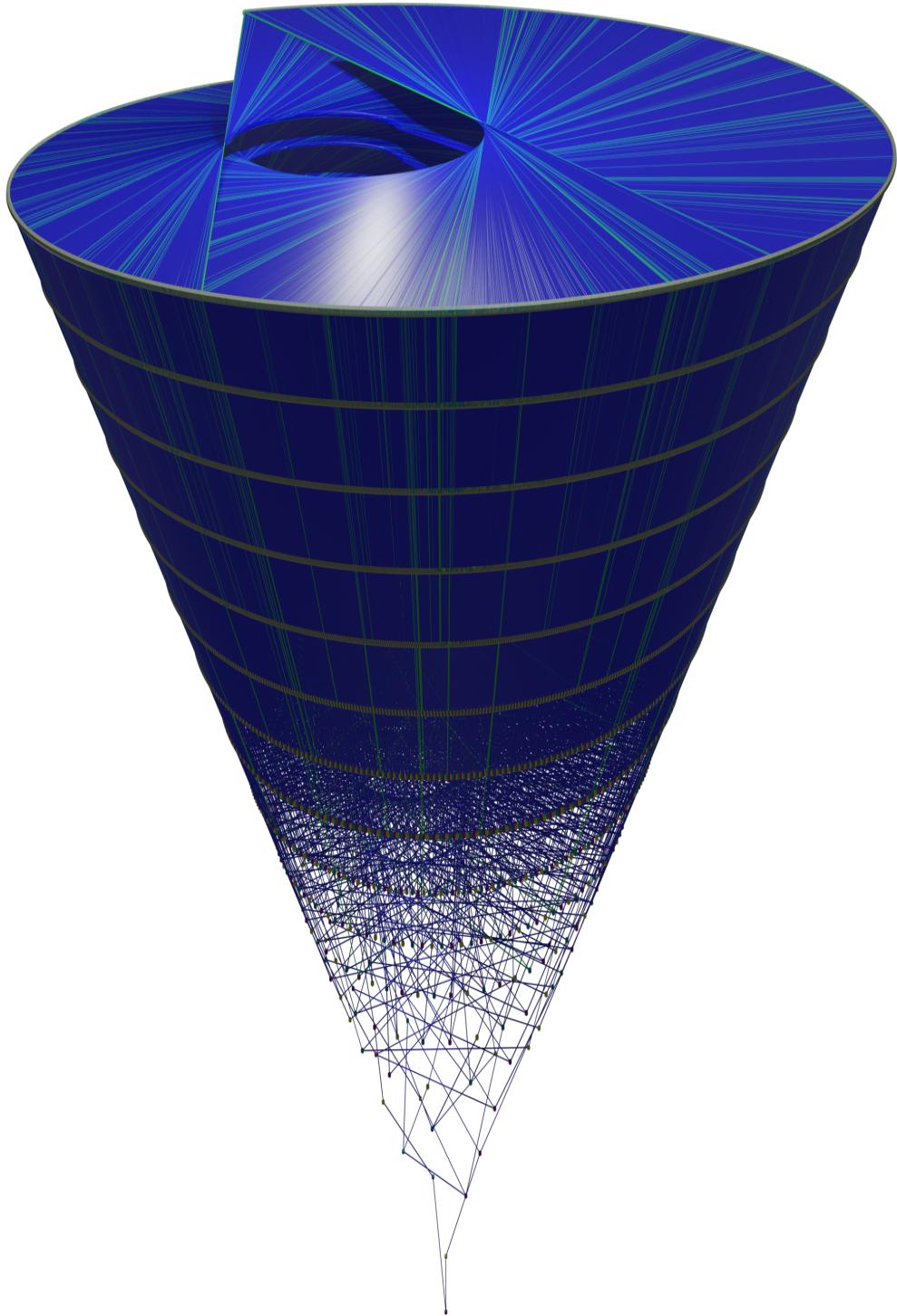


FIGURE 19. The Golden Automaton starting from 1 (blue). All its intersections with the one independently starting from 1457 are shown in green: it is fully overlapping it.

The following Figure 20 provides an orthogonal view of the Golden Automaton starting from 1 (blue) merged with one starting from 161 (green), which is the first A_g of number 31. As 31 is both lower than 127 in the binary tree and is displaying a higher orbit inflation propensity its overlap ends up much larger than that of 127, meaning its basin of attraction inflates along with its orbit.

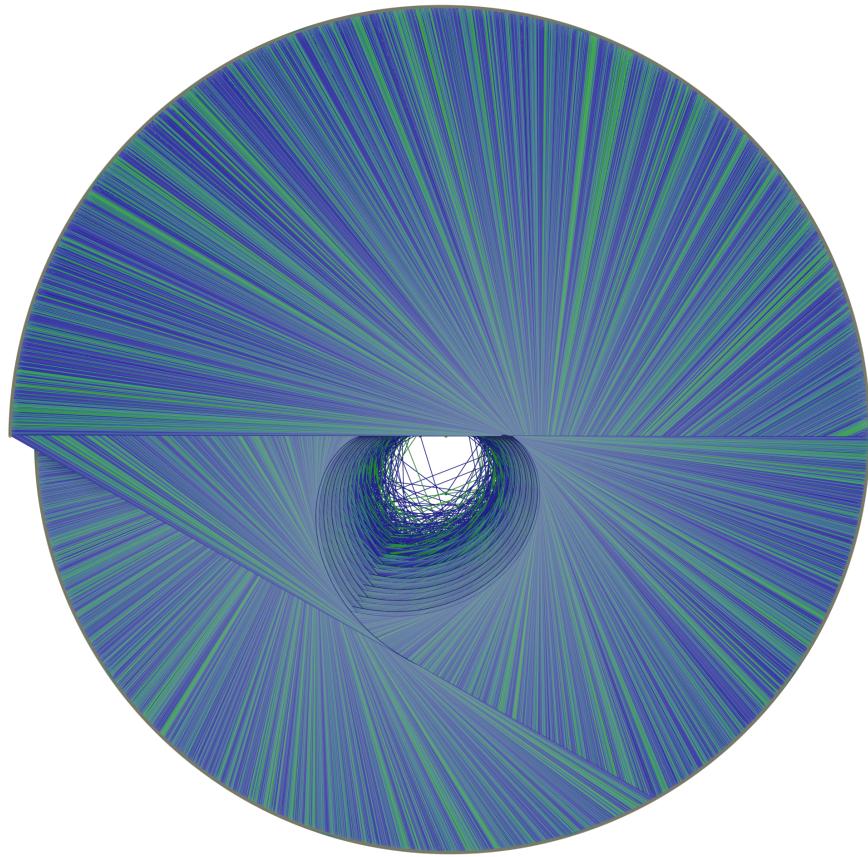


FIGURE 20. Orthogonal view of the Golden Automaton starting from 1 (blue), which is overlapping the one starting from 161 (green).

A close-up of the top of the Golden Automaton starting from 1 merged with the one starting from 161 is shown in (Figure 21). It is showing the particular trajectories of the Golden Automaton and the periodical changes from one row to the next one. The B-type number (in yellow) marking the end of the spiral on the left is Mersenne number 16383.

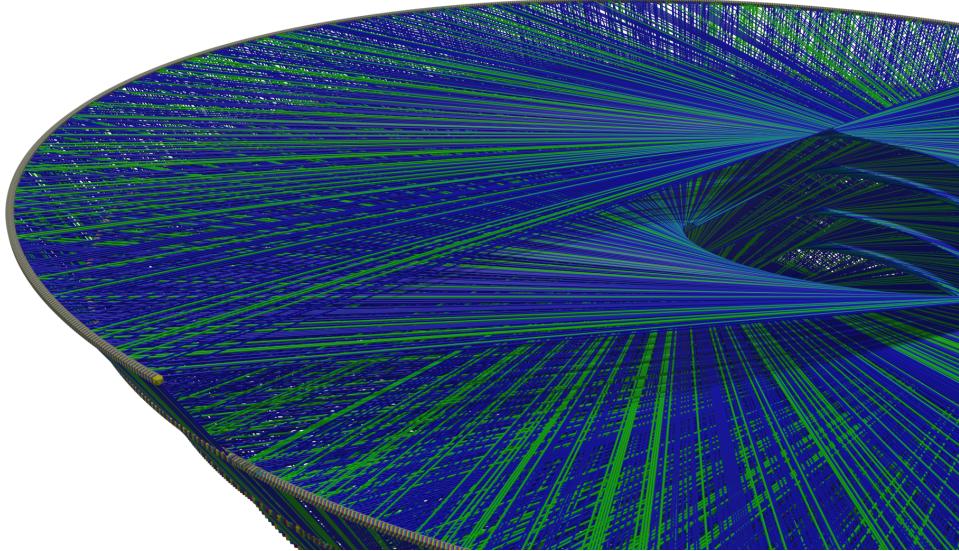


FIGURE 21. Close-up of the top of the Golden Automaton starting from 1, overlapping the one starting from 161.

In Figure 22 we can see the Golden Automaton starting from 161 overlapping the one starting from 1. 161 is marking the end of the first monotonous increase of Mersenne number 31. This number however is more diverging than the next Mersenne 127 (1457), and therefore, **as its orbit inflates, so does its basin of attraction**. Another important property of Mersenne 31 is that, as defined by OEIS [39], it is "self-contained", meaning its orbits contains multiples of itself (ie. the number 155).

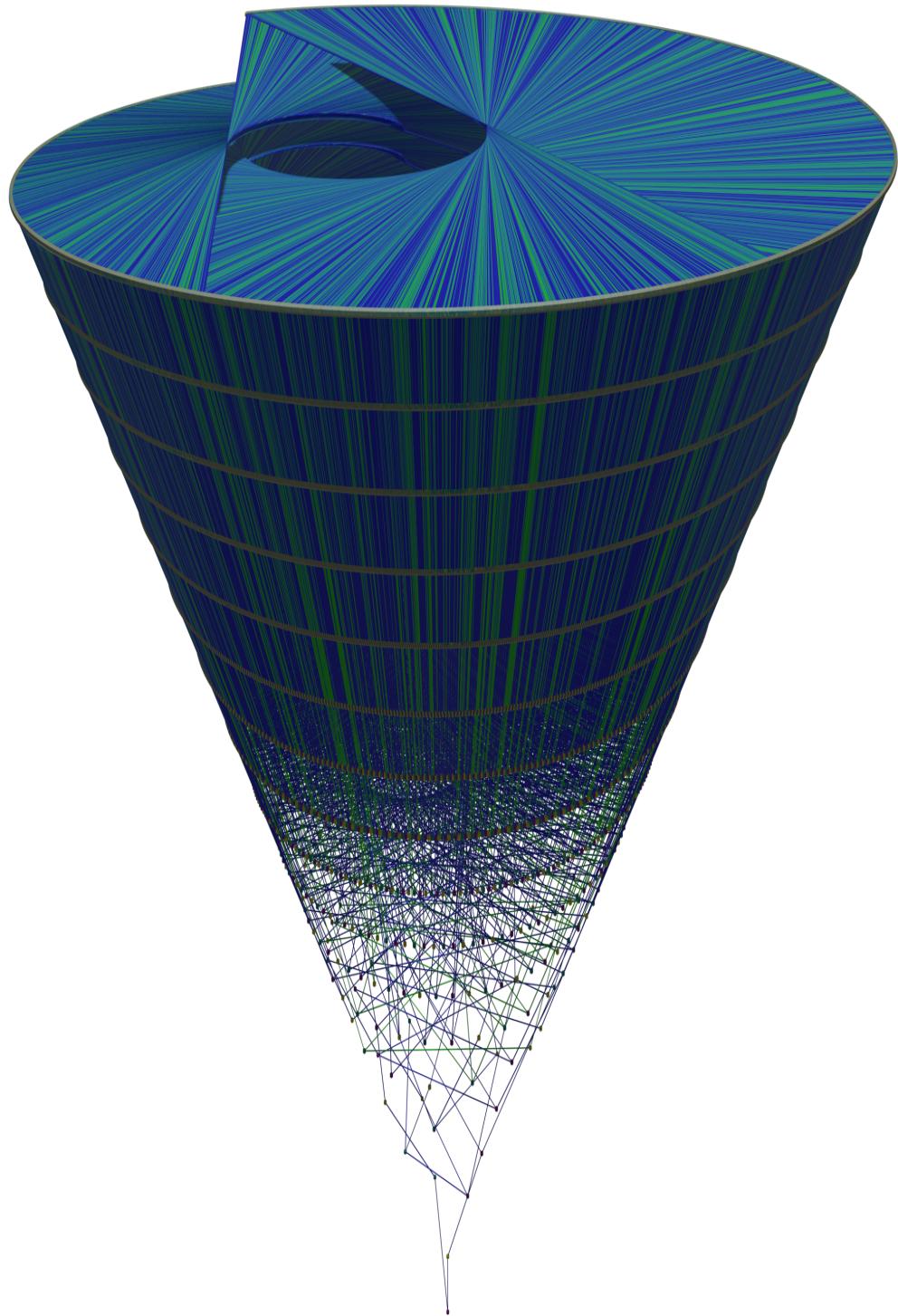


FIGURE 22. The Golden Automaton starting from 161 overlapping the one starting from 1.

The convexoid that is structuring the center of the basin of attraction of number 1 appears to be the truncated caustic generated by the multiplication by 3 on the binary tree projected on the unit circle. In the next figure we thus implement the fundamental operations of the ternary tree, $\{\cdot 3; \cdot 3 - 2; \cdot 3 + 2\}$.

Figure 23 visualizes the ternary tree over the binary tree projected on the unit circle. Operation $\cdot 3$ is figured in yellow, $\cdot 3 - 2$ in purple and $\cdot 3 + 2$ in teal. Number 1 is at exactly π , number 3 at 2π , number 7 at $\frac{\pi}{2}$ and number 5 at $\frac{3\pi}{2}$.

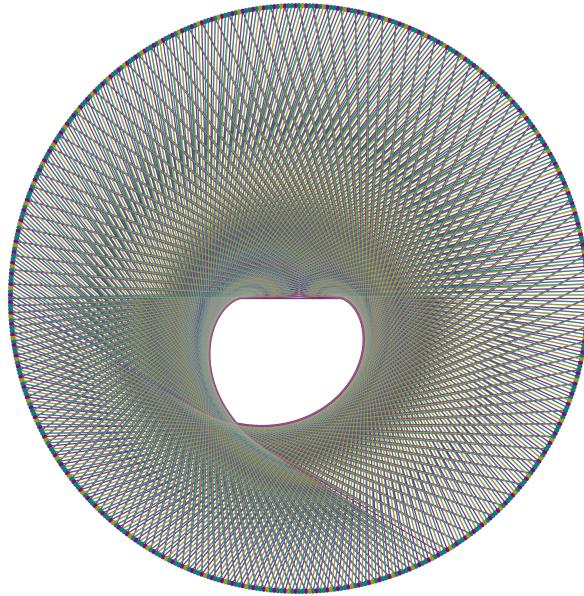


FIGURE 23. The ternary tree over the binary tree projected on the unit circle.

And then, with the purpose of better studying the impact of the multiplication by 3 on the non-ergodicity of the Collatz orbits, we implement a version with a better outline of the departure and arrival point of operation $\cdot 3$.

Figure 24 shows the truncated caustic of operation $\cdot 3$ alone, this time the color gradient indicates departure (red) and arrival (yellow), thus further outlining the non-ergodicity of the implied trajectories.

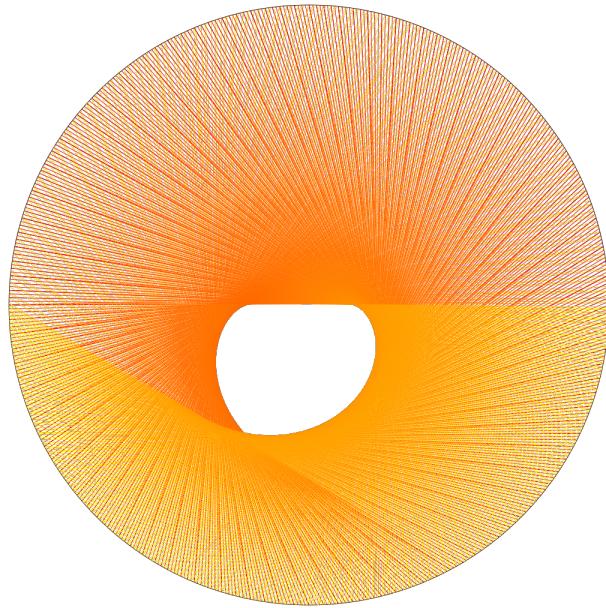


FIGURE 24. Truncated caustic of operation $\cdot 3$, with a color gradient from preimage (orange) to image (yellow).

The following graphs provide some measurements of the non-ergodicity generated by the truncated caustic (Figure 25). Clockwise, from the upper left: number of points on each side of the unit circle (Side 7 is "top", side 5 is "bottom") after iterations of the multiplication by 3 on number 1. The Top/Bottom ratio, on the next figure, converges to approximately 0,7. The next two figures display respectively the cosine and sine of the multiplication by 3 of each point of the unit circle.

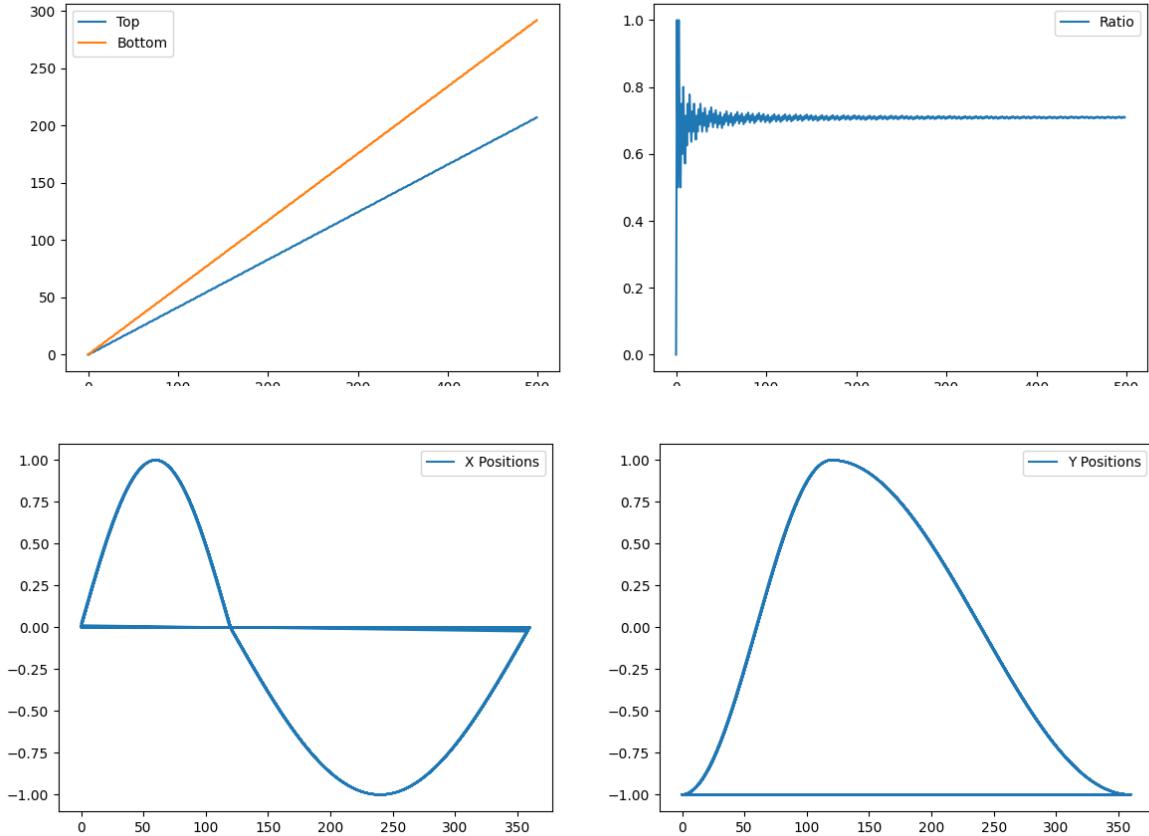


FIGURE 25. Non-ergodic measurements generated by the truncated caustic.

To provide more information about how many numbers the five rules solve, Figure 26 gives further information about the offspring generated by a number. Plotted are the numbers of points in the basin of attraction of two Mersenne numbers (upper half: 31 on the left and 511 on the right) with or without counting the points generated by their divergence (lower half: starting from 161 on the left, 13121 on the right) against how high the basin is calculated. Function 2^n is also plotted as a reference. The purpose is to show that the longer the five Rules are iterated, the more the amount of dots within the basin of attraction increases above 2^n . The top-left figure indicates the number of dots in the basin of 31, the top-right one indicates those in 161, that is, taking into account the divergence from 31 to 161. The bottom plots respectively represent the basins of attraction of 511 and 13121, which is the first A_g in the trajectory of 511, again, to take its divergence into account. The basin of 13121 does not depart from

2^n as fast as that of 511 but starts from a larger number of dots, generated by the 511-13121 divergence.

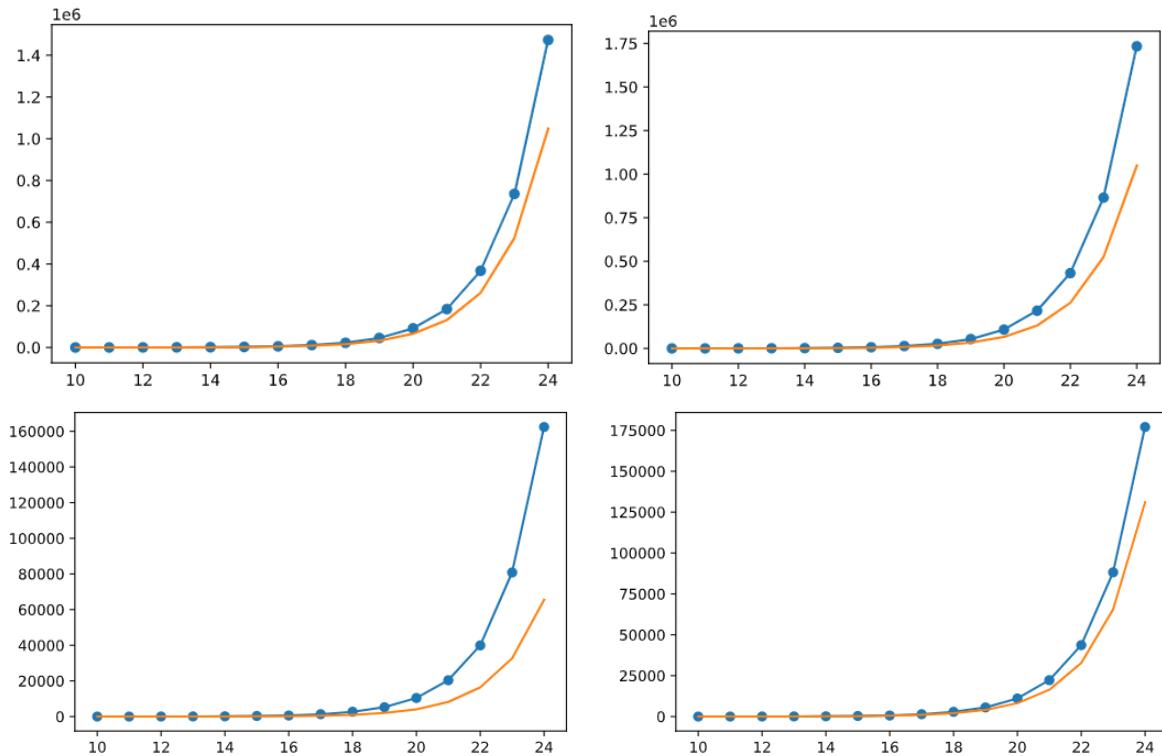


FIGURE 26. Amount of numbers proven by applying the five rules, starting from (clockwise, from the upper left corner) numbers 31, 511, 13121 and 161 respectively, by row of the binary tree. Function 2^n , implemented from the starting point, is shown in comparison.

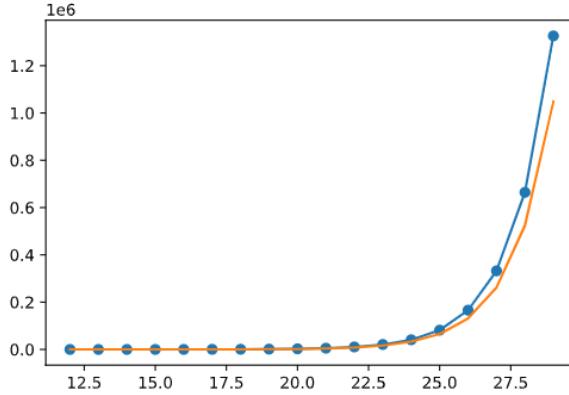


FIGURE 27. Starting point: 118097, First row of graph: 12, Last row of graph: 29

Remark 5.

The apparent growth rate of all the Mersenne numbers from 31 to 8191 is now calculated as the solution for x to $\sum_{k=1}^n x^k = N$ where N is the number of dots in the basin of the number that are found between itself and its first A_g (for example: 161 is the first A_g of 31) and n is the number of multiplications by 2 from the initial number that are needed to reach the row of this first A_g in the binary tree (for example, $n=3$ to go from 31 to 161). All the growth rates are larger than 2 (orange line)(Figure 28).

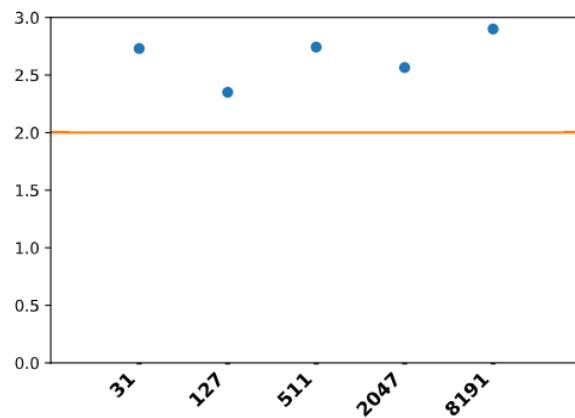


FIGURE 28. Growth rate of Mersenne numbers

We have already demonstrated in section 4 that any A_g can be written $G(x3^n)$ and it is precisely the catching of A_g numbers with a large factor n by the Golden Automaton that increases the quantity of dots in its offspring per given finite series of rows of the binary tree. Specifically, those A_g numbers, which are by definition the ones that can be iterated the most upon by the Golden Automaton are not evenly distributed on the unit circle, and we postulate that this is the most fundamental reason behind the apparent branching factor of the Golden Automaton being strictly greater than 2 in any point we calculated. And if indeed the branching factor of the Golden automaton tends to always be greater than two, it is impossible for two separate basins of attraction to cohabituate on the binary tree.

10. CONCLUSION

Whenever the Collatz conjecture is being studied, one cannot fail to quote Paul Erdős' famous claim that "mathematics may not be ready for such problems"; depending on one's epistemological attitude, the quote may either seem discouraging or the incentive to achieve a novel theoretical breakthrough. This is what we have attempted in this article, primarily by establishing an *ad hoc* equivalent of modular arithmetic for Collatz sequences to automatically demonstrate the convergence of infinite quivers of numbers, based on five arithmetic rules we proved apply on the entire dynamical system and which we further simulated to gain insight on their graph geometry and computational properties. This endeavor has led us to focus on the origins of the non-ergodicity of the Collatz dynamical system, which we found in the geometric properties of the multiplication by 3 on the complete binary tree over odd numbers. These symmetry-breaking properties, indeed, could be further studied in other contexts such as cryptography, harmonic analysis, or the study of L-functions. In particular, following Bocart 2018 [16] one can now gain a better insight at the geometric properties of the pseudorandomness generated by Collatz series, and even more, by the Collatz basins.

11. ATTRIBUTION

I. Aberkane created the framework of studying the Collatz dynamical system in the coordinates defined by the intersection of the binary and ternary trees over $2\mathbb{N} + 1$, identified and demonstrated the five rules and predicted they would be isomorphic to a Hydra game over the set of undecided Collatz numbers, which he defined as well. He directed the 3D visualization of the Golden Automaton and its 2D projection on the unit circle, and the search for its comparative growth rate. Contributing equally, A. Rahn and E. Sultanow designed and coded an optimised, highly scalable 2D graphical implementation of the five rules and ran all the simulations, confirming the Hydra game isomorphism and computing the first ever dot plot of the Golden Automaton

over odd numbers, which they optimised as well. They were also the first team to ever simulate the five rules to the level achieved in this article, and to confirm their emerging geometric properties on such a scale, including the linearity of their logarithmic scaling and the limit reproductive rates of single dots of the golden automaton. A. Rahn also designed, coded and optimized the 3D generated feather (Figure 3). Max Henkel was in charge of generating the other 3D figures and related simulations and was the first to outline the convexoid at the center of the Golden Automaton's basin of attraction. Sourangshu Ghosh was in charge of giving an overview of the topic described and also provided editorial guidance for the organization of the results' communication.

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REFERENCES

- [1] N. J. A. Sloane. A006370: The collatz or $3x+1$ map: $a(n) = n/2$ if n is even, $3n + 1$ if n is odd. <https://oeis.org/A006370>, 4 2021.
- [2] J.H. Conway. Unpredictable iterations. *Proc. 1972 Number Th. Conf.*, pages 49–52, 1972.
- [3] J.C. Lagarias. The $3x + 1$ problem and its generalizations. *Amer. Math. Monthly*, 92:3–23, 1985.
- [4] S. A. Kurtz and J. Simon. The undecidability of the generalized collatz problem. *Theory and Applications of Models of Computation: Proceedings of the 4th International Conference (TAMC 2007) held in Shanghai, May 22-25, 2007*, pages 542–553, 2007.
- [5] G. T. Leavens and M. Vermeulen. $3x + 1$ search programs. *Comput. Math. Appl.*, 24:79–99, 1992.
- [6] I. Vardi. The $3x + 1$ problem. *Computational Recreations in Mathematica*. Redwood City, CA: Addison-Wesley, pages 129–137, 1991.
- [7] T. Oliveira e Silva. Computational verification of the $3x + 1$ conjecture. <https://www.ieeta.pt/tos/3x+1.html>, 2008.
- [8] R. Terras. A stopping time problem on the positive integers. *Acta Arith.*, 30:241–252, 1976.
- [9] R. Terras. On the existence of a density. *Acta Arith.*, 35:101–102, 1979.
- [10] E.; Cloney, T.; Goles and G. Y. Vichniac. The $3x + 1$ problem: A quasi cellular automaton. *Complex Sys.*, 1:349–360, 1987.
- [11] E. W. Weisstein. Collatz problem. *From MathWorld—A Wolfram Web Resource*. <https://mathworld.wolfram.com/CollatzProblem.html>.
- [12] S. Wolfram. A new kind of science. Champaign, IL: Wolfram Media, pages 100, 122, and 904, 2002.
- [13] M. Bruschi. Two cellular automata for the $3x + 1$ map. *arXiv*, arXiv:nlin/0502061 [nlin.CG], 2005.
- [14] E. Zeleny. Collatz problem as a cellular automaton. <https://demonstrations.wolfram.com/CollatzProblemAsACellularAutomaton/>.
- [15] José A. Tenreiro Machado, Alexandra Galhano, and Daniel Cao Labora. A clustering perspective of the collatz conjecture. *Mathematics*, 9(4):314, 2021.
- [16] F. Bocart. Inflation propensity of collatz orbits: a new proof-of-work for blockchain applications. *Journal of Risk and Financial Management*, 11(4):83, 2018.
- [17] I. Aberkane. On the syracuse conjecture over the binary tree. *HAL*, 2017.
- [18] I. Aberkane. Endomorphisms of the collatz quiver. *HAL*, 2020.
- [19] C. Koch, E. Sultanow, and S. Cox. Divisions by two in collatz sequences: A data science approach. *International Journal of Pure Mathematical Sciences*, 21, 2020.
- [20] E. Sultanow, C. Koch, and S. Cox. Collatz sequences in the light of graph theory. Technical report, Potsdam University, 2020.
- [21] S. Feferman. Is the continuum hypothesis a definite mathematical problem. <https://math.stanford.edu/~feferman/papers/BernaysLecture2.pdf>, 2012.
- [22] D. E. Knuth. *Surreal Numbers: How Two Ex-students Turned on to Pure Mathematics and Found Total Happiness*. Addison-Wesley, Boston, MA, 12 edition, 1974.
- [23] L. Kirby and J. Paris. Accessible independence results for peano arithmetic. *Bulletin of the London Mathematical Society*, 1982.
- [24] M. Artigue, F. Arzarello, and S. Epp. Goodstein sequences: The power of a detour via infinity. <http://blog.kleinproject.org/?p=674>, 2015.
- [25] R. Goodstein. On the restricted ordinal theorem. *Journal of Symbolic Logic*, 9(2):33–41, 1944.

- [26] G. Gentzen. Die widerspruchsfreiheit der reinen zahlentheorie. *Mathematische Annalen*, 112:493–565, 1936.
- [27] K. Gödel. Über formal unentscheidbare sätze der principia mathematica und verwandter systeme, i. *Monatshefte für Mathematik und Physik*, 38(1):173–198, 1931.
- [28] E. A. Cichon. A short proof of two recently discovered independence results using recursion theoretic methods. *Proc. Amer. Math. Soc.*, 87:704–706, 1983.
- [29] B. Hodgson. Herculean or sisyphean tasks. *EMS Newsletter*, 51:11–16, 2004.
- [30] A. Caicedo. Goodstein’s function. *Revista Colombiana de Matemáticas*, 41(2):381–391, 2007.
- [31] D. Kaplan. A classification of non-standard models of peano arithmetic by goodstein’s theorem, 2012.
- [32] J. Miller. On the independence of goodstein’s theorem, 2001.
- [33] T. J. Stępień and Ł. T. Stępień. On the consistency of the arithmetic system. *Journal of Mathematics and System Science*, 7, 2017.
- [34] David Barina. Convergence verification of the collatz problem. *The Journal of Supercomputing*, 77(3):2681–2688, 2021.
- [35] Jan Kleinnijenhuis, Alissa M. Kleinnijenhuis, and Mustafa G. Aydogan. The collatz tree as a hilbert hotel: a proof of the $3x + 1$ conjecture. <http://arxiv.org/pdf/2008.13643v3.pdf>.
- [36] Christian Koch. Collatz python library. <https://github.com/c4ristian/collatz>, 2021.
- [37] L.A. Clark, F. Torzewska, B. Maybee, and A. Beige. Non-ergodicity in open quantum systems through quantum feedback. *Europhysics Letters*, 130, 2020.
- [38] Eldar Sultanow. Sources for the exploration of collatz sequences: Tex, mathematica and python. <https://github.com/Sultanow/collatz>, 2020.
- [39] N. J. A. Sloane. A005184: Self-contained numbers: odd numbers n whose collatz sequence contains a higher multiple of n . <https://oeis.org/A005184>, 4 2021.

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