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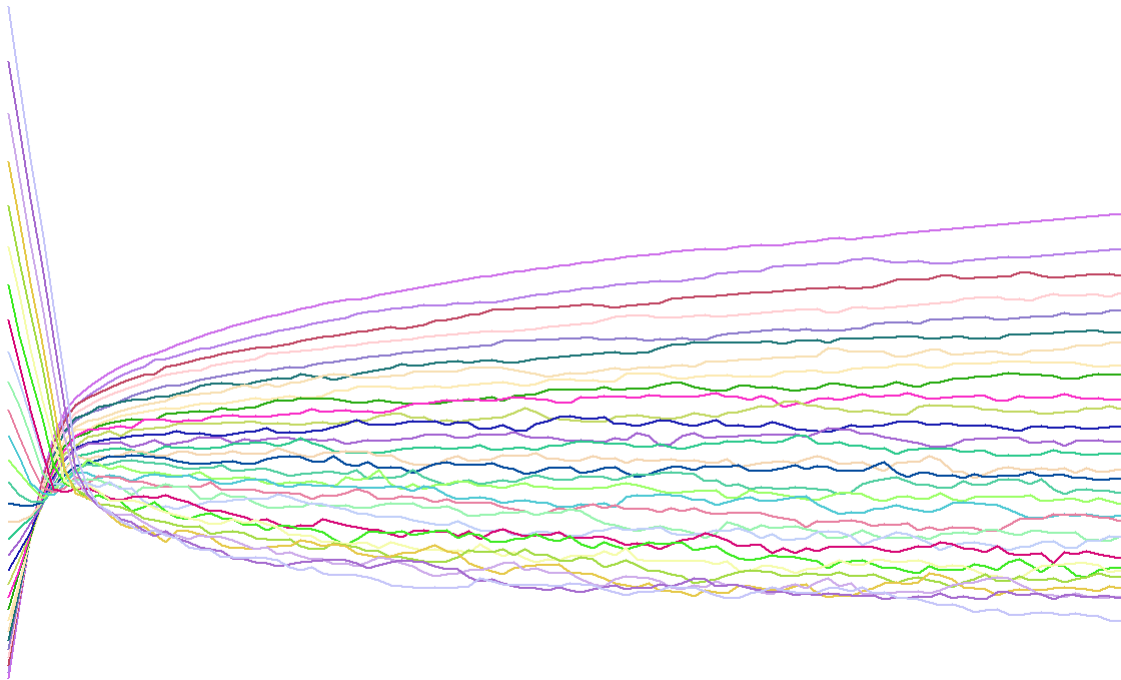
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On the convergence of the Elo rating system for a Bernoulli model and round-robin tournaments

Adrien Krifa, Florian Spinelli, Stéphane Junca



Abstract

This report deals with limit properties of the Elo rating system. It is commonly believed that 30 games are enough to accurately rank a player but it is not always the case [1]. The Bernoulli model is first presented for 2 players. Then we'll study the $\mathcal{N} > 2$ players case by numerically simulating Round-robin tournaments between the players. In all the case the Elo rating system correctly sorts people's strength.

AMS Classification: 60J20, 91D30

Key words: Ranking, Elo, rate, Markov chain, law of large numbers, ergodicity, least squares.

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1 Introduction

The Elo rating system is a famous ranking system created by Physicist Arpad Elo around 1970 to rank chess players [5]. It is now widely used to classify participants, teams, objects or anything that can be compared in pairs. Many online games use this system to rank players and match players of similar strength in order to have more interesting games.

In this report we mainly study the properties of the Elo model for a fixed finite number of players of constant strengths. That means that player improvement or regression is

not taken into account. In this simple case, a natural Bernoulli model suffices to classify the players. The Bernoulli model is used to simulate many games when "everyone plays everyone" and to study the convergence and the convergence rate of the Elo ratings. The Elo rating is then seen as a random variable updated after each tournament. The Elo rating system does rank players and is an easy tool to use as its formula is very basic. In this report, we have shown multiple results numerically.

- For 2 players we obtain numerical and theoretical results.
 - the logistic function of the expected values of the Elo difference of two players \mathbb{D}_{ij} = the expected values of the logistic function of \mathbb{D}_{ij} which converges towards p_{ij} the probability that player i wins against player j
 - we found the function towards which the Elo seems to converge to when $n \rightarrow \infty$
- For 3 or more players, many numerical tests show the right convergence of the Elo.

The report is organized as follows. First, the Elo rating system is recalled for two players. Then the natural Bernoulli model is given and used in round-robin tournaments. The section 2 deals with 2 players, the section 3 with 3 players and the section 4 with more, typically 100 players. Some conclusions are listed at the end.

1.1 The Elo system for 2 players

Consider two players i and j , $i \neq j$ with respective Elo rankings R_i^n and R_j^n at the discrete time n . The result or score of the match is S_{ij}^n . The update of the ranking proceeds as follows :

$$R_i^{n+1} = R_i^n + K (S_{ij}^n - b(R_i^n - R_j^n)), \quad (1)$$

$$R_j^{n+1} = R_j^n + K (S_{ji}^n - b(R_j^n - R_i^n)). \quad (2)$$

- K is the K-factor, $K = 20$ is commonly used. It satisfies the contractivity property to insure some convergence of the Elo,

$$K \sup_{x \in \mathbb{R}} b'(x) \ll 1. \quad (3)$$

- The scores satisfy $S_{ij}^n \in [0, 1]$. We use $S_{ij}^n = 0$ when i loses n th and $S_{ij}^n = 1$ when i wins. Thus, there is a skew-symmetric relation,

$$S_{ij}^n + S_{ji}^n = 1. \quad (4)$$

- The bonus-malus function $b(\cdot)$ which predicts the mean score of the match :

$$b \in [0, 1]. \quad (5)$$

Usually it is a distribution function. This function is also increasing, $\forall x \in \mathbb{R}$,

$$0 < b'(x). \tag{6}$$

The bigger difference in Elo ratings two players have, the more important is the mean score expected.

This function only depends on the difference in players' ratings, which is the main ingredient of the Elo.

The conservation of the total sum of the Elo rating is granted and a constant,

$$R_i^{n+1} + R_j^{n+1} = R_i^n + R_j^n, \tag{7}$$

since $b()$ satisfies a similar relation to the score (4),

$$b(x) + b(-x) = 1. \tag{8}$$

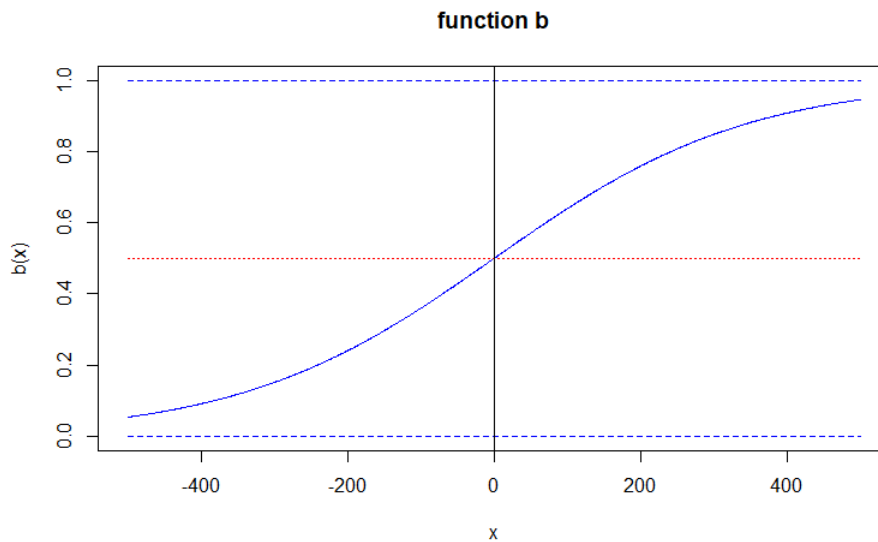
The function usually used is the logistic function,

$$b(x) = \frac{1}{1 + 10^{-x/400}}. \tag{9}$$

This choice satisfies a transitivity property for more than 2 players and will be discussed later.

$$\sup b' = b'(0) = \ln(10)/1600 \approx 0.0014 \tag{10}$$

Here is a plot of the b function on a $[-500, 500]$ interval.



1.2 The Bernoulli model

The natural mathematical way, without using the Elo model to measure the strength of the players is to know p_{ij} the probability that the player i beats the player j , $1 \leq i, j \leq \mathcal{N}$. A natural convention is to set $p_{ii} = 1/2$.

These quantities are easy to estimate, a priori, using the Law of Large Numbers (LLN). This $\mathcal{N} \times \mathcal{N}$ matrix \mathbb{P} satisfies a symmetry.

$$p_{ij} + p_{ji} = 1, \quad \mathbb{P} + \mathbb{P}^\top = \mathbb{U}, \quad (11)$$

where $\mathbb{U}_{ij} = 1$ (for every i, j) is the matrix of the orthogonal projector on the line $x_1 = x_2 = \dots = x_{\mathcal{N}}$. In other words $\mathbb{P} - (1/2)\mathbb{U}$ is a skew-symmetric matrix.

There are only $(\mathcal{N}^2 - \mathcal{N})/2$ free parameters for the matrix \mathbb{P} . The other parameters can be deduced by symmetry.

The ranking of the players in a round-robin tournament (everyone meets everyone) can be predicted by knowing what is their **power**, the expected number of wins for each player, that is for player i ,

$$p_{i.} = \sum_{k \neq i} p_{ik}. \quad (12)$$

Notice that $p_{i.}/(N - 1)$ represents a mean probability of win in a round-robin tournament (RRT). And we can assume that the label of any player corresponds to their ranking, (else the players are sorted according to their power)

$$p_1. \leq p_2. \leq \dots \leq p_{\mathcal{N}}.$$

When player i meets player j for the n th encounter, the score is the random variable S_{ij}^n following the Bernoulli law $\mathcal{B}(p_{ij})$. All the scores are assumed to be independent except for the symmetric relation $S_{ij}^n + S_{ji}^n = 1$. Thus, the matrix $S^n = (S_{ij}^n)_{1 \leq i, j \leq \mathcal{N}}$ has the same symmetry as the matrix \mathbb{P} , $S^n + (S^n)^\top = \mathbb{U}$.

1.3 The round-robin tournament

In this report, we would like to study the Elo rating system with the most stable and fairest tournament type that can possibly represent the strengths of each player. The elimination tournament type is not considered here for multiple reasons (e.g. a lot of match-ups wouldn't be covered, the best player could very well lose early in the tournament, etc). The round-robin tournament is considered as the fairest tournament to rank all players without using any ranking system (such as the Elo rating system). Counting the number of wins of each player is enough as long as we have a big enough sample. Indeed the expected number of wins is exactly the power p_i . as discussed with the Bernoulli model before.

Here everyone plays everyone else once. Then the Elo rating is updated,

$$R_i^{n+1} = R_i^n + K \sum_{j \neq i} (S_{ij}^n - b(R_i^n - R_j^n)) \quad (13)$$

For online video games, the Elo is often updated after each match. For a large number of matches it is expected that it corresponds to updating the Elo after each tournament as long as we have as many matches [7], we have also shown this graphically in section 3. Moreover, our numerical simulations show that the Elo is less fluctuating when it is updated after each match than after each tournament.

The report is organized as follows. In section 2 the Elo model is presented for 2 players. In particular the important property of mass conservation is presented. In section 3 the problem of whether the Elo has a transitivity property or not is pointed out for 3 players. In section 4 the case of many players ($\mathcal{N} \geq 4$) is studied.

2 ‘2’ players

2.1 Mathematical remarks for 2 players

Here we mainly study the simplest case, that is when the ‘strength’ of the players is constant and the results of the matches are independent. Therefore, p_{ij} , the probability that player i wins against player j is assumed to be constant. Of course, this quantity is enough to know who is the stronger player, that is player i if $p_{ij} > 1/2$, player j otherwise. The point of the Elo ranking is, in practice, to predict the outcome of a given match. In this report, we will check this expectation on the Elo ranking.

That means that $(S_{ij}^n)_n$ are independent Bernoulli random variables $\mathcal{B}(p_{ij}^n)$ and (R_i^n, R_j^n) is a Markov chain [11].

Reduction to a scalar discrete process (D_{ij}^n) For two players, the 2-dimensional Elo process can be reduced to a scalar one. Let $D_{ij}^n = R_i^n - R_j^n$, due to the fact that the expected result of an encounter depends only on the strength difference, the process (D_{ij}^n) satisfies the relation, using (1), (2), (4), (8),

$$D_{ij}^{n+1} = D_{ij}^n + 2K(S_{ij}^n - b(D_{ij}^n)), \quad D_{ij}^0 = R_i^0 - R_j^0. \quad (14)$$

All of the information is contained in the process (D_{ij}^n) , through the conservation of the sum of the Elos (7),

$$2R_i^n = R_i^0 + R_j^0 + D_{ij}^n, \quad (15)$$

$$2R_j^n = R_i^0 + R_j^0 - D_{ij}^n. \quad (16)$$

The set of the difference values \mathcal{D} Let \mathcal{D} be the set of all possible values of the process. This set can be built by recurrence. Let \mathcal{D}^n be the set of all values of the random variable D_{ij}^n . That means that all values in this set have a non-zero probability to be a value of D_{ij}^n (if $0 < p = p_{ij} < 1$).

$$\mathcal{D} := \bigcup_{n \geq 0} \mathcal{D}^n. \quad (17)$$

Here \mathcal{D} is a countable set as it is a countable union of countable sets since $\{D_{ij}^0\}$ is a singleton. If D_{ij}^0 is a continuous law (its distribution function is continuous) then \mathcal{D} is not countable.

Let us define the two increasing functions

$$g_1(x) = x + 2K(1 - b(x)), \quad (18)$$

$$g_0(x) = x - 2Kb(x). \quad (19)$$

under the contractivity assumption,

$$2K \sup b' < 1. \quad (20)$$

Now, the set \mathcal{D}^n is built by recurrence,

$$\mathcal{D}^0 = \{D_{ij}^0\}, \quad (21)$$

$$\mathcal{D}^1 = \{D_{ij}^0 + 2K(1 - b(D_{ij}^0)), D_{ij}^0 + 2K(0 - b(D_{ij}^0))\} = \{g_1(D_{ij}^0), g_0(D_{ij}^0)\}, \quad (22)$$

$$\mathcal{D}^{n+1} = g_1(\mathcal{D}^n) \cup g_0(\mathcal{D}^n). \quad (23)$$

That means that the size of this set is (usually) multiplied by 2 every time,

$$\text{cardinal } \mathcal{D}^n \leq 2^n$$

(and usually with an equality). As a consequence, the Elo ranking lives in an infinite, countable and unbounded set.

Proposition 2.1 *The set \mathcal{D} is infinite and unbounded.*

Proof: Notice that it is possible that a new value in \mathcal{D}^{n+1} can be an ancient value in \mathcal{D}^n . Indeed, \mathcal{D} is infinite because it is unbounded

Now, it comes down to proving that \mathcal{D} is unbounded. Let $M^n = \max \mathcal{D}^n$, $M^0 = D_{ij}^0$, $M^1 = g_1(M^0)$ and

$$M^{n+1} = g_1(M^n).$$

To have the maximum at time n , the player i has to win all of the n matches. Therefore, $P(D^n = M^n) = p^n > 0$ becomes a very rare event (with $0 < p < 1$). Since b is increasing (6) and $b \in [0, 1]$ then, $\forall x$, $0 < b(x) < 1$. Hence, $\forall x$, $g_1(x) > x$, g_1 has no fixed point, thus, $(M^n)_n$ is increasing and $M^n \rightarrow +\infty$. Which means that \mathcal{D} has no upper bound which concludes the proof. Notice that, in the same way, $\forall x$, $g_0(x) < x$ and \mathcal{D} has no lower bound. \square

Proposition 2.2 (Logarithmic growth of $\|R_i^n\|_{L^\infty(\Omega)}$)

$$\max_{\omega \in \Omega} R_i^n(\omega) \sim \frac{\ln n}{2c}$$

Remark 2.1 *The expansion of the set \mathcal{D}^n is very slow. Usually, b is concave on \mathbb{R}^+ and $b'(x) \rightarrow 0$ when $x \rightarrow +\infty$, thus, for the logistic function (9) $g_1(x) - x \sim \exp(-cx)$, $c = \ln 10/400 \simeq 1/200$. It suggests that getting a very big Elo by chance requires an extraordinarily large number of games.*

Proof: We can estimate $\max D_{ij}^n$. The maximal growth corresponds to player i winning all of their matches. This occurs with probability p_{ij}^n which is exponentially small except if $p_{ij} = 1$. Thus, the deterministic sequence $d_{n+1} = g_1(d_n)$ is considered using the notations of the proof of the proposition 2.1. The sequence (d_n) diverges towards $+\infty$, since $g_1(x) > x, \forall x$. At $+\infty$, $g_1(x) = x + 2K \exp(-cx) + O(\exp(-2cx))$, so $d_{n+1} \leq d_n + C \exp(-cd_n)$, with a constant $C > 2K$. Notice that $d_n = o(n)$ since $C \exp(-cd_n) \rightarrow 0$. It can be also proved that $d_n = o(n^a)$ for all $a > 0$ since $(n+1)^a - n^a \sim a/n^{1-a} \gg \exp(-cn^a)$. Let r_n be $d_n/\ln n$, $n > 1$, so $r_n = o(n^a/\ln n)$. We claim that (r_n) is bounded. The sequence is already positive for large n since $d_n \rightarrow +\infty$, only an upper bound for (r_n) is necessary. The sequence r_n satisfies the inequality $\frac{\ln(n+1)}{\ln n} r_n \leq r_n + \frac{C \exp(-cd_n)}{\ln n}$ which can be rewritten

$$\begin{aligned} r_{n+1} - r_n &\leq \frac{1}{\ln n} \left(C \exp(-cd_n) - \frac{r_n}{n} + O\left(\frac{r_n + \exp(-cd_n)}{n^2}\right) \right) \\ &\leq \frac{1}{\ln n} \left(C \exp(-cd_n) - \frac{r_n}{n} + O(n^{-3/2}) \right), \end{aligned}$$

taking $0 < a = 1/2$. Now, let A such that $cA > 1$, assume that n is large enough and $r_n > A$, then $d_n > A \ln n$ and $\exp(-cd_n) < 1/n^{cA}$. That means that $r_{n+1} - r_n < 0$. Moreover the right side is equivalent to a term of a divergent sum since $r_n/n > A/n$, thus in finite time r_n becomes less than A . This is enough to conclude that $r_n = O(1)$.

This asymptotic bound is valid for all $A > 1/c$ which means that $\limsup r_n \leq 1/c$. Thus, estimating $\liminf r_n$ in a similar way, we can prove that $r_n \rightarrow 1/c$ and

$$\max D_{ij}^n \sim \frac{\ln(n)}{c}.$$

The Factor $1/2$ in the proposition comes from (15). \square

Ergodicity The ergodicity reduces the cost of numerical simulations. A first ergodic result is the following :

Proposition 2.3 (Ergodicity of $b(R_i^n - R_j^n)$)

$$\frac{1}{N} \sum_{0 \leq n < N} b(R_i^n - R_j^n) = p_{ij} + O\left(\sqrt{\frac{\ln \ln N}{N}}\right) \quad a.s. \quad (24)$$

The temporal mean of the expected scores defined by the rating difference of the players converges towards the real expected score.

Proof: The recurrence relation (14) can be rewritten as follows,

$$D_{ij}^{n+1} - D_{ij}^n = 2K(S_{ij}^n - b(D_{ij}^n)).$$

Adding this relation for $n = 0, \dots, N - 1$ yields,

$$D^N - D_{ij}^0 = 2K \sum_{0 \leq n < N} (S_{ij}^n - b(D_{ij}^n)).$$

Using the law of iterated logarithm for $\sum_{n < N} S_{ij}^n$ (or the central limit theorem) [6], the previous bound for D_{ij}^n and dividing by N yields the announced result,

$$\frac{1}{N} \sum_{0 \leq n < N} b(D_{ij}^n) = p_{ij} + O\left(\sqrt{\frac{\ln \ln N}{N}}\right) + \frac{O(\ln N)}{2K N}.$$

With the Central Limit Theorem, the error has the order a little bit smaller $O(N^{-1/2})$, but, with a confidence less than 100%. \square

We also expect to have ergodicity to compute the limit of $E(R_i^n)$, [11].

Conjecture: Let D_{ij}^∞ be the limit in law of D_{ij}^n and $p_{ij} = b(\rho_i - \rho_j)$, then

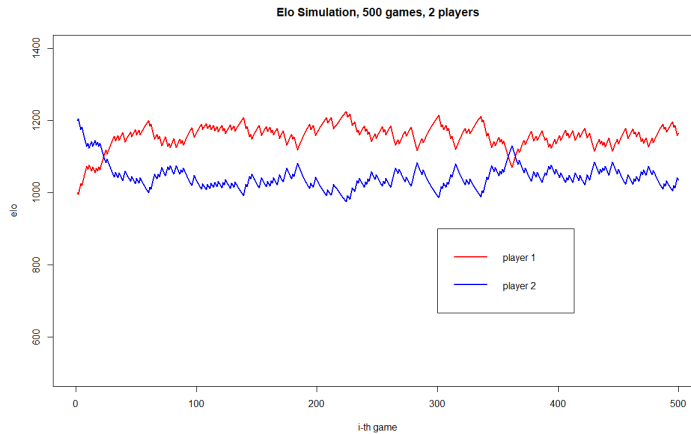
$$b(E(D_{ij}^\infty)) = E(b(D_{ij}^\infty)), \tag{25}$$

$$E(D_{ij}^\infty) = \rho_i - \rho_j \tag{26}$$

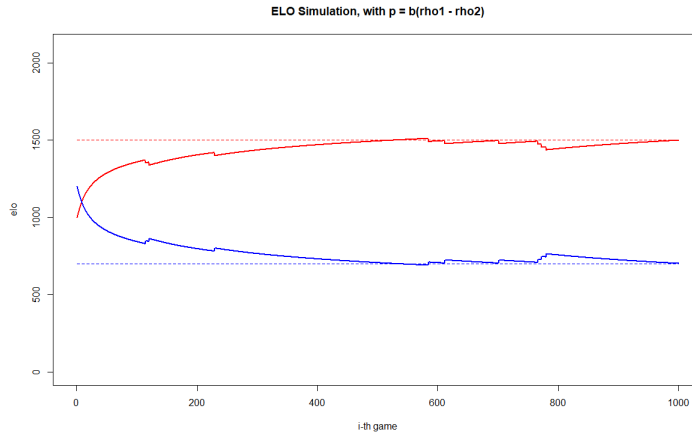
Thus, if $R_i^0 + R_j^0 = \rho_i + \rho_j$ then, $\lim_n E(R_i^n) = \rho_i$.

Notice that Proposition 2.3 suggests $E(b(D_{ij}^\infty)) = p_{ij}$. The commutation $b \circ E = E \circ b$ applied to D_{ij}^∞ for the nonlinear function $b(\cdot)$ is surprising. The law limit of D_{ij}^∞ has a symmetry with respect to the function $b(\cdot)$.

2.2 Simulations for 2 Players

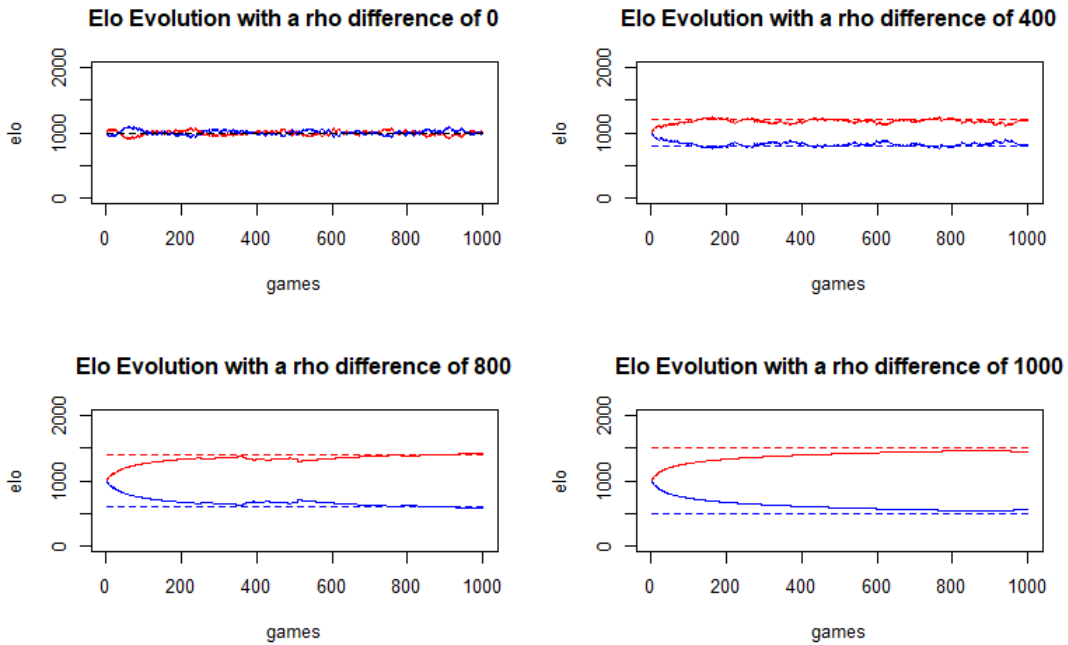


This graph shows the Elo evolution over 500 games of two players A and B with 1200 and 1000 initial Elo respectively, player A has a 60% chance of winning against player B



This graph shows the Elo evolution over 1000 games of two players A and B, A has a $p = b(\rho_1 - \rho_2)$ chance of winning against B. ρ is meant to be the "true" (theoretical) force of a player; $b(\rho_1 - \rho_2)$ is what we will name the "natural" probability of A winning against B. Here we have $\rho_1 = 1500$, $\rho_2 = 700$ and the players start with 1000 and 1200 Elo respectively.

Let's now see how the convergence rate evolves if ρ varies for both players, starting with the same initial condition : a 1500 Elo rating.



We notice that the bigger the difference in ρ_1 and ρ_2 is, the slower the convergence rate becomes.

Some convergences in law were proven in [1, 2, 8, 10]. **Let's assume that** $\lim_n E(R_i^n)$ **converges.** Since the Elo formula of the difference $D_{ij}^n = R_i^n - R_j^n$ only depends on $K, p_{ij}, b(\cdot), D_{ij}^0$. We guess that the limit does not depend on the initial data D_{ij}^0 , but due to the total sum conservation (7) it does depend on $M^0 = R_i^0 + R_j^0$. Thus, the limit only is determined by K, p_{ij} and M^0 . For $K = 20$ fixed we conjecture formula (27). The other equalities are then a direct consequence of (27).

$$\lim_n E(D_{ij}^n) = 2G(p_{ij}), \quad (27)$$

$$\lim_n E(R_i^n) = M^0/2 + G(p_{ij}), \quad (28)$$

$$\lim_n E(R_j^n) = M^0/2 - G(p_{ij}), \quad (29)$$

Notice that for the equality (29) the term $+G(p_{ij})$ is expected. However, it can be replaced by $-G(p_{ij})$ since, from the conservation of the total Elo (7) and $p = p_{ij}$ we have

$$G(p) + G(1 - p) = 0. \quad (30)$$

This relation is valid for all $p \in [0, 1]$. Indeed, $G(p_{ij})$ is the limit of $E(R_i^n)$ when $M^0 = 0$.

We guess that $G(1/2) = 0, G(1) = +\infty, G(0) = -\infty$

We want to estimate the unknown function G . From the symmetry (34), and the expected singular behavior the study is limited to $]1/2, 1[$.

A parameter \mathcal{N} of discretisation of this interval $]1/2, 1[$ is chosen and $h = 2^{-\mathcal{N}-1}$. $p_k = 1/2 + kh, k = 1, \dots, 2^{\mathcal{N}}$. This choice is motivated to have nested grid.

Now, ρ_i^k will be estimated for all $p_{ij} = p_k$.

Initial Elo The convergence of this process is slow. To improve the convergence, it is important to cautiously choose the initial data.

Taking the expectation in the Elo formula (1) yields,

$$E(R_i^{n+1}) - E(R_i^n) = K (p_{ij} - E(b(R_i^n - R_j^n))). \quad (31)$$

Passing to the limit, the left side vanishes and

$$p_{ij} = \lim_n E(b(D_{ij}^n)) \quad (32)$$

$\text{Var } D_{ij}^n$ is related to K , indeed $Kb'(0) \ll 1$. A relatively 'small' variance is expected, a natural choice is to take

$$b(D_{ij}^0) = b(\rho_i - \rho_j) = p_{ij}. \quad (33)$$

Notice that for $p_{ij} = 1/2$ we recover the optimal choice $D_{ij}^0 = 0 = G(p_{ij})$ But for p_{ij} not limited to $1/2$, this choice may be worse.

Another good possibility is to start with $D_0 = 0$ for $1/2$ and p_1 and then to proceed with the well known **continuation method**. $p = 1/2, R_i^0 = R_j^0$ is the right choice. After the computation of ρ_i^k , to compute ρ_i^{k+1} for the initial data we use the final limit of the previous case. $R_i^0 = \rho_i^k, R_j^0 = \rho_j^k$.

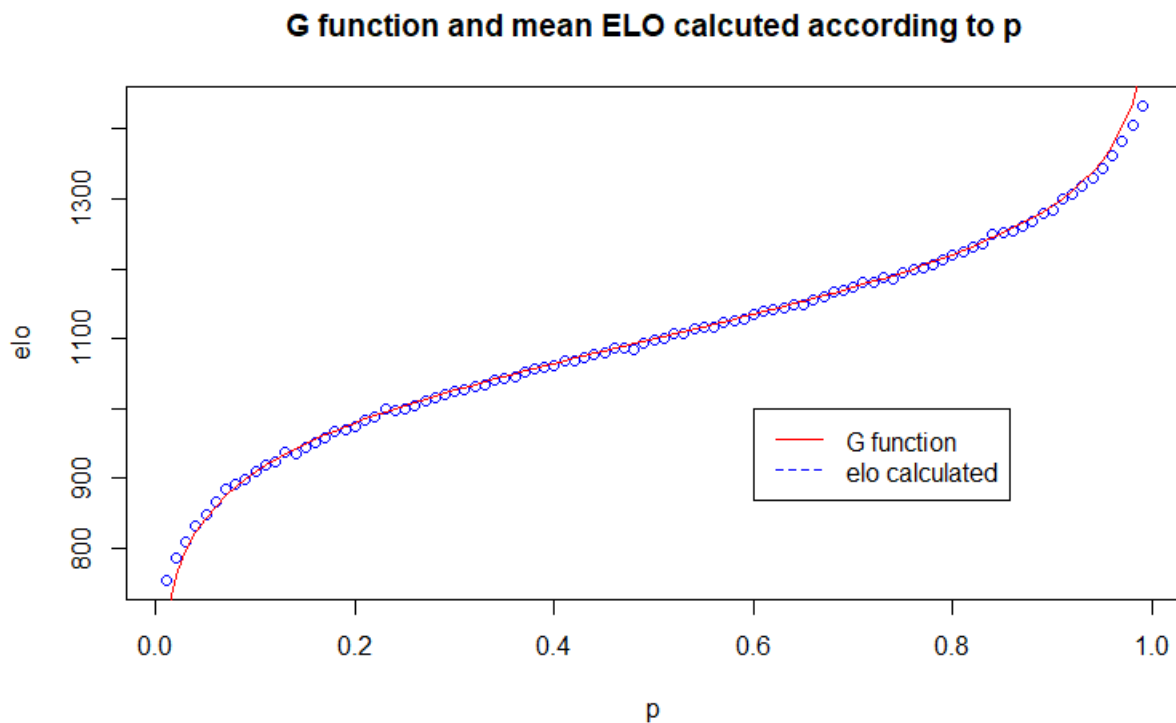
The critical non random case $p_{ij} = 1$ This case is critical. For the Elo ranking we simply have $R_i^n = R_i^0$ thus the convergence is effective. But for $p_{ij} \simeq 1$ and $p_{ij} < 1$, we expect that $G(p_{ij}) \gg 1$.

If the G function that we will discuss in a few paragraphs is the correct one, then $G(1 - 0) = +\infty$

Mean estimates Temporal mean I: A nice way to calculate $\lim_n E(R_i^n)$ is described here :

$$\lim_n E(R_i^n) \simeq \frac{1}{m} \sum_{n=1}^m R_i^n$$

Same approximation, with a smaller cost $2m$. This is valid if ergodicity occurs for the Elo ranking [11].

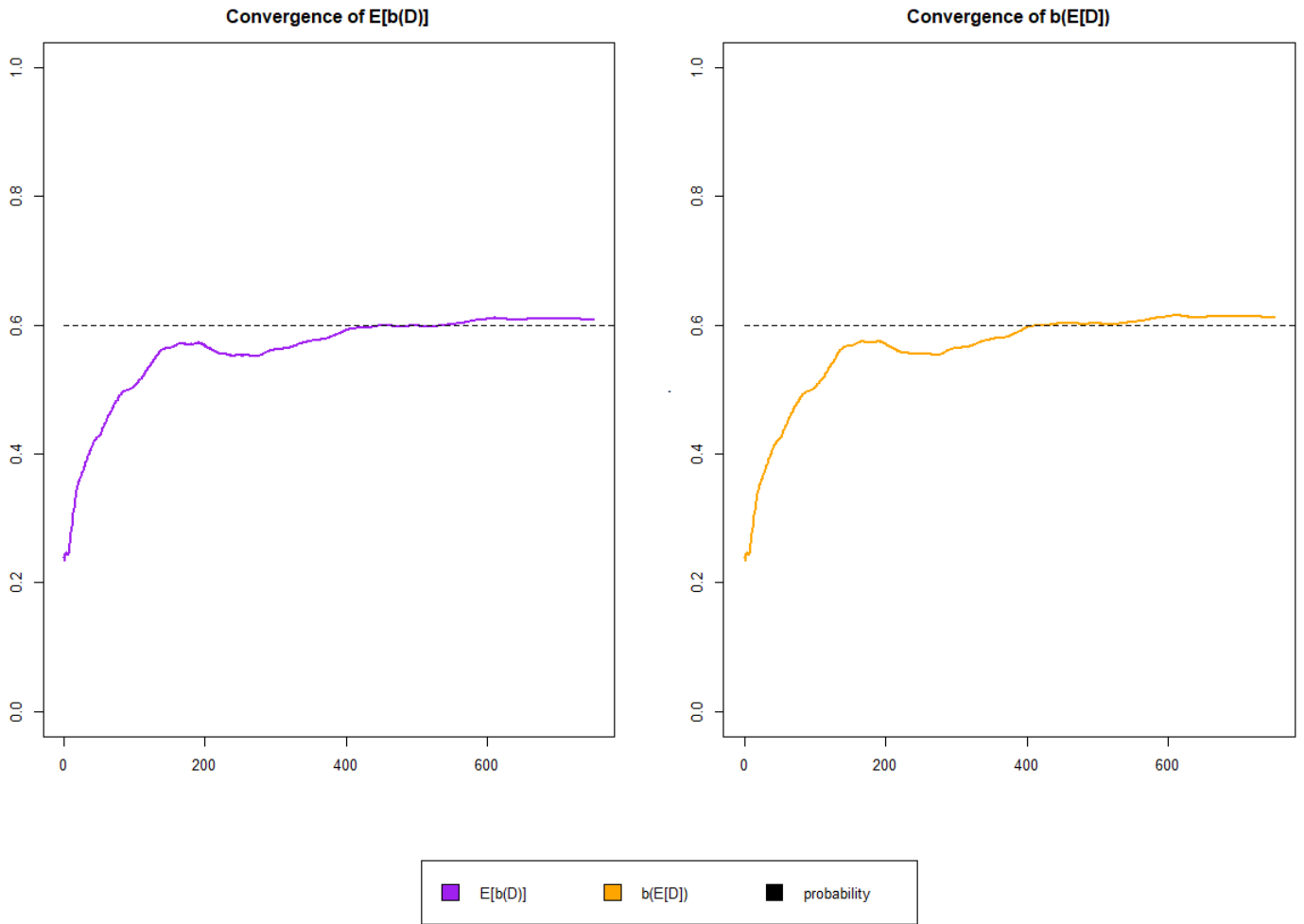


The G function should predict the Elo a player gets after a big sample of matches, given their initial elo and their opponent's elo. To do that we first simulated games between two players A and B of 1200 and 1000 elo ratings respectively, those players would play each other over 1000 games 10 times with a probability of $p = 0.01$ being the chance of A beating B. We would then average those 10 final values to get a good idea of what player A and B's Elos would be. Then the probability would increase by 0.01 and the process would

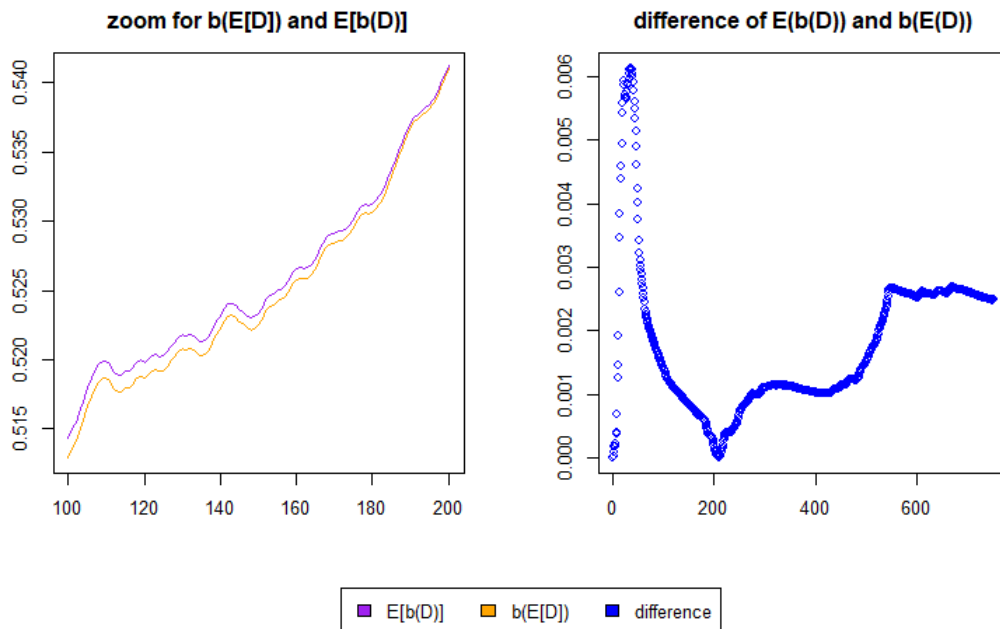
repeat itself until we reach $p = 0.99$. We tried approaching the G function in different ways but what seemed to work for us was first finding the reciprocal function of b . We saw that it almost matched the results that we had gotten, it turns out that by dividing the reciprocal of the logistic function b we would get a function that matched almost perfectly our results. Thus we concluded that

$$G(x) = -200 \log_{10}((1-x)/x) = b^{-1}(x)/2 \tag{34}$$

2.3 $\mathbb{E}[b(D)] = b(\mathbb{E}[D])$



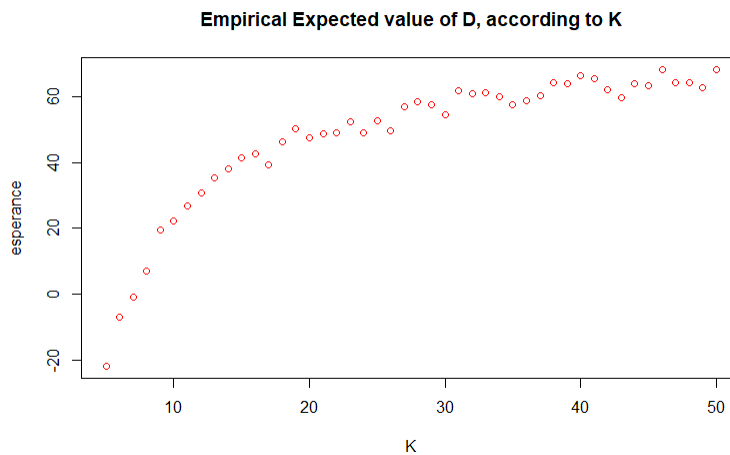
We noticed an equality between $E(b(D))$ and $b(E(D))$; in order to do find this result, we empirically simulated these (while also verifying that they converge towards p) by using the Law of Large Numbers.



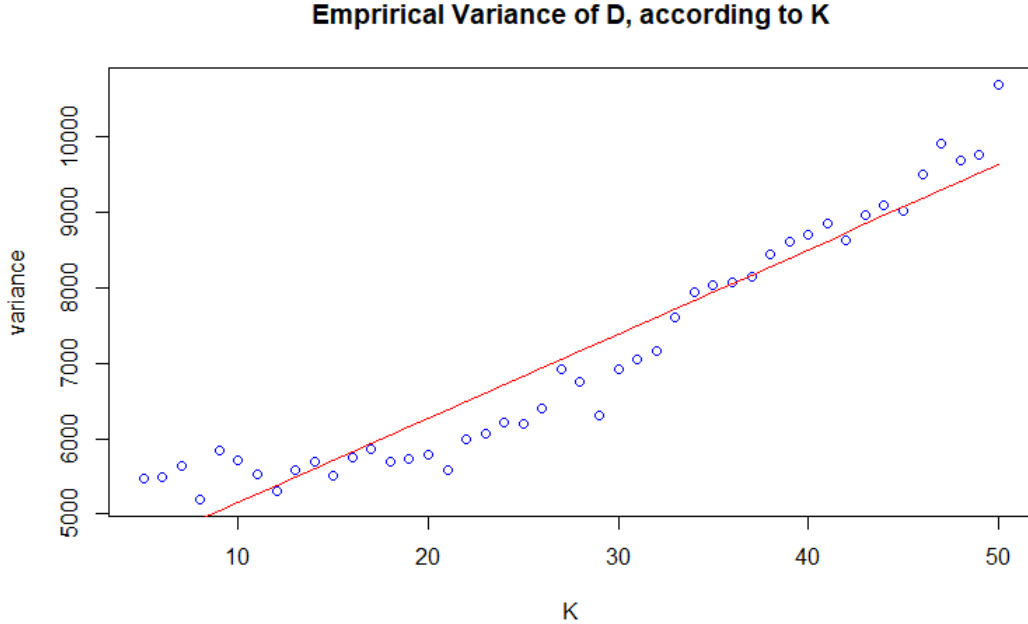
In order to make sure that the equality is true we zoomed into a range of 100 games and we noticed that the graphs were very similar to each other. To be more rigorous we then plotted the difference between $E(b(D))$ and $b(E(D))$ and we noticed that the maximum error over that 750 games sample was of about 0.006 .

2.4 Variance and expected value of the difference D

We were interested in the expected value and the variance of D varying with K . To do so, we simulated results with the use of the LLN, and here are the plots we got :



For small values of K , the expected value seems to vary, this is due to the convergence rate which is too slow. Once we get a big enough K the expectancy doesn't seem to depend on K anymore. We conjecture that for an infinite number of games, the expected value of D stays the same no matter which K is chosen.



The variance seems to be increasing in a linear way while K varies. To check this hypothesis in a numerical way, we used the Least Squares Method and it does seem to be the case.

3 $\mathcal{N} = 3$ players

In this section the case of 3 players is considered. Now, we want to test the validity of the Elo rating system. After many encounters the Elo rating has to correspond to the real strengths of the players in order to be practical. Just like before we will test the Elo rating system with the use of round-robin tournaments.

The Elo is initialized at time $n = 0$ so $R_i^0; i = 1, 2, 3$ are fixed. Then players play a round robin tournament at each time n . The Elo ranking is updated after each round-robin tournament.

$$R_i^{n+1} = R_i^n + K (S_{ij}^n + S_{ik}^n - b(R_i^n - R_j^n) - b(R_i^n - R_k^n)), \quad (35)$$

where $i \neq j \neq k \neq i, i, j, k \in \{1, 2, 3\}$.

Due to the symmetrical relations the total sum of the ranking is conserved,

$$\sum_k R_k^n = \sum_k R_k^0. \quad (36)$$

In order to study Elo rankings for more than 2 players, we decided to use three kinds of simulations :

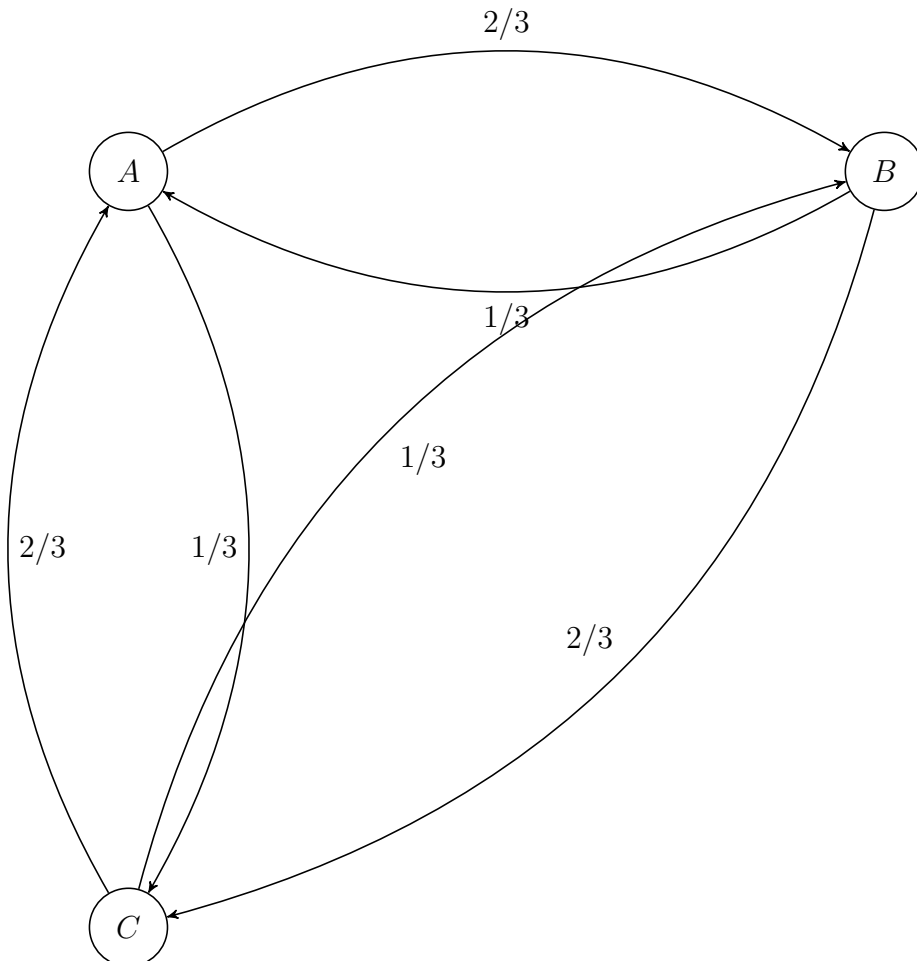
- Round-Robin tournament : updated after each tournament
- Round-Robin tournament : updated after each game
- Matchmaking : a random pair of players play against each other while all of the others don't.

In the end we mainly used the Round Robin tournament type in which rankings are updated after each tournament is over.

We now use the probability matrix defined in 1.2.

Out of convention, we will have $p_{ii} = 0.5$, even though we won't make players face themselves.

To illustrate our different modelings for the $N = 3$ players case, we will use the matrix generated by the following graph :

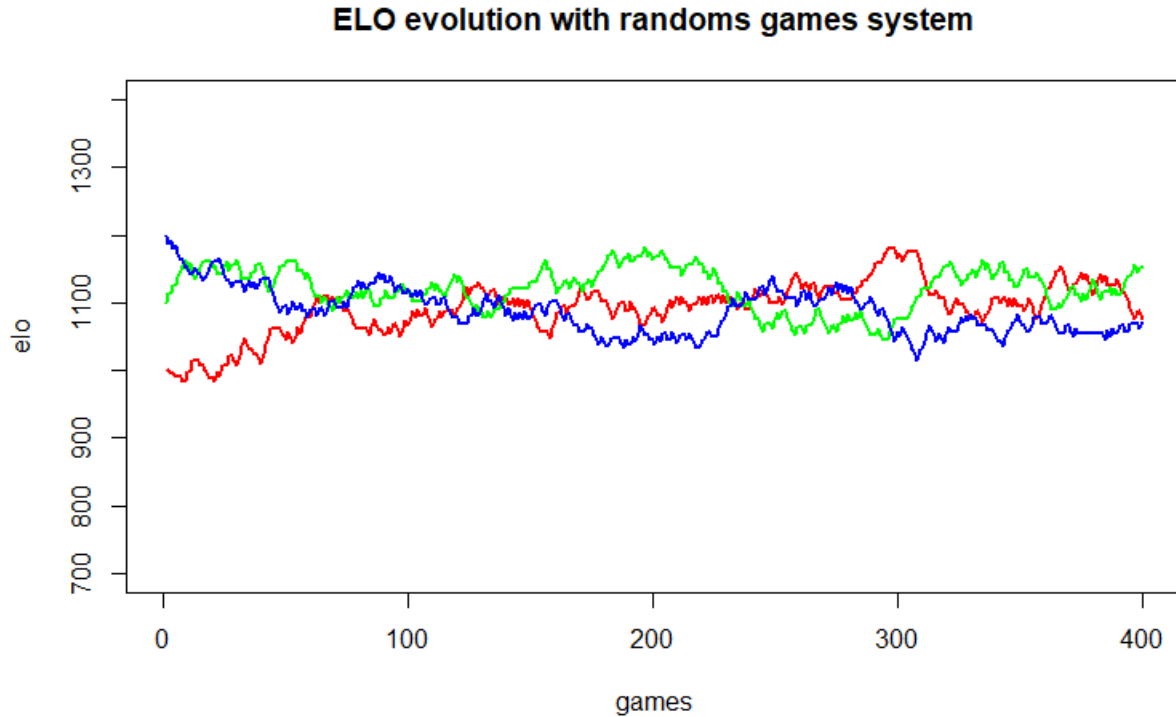


With this \mathbb{P} matrix we hope that players will end up with the same strength, as they all have the same ρ .

In this example $p_{12} = p_{13} = p_{23} = 2/3$. This case is contradictory to the Arpad Elo assumption (37). This case is critical since, for a round robin tournament, all the players have the same power, $p_i = 4/3, i = 1, 2, 3$.

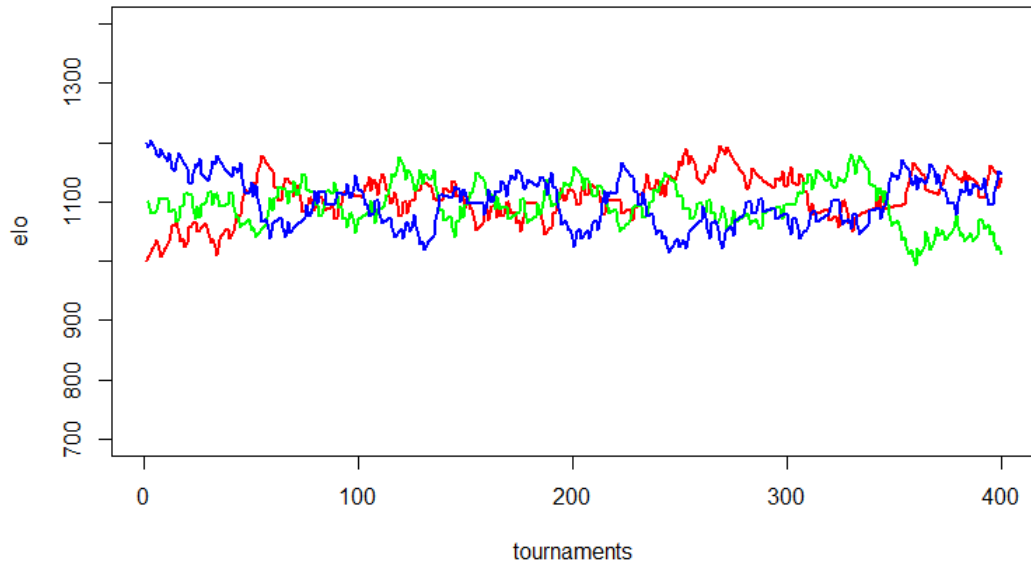
The next graphs show the Elo evolution with the use of 3 different methods over 400 games of three players A, B and C with 1200, 1100 and 1000 initial Elo respectively, probabilities are determined by the \mathbb{P} matrix, here we have :

3.1 Matchmaking



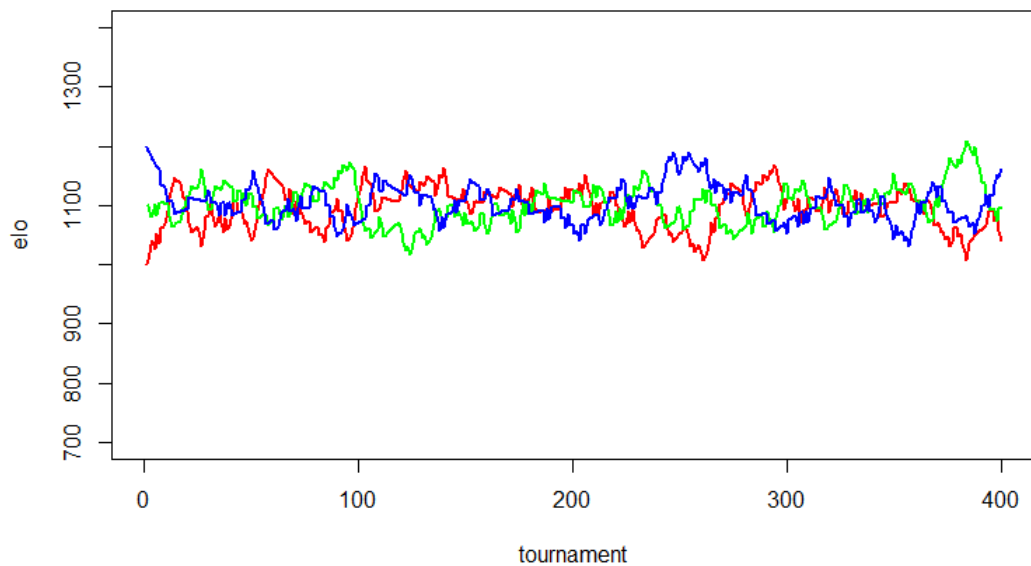
3.2 Round-Robin tournament : updated after each tournament

ELO evolution after 400 RR tournaments (updated after each tournament ends)



3.3 Round-Robin tournament : updated after each match

ELO evolution after 400 RR tournaments (updated after each single match)



3.4 \mathbb{P} transitive

We consider that the \mathbb{P} matrix satisfies Arpad Elo's assumption (37). The simplest case is considered. The real strengths of the 3 players are fixed and ordered. The Arpad Elo assumption means that the probability that player i beats player j only depends on the difference of the 'real' (theoretical) ratings, $\rho_1 \geq \rho_2 \geq \rho_3$, fix the probabilities,

$$p_{ij} = b(\rho_i - \rho_j). \quad (37)$$

The Elo system is invariant by translation so, in our simulations, we always assume that

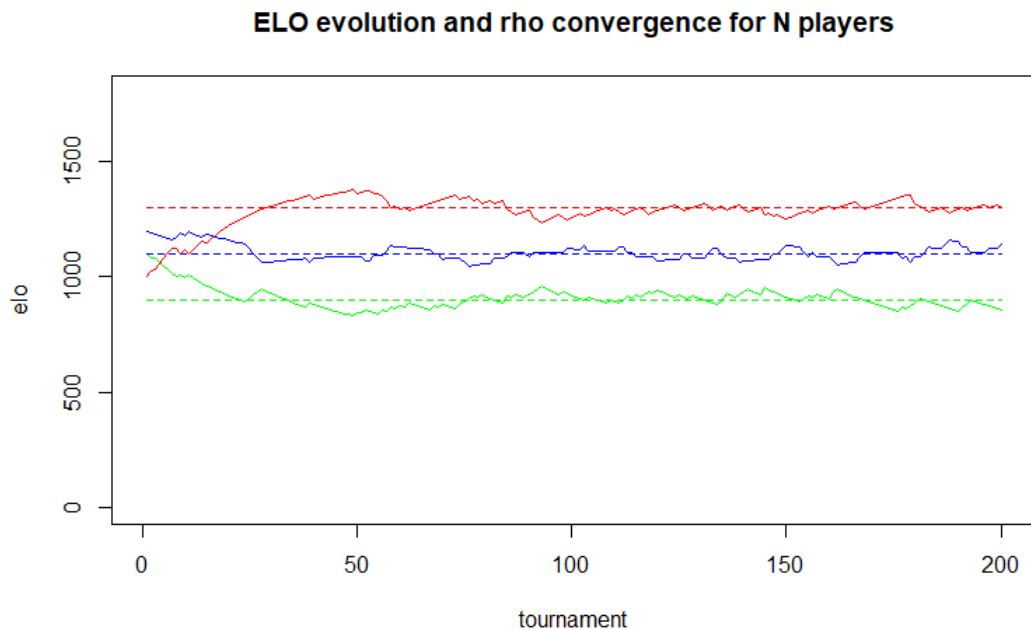
$$R_1^0 + R_2^0 + R_3^0 = \rho_1 + \rho_2 + \rho_3. \quad (38)$$

That means that there are only 2 free parameters (ρ_1, ρ_2, ρ_3) (2=3-1 due to the invariance by translation) and not 3 for the \mathbb{P} matrix. For instance, we can fix only 2 differences δ_{12}, δ_{13} where $\delta_{ij} = \rho_i - \rho_j$, (then $\delta_{ji} = -\delta_{ij}$) to know all the other possible differences of 'theoretical' ratings $\delta_{23} = \delta_{21} + \delta_{13} = \delta_{13} - \delta_{12}$.

Therefore the Elo hypothesis (37) reduces the natural choice of the \mathbb{P} matrix from a set of dimension 3 to a set of dimension 2.

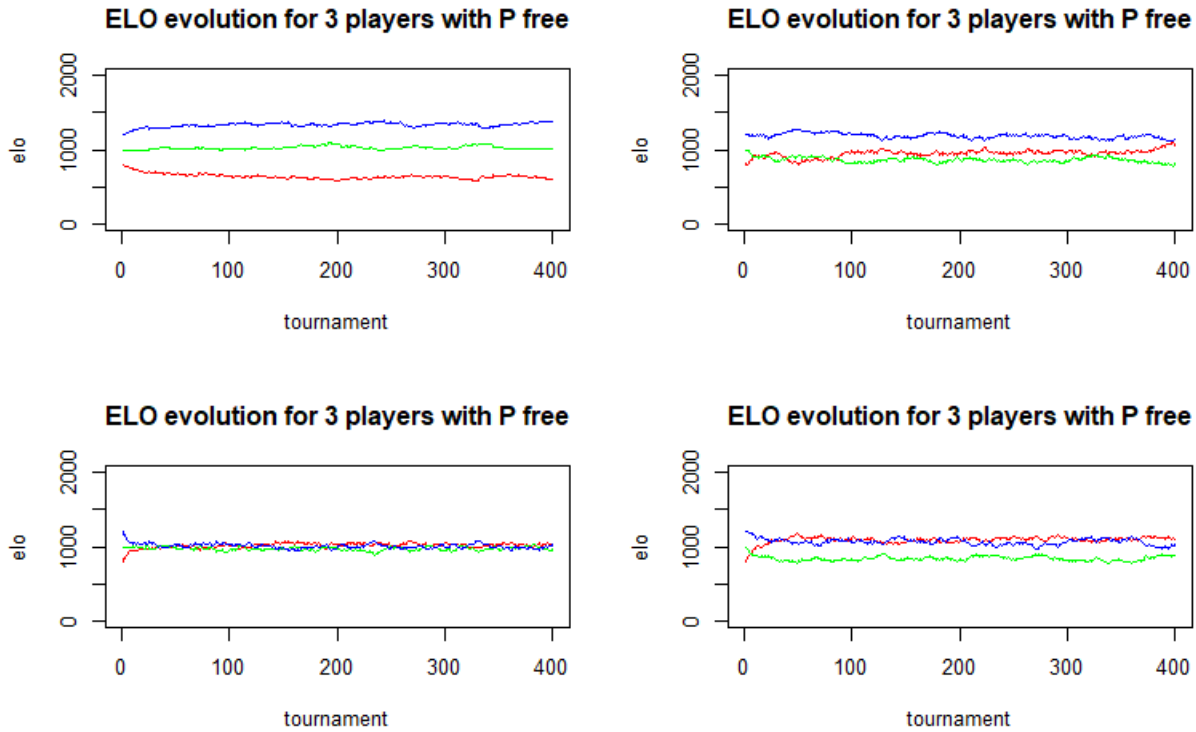
For instance, due to (37) $p_{12} \geq p_{13}$, which is not always the case in reality. The realest case will be considered later in section 3.5.

Numerical simulations



3.5 No constraint on \mathbb{P}

We will now observe the Elo evolution without the transitivity relation. Here we generate the \mathbb{P} probabilities by using a uniform law on $[0, 1]$, with the symmetrical relationship $p_{ij} = 1 - (p_{ji})$ and $p_{ii} = 0.5$.



We have 3 players A, B and C who respectively start from 800, 1000 and 1200 elo. Then we observe the different simulations over 4 random selections of independent \mathbb{P} matrices. We can observe tendencies varying with the results, and we get a good estimate of every player's ρ value.

4 Round-robin tournament for $\mathcal{N} \gg 1$

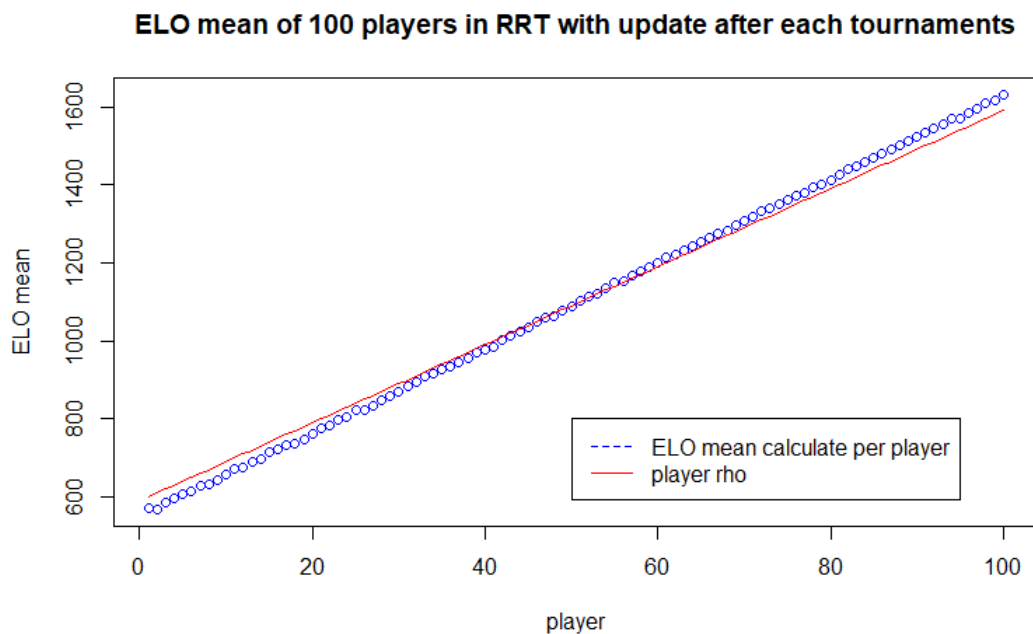
In this section, the case for N players is studied.

The round-robin tournament is still studied.

Transitivity relation satisfied by the matrices $(p_{ij})_{1 \leq i, j \leq N}$

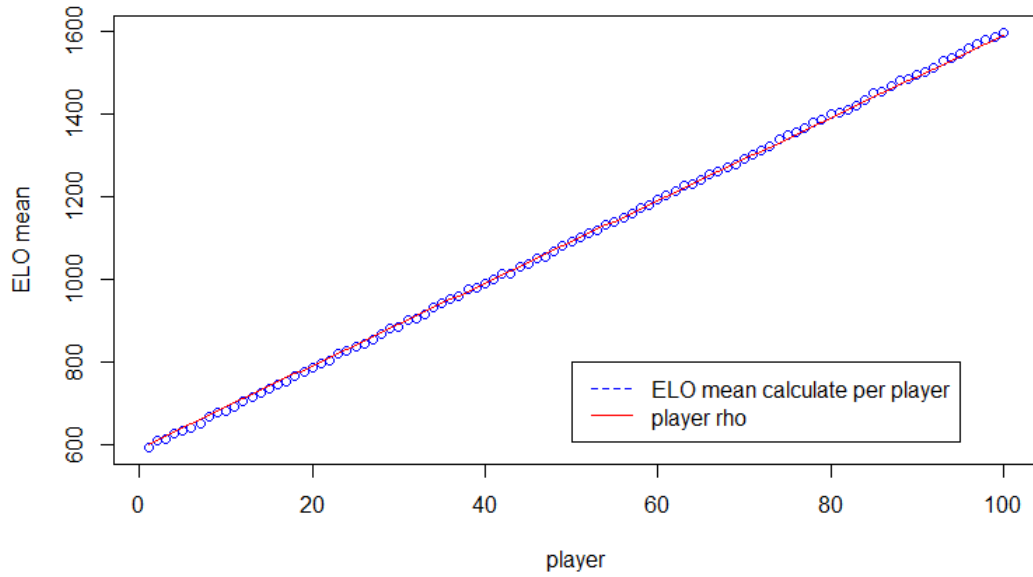
100 players

Now we generate 100 different ρ values, going from 600 to 1600, with a step of 10 between each value. Here the win probability of a given player is defined as $p = b(\rho_i - \rho_j)$ as defined earlier.



500 tournaments take place in which everyone plays everyone, Elos are updated after each tournament and we calculate the average Elo each player gets. Above is the graph obtained. We also plotted the ρ values to compare them with our results. We observe that the players are correctly ordered.

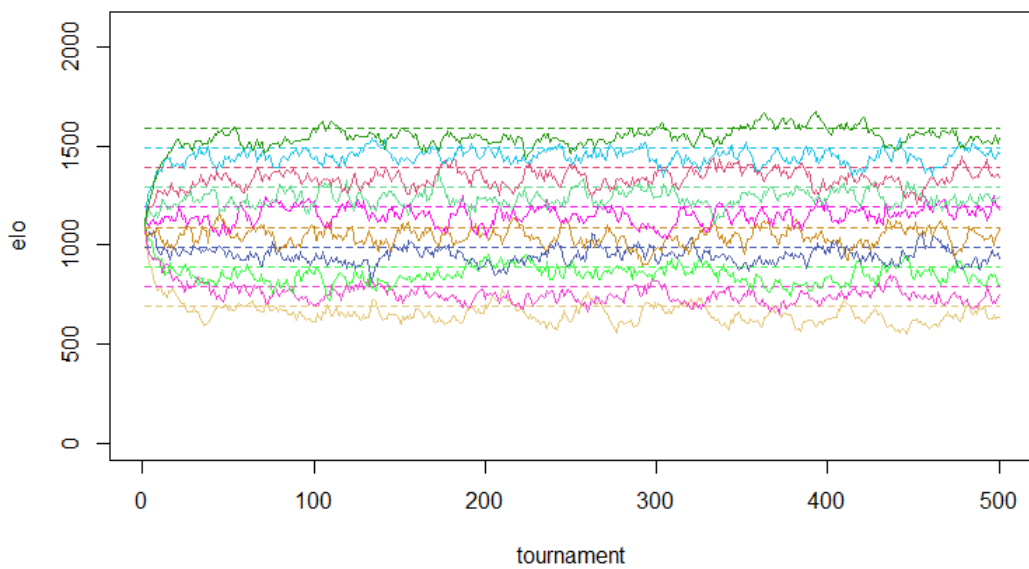
ELO mean of 100 players, calculated by natural probability



Analogously the same results were obtained when we updated Elos after every match.

Here is a small example of the Elo convergence for ten players of the aforementioned process.

ELO evolution and rho convergence for N players



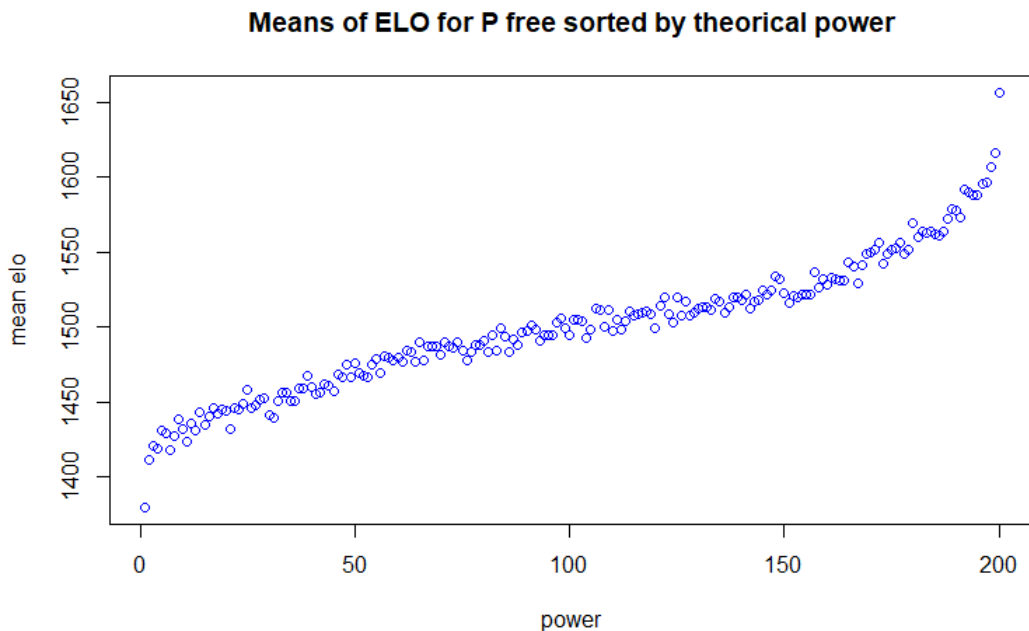
We notice that all of the players have their final elo close to their initial ρ , they fluctuate around this value. Also, the worse our initial condition, the more time we need in order to converge towards ρ .

No transitivity In reality, the transitivity is not satisfied.

In order to verify that Elo correctly ranks people, we tried using the \mathbb{P} matrix previously defined 1.2 with the difference that its probabilities are generated by a uniform law. We used 200 different players all starting at 1500 initial Elo.

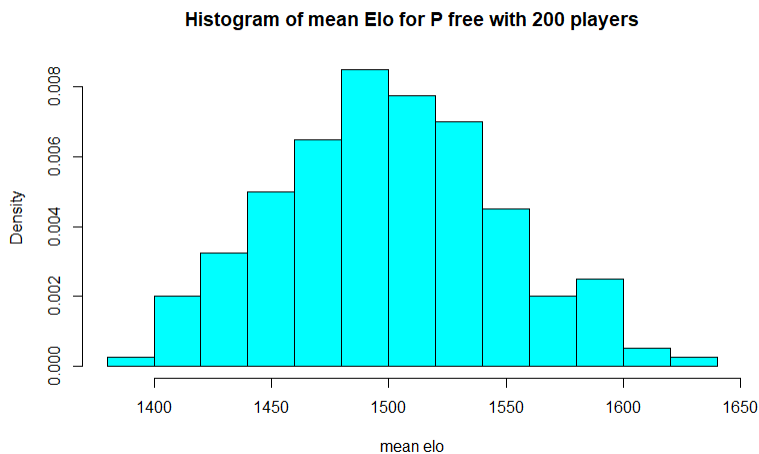
Just like previously, the theoretical strength of a player is defined by the sum of a player's probabilities to win against all of the other players. Thus, we ordered them as we did in 1.2

$$p_1. \leq p_2. \leq \dots \leq p_N.$$



Then we make them play over 500 tournaments where everyone plays everyone and we calculate the average Elo every player gets after these tournaments. It turns out that players' order is very close to what they were expected to get and the graph has an increasing tendency which is expected. From this simulation we can conclude that Elo correctly orders players.

Let's now get interested in the Elo repartition with the use of a histogram.



Most of the players are close to the initial elo (1500), the further away we are from this elo the more extreme of a case we are in and the more we stand out compared to the other players.

Conclusion

In this paper we studied the Elo convergence from 2 to N players. This report was mainly made from R simulations to illustrate some properties of the Elo system. We numerically verified some known and expected Elo properties. Elo properly sorts people's strength also when the assumption $p_{ij} = b(\rho_i - \rho_j)$ is not fulfilled. We also numerically found the limit of the mean Elo for two players:

- the G function ($b^{-1}/2$) which determines a player's Elo's limit
- $\mathbb{E}[b(D)] = b(\mathbb{E}[D]) = p_{ij}$

The equality $\mathbb{E}[b(D)] = p_{ij}$ is proven here in an ergodic context and it is easily seen when the expectations converge. The other identity is more difficult $b(\mathbb{E}[D]) = p_{ij}$ and not proven anywhere. These results were expected by Arpad Elo himself [5] but such explicit limits are still an open mathematical problem. Another interesting study could be made with a different "bonus" function used instead of the logistic one [10], and an amazing result would also be to numerically find a G function for any given N players.

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