

# Supplementary Material to: Epidemic models in measure spaces: persistence, concentration and oscillations.

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In this document, to distinguish them from internal references, references to the labels from the main text are preceded by the letter M.

## 1 More numerical illustrations

### 1.1 Illustrations of the main Theorem

We start with an illustration of the long-time behavior of the solution to (M2.1) when the initial mass of the fitness maximum is positive ( $I_0(\alpha^{-1}(\alpha^*)) > 0$ ). To help with the visual representation, in our example,  $I_0$  is chosen absolutely continuous with respect to the Lebesgue measure, *i.e.* carried by a  $L^1(\mathbb{R}^N)$  function. In Figure 1 we provide a plot of the fitness function (left-hand side) and the initial data (right-hand side). The fitness function attains its maximum on a rectangle with positive Lebesgue measure and the support of the initial data intersects this rectangle with non-negligible intersection.

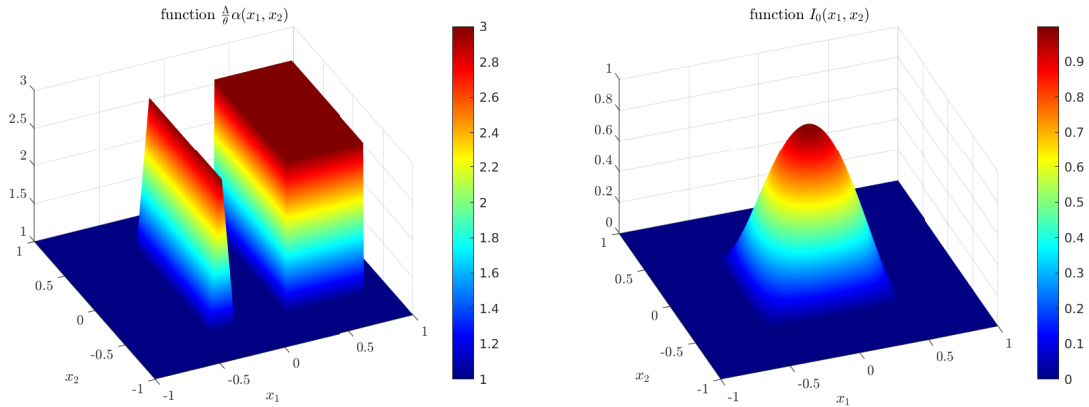


Figure 1: Illustration of Theorem M2.2 in the case i), *i.e.*, when  $I_0(\alpha^{-1}(\alpha^*)) > 0$ . Parameters of this simulation are:  $\Lambda = 2$ ,  $\theta = 1$ ,  $\alpha(x) = 0.5 + (\mathbb{T}_{[-0.4, -0.2]}(x_1) + \mathbb{1}_{[0.2, 0.8]}(x_1)) \mathbb{1}_{[-0.6, 0.6]}(x_2)$  where  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $\mathbb{T}_{[-0.4, -0.2]}$  is the triangular function of height one and support  $[-0.4, -0.2]$ , and  $\gamma = \frac{1}{2\alpha}$ . Initial condition is given by  $I_0(dx) = I_0(x_1, x_2) dx$  where  $I_0(x_1, x_2) = \mathbb{1}_{[-0.5, 0.5]}(x_1) \cos(\pi x_1) \mathbb{1}_{[-0.5, 0.5]}(x_2) \cos(\pi x_2)$ . In particular, we have  $\alpha^* = 3/2$  and  $\alpha^{-1}(\alpha^*) = (\{-0.3\} \cup [0.2, 0.5]) \times [-0.5, 0.5]$ .

In Figure 2 we plot the time evolution of  $S(t)$  (left-hand side) and a snapshot of the distribution  $I(t, x)$  at  $t = 50$ . We observe that the mass that was initially located outside of the fitness maximum has vanished. What remains is a distribution of mass in the initial rectangle of maximal fitness (according to Theorem M2.2, the distribution precise can be computed). The distribution located at  $\{x_1 = -0.3\}$  is still positive, but is negligible with respect to the Lebesgue measure and does not contribute to the mass. In Figure 3 we present four snapshots of the distribution  $I(t, x)$  to monitor the time evolution of this distribution with the same initial distribution.

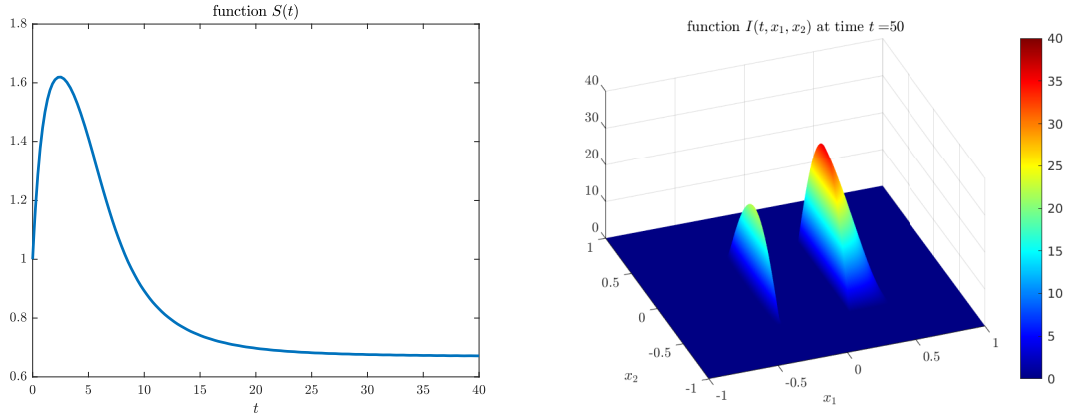


Figure 2: (continued from Fig. 1) Illustration of Theorem M2.2 in the case i), *i.e.*, when  $I_0(\alpha^{-1}(\alpha^*)) > 0$ . Function  $t \rightarrow S(t)$  converges towards  $1/\alpha^* = 2/3$  and function  $x \rightarrow I(t, x)$  at time  $t = 50$  is asymptotically concentrated on  $\alpha^{-1}(\alpha^*) = (\{-0.3\} \cup [0.2, 0.5]) \times [-0.5, 0.5]$ .

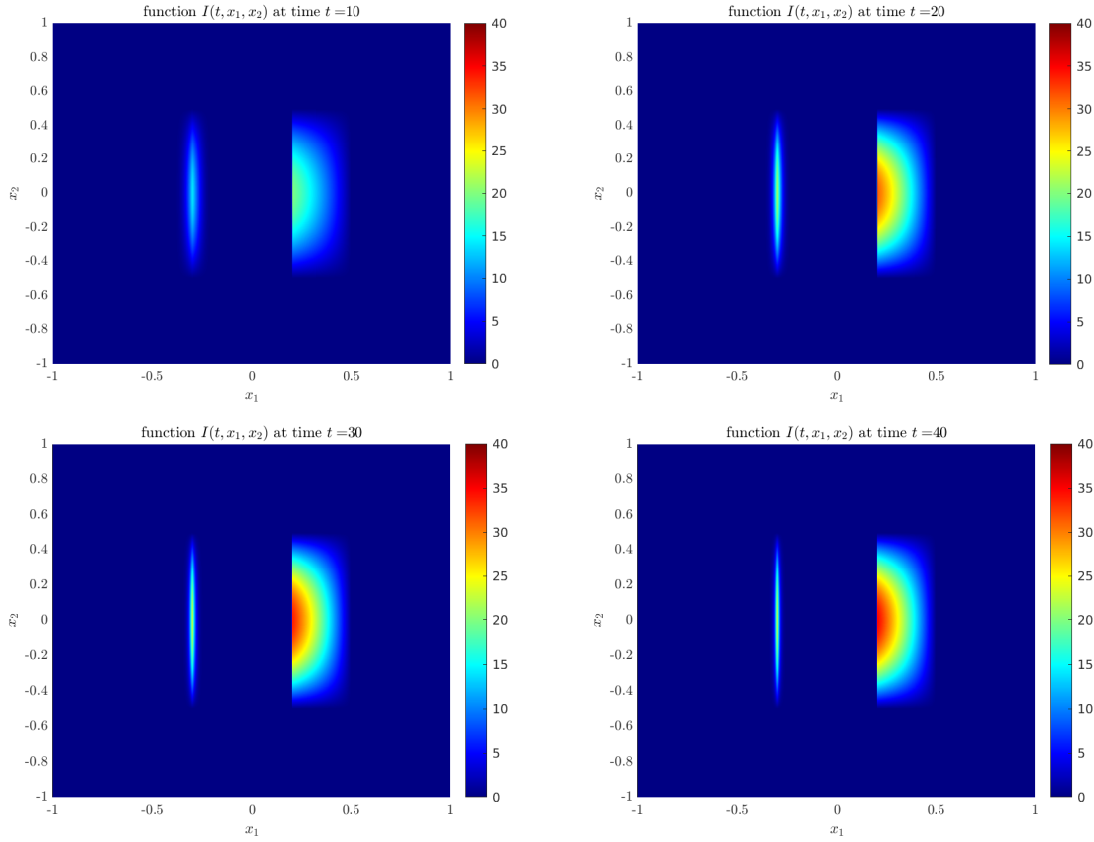


Figure 3: Illustration of Theorem M2.2 in the case i), *i.e.*, when  $I_0(\alpha^{-1}(\alpha^*)) > 0$ . Function  $x \rightarrow I(t, x)$  at time  $t = 10, 20, 30$  and  $40$ . The function  $I$  remains bounded in this case.

Figure 4 consists of four snapshots of the example presented in Figure M1 of the Main text.

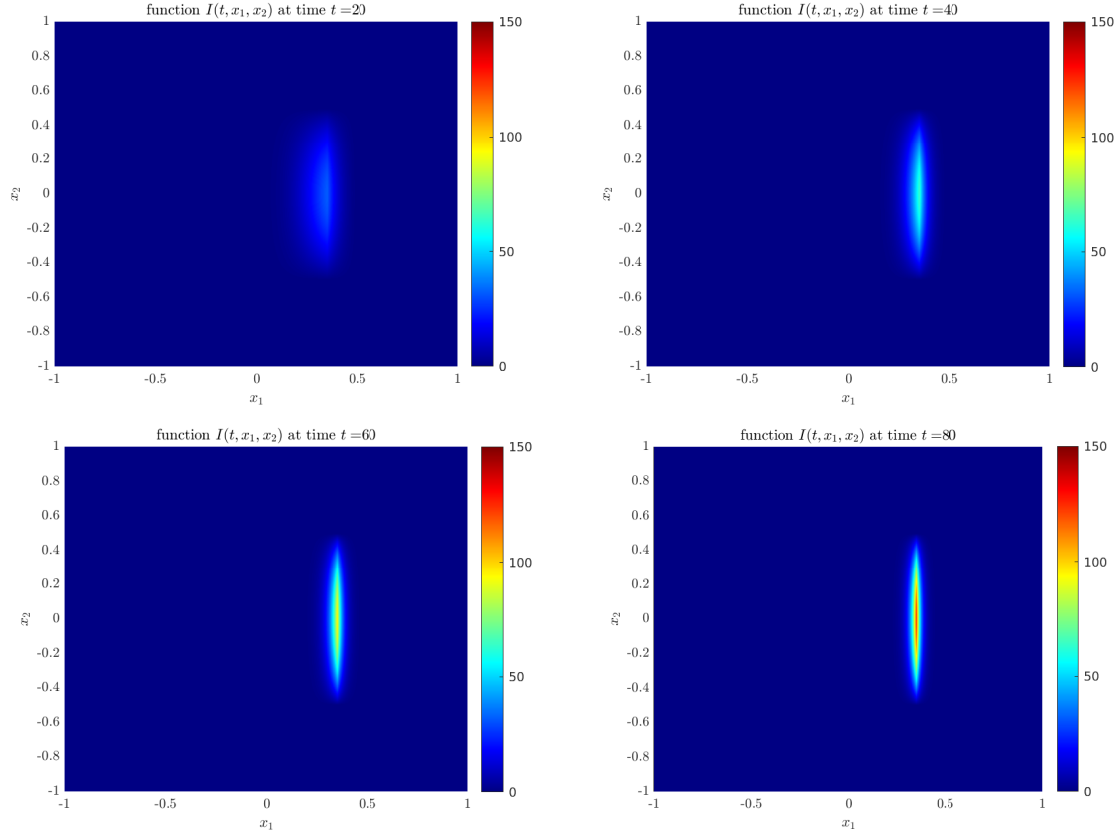


Figure 4: Illustration of Theorem M2.2 in the case ii), *i.e.*, when  $I_0(\alpha^{-1}(\alpha^*)) = 0$ . Function  $x \rightarrow I(t, x)$  at time  $t = 20, 40, 60$  and  $80$ . The function  $I$  asymptotically converges towards a singular measure.

## 1.2 Transient dynamics on local maxima: a numerical example

In many biologically relevant situations it may be more usual to observe situations involving a fitness function with one global maximum and several (possibly many) local maxima, whose values are not exactly equal to the global maximum but very close. In such a situation, while the long-term distribution will be concentrated on the global maximum, one may observe a transient behavior in which the orbits stay close to the equilibrium of the several global maxima situation (corresponding to Theorem M2.7), before it concentrates on the eventual distribution. We leave the analytical treatment of such a situation open for future studies, however, we present a numerical experiment in Figure 5 which shows such a transient behavior.

In this simulation, we took a fitness function presenting one global maximum at  $x_2 = +0.5$  and a local maximum at  $x_1 = -0.5$ , whose value is close to the global maximum. The precise definition of  $\alpha(x)$  is

$$\alpha(x) = 0.95 \times \mathbb{P}_{[x_1 - \delta, x_1 + \delta]}(x) + \mathbb{P}_{[x_2 - \delta, x_2 + \delta]}(x) \text{ with } \delta = 0.2. \quad (1)$$

The function  $I_0(x)$  is chosen as

$$I_0(x) = \min\left(1, 4(x - x_1)^2\right) \min\left(1, 4(x - x_2)^2\right) \mathbb{1}_{[-1, 1]}(x), \quad (2)$$

so that  $\kappa_1 = 2$  and  $\kappa_2 = 2$ . Finally,

$$\gamma(x) = \frac{1}{1 + \mathbb{P}_{[x_1 - \delta, x_1 + \delta]}(x) + 3\mathbb{P}_{[x_2 - \delta, x_2 + \delta]}(x)} \quad (3)$$

so that  $\gamma(x_1) = \frac{1}{2}$  and  $\gamma(x_2) = \frac{1}{4}$ . Summarizing, we have

$$\frac{N + \kappa_1}{2\gamma(x_1)} = 3 < 6 = \frac{N + \kappa_2}{2\gamma(x_2)}.$$

If  $\alpha(x)$  had two global maximum at the same level, Theorem M2.7 would predict that the mass  $I(t, dx)$  vanishes near  $x_2$  and concentrates on  $x_1$ . Since the value of  $\alpha(x_2)$  is slightly higher than the value of  $\alpha(x_1)$ , however, it is clear that the eventual distribution will be concentrated on  $x_2$ . We observe numerically (see Figure 5) that the distribution first concentrates on  $x_1$  on a transient time scale, before the dynamics on  $x_2$  takes precedence. We refer to Burie, Djidjou-Demasse, and Ducrot [3] for a related model with mutations where these transient behaviors are analytically characterized.

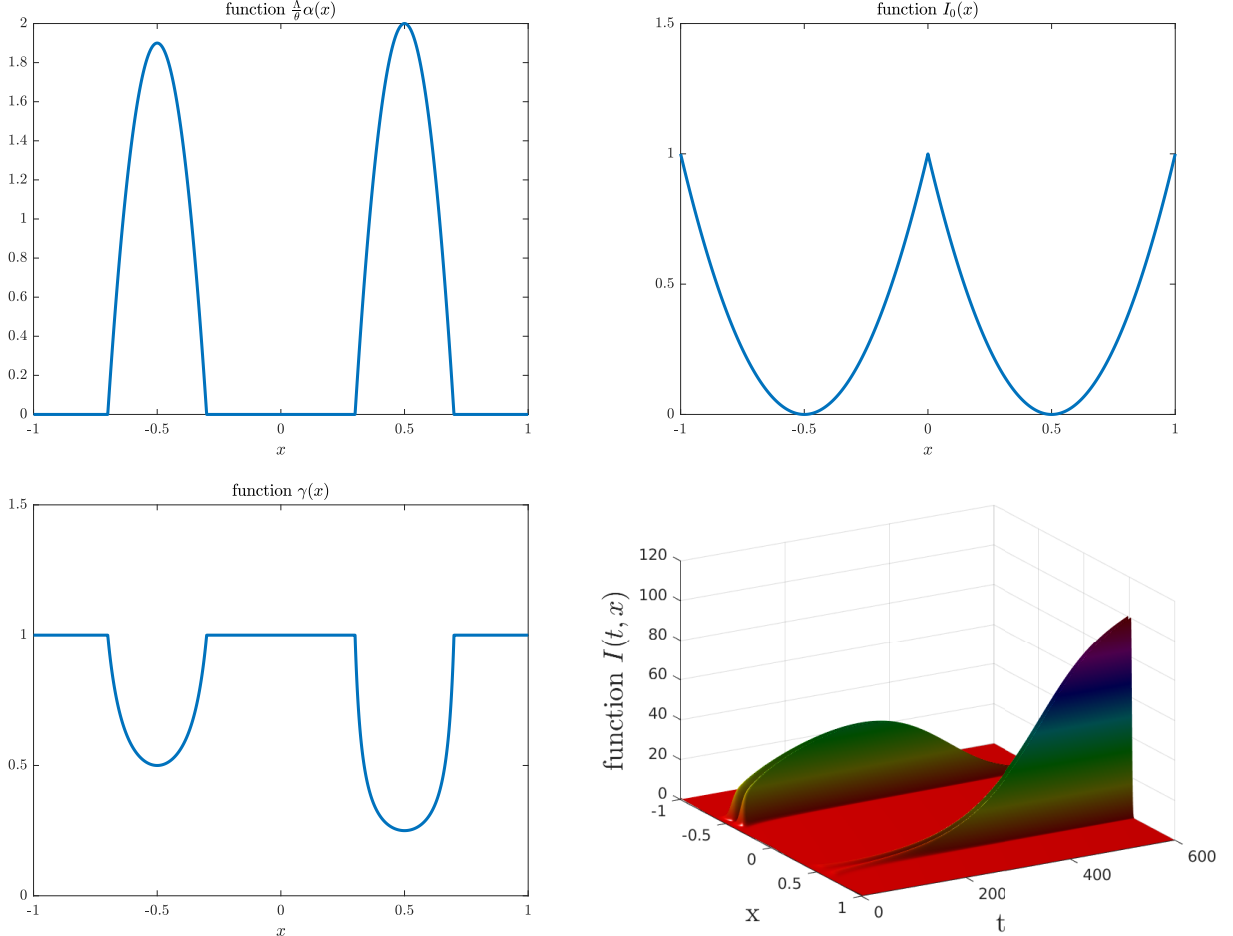


Figure 5: Illustration of a transient behavior for (M2.1). Parameters of this simulation are:  $\Lambda = 2$ ,  $\theta = 1$ ,  $\alpha(x)$  is given by (1),  $I_0(x)$  by (2) and  $\gamma(x)$  by (3). In particular,  $\alpha^* = 1$ ,  $\alpha^{-1}(\alpha^*) = \{x_2\}$  with  $x_1 = -0.5$ ,  $x_2 = 0.5$ . Other parameters are  $\kappa_1 = 2$ ,  $\kappa_2 = 2$ ,  $\gamma(x_1) = 1/2$ ,  $\gamma(x_2) = 1/4$ . The value of the local maximum at  $x_1$ ,  $\alpha(x_1) = 0.95$ , being very close to  $\alpha^*$ , observe that the distribution  $I(t, x)$  first concentrates around  $x_1$  before the global maximum  $x_2$  becomes dominant (bottom right plot).

### 1.3 Oscillations

In Figure 6 we present a numerical simulation that illustrates the example provided in section M2.3 of the main text. The parameter  $L$  is set to 1. A supplementary movie is also available ([spiraling.avi](#)).

## 2 Existence of a regular metric projection

In this Section we let  $(M, d)$  be a complete metric space. Let  $\mathcal{K}(M)$  be the set of compact subsets in  $M$  and let  $K \in \mathcal{K}(M)$ . We first recall that we can define a kind of frame of reference, internal to  $K$ , which allows to identify each point in  $K$ .

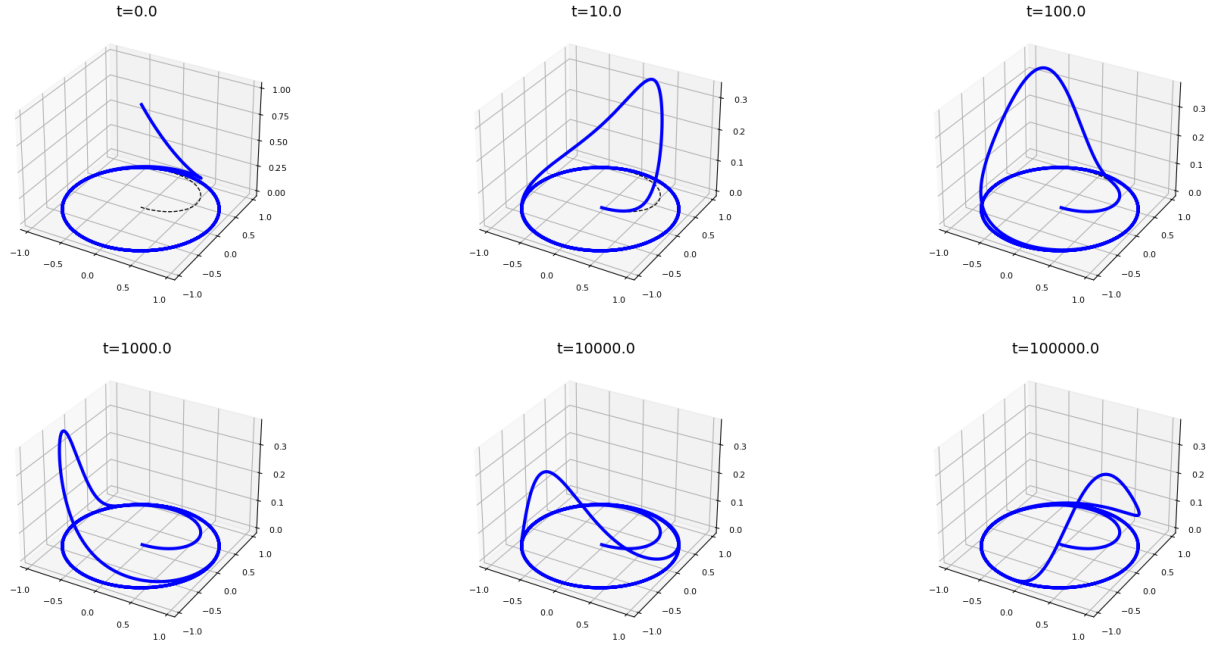


Figure 6: Numerical solution of the example provided in Section M2.3 of the main text with  $L = 1$ . The  $x, y$  axes correspond to the underlying physical space and we plot  $I(t, \tau)$  on the  $z$  axis at every  $x = (1 - e^{-\tau}) \cos(\tau)$  and  $y = (1 - e^{-\tau}) \sin(\tau)$ . The observation times are spaced exponentially from one another ( $t = 0, 10, 10^2, 10^3, 10^4, 10^5$ ) to observe constant shifts of the fixed asymptotic profile. These plots are snapshots of the supplementary movie `spiraling.avi`.

Let us denote  $\mathcal{K}(M)$  the set formed by all compact subsets of  $M$ . Recall that  $(\mathcal{K}(M), d_H)$  is a complete metric space, where  $d_H$  is the Hausdorff distance

$$d_H(K_1, K_2) = \max \left( \sup_{x \in K_1} d(x, K_2), \sup_{x \in K_2} d(x, K_1) \right).$$

**Proposition 2.1** (Metric coordinates). *There exists a finite number of points  $x_1, \dots, x_n \in K$  with the property that each  $y \in K$  can be identified uniquely by the distance between  $y$  and  $x_1, \dots, x_n$ . In other words the map*

$$y \xrightarrow{c_K} \begin{pmatrix} d(y, x_1) \\ \vdots \\ d(y, x_n) \end{pmatrix} \in \mathbb{R}_+^n,$$

*is one-to-one. Moreover  $c_K$  is continuous and its reciprocal function  $c_K^{-1} : c_K(K) \rightarrow K$  is also continuous.*

*Proof.* Let us choose  $x_1 \in K$  and  $x_2 \in K$  such that  $x_1 \neq x_2$ . We recursively construct a sequence  $x_n$  and a compact set  $K_n$  such that

$$K_n = \{y \in K : d(y, x_i) = d(y, x_1) \text{ for all } 1 \leq i \leq n\},$$

$$x_{n+1} \in K_n,$$

the choice of  $x_{n+1}$  being arbitrary. Clearly  $K_n$  is a compact set and  $K_{n+1} \subsetneq K_n$ . Suppose by contradiction that  $K_n \neq \emptyset$  for all  $n \in \mathbb{N}$ , then (because  $K$  is compact) one can construct a sequence  $x_{\varphi(n)}$ , extracted from  $x_n$ , and which converges to a point

$$x = \lim_{n \rightarrow +\infty} x_{\varphi(n)} \in \bigcap_{n \in \mathbb{N}} K_n =: K_\infty.$$

In particular  $K_\infty$  is not empty. However we see that, by definition of  $K_\infty$ , we have  $d(x, x_n) = d(x, x_1) > 0$  for all  $n \in \mathbb{N}$ , which contradicts the fact that

$$\lim_{n \rightarrow +\infty} d(x, x_{\varphi(n)}) = 0.$$

Hence we have shown by contradiction that there exists  $n_0 \in \mathbb{N}$  such that  $K_{n_0} = \emptyset$  and  $K_{n_0-1} \neq \emptyset$ . This is precisely the injectivity of the map  $c_K : K \rightarrow \mathbb{R}^{n_0}$ .

To show the continuity, we remark that  $c_K$  is continuous, and therefore for each closed set  $F \subset K$ ,  $F$  is compact so that  $c_K(F)$  is compact and therefore closed. Therefore  $(c_K^{-1})^{-1}(F) = c_K(F)$  is closed in  $c_K(K)$ . The proposition is proved.  $\square$

Recall that the Borel  $\sigma$ -algebra  $\mathcal{B}(M)$  is the closure of the set of all open sets in  $\mathfrak{P}(M) = 2^M$  under the operations of complement and countable union. A function  $\varphi : M \rightarrow N$  is Borel measurable if the reciprocal image of any Borel set is Borel, *i.e.*  $\varphi^{-1}(B) \in \mathcal{B}(M)$  for all  $B \in \mathcal{B}(N)$ .

**Proposition 2.2** (Borel function of choice). *There exists a Borel measurable map  $c : (\mathcal{K}(K), d_H) \rightarrow (K, d)$  such that*

$$c(K') \in K' \text{ for all } K' \in \mathcal{K}(K).$$

*Proof.* Let  $c_K : K \rightarrow \mathbb{R}^{n_0}$  be the map constructed in Proposition 2.1. For a compact  $K' \subset K$  we define

$$c(K') := c_K^{-1} \left( \min_{y \in c_K(K')} y \right),$$

where the minimum is taken with respect to the lexicographical order in  $\mathbb{R}^{n_0}$  (which is a total order and therefore identifies a unique minimum for each  $K' \in \mathcal{K}(K)$ ). Since the map  $\tilde{K} \subset \mathbb{R}^{n_0} \rightarrow \min_{y \in K} y$  is Borel for the topology on  $\mathcal{K}(\mathbb{R}^{n_0})$  induced by the Hausdorff metric, so is  $c$ . The proposition is proved.  $\square$

**Proposition 2.3** (Borel measurability of the metric projection). *Let  $K \subset M$  be compact. The map  $P_K : M \rightarrow \mathcal{K}(K)$  defined by*

$$P_K(x) = \{y \in K : d(x, y) = d(x, K)\},$$

*is Borel measurable.*

*Proof.* First we remark that the map

$$P_K(x) := \{y \in K : d(x, y) = d(x, K)\} \in \mathcal{K}(M),$$

is well-defined for each  $x \in M$ , and therefore forms a mapping from  $M$  into  $\mathcal{K}(K) \subset \mathcal{K}(M)$ . Indeed  $P_K(x)$  is clearly closed in the compact space  $K$ , therefore is compact.

To show the Borel measurability of  $P_K$ , we first remark that, given a compact space  $K' \subset K$ , the set

$$\widetilde{P_K^{-1}}(K') := \{x \in M : P_K(x) \cap K' \neq \emptyset\}$$

is closed. Indeed let  $x_n \rightarrow x$  be a sequence in  $\widetilde{P_K^{-1}}(K')$ , then by definition there exists  $y_n \in K'$  such that  $d(x_n, y_n) = d(x_n, K)$ . By the compactness of  $K'$ , there exists  $y \in K'$  and a subsequence  $y_{\varphi(n)}$  extracted from  $y_n$  such that  $y_{\varphi(n)} \rightarrow y$ . Because of the continuity of  $z \mapsto d(z, K)$ , we have

$$d(x, y) = \lim_{n \rightarrow +\infty} d(x_{\varphi(n)}, y_{\varphi(n)}) = \lim_{n \rightarrow +\infty} d(x_{\varphi(n)}, K) = d(x, K),$$

therefore  $y \in P_K(x) \cap K'$ , which shows that  $x \in \widetilde{P_K^{-1}}(K')$ . Hence  $\widetilde{P_K^{-1}}(K')$  is closed.

We are now in a position to show the Borel regularity of  $P_K$ . Let  $C \in \mathcal{K}(K)$  and  $R > 0$  be given. We define  $B_H(C, R)$  the ball of center  $C$  and radius  $R$  in the Hausdorff metric:

$$B_H(C, R) = \{C' \in \mathcal{K}(K) : d_H(C, C') \leq R\}.$$

Then

$$P_K^{-1}(B_H(C, R)) = \{x \in M : d_H(P_K(x), C) \leq R\} = B_1 \cap B_2,$$

where

$$B_1 := \{x \in M : d(y, C) \leq R \text{ for all } y \in P_K(x)\}, \text{ and}$$

$$B_2 := \{x \in M : d(z, P_K(x)) \leq R \text{ for all } z \in C\}.$$

It can be readily seen that  $B_1$  is a Borel set by writing

$$B_1 = \widetilde{P_K^{-1}}(V_R(C)) \bigcap_{n \geq 1} \left( M \setminus \left( \widetilde{P_K^{-1}}(K \setminus V_{R+\frac{1}{n}}(C)) \right) \right),$$

where  $V_R(C) := \{y \in K : d(y, C) \leq R\}$ . To see that  $B_2$  is a Borel set, we choose a sequence  $z_n$  which is dense in  $C$  and write

$$B_2 = \bigcap_{k \geq 1} \bigcap_{n \geq 1} \widetilde{P_K^{-1}}(B(z_n, R + 1/k)).$$

Indeed if  $x \in B_2$  then  $P_K(x)$  intersects every ball of radius  $R$  and center  $z \in C$ ; in particular  $P_K(x)$  intersects every ball of radius  $R + 1/k$  and center  $z_n$ . Conversely suppose that  $P_K(x)$  intersects every ball  $B(z_n, R + 1/k)$  for  $n \geq 1$  and  $k \geq 1$ . If  $z \in C$  then there is a sequence  $z_{\varphi(k)}$  such that  $z = \lim z_{\varphi(k)}$ , and (by assumption) we have  $P_K(x) \cap B(z_{\varphi(k)}, R + 1/k) \neq \emptyset$ . Therefore

$$d(z, P_K(x)) = \lim_{k \rightarrow +\infty} d(z_{\varphi(k)}, P_K(x)) \leq \lim_{k \rightarrow +\infty} R + \frac{1}{k} = R.$$

Thus  $x \in B_2$ . The equality is proved.

We conclude that  $\widetilde{P_K^{-1}}(B_H(C, R))$  is a Borel set for all  $C \in \mathcal{K}(K)$  and  $R > 0$ , and since those sets form a basis of the Borel  $\sigma$ -algebra,  $P_K$  is indeed Borel measurable. The Lemma is proved.  $\square$

**Theorem 2.4** (Existence of a regular metric projection). *Let  $K \subset M$  be compact. There exists a Borel measurable map  $P_K : M \rightarrow K$  such that*

$$d(x, P_K(x)) = d(x, K).$$

*Proof.* The proof is immediate by combining Proposition 2.3 Proposition 2.2.  $\square$

**Proposition 2.5** (Metric projection on measure spaces). *Let  $K \in \mathcal{K}(M)$  be a given compact set. Let  $\mu \in \mathcal{M}_+(M)$  be a given nonnegative Borel measure on  $M$ . Then the Kantorovitch-Rubinstein distance between  $\mu$  and  $\mathcal{M}_+(K)$  can be bounded by the distance between  $K$  and the furthest point in  $\text{supp } \mu$ :*

$$d_0(\mu, \mathcal{M}_+(K)) \leq \|\mu\|_{AV} \sup_{x \in \text{supp } \mu} d(x, K).$$

*Proof.* Indeed, let us choose a Borel measurable metric projection  $P_K$  on  $K$  as in Theorem 2.4. Let  $\mu^K$  be the image measure defined on  $\mathcal{B}(K)$  by

$$\mu^K(B) := \mu(P_K^{-1}(B)), \text{ for all } B \in \mathcal{B}(K).$$

Then in particular for all  $f \in BC(M)$  we have

$$\int_{\mathbb{R}^N} f(P_K(x)) d\mu(x) = \int_K f(x) d\mu^K(x).$$

Let  $f \in \text{Lip}_1(M)$ , then we have

$$\begin{aligned} \int_{\mathbb{R}^N} f(x) d(\mu - \mu^K)(x) &= \int_{\mathbb{R}^N} f(x) d\mu(x) - \int_{\mathbb{R}^N} f(x) d\mu^K(x) \\ &= \int_{\mathbb{R}^N} f(x) d\mu(x) - \int_{\mathbb{R}^N} f(P_K(x)) d\mu(x) \\ &= \int_{\mathbb{R}^N} (f(x) - f(P_K(x))) d\mu(x) \\ &\leq \int_{\text{supp } \mu} |x - P_K(x)| d\mu(x) \\ &\leq \sup_{y \in \text{supp } \mu} d(y, K) \int_{\text{supp } \mu} 1 d\mu = \|\mu\|_{AV} \sup_{x \in \text{supp } \mu} d(x, K). \end{aligned}$$

Therefore  $d_0(\mu, \mu^K) \leq \|\mu\|_{AV} \sup_{x \in \text{supp } \mu} d(x, K)$  and, since  $\mu^K \in \mathcal{M}_+(K)$ ,

$$d_0(\mu, \mathcal{M}_+(K)) \leq d_0(\mu, \mu^K) \leq \|\mu\|_{AV} \sup_{x \in \text{supp } \mu} d(x, K).$$

The Proposition is proved.  $\square$

### 3 Disintegration of measures

#### 3.1 Bourbaki's disintegration theorem

We recall the disintegration theorem as stated in [2, VI, §3, Theorem 1 p. 418]. We use Bourbaki's version, which is proved by functional analytic arguments, for convenience, although other approaches exist which are based on measure-theoretic arguments and may be deemed more intuitive. We refer to Ionescu Tulcea and Ionescu Tulcea for a disintegration theorem resulting from the theory of (strong) liftings [7, 8].

Let us first recall some background on adequate families. This is adapted from [2, V.16 §3] to the context of finite measures of  $\mathbb{R}^N$ . We let  $T$  and  $X$  be locally compact topological spaces and  $\mu \in \mathcal{M}_+(T)$  be a fixed Borel measure.

**Definition 3.1** (Scalarly essentially integrable family). Let  $\Lambda : t \mapsto \lambda_t$  be a mapping from  $T$  into  $\mathcal{M}_+(X)$ .  $\Lambda$  is *scalarly essentially integrable* for the measure  $\mu$  if for every compactly supported continuous function  $f \in C_c(X)$ , the function  $t \mapsto \int_X f(x)\lambda_t(dx)$  is in  $L^1(\mu)$ . Setting  $\nu(f) = \int_T \int_X f(x)\lambda_t(dx)\mu(dt)$  defines a linear form on  $C_c(X)$ , hence a measure  $\nu$ , which is the *integral* of the family  $\Lambda$ , and we denote

$$\int_T \lambda_t \mu(dt) := \nu.$$

Recall that every positive Borel measure  $\mu$  on a locally compact space  $X$  defines a positive bounded linear functional on  $C_c(X)$  equipped with the inductive limit of the topologies on  $C_c(K)$  when  $K$  runs over the compact subsets of  $X$ . Conversely if  $\mu$  is a positive bounded linear functional on  $C_c(X)$ , there are two canonical ways to define a measure on the Borel  $\sigma$ -algebra.

1. Outer-regular construction. Let  $U \subset X$  be an open, then one can define

$$\mu^*(U) := \sup \{ \mu(f) : f \in C_c(X), 0 \leq f(x) \leq \mathbb{1}_U(x) \},$$

then for an arbitrary Borel set  $B$ ,

$$\mu^*(B) := \inf \{ \mu^*(U) : U \text{ open}, B \subset U \}.$$

This notion corresponds to that of the *upper integral* discussed in [2, IV.1 §1].

2. Inner-regular construction. If  $U \subset X$  is open, we define  $\mu^\bullet(U) := \mu^*(U)$  and similarly if  $K \subset X$  is compact, then  $\mu^\bullet(K) := \mu^*(K)$ . Then for an arbitrary Borel set  $B$  which is contained in an open set of finite measure:  $B \subset U$  with  $\mu^\bullet(U) < +\infty$ , we define

$$\mu^\bullet(B) := \sup \{ \mu^\bullet(K) : K \text{ compact}, K \subset B \}.$$

Else  $\mu^\bullet(B) = +\infty$ . This corresponds to the *essential upper integral* discussed in [2, V.1, §1].

It is always true that  $\mu^\bullet \leq \mu^*$ , however it may happen that  $\mu^* \neq \mu^\bullet$  when  $\mu^*$  is not finite, see e.g. [1, II§7.11 p.113] or [2, V.1, §1]. If  $\mu$  is a Borel measure, then we define the corresponding notions of  $\mu^\bullet$  and  $\mu^*$  associated with the linear functional  $f \mapsto \int_X f(x)\mu(dx)$ . Note that if  $\mu$  is Radon, then  $\mu^* = \mu = \mu^\bullet$ .

**Definition 3.2** (Pre-adequate and adequate families). We follow [2, Definition 1, V.17§3]. Let  $\Lambda : t \mapsto \lambda_t$  be a scalarly essentially  $\mu$ -integrable mapping from  $T$  into  $\mathcal{M}_+(X)$ ,  $\nu$  the integral of  $\Lambda$ .

We say that  $\Lambda$  is  $\mu$ -pre-adequate if, for every lower semi-continuous function  $f \geq 0$  defined on  $X$ , the function  $t \mapsto \int f(x)\lambda_t^\bullet(dx)$  is  $\mu$ -measurable on  $T$  and

$$\int_X f(x)\nu^\bullet(dx) = \int_T \int_X f(x)\lambda_t^\bullet(dx)\mu^\bullet(dt).$$

We say that  $\Lambda$  is  $\mu$ -adequate if  $\Lambda$  is  $\mu'$ -pre-adequate for every positive Borel measure  $\mu' \leq \mu$ .

The last notion we need to define is the one of  $\mu$ -proper function.

**Definition 3.3** ( $\mu$ -proper function). We say that a function  $p : T \rightarrow X$  is  $\mu$ -proper if it is  $\mu$ -measurable and, for every compact set  $K \subset X$ , the set  $p^{-1}(K)$  is  $\mu^\bullet$ -measurable and  $\mu^\bullet(p^{-1}(K)) < +\infty$ .



If  $\mu$  is Radon, in particular, then every  $\mu$ -measurable mapping  $p : T \rightarrow X$  ( $X$  being equipped with the Borel  $\sigma$ -algebra) is  $\mu$ -proper. The following Theorem is taken from [2, Theorem 1, VI.41 No.1, §3].

**Theorem 3.4** (Disintegration of measures). *Let  $T$  and  $X$  be two locally compact spaces having countable bases,  $\mu$  be a positive measure on  $T$ ,  $p$  be a  $\mu$ -proper mapping of  $T$  into  $X$ , and  $\nu = p(\mu)$  the image of  $\mu$  under  $p$ . There exists a  $\nu$ -adequate family  $x \mapsto \lambda_x$  ( $x \in X$ ) of positive measures on  $T$ , having the following properties:*

- a)  $\|\lambda_x\| = 1$  for all  $x \in p(T)$ ;
- b)  $\lambda_x$  is concentrated on the set  $p^{-1}(\{x\})$  for all  $x \in p(T)$ , and  $\lambda_x = 0$  for  $x \notin p(T)$ ;
- c)  $\mu = \int \lambda_x \nu(dx)$ .

Moreover, if  $x \mapsto \lambda'_x$  ( $x \in X$ ) is a second  $\nu$ -adequate family of positive measures on  $T$  having the properties b) and c), then  $\lambda'_x = \lambda_x$  almost everywhere in  $B$  with respect to the measure  $\nu$ .

### 3.2 Explicit disintegration in Euclidean spaces

Suppose that  $X = \mathbb{R}^N$  and  $I_0 \in L^1(\mathbb{R}^n)$ . Since we restrict to measures which are absolutely continuous with respect to the Lebesgue measure here, with a small abuse of notation we will omit the element  $dx$  when the context is clear. Assume that  $\alpha$  is Lipschitz continuous on  $\mathbb{R}^N$  and that

$$\frac{I_0(x)}{|\nabla\alpha(x)|} \in L^1(\mathbb{R}^N). \quad (4)$$

The coarea formula implies that, for all  $g \in L^1(\mathbb{R}^N)$ , we have

$$\int_{\mathbb{R}^N} g(x) |\nabla\alpha(x)| dx = \int_{\mathbb{R}} \int_{\alpha^{-1}(y)} g(x) \mathcal{H}_{N-1}(dx) dy,$$

where  $\mathcal{H}_{N-1}(dx)$  is the  $(N-1)$ -dimensional Hausdorff measure (see Federer [4, §3.2]).

Therefore if  $g(x) = f(x) \frac{I_0(x)}{|\nabla\alpha(x)|}$  we get

$$\int_{\mathbb{R}^N} f(x) I_0(x) dx = \int_{\mathbb{R}} \int_{\alpha^{-1}(y)} f(x) \frac{I_0(x)}{|\nabla\alpha(x)|} \mathcal{H}_{N-1}(dx) dy, \quad (5)$$

and if moreover  $f(x) = \varphi(\alpha(x))$  we get

$$\int_{\mathbb{R}} \varphi(y) A(dy) = \int_{\mathbb{R}^N} \varphi(\alpha(x)) I_0(x) dx = \int_{\mathbb{R}} \varphi(y) \int_{\alpha^{-1}(y)} \frac{I_0(x)}{|\nabla\alpha(x)|} \mathcal{H}_{N-1}(dx) dy,$$

where we recall that  $A(dy)$  is the image measure of  $I_0(dx)$  through  $\alpha$ . Therefore we have an explicit expression for  $A(dy)$ :

$$A(dy) = \int_{\alpha^{-1}(y)} \frac{I_0(x)}{|\nabla\alpha(x)|} \mathcal{H}_{N-1}(dx) dy \quad (6)$$

and (recalling (5)) we deduce the following explicit disintegration of  $I_0$ :

$$I_0(y, dx) = \frac{\mathbb{1}_{x \in \alpha^{-1}(y)} \frac{I_0(x)}{|\nabla\alpha(x)|} \mathcal{H}_{N-1}(dx)}{\int_{\alpha^{-1}(y)} \frac{I_0(z)}{|\nabla\alpha(z)|} \mathcal{H}_{N-1}(dz)}. \quad (7)$$

Equations (6) and (7) give an explicit formula for the disintegration introduced in (M2.6).

**Remark 3.5.** In particular, if  $\alpha$  is a  $C^2$  function with no critical point in  $\text{supp } I_0$  except for a finite number of regular maxima (in the sense that the bilinear form  $D^2\alpha(x)$  is non-degenerate at each maximum; this is a typical situation), then the assumption (4) above is automatically satisfied if  $N \geq 3$  and  $I_0 \in L^\infty(\mathbb{R}^N)$ . If  $N = 2$  then a sufficient condition to satisfy (4) with  $I_0 \in L^\infty(\mathbb{R}^N)$  should involve  $I_0$  vanishing sufficiently fast in the neighborhood of each maximum of  $\alpha$ .

**Remark 3.6.** The assumption M2.3 in the main text means that the initial measure  $I_0(dx)$  is uniformly positive in a small neighborhood of  $\alpha^{-1}(\alpha^*)$ . For instance, if the assumptions (4) is satisfied and if there exists an open set  $U$  containing  $\alpha^{-1}(\alpha^*) \cap \gamma^{-1}(\gamma^*)$  on which  $I_0(x)$  is almost everywhere uniformly positive, then Assumption M2.3 is automatically satisfied.

## 4 Epidemic models with discrete phenotypic spaces

In this section we propose an application of the general result, namely Theorem M2.2, to the case of discrete (and finite) systems. This corresponds to a choice of  $X$  as a discrete spaces, say  $X = \mathbb{N}$  equipped with the natural distance.

We start with the case of *finite* systems, *i.e.*, the case when system (M2.1) can be written as follows:

$$\begin{cases} \frac{d}{dt}S(t) = \Lambda - \theta S(t) - S(t)(\alpha_1\gamma_1 I^1(t) + \alpha_2\gamma_2 I^2(t) + \dots + \alpha_n\gamma_n I^n(t)) \\ \frac{d}{dt}I^1(t) = (\alpha_1 S(t) - 1)\gamma_1 I^1(t) \\ \frac{d}{dt}I^2(t) = (\alpha_2 S(t) - 1)\gamma_2 I^2(t) \\ \vdots \\ \frac{d}{dt}I^n(t) = (\alpha_n S(t) - 1)\gamma_n I^n(t). \end{cases} \quad (8)$$

For the above system, we can completely characterize the asymptotic behavior of the population. To that aim we define the basic reproductive number (in the ecological or epidemiological sense) for species  $i$  as follows:

$$\mathcal{R}_0^i := \frac{\Lambda}{\theta} \alpha_i, \quad i = 1, \dots, n.$$

Then we can show that the only species that do not get extinct are the ones for which  $\mathcal{R}_0^i$ ,  $i = 1, \dots, n$ , is maximal and strictly greater than one.

In the case when several species have the same basic maximal reproductive number, then these species all survive and the asymptotic distribution can be computed explicitly as a function of the initial data  $I_0^i$  and of the values  $\gamma_i$ .

**Theorem 4.1** (Asymptotic behavior of finite systems). *Let  $n \geq 1$  and  $\alpha_1, \dots, \alpha_n$  and  $\gamma_1 > 0, \dots, \gamma_n > 0$  be given. Set*

$$\alpha^* := \max\{\alpha_1, \dots, \alpha_n\},$$

*and assume that*

$$\mathcal{R}_0^* := \frac{\Lambda}{\theta} \alpha^* = \max\{\mathcal{R}_0^1, \dots, \mathcal{R}_0^n\} > 1.$$

*Then, for any initial data  $S_0 \geq 0, I_0^1 > 0, \dots, I_0^n > 0$ , the corresponding solution to (8) converges in the large times to  $(S_\infty, (I_\infty^1, \dots, I_\infty^n))$  given by*

$$S_\infty = \frac{1}{\alpha^*} \text{ and } I_\infty^i = \begin{cases} 0 & \text{if } \mathcal{R}_0^i < \mathcal{R}_0^*, \\ e^{\tau\gamma_i} I_0^i & \text{if } \mathcal{R}_0^i = \mathcal{R}_0^*, \end{cases} \text{ for all } i = 1, \dots, n,$$

*wherein the constant  $\tau \in \mathbb{R}$  is defined as the unique solution of the equation:*

$$\sum_{\{i: \mathcal{R}_0^i = \mathcal{R}_0^*\}} \gamma_i I_0^i e^{\tau\gamma_i} = \frac{\theta}{\alpha^*} (\mathcal{R}_0 - 1).$$

Note that in the case when the interaction of the species with the resource is described by the Michaelis-Menten kinetics instead of the mass action law, or when the growth of the resource obeys a logistic law (Hsu [5]), a similar result was already present in Hsu, Hubbell and Waltman [6] and Hsu [5], including the case when several species have the exact same reproduction number (or fitness)  $\mathcal{R}_0^i$ .

In the case of a *countable* system we can still provide a complete description when both  $\alpha$  and  $\gamma$  converge to a limit near  $+\infty$ . We now investigate the following system

$$\begin{cases} \frac{d}{dt}S(t) = \Lambda - \theta S(t) - S(t) \sum_{i \in \mathbb{N}} \alpha_i \gamma_i I_i(t), \\ \frac{d}{dt}I^i(t) = (\alpha_i S(t) - 1)\gamma_i I^i(t), \end{cases} \quad \text{for } i \in \mathbb{N}, \quad (9)$$

supplemented with some initial data

$$S(0) = S_0 > 0 \text{ and } I^i(0) = I_0^i > 0, \forall i \in \mathbb{N} \text{ with } \sum_{i \in \mathbb{N}} I_0^i < \infty. \quad (10)$$

Since components of (9) starting from a zero initial data will stay equal to zero in positive time, they can be removed from the system without impacting the dynamics and we may without loss of generality assume that  $I_0^i > 0$  for all  $i \in \mathbb{N}$  (as we did in (10)).

In the sequel we denote by  $(S(t), I^i(t))$  be the corresponding solution to (9) with initial data  $S(0) = S_0$  and  $I^i(0) = I_0^i$  for all  $i \in \mathbb{N}$ .

**Theorem 4.2** (Asymptotic behavior of discrete systems). *Let  $(\alpha_i)_{i \in \mathbb{N}}$  and  $(\gamma_i)_{i \in \mathbb{N}}$  be bounded sequences with  $\gamma_i > 0$  for all  $i \in \mathbb{N}$ . Set*

$$\alpha^* := \sup\{\alpha_i, i \in \mathbb{N}\},$$

and assume that

$$\mathcal{R}_0^* := \frac{\Lambda}{\theta} \alpha^* > 1.$$

We distinguish two cases.

- i) *If the set  $\{i \in \mathbb{N} : \alpha_i = \alpha^*\}$  is not empty, then  $(S(t), I^i(t))$  converges to the following asymptotic stationary state*

$$S_\infty = \frac{1}{\alpha^*}, \text{ and } I_\infty^i = \begin{cases} 0 & \text{if } \mathcal{R}_0^i < \mathcal{R}_0^*, \\ e^{\tau \gamma_i} I_0^i & \text{if } \mathcal{R}_0^i = \mathcal{R}_0^*, \end{cases} \text{ for all } i \in \mathbb{N} \cup \{\infty\}.$$

where the constant  $\tau \in \mathbb{R}$  is the unique solution of the equation:

$$\sum_{\{i \in \mathbb{N} : \mathcal{R}_0^i = \mathcal{R}_0^*\}} \gamma_i I_0^i e^{\tau \gamma_i} = \frac{\theta}{\alpha^*} (\mathcal{R}_0^* - 1).$$

- ii) *If the set  $\{i \in \mathbb{N} : \alpha_i = \alpha^*\}$  is empty, then one has  $S(t) \rightarrow \frac{1}{\alpha^*}$  and  $I^i(t) \rightarrow 0$  for all  $i \in \mathbb{N}$  as  $t \rightarrow \infty$ , while*

$$\liminf_{t \rightarrow +\infty} \sum_{i \in \mathbb{N}} I^i(t) > 0.$$

*If moreover one has  $\alpha_n \rightarrow \alpha^*$  and  $\gamma_n \rightarrow \gamma_\infty > 0$  as  $n \rightarrow \infty$  with  $n \in \mathbb{N}$  then the total mass converges to a positive limit*

$$\lim_{t \rightarrow +\infty} \sum_{i \in \mathbb{N}} I^i(t) = \frac{\theta}{\alpha^* \gamma_\infty} (\mathcal{R}_0^* - 1). \quad (11)$$

Note that for more complex countable systems, such as if the  $\omega$ -limit set of  $\alpha_n$  or  $\gamma_n$  contains two or more distinct values, then it is no longer possible in general to state a result independent of the initial data. We discuss a similar phenomenon for measures with a density with respect to the Lebesgue measure in Section M2.2.

#### 4.1 Proof of Theorem 4.1 and 4.2 by using $X = \mathbb{R}^N$

In this section we show how the theory for discrete systems can be included in the theory for measure-valued solutions to (M2.1) in with  $X = \mathbb{R}^N$ . The theorem are trivially deduced from Theorem M2.2 in the Alexandroff compactification of  $\mathbb{N}$ ,  $X = \mathbb{N} \cup \{\infty\}$ , equipped with the distance  $d(n, m) = (\frac{1}{n} + \frac{1}{m}) \mathbb{1}_{n \neq m}$ . We provide here a somewhat different proof which uses well-chosen Dirac masses to embed the systems in  $X = \mathbb{R}^N$ .

*Proof of Theorem 4.1.* Let us choose  $n$  distinct real numbers  $x_1, \dots, x_n$ . Then there exist continuous functions  $\alpha(x)$  and  $\gamma(x)$  such that

$$\alpha(x_i) = \alpha_i \text{ and } \gamma(x_i) = \gamma_i \text{ for all } i = 1, \dots, n.$$

There are many ways to construct  $\alpha(x)$  and  $\gamma(x)$ ; for instance one can work with Lagrange polynomials and interpolate with a constant value outside of a ball and when the values of  $\gamma(x)$  become close to 0. In particular one can impose that  $\alpha(x)$  and  $\gamma(x)$  are bounded and that  $\gamma(x) > 0$  for all  $x \in \mathbb{R}^N$ , thus meeting Assumption M2.1. Now define the initial data

$$I_0(dx) := \sum_{i=1}^n I_0^i \delta_{x_i}.$$

Clearly, the solution  $(S(t), I(t, dx))$  to (M2.1) can be identified with the solution  $(S(t), I^i(t))$  to (8) by the formula

$$I(t, dx) = \sum_{i=1}^n I^i(t) \delta_{x_i}.$$

Since the set  $\{x : \alpha(x) = \alpha^*\}$  has non-zero measure for  $I_0$ , we are in the situation i) of Theorem M2.2 and Theorem 4.1 can therefore be deduced from Theorem M2.2.  $\square$

*Proof of Theorem 4.2.* As in the proof of Theorem 4.1, we identify the solutions to (9) with the solutions  $I(t, dx)$  to (M2.1) through the formula

$$I_0(dx) := \sum_{i=1}^{+\infty} I_0^i \delta_{x_i}, \quad I(t, dx) = \sum_{i=1}^{+\infty} I^i(t) \delta_{x_i},$$

only this time we choose  $x_i = (\alpha_i, \gamma_i) \in \mathbb{R}^2$ . Because of this particular choice, it is fairly easy to construct  $\alpha(x)$  and  $\gamma(x)$  by the formula

$$\alpha(x_1, x_2) := \alpha^\infty f\left(\frac{x_1}{\alpha^\infty}\right), \quad \gamma(x_1, x_2) := \left| \gamma^\infty f\left(\frac{x_2 - \gamma_0}{\gamma^\infty}\right) \right| + \gamma_0,$$

where  $\alpha^\infty := \sup |\alpha_i|$ ,  $\gamma^\infty := \sup |\gamma_i|$ ,  $\gamma_0 = \inf \gamma_i$  and  $f(x) := \min(\max(x, -1), 1)$ . Then the conclusions of Theorem 4.2 in case i) are given by a direct application of Theorem M2.2. Suppose that the set  $\{i : \alpha_i = \alpha^*\}$  is empty, then we are in case ii) of Theorem M2.2, and we can readily conclude that

$$S(t) \xrightarrow[t \rightarrow +\infty]{} \frac{1}{\alpha^*} \text{ and } \liminf_{t \rightarrow +\infty} \sum_{i=1}^{+\infty} I^i(t) = \liminf_{t \rightarrow +\infty} \int_{\mathbb{R}^2} I(t, dx) > 0.$$

If we assume moreover that  $\alpha_n \rightarrow \alpha^*$  and  $\gamma_n \rightarrow \gamma^*$ , then the maximum of  $\alpha(x)$  on  $\text{supp } I_0$  is attained at a single point  $(\alpha^*, \gamma^*)$  and we can apply Theorem 5.1 to find that

$$I(t, dx) \xrightarrow[t \rightarrow +\infty]{} I^\infty \delta_{(\alpha^*, \gamma^*)},$$

where  $I^\infty = \frac{\theta}{\gamma^*} (\mathcal{R}_0^* - 1)$ . This finishes the proof of Theorem 4.2.  $\square$

## 5 The case of a unique fitness maximum

If the function  $\alpha(x)$  has a unique global maximum in the support of the initial data, then our analysis leads to a complete description of the asymptotic state of the population. This may be the unique case when the behavior of the orbit is completely known, independently on the positivity of the initial mass of the fitness maximizing set  $\{\alpha(x) = \alpha^*\}$ .

**Theorem 5.1** (The case of a unique global maximum). *Let Assumption M2.1 be satisfied. Suppose that the function  $\alpha = \alpha(x)$  has a unique maximum  $\alpha^*$  on the support of  $I_0$  attained at  $x^* \in \text{supp } I_0$ , and that*

$$\mathcal{R}_0(I_0) := \frac{\Lambda}{\theta} \alpha^* > 1.$$

Then it holds that

$$S(t) \xrightarrow[t \rightarrow +\infty]{} \frac{1}{\alpha^*}, \quad d_0(I(t, dx), I^\infty \delta_{x^*}(dx)) \xrightarrow[t \rightarrow +\infty]{} 0,$$

where  $\delta_{x^*}(dx)$  denotes the Dirac measure at  $x^*$  and

$$I^\infty := \frac{\theta}{\alpha^* \gamma(x^*)} (\mathcal{R}_0(I_0) - 1).$$

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