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# van der Corput inequality for real line and Wiener-Wintner theorem for amenable groups.

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**Abstract.** We extend the classical van der Corput inequality to the real line. As a consequence, we obtain a simple proof of the Wiener-Wintner theorem for the  $\mathbb{R}$ -action which assert that for any family of maps  $(T_t)_{t \in \mathbb{R}}$  acting on the Lebesgue measure space  $(\Omega, \mathcal{A}, \mu)$  where  $\mu$  is a probability measure and for any  $t \in \mathbb{R}$ ,  $T_t$  is measure-preserving transformation on measure space  $(\Omega, \mathcal{A}, \mu)$  with  $T_t \circ T_s = T_{t+s}$ , for any  $t, s \in \mathbb{R}$ . Then, for any  $f \in L^1(\mu)$ , there is a a single null set off which  $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(T_t \omega) e^{2i\pi\theta t} dt$  exists for all  $\theta \in \mathbb{R}$ . We further present the joining proof of the amenable group version of Wiener-Wintner theorem due to Weiss and Ornstein.

## 1 Introduction

In this paper, using our generalization of van der Corput inequality for the real line, we present a simple proof of Wiener-Wintner theorem for the flows. We further present the joining proof of the amenable groups version of it due to Ornstein and Weiss [O-W]. This accomplished by applying the Furstenberg joinings machinery. The classical Wiener-Wintner theorem [W-W] assert the following.

**Theorem.** *Let  $(\Omega, \mathcal{A}, \mu, T)$  be a dynamical system where  $\mu$  is a probability measure. Then, for any  $f$  in  $L^1(\mu)$ , There is a set  $\Omega'$  of full measure such that for any  $\omega \in \Omega'$  the sums*

$$\frac{1}{N} \sum_0^{N-1} f(T^n \omega) z^n$$

*converge for all  $z$  in the unit circle  $C = \{z \in \mathbb{C} : |z| = 1\}$ .*

The original proof can be found in [W-W]. Subsequently, Furstenberg in [F1] obtain a joining proof of Wiener-Wintner theorem. Later, I. Assani [A], A. Below & V. Losert [B-L] proved the stronger version of this theorem. This stronger version is due to Bourgain [B]. Theirs proofs is based on the Hellinger integral (Known also as affinity principale). In [L1], E. Lesigne generalize Wiener-Wintner theorem to the polynomial case. His proof is based on the Furstenberg's joinings technique. Afterwards, in [L2], using van der Corput inequality and the spectral theory of skew products, he extended the stronger version of polynomial Wiener-Wintner theorem to the case of weak-wixing dynamical systems<sup>1</sup>.

In this paper, we extend van der Corput inequality to the continuous time and we give a simple proof of the flow version of Wiener-Wintner theorem. We further present the Ornstein-Weiss's joining of the amenable group version of this fundamental theorem in ergodic theory. The proof is based on Furstenberg's joinings machinery combined with the recent result of E. Lindenstrauss [Li].

The plan of the paper is as follows. In Section 2, we state and prove the continuous van der Corput inequality and the flow version of Wiener-Wintner theorem. In section 2, we state and prove the amenable group version of Wiener-Wintner theorem.

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<sup>1</sup>Seven year after this note was written , M. Lacey and E. Terwilleger [L-T] produce an oscillation proof of the Hilbert version of Wiener-Wintner theorem.

## 2 van der Corput for real line

In this section, we state our first main result.

**Theorem 2.1 (Theorem (van der Corput).)** *Let  $(u(t))_{t \in [0, T]}$  be an integrable complex valued function and let  $S \in (0, T]$ . Then*

$$\left| \int_0^T u(t) dt \right|^2 \leq \frac{S+T}{S^2} \int_0^S \int_0^S \int_0^T u(t+s'-s) \bar{u}(t) ds ds' dt. \quad (1)$$

**Proof:** We start by noticing that we have

$$S \int_0^T u(t) dt = \int_0^{T+S} \int_0^S \tilde{u}(t-s) ds dt, \quad (2)$$

where  $\tilde{u}$  stand for

$$\tilde{u}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ u(t) & \text{if } 0 \leq t \leq T, \\ 0 & \text{if not.} \end{cases}$$

Indeed, we have

$$\int_0^{T+S} \int_0^S \tilde{u}(t-s) ds dt = \int_0^S \int_{-s}^{T+S-s} \tilde{u}(t) dt ds \quad (3)$$

$$= \int_0^S \int_0^T u(t) dt ds \quad (4)$$

$$= S \int_0^T u(t) dt. \quad (5)$$

Whence,

$$S^2 \left| \int_0^T u(t) dt \right|^2 = \left| \int_0^{T+S} \int_0^S \tilde{u}(t-s) ds dt \right|^2. \quad (6)$$

Now, applying Cauchy-Schwarz inequality, we obtain

$$S^2 \left| \int_0^T u(t) dt \right|^2 \leq (T+S) \left( \int_0^{T+S} \left| \int_0^S \tilde{u}(t-s) ds \right|^2 dt \right). \quad (7)$$

But

$$\left| \int_0^S \tilde{u}(t-s) ds \right|^2 = \int_0^S \tilde{u}(t-s) \overline{\tilde{u}(t-s')} ds ds' \quad (8)$$

$$= \int_0^S \tilde{u}(t-s) \overline{\tilde{u}(t-s')} ds ds' \quad (9)$$

$$= \int_0^S \tilde{u}(t+s-s) \overline{\tilde{u}(t)} ds ds' \quad (10)$$

Whence

$$\left| \int_0^T u(t) dt \right|^2 \leq \frac{S+T}{S^2} \int_0^S \int_0^S \int_0^T u(t+s-s) \overline{u(t)} ds ds' dt. \quad (11)$$

This achieve the proof of the theorem. ■

**Theorem 2.2 (Limit version of continuous van der Corput theorem.)**

*Let  $(u(t))_{t \in \mathbb{R}}$  be a bounded complex valued function. Then*

$$\begin{aligned} & \limsup_{T \rightarrow \infty} \left| \frac{1}{T} \int_0^T u(t) dt \right|^2 \\ & \leq \limsup_{S \rightarrow \infty} \frac{1}{S^2} \int_0^S \int_0^S \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T u(t+s-s) \overline{u(t)} ds ds' dt. \end{aligned} \quad (12)$$

**Proof:** Straightforward from Theorem 2.1. ■

Now, let us state the continuous version of Wiener-Wintner theorem.

**Theorem 2.3 (Continuous version of Wiener-Wintner theorem.)**

*Let  $(T_t)_{t \in \mathbb{R}}$  be a maps acting on the Lebesgue measure space  $(\Omega, \mathcal{A}, \mu)$  where  $\mu$  is a probability measure and for any  $t \in \mathbb{R}$ ,  $T_t$  is measure-preserving transformation on measure space  $(\Omega, \mathcal{A}, \mu)$  with  $T_t \circ T_s = T_{t+s}$ , for any  $t, s \in \mathbb{R}$ . Then, for any  $f \in L^1(\mu)$ , there is a single null set off which  $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(T_t \omega) e^{2i\pi\theta t} dt$  exists for all  $\theta \in \mathbb{R}$ .*

We will assume without loss of generality that  $\mu$  ergodic. Indeed, otherwise, on can use the ergodic decomposition of  $\mu$ . So, it is sufficient to prove the following :

**Theorem 2.4** For any  $f$  in  $L^2(\mu)$ , there is a set  $\Omega'$  of full measure such that the sums  $\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T f(T_t \omega) e^{2i\pi t \theta} dt$  converge to 0 for all  $\theta$  in  $\mathbb{R}$ , where  $e^{2\pi i \theta} \notin e(T)$  and  $\omega \in \Omega'$ .  $e(T)$  stand for the set of eigenvalue of the Koopman operator  $U_T : g \mapsto g \circ T$ .

Before proceeding to the proof of Theorem 2.4, let us notice that it suffices to prove it for a dense class of functions ( $L^2$  functions for instance). Indeed, Put

$$R(\omega, f) = \limsup_{T \rightarrow +\infty} \left| \int_0^T f(T_t(\omega)) e^{2\pi i t \theta} dt \right|,$$

and assume that  $g$  in the dense class for which theorem holds. Then

$$R(\omega, f) = R(\omega, f - g).$$

and then

$$\mu\{\omega : R(\omega, f - g) > \epsilon\} \leq \frac{\|f - g\|_1}{\epsilon}.$$

We thus get by the density of  $L^2(\mu)$  in  $L^1(\mu)$ , that there exist  $g$  in  $L^2(\mu)$  such that :  $\|f - g\|_1 < \epsilon^2$ . Then

$$\mu\{\omega : R(\omega, f - g) > \epsilon\} \leq \epsilon.$$

Since  $\epsilon$  is arbitrary, we see  $R(\omega, f) = 0$  a.e., where the null set excluded is independent of  $z$ .

We start by recalling that by Bochner theorem, for any  $f \in L^2(X)$ , there exists a unique finite Borel measure  $\sigma_f$  on  $\mathbb{R}$  such that

$$\widehat{\sigma}_f(t) = \int_{\mathbb{R}} e^{-it\xi} d\sigma_f(\xi) = \langle U_t f, f \rangle = \int_{\Omega} f \circ T_t(\omega) \cdot \overline{f}(\omega) d\mu(\omega).$$

$\sigma_f$  is called the *spectral measure* of  $f$ . If  $f$  is eigenfunction with eigenfrequency  $\lambda$  then the spectral measure is the Dirac measure at  $\lambda$ .

We need also the following fundamental results.

**Theorem 2.5** *Let  $(\Omega, \mathcal{A}, \mu, (T_t)_{t \in \mathbb{R}})$  be an ergodic dynamical flow. Then, for any  $S > 0$  and all  $f, g \in L^2(X)$ , for almost all  $\omega \in \Omega$ , we have*

$$\lim_{\tau \rightarrow +\infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot g(T_t\omega) dt = \int_\Omega f \circ T_s \cdot g d\mu$$

uniformly for  $s$  in the interval  $[-S, S]$ .

This yields the exact result need it.

**Corollary 2.6** *Let  $f \in L^2(\mu)$ . There exist a full measure subset  $\Omega_f$  of  $\Omega$  such that, for any  $\omega \in \Omega_f$  and any  $s \in \mathbb{R}$ , we have*

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot \bar{f}(T_t\omega) dt = \int_X f \circ T_s \cdot \bar{f} d\mu.$$

**Proof:** [ of Theorem 2.4] Let  $f$  in  $L^\infty(\mu)$  and  $\omega \in \Omega_f$  as in Corollary 2.6, then we have

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s}\omega) \cdot \bar{f}(T_t\omega) dt = \int_X f \circ T_s \cdot \bar{f} d\mu \quad (13)$$

$$\stackrel{\text{def}}{=} \langle f \circ T_s, f \rangle. \quad (14)$$

Put

$$u(t) = f(T_t\omega)e^{2\pi i t \theta},$$

and apply further van der Corput's inequality (Theorem 2.1) to get

$$\begin{aligned} & \left| \frac{1}{\tau} \int_0^\tau f(T_t\omega)e^{2\pi i t \theta} dt \right|^2 \\ & \leq \frac{S+\tau}{\tau S^2} \int_0^S \int_0^S e^{2\pi i (s-s')\theta} \frac{1}{\tau} \int_0^\tau f(T_{t+s-s'}) \bar{f}(T_t\omega) dt ds ds'. \end{aligned} \quad (15)$$

We thus deduce that for almost all  $\omega$  and all  $\theta \in \mathbb{R}$ , we have

$$\begin{aligned} & \limsup_{\tau \rightarrow \infty} \left| \frac{1}{\tau} \int_0^\tau f(T_t\omega)e^{2\pi i t \theta} dt \right|^2 \\ & \leq \frac{1}{S^2} \int_0^S \int_0^S e^{2\pi i (s-s')\theta} \left( \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(T_{t+s-s'}) \bar{f}(T_t\omega) dt \right) ds ds'. \end{aligned} \quad (16)$$

This combined with Corollary 2.6 gives

$$\begin{aligned} & \limsup_{\tau \rightarrow \infty} \left| \frac{1}{\tau} \int_0^\tau f(T_t\omega)e^{2\pi i t \theta} dt \right|^2 \\ & \leq \frac{1}{S^2} \int_0^S \int_0^S \left( \int_{\mathbb{R}} e^{2\pi i (s-s')(\theta-\gamma)} d\sigma_f(s-s') \right) ds ds', \end{aligned} \quad (17)$$

where  $\sigma_f$  stand for the spectral measure of  $f$ . But, since

$$\frac{1}{S^2} \int_0^S \int_0^S e^{2\pi i(s-s')(\theta-\gamma)} ds ds' = \left| \frac{1}{S} \int_0^S e^{2\pi i s(\theta-\gamma)} ds \right|^2 \quad (18)$$

if  $\theta \neq \gamma$ , we have

$$\lim_{S \rightarrow \infty} \frac{1}{S^2} \int_0^S \int_0^S e^{2\pi i(s-s')(\theta-\gamma)} ds ds' = 0. \quad (19)$$

Whence, if  $e^{2\pi i\theta}$  is not a eigenvalue of  $(T_t)$ , we have

$$\lim_{S \rightarrow \infty} \frac{1}{S^2} \int_0^S \int_0^S \left( \int_{\mathbb{R}} e^{2\pi i(s-s')(\theta-\gamma)} d\sigma_f(s-s') \right) ds ds' = 0.$$

Since all the sums are bounded, we deduce from Lebesgue theorem that for almost all  $\omega$ , and for all  $\theta$  in  $\mathbb{R}$ , where  $e^{2\pi i\theta} \notin e(T)$ ,

$$\lim_{\tau \rightarrow \infty} \frac{1}{\tau} \int_0^\tau f(T_t \omega) e^{2\pi i t \theta} dt = 0,$$

and this finish the proof of the theorem. ■

## 2. Joining's proof of Wiener-Wintner Theorem for action of amenable group

In this section, we deal with actions on Lebesgue spaces, that is, given a locally compact groupe  $G$  and the a Lebesgue space  $(X, \mathcal{A}, \mu)$ , a  $G$ -action is a measurable mapping  $G \times X \rightarrow X$ ,  $(g, x) \mapsto g.x$ , such that for all  $g, h \in G$ ,  $g.(h.x) = (gh).x$  and  $e.x = x$  for almost all  $x \in X$  ( where  $e$  is the identity in  $G$ ). Furthermore,  $T_g : x \mapsto g.x$  is measure -preserving for every  $g \in G$ . We will mainly concerned with  $G$  which is amenable group ( locally compact second countable) or the subclass of locally compact abelian groups.

We recall that  $G$  is an amenable group if for any compact  $K \subset G$  and  $\delta > 0$  there is a compact set  $F \subset G$  such that

$$h_L(F \Delta KF) < \delta h_L(F), \quad (20)$$

where  $h_L$  stand for the left Haar measure on  $G$ . It is well known that the amenability is equivalent to the existence of Følner sequence  $(F_n)$ , that is,

$(F_n)$  is a sequence of compact subsets of  $G$  for which for every compact  $K$  and  $\delta > 0$ , for all large enough  $n$  we have that  $F_n$  satisfy (20). Assume further that  $(F_n)$  satisfy the so-called Shulman Condition, that is, for some  $C > 0$  and all  $n$

$$h_L\left(\bigcup_{k \leq n} F_k^{-1} F_n\right) \leq C \cdot h_L(F_n). \quad (21)$$

Under this assumptions, E. Lindenstrauss proved that the Birkhoff pointwise ergodic theorem holds, that is, Then for any  $f \in L^1(\mu)$ , there is a  $G$ -invariant  $f^* \in L^1(\mu)$  such that

$$\lim \frac{1}{h_L(F_n)} \int_{F_n} f(g\omega) dh_L(g) = f^*(\omega) \text{ a.e.}$$

To formulate the  $G$ -version of Wiener-Wintner theorem, we replace the group rotations by homomorphisms  $\Theta$  from  $G$  to a finite dimensional unitary group  $U_d$ . The canonical action in this case is given by  $g.u = \Theta(g).u$ ,  $u \in U_d$  and  $g \in G$ . The invariant measure is the Haar measure on  $U_d$ . In this setting, we formulate the Wiener-Wintner theorem as follows:

**Theorem 2.7 (Group version of Wiener-Wintner theorem)** *Let  $G$  be an amenable group acting on a Lebesgue space  $(\Omega, \mathcal{A}, \mu)$  and assume that  $G$  satisfy Shulman condition. Let  $f \in L^\infty(\mu)$ . Then, there is a set  $\Omega_f$  of full measure such for any  $\omega \in \Omega_f$*

$$\frac{1}{h_L(F_n)} \int_{F_n} f(g\omega) \phi(\Theta(a)u) dh_L(g)$$

*converge for all finite dimensional unitary representation  $\Theta$  of  $G$  into  $U_d$  (all  $d$ ), all continuous function  $\phi$  on  $U_d$  and all  $u \in U_d$ . We further have that the limit on the orthocomplement of the space spanned by the finite dimensional invariant subspaces is zero.*

Before proceeding to the proof let us recall some important tools.

A *joining* of two actions of the same group  $\mathcal{X} = (X, \mathcal{A}, \mu, G)$  and  $\mathcal{Y} = (Y, \mathcal{B}, \nu, G)$  is the probability measure  $\lambda$  on  $(X \times Y, \mathcal{A} \times \mathcal{B})$  which is invariant under the diagonal action of  $G$  ( $g.(x, y) = (g.x, g.y)$ ) and whose marginals on  $(\mathcal{A} \times Y)$  and  $(X \times \mathcal{B})$  are  $\mu$  and  $\nu$  respectively (i.e. if  $A \in \mathcal{A}$ ,  $\lambda(A \times Y) = \mu(A)$ ;

and if  $B \in \mathcal{B}$ ,  $\lambda(X \times B) = \nu(B)$ ). The set of joinings is never empty. (Take  $\mu \times \nu$ ). As we deal with Lebesgue spaces, a joining  $\lambda$  of two ergodic  $G$ -actions  $\mathcal{X}$  and  $\mathcal{Y}$  has the property that there exists a Lebesgue space  $\Omega$  and the probability  $\mathbb{P}$  on  $\Omega$  such that  $\lambda = \int \lambda_\omega d\mathbb{P}(\omega)$ , where  $\lambda_\omega$  is ergodic. (This is just the ergodic decomposition of  $\lambda$ , and as the marginals of  $\lambda$  are ergodic a.e.,  $\lambda_\omega$  is joining). Therefore the set of ergodic joinings is never empty.<sup>2</sup>

Historically, joinings were introduced by H. Furstenberg in his paper [F2] on disjointness. In particular, he defined the important notion of disjointness for  $\mathbb{Z}$ -action in the following way :  $(X, \mathcal{A}, \mu, T)$  and  $(Y, \mathcal{B}, \nu, S)$  is *disjoint* if the only joining between them is the product joining. In the case of  $G$ -action we have the following definition.

**Definition 2.8** *Let  $\mathcal{X}$  and  $\mathcal{Y}$  be two actions of the same group  $G$ .  $\mathcal{X}$  and  $\mathcal{Y}$  are disjoint if the only joining between them is the product joining. We denote this disjointness by  $\mathcal{X} \perp \mathcal{Y}$ .*

In the case of  $\mathbb{Z}$ -actions, H. Hahn & W. Parry obtain in [H-P] that if two transformations have mutually singular maximal types, then they are disjoint. But, As for the joinings theory, the spectral theory of  $\mathbb{Z}$ -actions can be extended to the case of locally abelian  $G$ -actions. Therefore, we have the following group version of Hahn-Parry theorem.

**Theorem 2.9 ((Hahn & Parry))** *If two  $G$ -actions  $\mathcal{X}$  and  $\mathcal{Y}$  have mutually singular maximal spectral types, then they are disjoint.*

**Proof:** Let recall that the spectral measure of a function  $f \in L^2(X)$  under the operators  $U_g$  ( defined on  $L^2(X)$  by  $U_g(f) = f \circ T_g$ ) is the measure  $\sigma_f$  on  $\hat{G}$  (dual group of  $G$ , i.e., the set of all continuous characters of  $G$ ) where its Fourier transform  $\hat{\sigma}_f$  is given by  $\hat{\sigma}_f(g) = \langle U_g f, f \rangle$ . Now, we follows the proof given in [Th]. In  $X \times Y$  endowed with a joining measure  $\lambda$ , consider  $f_1 \in L^2(X)$  and  $f_2 \in L^2(Y)$  and consider  $H_{f_1}$  the  $L^2(\lambda)$  closure of the linear span of the functions  $(U_g(f_1) - \int f_1 d\mu) \times 1_Y$ ,  $g \in G$ . The projection

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<sup>2</sup>see [D-R], for instance.

of  $1_X \times f_2$  on  $H_{f_1}$  will have a spectral measure absolutely continuous with respect to the spectral type of  $U_g$  on  $L^2(X)$  and thus has to be 0. Therefore  $1_X \times f_2 \perp (f_1 - \int f_1) \times 1_Y$ , and  $\int f_1(x)f_2(y)d\lambda(x, y) = \int f_1 d\mu \int f_2 d\nu$ . ■  
From this theorem we have the following.

**Corollary 2.10** *Let  $\chi_0$  be a non trivial character and define the action of  $G$  on torus  $\mathbb{T}$  by  $(g, e^{ix}) \mapsto \chi_0(g)e^{ix}$ . Assume that for any  $n \in \mathbb{Z}$ , the character  $\chi_0^n$  define on  $G$  by  $g \mapsto \chi_0(g^n)$  is not eigenvalue of the  $G$ -action on  $\mathcal{X}$ . The the  $G$ -action on  $\mathbb{T}$  and the  $G$ -action on  $\mathcal{X}$  are disjoint.*

**Proof:** Let recall that  $\chi_0$  is a eigenvalue of  $G$ - action if there exist a eigenfunction  $f \in L^2(X, \mu)$  such that  $f \circ T_g = \chi_0(g) f$ . We deduce that the spectral measure of  $f$  is  $\|f\|_2^2 \delta_{\chi_0}$  ( $\delta_{\chi_0}$  is the Dirac measure on  $\chi_0$ ). Since for any  $n \in \mathbb{Z}$ ,  $\chi_0^n$  is not eigenvalue of  $G$ -action on  $\mathcal{X}$ , we conclude that the maximal spectral types of this two  $G$ -actions are mutually singular. Now apply the Hahn-Parry theorem to complete the proof. ■

For the general case of amenable group which satisfy Shulman condition, we have the following lemma from [O-W].

**Lemma 2.11** *Let  $U$  be the closure of  $\Theta(G)$  in  $U_d$ . Then, if the product  $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, \mathbf{G})$  is ergodic then there is only on  $G$ -invariant measure on  $U \times \Omega$  that projects onto  $\mu$  on  $\Omega$ .*

**Proof:** [ of Theorem 2.7 ] We start by assuming without lost of generality that the action on  $(\Omega, \mathbf{A}, \mu, )$  is ergodic and by presenting the proof for the case when  $G$  is locally abelian group. Let  $f \in L^\infty(\mu)$  and  $\phi$  continuous function. Then, by the pointwise theorem there is a set of full measure of  $\omega$ . Assume that  $\omega$  is in this subset and let  $\chi_0 \in \hat{G}$  such  $\chi_0$  is not eigenvalue. Then, the product  $(U, \Theta, G) \times (\Omega, \mathbf{A}, \mu, \mathbf{G})$  is ergodic. Moreover, by taking a subsequence  $(n_k)$ , we can assume that

$$\lim_{k \rightarrow +\infty} \frac{1}{h_L(F_{n_k})} \int_{F_{n_k}} f(g\omega)\phi(\Theta(a)u)dh_L(g) = \lambda(f \otimes \phi).$$

It follows that  $\lambda$  is a joining and by Corollary 2.10  $\lambda = dh \times \mu$ . We end the proof by noticing that there is a countable of eigenvalue. The general case

follows in the same manner by taking

$$F(\omega) = \int \psi(u)I(u_1\omega)du,$$

where  $I$  is a bounded invariant functions on  $U \times \Omega$  and  $\psi$  is any positive continuous function on  $u$ . Therefore, transforming  $F$  by  $g$  is the same as transforming  $\psi$  by  $\Theta(g)$ . We thus have that a non-constant  $I$  will give rise to finite dimensional invariant subspaces for  $G$  on  $\Omega$ . Moreover, by taking  $(U, \Theta, G)$  not in in the list of countable representations  $(U_j, \Theta_j, G)$ , the condition of Lemma 2.11 is satisfied and therefore as before the only joining is the product measure, and we are done. ■

**Question 2.12** *We ask on the possible extension of van der Corput inequality to the locally compact group and its application to obtain produce a direct proof of the group version of Wiener-Wintner theorem.*

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