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Abstract: How to compute (super) hedging costs in rather general financial market models with transaction costs in discrete-time ? Despite the huge literature on this topic, most of results are characterizations of the super-hedging prices while it remains difficult to deduce numerical procedure to estimate them. We establish here a dynamic programming principle and we prove that it is possible to implement it under some conditions on the conditional supports of the price and volume processes for a large class of market models including convex costs such as order books but also non convex costs, e.g. fixed cost models.

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1. Introduction

The problem of characterizing the set of all possible prices hedging a European claim has been extensively studied in the literature under classical no-arbitrage conditions. In discrete-time and without transaction costs, a dual characterization is deduced through dual elements, the equivalent martingale measures, whose existence characterizes the well known no-arbitrage condition NA, see the FTAP theorem of [6]. In continuous time, similar characterizations are obtained under the NFLVR condition of Delbaen and Schachermayer [7], [8] for instance. The Black and Scholes model [3] is the canonical example of complete market in mathematical finance such that the equivalent probability measure is unique. The advantage of this simple model is that hedging prices are explicitly given. Unfortunately, for incomplete market models, it is difficult to establish numerical procedures to estimate the super-hedging prices from the dual characterization. This is why it is usual to specify a particular martingale measure, see [27], [10] and [12].

In presence of transaction costs, the financial market is a priori incomplete and computing the infimum super-hedging prices remains a challenge. In the Kabanov model with transaction costs [14], the main result is a dual characterization [14][Theorem 3.3] through the so-called consistent price systems (CPS) that characterize various kinds of no-arbitrage conditions for these models, see [14][Section 3.2]. Unfortunately, it is difficult to characterize the consistent price systems and deduce a numerical estimation of the prices. A first attempt (and the only one) is proposed in [21] for finite probability space. More generally, vector optimization methods are proposed for risk measures as in [4] still for finite probability spaces. Also, various asymptotic results are obtained for small transaction costs by Schachermayer [28], [11] and others [15], [16], still for conic models.

For non conic models, in the presence of an order book for instance, more generally with convex cost, or with fixed costs, few results are available in the literature. Well known papers such as [13], [24], [22] , [19], [20] only formulate characterizations of the super-hedging prices. The very question we aim to address in this paper is how to numerically compute the infimum super-hedging cost of a European claim.

To do so, we first provide a dynamic programming principle in a very general setting in discrete time, see Theorem 3.1. Notice that we do not need any no-arbitrage condition to formulate it. Secondly, we propose some conditions under which it is possible to implement the dynamic programming principle. Actually, we shall see that we only need to have an insight on the conditional supports of the increments of the process describing the financial market, mainly the price and volume process.

Our main results are formulated under some weak non-arbitrage conditions such that the minimal super-hedging costs are non negative for non negative payoffs, as in [5], [2]. These conditions avoid the unrealistic case of infinitely negative prices. The main problem is how to compute an essential supremum and an essential infimum. We show that they may coincide with pointwise supremum and infimum respectively. This is sufficient to compute backwardly the hedging costs as solutions to pointwise (random) optimization problem.

The paper is organized as follows. The financial market is defined by a cost process, which is not necessarily convex, as described in Section 2. Then, the dynamic programming principle is established in Section 3, see Theorem 3.1. The last Section 4 is devoted to the implementation of the dynamic programming principle. Precisely, we formulate results that ensure the propagation of the lower semicontinuity to the minimal hedging cost at any time, e.g. with respect to the spot price, see Theorem 4.5, Corollary 4.9, Theorem 4.14, Theorem 4.16 and Theorem 4.26. In Subsection 4.3, fixed costs models are considered. Theorem 4.20 also states the propagation of the lower semicontinuity that allows to numerically compute the minimal hedging cost backwardly. It is formulated under a no-arbitrage condition on the enlarged market only composed of linear transaction costs in the spirit of [19] but also [22] in the context of utility maximization.

2. Financial market model defined by a cost process

We consider a stochastic basis in discrete-time $(\Omega, (\mathcal{F}_t)_{t=0}^T, \mathrm{P})$ where the filtration $(\mathcal{F}_t)_{t=0}^T$ is complete, i.e. $\mathcal{F}_0$ contains the negligible sets for P. By convention, we also define $\mathcal{F}_{-1} := \mathcal{F}_0$. If A is a random subset of $\mathbf{R}^d$, $d \geq 1$, we denote by $L^0(A, \mathbf{R}^d)$ the family of (equivalence classes of) all random variables X (defined up to a negligible set) such that $X(\omega) \in A(\omega)$, $\mathrm{P}(\omega)$ a.s. It is well known that, if $A(\omega) \neq \emptyset$ $\mathrm{P}(\omega)$ a.s. and if A is graph-measurable, see [23], then $L^0(A, \mathbf{R}^d) \neq \emptyset$. When using this property, we refer it by saying *by measurable selection arguments*, as it is usual to do when claiming the existence of $X \in L^0(\mathbf{R}, \mathcal{F})$ such that $X \in A$ a.s..

We also adopt the following notations. We denote by int$A$ the interior of any $A \subseteq \mathbf{R}^d$ and cl$A$ is its closure. The positive dual of A is defined as $A^* := \{x \in \mathbf{R}^d : \; ax \geq 0, \forall a \in A\}$ where ax designates the Euclidean scalar product of $\mathbf{R}^d$. At last, if $r \geq 0$, we denote by $\bar{B}(0, r) \subseteq \mathbf{R}^d$ the closed ball of all $x \in \mathbf{R}^d$ such that the norm satisfies $|x| \leq r$.

We consider a financial market where transaction costs are charged when the agents buy or sell risky assets. The typical case is a model defined by a bond whose discounted price is $S^1 = 1$ and $d-1$ risky assets that may be traded at some bid and ask discounted prices S^b and S^a, respectively, when selling or buying. We refer the readers to the huge literature on models with transactions costs, in particular see [14].

Our general model is defined by a set-valued process $(\mathbf{G}_t)_{t=0}^T$ adapted to the filtration $(\mathcal{F}_t)_{t=0}^T$. Precisely, we suppose that for all $t \leq T$, $\mathbf{G}_t$ is $\mathcal{F}_t$-measurable in the sense of the graph $\mathrm{Graph}(\mathbf{G}_t) = \{(\omega, x) : \ x \in G_t(\omega)\}$ that belongs to $\mathcal{F}_t \times \mathcal{B}(\mathbf{R}^d)$, where $\mathcal{B}(\mathbf{R}^d)$ is the Borel σ-algebra on $\mathbf{R}^d$ and $d \geq 1$ is the number of assets.

We suppose that $\mathbf{G}_t(\omega)$ is closed for every $\omega \in \Omega$ and $\mathbf{G}_t(\omega) + \mathbf{R}^d_+ \subseteq \mathbf{G}_t(\omega)$, for all $t \leq T$. The cost value process $\mathrm{C} = (\mathrm{C}_t)_{t=0}^T$ associated to $\mathbf{G}$ is defined as:

$$\mathrm{C}_t(z) = \inf\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\} = \min\{\alpha \in \mathbf{R} : \ \alpha e_1 - z \in \mathbf{G}_t\}, \quad z \in \mathbf{R}^d.$$

We suppose that the right hand side in the definition above is non empty a.s. and $-e_1$ does not belong to $\mathbf{G}_t$ a.s. where $e_1 = (1, 0, \cdots, 0) \in \mathbf{R}^d$. Moreover, by assumption, $\mathrm{C}_t(z)e_1 - z \in \mathbf{G}_t$ a.s. for all $z \in \mathbf{R}^d$. Note that $\mathrm{C}_t(z)$ is the minimal amount of cash one needs to get the financial position $z \in \mathbf{R}^d$ at time t. In particular, we suppose that $\mathrm{C}_t(0) = 0$.

Similarly, we may define the liquidation value process $\text{Ł} = (\text{Ł}_t)_{t=0}^T$ associated to $\mathbf{G}$ as:

$$\text{Ł}_t(z) := \sup\left\{\alpha \in \mathbf{R} : z - \alpha e_1 \in \mathbf{G}_t\right\}, \quad z \in \mathbf{R}^d.$$

We observe that $\text{Ł}_t(z) = -\mathrm{C}_t(-z)$ and $\mathbf{G}_t = \{z \in \mathbf{R}^d : \ \text{Ł}_t(z) \geq 0\}$ so that our model is equivalently defined by Ł or C. Note that $\mathbf{G}_t$ is closed if and only if $\text{Ł}_t(z)$ is upper semicontinuous (u.s.c.) in z, see [19], or equivalently $\mathrm{C}_t(z)$ is lower semicontinuous (l.s.c.) in z. Naturally, $\mathrm{C}_t(z) = \mathrm{C}_t(S_t, z)$ depends on the available quantities and prices for the risky assets, described by an exogenous vector-valued $\mathcal{F}_t$-measurable random variable S_t of $\mathbf{R}^m_+$, $m \geq d$, and on the quantities $z \in \mathbf{R}^d$ to be traded. Here, we suppose that $m \geq d$ as an asset may be described by several prices and quantities offered by the market, e.g. bid and ask prices, or several pair of bid and ask prices of an order book and the associated quantities offered by the market.

In the following, we suppose the following assumptions on the cost process C. For any $t \leq T$, the cost function C_t is a lower-semi continuous Borel function defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that

$$
\begin{aligned}
&\mathrm{C}_t(s,0) = 0,\ \forall s \in \mathbf{R}^m_+,\\
&\mathrm{C}_t(s, x+\lambda e_1) = \mathrm{C}_t(s,x) + \lambda,\ \lambda \in \mathbf{R},\ x \in \mathbf{R}^d,\ s \in \mathbf{R}^m_+\ \text{(cash invariance)},\\
&\mathrm{C}_T(s,x_2) \geq \mathrm{C}_T(s,x_1),\ \forall x_1, x_2 \text{ s.t. } x_2 - x_1 \in \mathbf{R}^d_+\ (\mathrm{C}_T \text{ is increasing w.r.t. } \mathbf{R}^d_+),\\
&|\mathrm{C}_t(s,x)| \leq h_t(s,x),
\end{aligned}
$$

where h_t is a deterministic continuous function. Note that C_T is increasing w.r.t. $\mathbf{R}^d_+$ is equivalent to $\mathbf{G}_T + \mathbf{R}^d_+ \subseteq \mathbf{G}_T$. Moreover, if δ is an increasing bijection from $[0,+\infty]$ to $[0,+\infty]$ such that $\delta(0)=0$ and $\delta(\infty)=\infty$, we say that C_t is positively super δ-homogeneous if the following property holds:

$$
\mathrm{C}_t(s, \lambda x) \geq \delta(\lambda) C_t(s,x), \forall \lambda \geq 1,\ s \in \mathbf{R}^m_+,\ x \in \mathbf{R}^d.
$$

A classical case is when $\delta(x) = x$ and the positive homogeneous property holds, e.g. for models with proportional transaction costs, as the solvency set process $\mathbf{G}$ is a positive cone, see [14]. More generally, if $\mathrm{C}_t(s,x)$ is convex in x and $\mathrm{C}_t(s,0) = 0$, it is clear that C_t is positively super δ-homogeneous with $\delta(x) = x$. Actually, in our definition, the domain of validity $\lambda \geq 1$ may be replaced by $\lambda \geq r$ where $r > 0$ is arbitrarily chosen. In that case, all the results we formulate in this paper are still valid. We now present a typical model that satisfies our assumptions:

Example 2.1 (Order book)**.** Suppose that the financial market is defined by an order book. In that case, we define S_t, at any time t, as

$$
S_t = ((S_t^{b,i,j}, S_t^{a,i,j}), (N_t^{b,i,j}, N_t^{a,i,j}))_{i=1,\cdots,d, j=1,\cdots,k},
$$

where k is the order book's depth and, for each $i = 1, \cdots, d$, $S_t^{b,i,j}, S_t^{a,i,j}$ are the bid and ask prices for asset i in the j-th line of the order book and $(N_t^{b,i,j}, N_t^{a,i,j}) \in (0,\infty)^2$ are the available quantities for these bid and ask prices. We suppose that $N_t^{b,i,k} = N_t^{a,i,k} = +\infty$ so that the market is somehow liquid. By definition of the order book, we have $S_t^{b,i,1} > S_t^{b,i,2} > \cdots > S_t^{b,i,k}$ and $S_t^{a,i,1} < S_t^{a,i,2} < \cdots < S_t^{a,i,k}$. We then define the cost function as

$$
\mathrm{C}_t(x) = x^1 + \sum_{i=2}^{d} \mathrm{C}_t^i(x^i), \quad x = (x^1, \cdots, x^d) \in \mathbf{R}^d.
$$

With the convention $\sum_{r=1}^{j}=0$ if $j=0$, we consider the cumulated quantities $Q_t^{a,i,j}:=\sum_{r=1}^{j}N_t^{a,i,r}$, $j=0,\cdots,k$, the same for $Q_t^{b,i,j}$. We have:

$$\begin{aligned}\mathrm{C}_t^i(y) &= \sum_{r=1}^{j}N_t^{a,i,r}S_t^{a,i,r}+(y-Q_t^{a,i,j})S_t^{a,i,j+1}, \quad \text{if } Q_t^{a,i,j}<y\le Q_t^{a,i,j+1},\\ \mathrm{C}_t^i(y) &= -\sum_{r=1}^{j}N_t^{b,i,r}S_t^{b,i,r}+(y+Q_t^{b,i,j})S_t^{b,i,j+1}, \quad \text{if } -Q_t^{b,i,j+1}<y\le -Q_t^{b,i,j}.\end{aligned}$$

Note that the first expression of $\mathrm{C}_t^i(z)$ above corresponds to the case where we buy $y>0$ units of asset i. The second expression is $\mathrm{C}_t^i(y)=-\mathrm{Ł}_t^i(-y)$ when $y<0$ so that $-\mathrm{C}_t^i(y)$ is the liquidation value of the position $-y$, i.e. by selling the quantity $-y>0$ at the bid prices. We observe that $\mathrm{C}_t^i(y)$ is a convex function in y satisfying the cash invariance, such that $\mathrm{C}_t^i(0)=0$ and, at last, we show that C_t^i is positively super homogeneous as defined above.

To do so, we first consider $y>0$ and we show that $\mathrm{C}_t^i(\lambda y)\ge\lambda\mathrm{C}_t^i(y)$ for $\lambda>1$ by induction on the interval $]Q_t^{a,i,j},Q_t^{a,i,j+1}]$ that contains y. For $j=1$, $\mathrm{C}_t^i(y)=S_t^{a,i,1}y$ and $\mathrm{C}_t^i(\lambda y)=\mathrm{C}_t^i(Q_t^{a,i,j_\lambda})+(\lambda y-Q_t^{a,i,j_\lambda})S_t^{a,i,j_\lambda+1}$ where j_λ is such that $\lambda y\in]Q_t^{a,i,j_\lambda},Q_t^{a,i,j_\lambda+1}]$. As $S_t^{a,i,1}$ is the smallest ask price, we get that $\mathrm{C}_t^i(Q_t^{a,i,j_\lambda})\ge Q_t^{a,i,j_\lambda}S_t^{a,i,1}$ and $(y-Q_t^{a,i,j_\lambda})S_t^{a,i,j_\lambda+1}\ge(\lambda y-Q_t^{a,i,j_\lambda})S_t^{a,i,1}$. We deduce that $\mathrm{C}_t^i(\lambda y)\ge\lambda yS_t^{a,i,1}$ hence $\mathrm{C}_t^i(\lambda y)\ge\lambda\mathrm{C}_t^i(y)$. More generally, if $y\in]Q_t^{a,i,j},Q_t^{a,i,j+1}]$, $\lambda y>\lambda Q_t^{a,i,j}$ hence $\mathrm{C}_t(\lambda y)\ge\mathrm{C}_t(\lambda Q_t^{a,i,j})+(\lambda y-\lambda Q_t^{a,i,j})S_t^{a,i,\tilde{j}}$ where $\tilde{j}$ is such that $Q_t^{a,i,\tilde{j}}<\lambda Q_t^{a,i,j}\le Q_t^{a,i,\tilde{j}+1}$. Indeed, the extra quantity $\lambda y-\lambda Q_t^{a,i,j}$ is bought at a price larger than or equal to the maximal ask price $S_t^{a,i,\tilde{j}}$ when buying the quantity $\lambda Q_t^{a,i,j}$. As $\lambda Q_t^{a,i,j}>Q_t^{a,i,j}$, we deduce that $\tilde{j}\ge j+1$. Using the induction hypothesis, we have $\mathrm{C}_t^i(\lambda Q_t^{a,i,j})\ge\lambda\mathrm{C}_t^i(Q_t^{a,i,j})$ and we deduce that

$$\mathrm{C}_t^i(\lambda y)\ge\lambda C_t^i(Q_t^{a,i,j})+(\lambda y-\lambda Q_t^{a,i,j})S_t^{a,i,j+1}=\lambda\mathrm{C}_t^i(y).$$

By the same reasoning, $\mathrm{Ł}_t^i(\lambda y)\le\lambda\mathrm{Ł}_t^i(y)$ if $y>0$ with $\mathrm{Ł}_t^i(y)=-\mathrm{C}_t^i(-y)$. Therefore, we also get that $\mathrm{C}_t^i(\lambda y)\ge\lambda\mathrm{C}_t^i(y)$ for $\lambda>1$ and $y<0$.

We finally conclude that the cost process C satisfies the conditions we impose above. In particular, notice that $\mathrm{C}_t(s,z)$ is continuous in (s,z). $\triangle$

A portfolio process is by definition a stochastic process $(V_t)_{t=-1}^{T}$ where $V_{-1}\in\mathbf{R}e_1$ is the initial endowment expressed in cash that we may convert immediately into $V_0\in\mathbf{R}^d$ at time $t=0$. By definition, we suppose that

$$\Delta V_t=V_t-V_{t-1}\in-\mathbf{G}_t,\ a.s.,\quad t=0,\cdots,T.$$

This means that any position $V_{t-1} = V_t + (-\Delta V_t)$ may be changed into the new position V_t, letting aside the residual part $(-\Delta V_t)$ that can be liquidated without any debt, i.e. $Ł_t(-\Delta V_t) \geq 0$.

3. Dynamic programming principle for pricing

Let $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$ be a contingent claim. Our goal is to characterize the set of all portfolio processes $(V_t)_{t=-1}^T$ such that $V_T = \xi$, as defined in the last section. We are mainly interested by the infimum cost one needs to hedge ξ, i.e. the infimum value of the initial capitals $V_{-1}e_1 \in \mathbf{R}$ among the portfolios $(V_t)_{t=-1}^T$ replicating ξ.

In the following, we use the notation $z = (z^1, z^2, ..., z^d) \in \mathbf{R}^d$ and we denote $z^{(2)} = (z^2, ..., z^d)$. We shall heavily use the notion of $\mathcal{F}_t$-measurable conditional essential supremum (resp. infimum) of a family of random variables, i.e. the smallest (resp. largest) $\mathcal{F}_t$-measurable random variable that dominates (resp. is dominated by) the family with respect to the natural order between $[-\infty, \infty]$-valued random variables, i.e. $X \leq Y$ if $\mathrm{P}(X \leq Y) = 1$, see [14, Section 5.3.1].

3.1. The one step hedging problem

Recall that $V_{T-1} \geq_{\mathbf{G}_T} V_T$ by definition of a portfolio process. Then, the hedging problem $V_T = \xi$ [1] is equivalent at time $T-1$ to:

$$\begin{aligned}
Ł_T(V_{T-1}) \geq \xi \quad & \Longleftrightarrow V_{T-1}^1 \geq \xi^1 - Ł_T((0, V_{T-1}^{(2)})), \\
& \Longleftrightarrow V_{T-1}^1 \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 - Ł_T((0, V_{T-1}^{(2)} - \xi^{(2)}))\right), \\
& \Longleftrightarrow V_{T-1}^1 \geq \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T((0, \xi^{(2)} - V_{T-1}^{(2)}))\right), \\
& \Longleftrightarrow V_{T-1}^1 \geq F_{T-1}^{\xi}(V_{T-1}^{(2)}),
\end{aligned}$$

where

$$F_{T-1}^{\xi}(y) \quad := \quad \text{ess sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T((0, \xi^{(2)} - y))\right). \tag{3.1}$$

[1] The problem $V_T \geq_{\mathbf{G}_T} \xi$ is equivalent to our one if $\mathbf{G}_T + \mathbf{G}_T \subseteq \mathbf{G}_T$. In general, any V_T such that $V_T \geq_{\mathbf{G}_T} \xi$ may be changed into ξ through an additional cost. So, the formulation $V_T = \xi$ is chosen as we are interested in minimal costs.

By virtue of Proposition 5.7 in Appendix, we may suppose that $F^{\xi}_{T-1}(\omega, y)$ is jointly $\mathcal{F}_{T-1} \times \mathcal{B}(\mathbf{R}^{d-1})$-measurable, l.s.c. as a function of y and convex if $\mathrm{C}_T(s, y)$ is convex in y. As $\mathcal{F}_{T-1}$ is supposed to be complete, we conclude that F^{ξ}_{T-1} is an $\mathcal{F}_{T-1}$ normal integrand, see Definition 5.1 and [26].

3.2. The multi-step hedging problem

We denote by $\mathcal{P}_t(\xi)$ the set of all portfolio processes starting at time $t \le T$ that replicates ξ at the terminal date T:

$$\mathcal{R}_t(\xi) := \left\{(V_s)_{s=t}^T, -\Delta V_s \in L^0(\mathbf{G}_s, \mathcal{F}_s),\ \forall s \ge t+1, V_T = \xi\right\}.$$

The set of replicating prices of ξ at time t is

$$\mathcal{P}_t(\xi) := \left\{V_t = (V_t^1, V_t^{(2)}) : (V_s)_{s=t}^T \in \mathcal{R}_t(\xi)\right\}.$$

The infimum replicating cost is then defined as:

$$c_t(\xi) := \operatorname{ess\,inf}_{\mathcal{F}_t} \left\{\mathrm{C}_t(V_t),\ V_t \in \mathcal{P}_t(\xi)\right\}.$$

By the previous section, we know that $V_{T-1} \in \mathcal{P}_{T-1}(\xi)$ if and only if

$$V^1_{T-1} \ge \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1})\right) \text{ a.s.}$$

Similarly, $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there exists $V^{(2)}_{T-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V^1_{T-2} \ge \operatorname{ess\,sup}_{\mathcal{F}_{T-2}} \left(\operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1})\right) + \mathrm{C}_{T-1}(0, V^{(2)}_{T-1} - V^{(2)}_{T-2})\right).$$

By the tower property satisfied by the conditional essential supremum, we deduce that $V_{T-2} \in \mathcal{R}_{T-2}(\xi)$ if and only if there is $V^{(2)}_{T-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{T-1})$ such that

$$V^1_{T-2} \ge \operatorname{ess\,sup}_{\mathcal{F}_{T-2}} \left(\xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V^{(2)}_{T-1}) + \mathrm{C}_{T-1}(0, V^{(2)}_{T-1} - V^{(2)}_{T-2})\right).$$

Recursively, we get that $V_t \in \mathcal{P}_t(\xi)$ if and only if, for some $V^{(2)}_s \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$, $s = t+1, \cdots, T-1$, and $V^{(2)}_T = \xi^{(2)}$, we have

$$V^1_t \ge \operatorname{ess\,sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t+1}^T \mathrm{C}_s(0, V^{(2)}_s - V^{(2)}_{s-1})\right).$$

In the following, for $u \leq T-1$, $\xi_{u-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{u-1})$, and $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, we introduce the sets

$$\Pi_u^T(\xi_{u-1}, \xi) := \{\xi_{u-1}^{(2)}\} \times \Pi_{s=u}^{T-1} L^0(\mathbf{R}^{d-1}, \mathcal{F}_s) \times \{\xi^{(2)}\}$$

of all families $(V_s^{(2)})_{s=u-1}^{t+1}$ such that $V_{u-1}^{(2)} = \xi_{u-1}^{(2)}$, $V_s^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_s)$ for all $s = u, \cdots, T-1$ and $V_T^{(2)} = \xi^{(2)}$. We set $\Pi_u^T(\xi) := \Pi_u^T(0, \xi) = \Pi_u^T(\xi_{u-1}, \xi)$ when $\xi_{u-1}^{(2)} = 0$. When $u = T$, we set $\Pi_T^T(\xi_{T-1}, \xi) := \{\xi_{T-1}^{(2)}\} \times \{\xi^{(2)}\}$. Therefore, the infimum replicating cost at time 0 is given by

$$c_0(\xi) = \underset{V^2 \in \Pi_0^T(\xi)}{\text{ess inf}_{\mathcal{F}_0}} \, \text{ess sup}_{\mathcal{F}_0} \left(\xi^1 + \sum_{s=0}^{T} \mathrm{C}_s(0, V_s^2 - V_{s-1}^2) \right).$$

For $0 \leq t \leq T$ and $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we define $\gamma_t^\xi(V_{t-1})$ as:

$$\gamma_t^\xi(V_{t-1}) := \underset{V^{(2)} \in \Pi_t^T(V_{t-1}, \xi)}{\text{ess inf}_{\mathcal{F}_t}} \, \text{ess sup}_{\mathcal{F}_t} \left(\xi^1 + \sum_{s=t}^{T} \mathrm{C}_s(0, V_s^{(2)} - V_{s-1}^{(2)}) \right).$$

Note that $\gamma_t^\xi(V_{t-1})$ is the infimum cost to replicate the payoff ξ when starting from the initial risky position $(0, V_{t-1}^{(2)})$ at time t. Observe that $\gamma_t^\xi(V_{t-1})$ does not depend on the first component V_{t-1}^1. Moreover,

$$\gamma_T^\xi(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}).$$

As $\mathbf{G}_T + \mathbf{R}_+^d \subseteq \mathbf{G}_T$, we also observe that $\gamma_T^\xi(V_{T-1}) \geq \gamma_T^0(V_{T-1})$. At last, observe that $c_0(\xi) = \gamma_0^\xi(0)$. Therefore, the main goal of our paper is to study the random functions $(\gamma_t^\xi)_{t=0,1,\cdots,T}$ and to propose conditions under which it is possible to compute them backwardly so that we may estimate $c_0(\xi)$. The main contribution of this section is the following:

Theorem 3.1 (Dynamic Programming Principle)**.** *For any* $0 \leq t \leq T-1$ *and* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$*, we have*

$$\gamma_t^\xi(V_{t-1}) = \underset{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{\text{ess inf}_{\mathcal{F}_t}} \, \text{ess sup}_{\mathcal{F}_t} \left(\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma_{t+1}^\xi(V_t) \right). \quad (3.2)$$

Proof. We denote the right hand side of (3.2) by $\bar{\gamma}_t^\xi(V_{t-1})$. We first verify (3.2) for $t = T-1$. Recall that $\gamma_T^\xi(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)})$ if V_{T-1} belongs to $L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$. It is clear that (3.2) holds for $t = T-1$ by definition of $\gamma^\xi_{T-1}(V_{T-1})$. By induction, let us show that (3.2) holds at time t if this holds at time $t+1$. Let us define

$$f_t(V_{t-1}, V_t) := \text{ess sup}_{\mathcal{F}_t}\left(\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)}) + \gamma^\xi_{t+1}(V_t)\right), t \le T-1.$$

We observe that the collection of random variables

$$\Gamma_t = \{f_t(V_{t-1}, V_t) :\ V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)\}$$

is directed downward, i.e. if $f_t^j = f_t(V_{t-1}, V_t^j) \in \Gamma_t,\ j = 1, 2$, then there exists $f_t \in \Gamma_t$ such that $f_t \le f_t^1 \wedge f_t^2$. Indeed, to see it, it suffices to consider $f_t = f_t(V_{t-1}, V_t)$ where $V_t = V_t^1 1_{\{f_t^1 \le f_t^2\}} + V_t^2 1_{\{f_t^1 > f_t^2\}}$. Therefore, there exists a sequence $(V_t^n)_{n\ge 1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\bar{\gamma}^\xi_t(V_{t-1}) = \inf_n f_t(V_{t-1}, V_t^n)$, see [14, Section 5.3.1]. We deduce for any $\epsilon > 0$, the existence of $\tilde{V}_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\bar{\gamma}^\xi_t(V_{t-1}) + \epsilon \ge f_t(V_{t-1}^{(2)}, \tilde{V}_t^{(2)})$. Similarly, by forward iteration, using the induction hypothesis $\gamma^\xi_r(\tilde{V}_{r-1}) = \bar{\gamma}^\xi_r(\tilde{V}_{r-1}),\ r \ge t+1$, we obtain the existence of $\tilde{V}_r \in L^0(\mathbf{R}^d, \mathcal{F}_r)$ such that $\gamma^\xi_r(\tilde{V}_{r-1}) + \epsilon \ge f_r(\tilde{V}_{r-1}^{(2)}, \tilde{V}_r^{(2)})$, for all $r = t+1, \cdots, T-1$. With $\tilde{V}_{t-1} = V_{t-1}$ and $\tilde{V}_T = \xi$, we deduce that

$$\bar{\gamma}^\xi_t(V_{t-1}) + \epsilon T \ge \text{ess sup}_{F_t}\left(\xi^1 + \sum_{s=t}^{T} \mathrm{C}_s(0, \tilde{V}_s^{(2)} - \tilde{V}_{s-1}^{(2)})\right) \ge \gamma^\xi_t(V_{t-1}).$$

As ϵ goes to 0, we conclude that $\bar{\gamma}^\xi_t(V_{t-1}) \ge \gamma^\xi_t(V_{t-1})$. The reverse inequality is easily obtained by induction and using the assumption that $\bar{\gamma}^\xi_r$ and γ^ξ_t coincide if $r \ge t$ with the tower property. The conclusion follows. □

4. Computational feasibility of the dynamic programming principle

The dynamic programming principle (3.2) allows to get $\gamma^\xi_t(V_{t-1})$ from the cost function C_t and from γ^ξ_{t+1}. In this section, our first main contribution is to show that γ^ξ_t is l.s.c. for any t and convex if the cost functions. Then, we formulate some results allowing to compute ω-wise the essential supremum and the essential infimum of (3.2).

As the term $\mathrm{C}_t(0, V_t^{(2)} - V_{t-1}^{(2)})$ in (3.2) is $\mathcal{F}_t$-measurable, it is sufficient to consider the conditional supremum

$$\theta_t^\xi(V_t) := \text{ess sup}_{\mathcal{F}_t}\, \gamma_{t+1}^\xi(V_t)$$

to compute the essential supremum of (3.2). In the following, we shall use the following notations:

$$D_t^\xi(V_{t-1}, V_t) = \mathrm{C}_t((0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(V_t), \tag{4.3}$$

$$D_t^\xi(S_t, V_{t-1}, V_t) = \mathrm{C}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(S_t, V_t). \tag{4.4}$$

The second notation is used when we stress the dependence on S_t.

4.1. Computational feasibility for convex costs

The following first result ensures the propagation of the lower semicontinuity and convexity of the random function γ_{t+1}^ξ to γ_t^ξ as we shall see in Theorem 4.5. This is a crucial property to compute pointwise the essential infimum in (3.2).

Proposition 4.1. *Suppose that there exists a random $\mathcal{F}_{t+1}$-measurable lower semi-continuous function $\bar{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \bar{\gamma}_{t+1}^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, there exists a random $\mathcal{F}_t$-measurable lower semi-continuous function $\bar{\theta}_t^\xi$ defined on $\mathbf{R}^d$ such that $\theta_t^\xi(V_t) = \bar{\theta}_t^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, the random function $y \mapsto \bar{\theta}_t^\xi(y)$ is a.s. convex if $y \mapsto \bar{\gamma}_{t+1}^\xi(y)$ is a.s. convex.*

Proof. We consider the random function

$$f(z) = z^1 + \bar{\gamma}_{t+1}^\xi((0, z^{(2)})) = z^1 + f((0, z^{(2)})), \quad z \in \mathbf{R}^d.$$

We have $\gamma_{t+1}^\xi(V_t) = f((0, V_t^{(2)}))$ so it suffices to apply Proposition 5.7. □

In order to numerically compute the minimal costs, we need to impose the finiteness of $\gamma_t^\xi(V_{t-1})$, i.e. $\gamma_t^\xi(V_{t-1}) > -\infty$, at any time t, and for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$. This is why we introduce the following condition:

Definition 4.2. *We say that the financial market satisfies the Absence of Early Profit condition (AEP) if, at any time $t \le T$, and for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, $\gamma_t^0(V_t) > -\infty$ a.s..*

Remark 4.3.

1.) Let us comment the condition AEP. Suppose that AEP does not hold, i.e. there is $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ such that $\Lambda_t = \{\gamma_t^0(V_t) = -\infty\}$ satisfies $P(\Lambda_t) > 0$.

Any arbitrarily chosen amount of cash $-n < 0$ allows to hedge the zero payoff at time t on Λ_t when starting from the initial position $(0, V_t^2)$ by definition of $\gamma_t^0(V_t) = -\infty$. Then, at time t, we may obtain an arbitrarily large profit on Λ_t as follows: We write $0 = ((0, V_t^2) - ne_1)\, 1_{\Lambda_t} + a_{t-1}^n$ where $a_{t-1}^n = (ne_1 - (0, V_t^2))\, 1_{\Lambda_t}$. The position $(0, V_t^2) - ne_1$ allows to get the zero claim at time T. Moreover, $Ł_t(a_{t-1}^n) = n1_{\Lambda_t} + Ł_t((0, V_t^2))1_{\Lambda_t}$ tends to $+\infty$ as $n \to \infty$ on Λ_t, i.e. it is possible to make an early profit at time t, as large as possible.
2.) If $\xi \in L^0(\mathbf{R}_+^d, \mathcal{F}_T)$, then $\gamma_t^\xi(V_{t-1}) \geq \gamma_t^0(V_{t-1}) > -\infty$ under AEP.
3.) Under Assumptions 4 and 5 below, condition AEP holds by Lemma 5.23. $\triangle$

Assumption 1. *The payoff ξ is hedgeable, i.e. there exists a portfolio process $(V_u^\xi)_{u=0}^T$ such that $\xi = V_T^\xi$.*

Lemma 4.4. *Under Assumption 1, $\gamma_t^\xi(V_{t-1}) < \infty$ for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$.*

Proof. We observe that the amount of capital $\alpha_t = \mathrm{C}_t(V_t^\xi - (0, V_{t-1}^{(2)}))$ allows one to get the position $V_t^\xi - (0, V_{t-1}^{(2)})$. Therefore, starting from the initial position $(0, V_{t-1}^{(2)})$, the capital $\mathrm{C}_t(V_t^\xi - (0, V_{t-1}^{(2)}))$ is enough to get V_t^ξ and then ξ at time T since $V_T^\xi = \xi$. We then deduce that

$$\gamma_t^\xi(V_{t-1}) \leq \alpha_t \leq h_t(S_t, V_t^\xi - (0, V_{t-1}^{(2)})) < \infty.$$

$\square$

The following theorem states that convexity and lower semicontinuity propagates backwardly from γ_{t+1}^ξ to γ_t^ξ.

Theorem 4.5. *Suppose that Assumption 1 and condition AEP hold. Suppose that there exists a random $\mathcal{F}_{t+1}$-measurable lower semi-continuous convex function $\bar{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \bar{\gamma}_{t+1}^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Suppose that the cost function $\mathrm{C}_t(s, z)$ is convex in z. Then, there exists a random $\mathcal{F}_t$-measurable lower semi-continuous convex function $\bar{\gamma}_t^\xi$ defined on $\mathbf{R}^d$ such that $\gamma_t^\xi(V_{t-1}) = \bar{\gamma}_t^\xi(V_{t-1})$ for all $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and we have:*

$$\bar{\gamma}_t^\xi(v_{t-1}) = \inf_{y \in \mathbf{R}^d} \left(\mathrm{C}_t(0, y^{(2)} - v_{t-1}^{(2)}) + \bar{\theta}_t^\xi(y) \right),$$

where $\bar{\theta}_t^\xi$ is given by Proposition 4.1.

Proof. By Proposition 4.1, we deduce that $\theta_t^\xi(V_t) = \bar{\theta}_t^\xi(V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ where $\bar{\theta}_t^\xi$ is an $\mathcal{F}_t$-measurable lower semi-continuous convex function. Therefore, $\bar{D}_t(v_{t-1}, v_t) := \mathrm{C}_t(0, v_t^{(2)} - v_{t-1}^{(2)}) + \bar{\theta}_t^\xi(v_t)$ is an $\mathcal{F}_t$-measurable l.s.c. convex function in (v_{t-1}, v_t). By Proposition 5.20, $\gamma_t^\xi(V_{t-1}) = \phi_t(V_{t-1})$ where $\phi_t(\omega, v_{t-1})$ is $\mathcal{F}_t \otimes \mathbf{R}^d$-measurable. We claim that $\phi_t(\omega, \cdot) > -\infty$ a.s.. Otherwise, by measurable selection argument, we may find an $\mathcal{F}_t$-measurable selection V_{t-1} such that $-\infty = \phi_t(V_{t-1}) = \gamma_t^\xi(V_{t-1})$ on a non null set. This is in contradiction with the AEP condition. Similarly, by Lemma 4.4, we deduce that $\phi_t(\omega, \cdot) < \infty$ a.s.. Therefore, the random function $\phi_t(\omega, \cdot)$ only takes finite values a.s.. By Proposition 5.20, we finally conclude that $\bar{\gamma}_t^\xi(v_{t-1}) = \phi_t(v_{t-1})$ is a real-valued random convex function. In particular, $\bar{\gamma}_t^\xi$ is continuous. □

Remark 4.6. Suppose that the cost functions $\mathrm{C}_t(s, z)$, $t \le T$, are convex in z. Under Assumption 1, as $\gamma_T^\xi(V_{T-1}) = \xi^1 + \mathrm{C}_T(0, \xi^{(2)} - V_{T-1}^{(2)}))$ is l.s.c. and convex in V_{T-1}, we deduce that Theorem 4.5 applies backwardly step by step. In particular, it is possible to compute $\gamma_t^\xi(v_{t-1})$ at any time t as a ω-wise infimum. △

In the following, we consider conditions under which it is possible to compute ω-wise the essential supremum θ_t^ξ. The main ingredient is the knowledge of the conditional support $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ of S_{t+1} knowing $\mathcal{F}_t$. Recall that $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ is the smallest $\mathcal{F}_t$-measurable random closed set that contains $S_{t+1}(\omega)$ a.s., see [9].

Assumption 2. *For each $t \le T-1$, there exists a family of Borel functions $(\alpha_t^m)_{m \ge 1}$ defined on $\mathbf{R}^m$ such that $\mathrm{supp}_{\mathcal{F}_t} S_{t+1}$ admits the Castaing representation $(\alpha_t^m(S_t))_{m \ge 1}$, i.e. $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \mathrm{cl}(\alpha_t^m(S_t))_{m \ge 1}$.*

Proposition 4.7. *Suppose that there exists a lower semi-continuous function $\tilde{\gamma}_{t+1}^\xi$ defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that $\gamma_{t+1}^\xi(V_t) = \tilde{\gamma}_{t+1}^\xi(S_{t+1}, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, $\theta_t^\xi(V_t) = \sup_{z \in supp_{\mathcal{F}_t} S_{t+1}} \tilde{\gamma}_{t+1}^\xi(z, V_t)$. Moreover, under Assumption 2, there exists a function $\tilde{\theta}_t^\xi(s, v)$ defined on $(s, v) \in \mathbf{R}^m \times \mathbf{R}^d$, which is l.s.c. in v, such that $\theta_t^\xi(V_t) = \tilde{\theta}_t^\xi(S_t, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and we have:*

$$\tilde{\theta}_t^\xi(s, v) := \sup_m \tilde{\gamma}_{t+1}^\xi(\alpha_m(s), v) \quad (s, v) \in \mathbf{R}^m \times \mathbf{R}^d.$$

At last, $\tilde{\theta}_t^\xi(s, v)$ is l.s.c. in (s, v) if the functions $(\alpha_m)_{m \ge 1}$ are continuous and, if $\tilde{\gamma}_{t+1}^\xi(s, v)$ is convex in v, then $\tilde{\theta}_t^\xi(s, v)$ is convex in v.

Proof. The proof is immediate by Proposition 5.6 and Lemma 5.8. □

Assumption 3. *For each $t \leq T-1$, there exists a family of Borel functions $(\alpha_t^m)_{m\geq 1}$ such that $S_{t+1} \in \{\alpha_t^m(S_t) : m \geq 1\}$ a.s. and $\mathrm{P}(S_{t+1} = \alpha_t^m(S_t)|\mathcal{F}_t) > 0$ a.s. for all $m \geq 1$.*

Proposition 4.8. *Suppose that there exists a Borel function $\tilde{\gamma}_{t+1}^{\xi}$ defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that $\gamma_{t+1}^{\xi}(V_t) = \tilde{\gamma}_{t+1}^{\xi}(S_{t+1}, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Then, under Assumption 3 , there exists a Borel function $\tilde{\theta}_t^{\xi}(s,v)$ defined on $(s,v) \in \mathbf{R}^m \times \mathbf{R}^d$ such that $\theta_t^{\xi}(V_t) = \tilde{\theta}_t^{\xi}(S_t, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ and we have:*

$$\tilde{\theta}_t^{\xi}(s,v) := \sup_m \tilde{\gamma}_{t+1}^{\xi}(\alpha_m(s), v) \quad (s,v) \in \mathbf{R}^m \times \mathbf{R}^d.$$

Proof. The proof is immediate by Lemma 5.19. Note that we do not suppose that C_t is convex to obtain this result. □

Corollary 4.9. *Assume that the assumptions of Proposition 4.7 or Proposition 4.8 hold and Condition AEP holds. Suppose that $\tilde{\gamma}_{t+1}^{\xi}(s,v)$ is convex in v. Then, $\gamma_t^{\xi}(V_{t-1}) = \tilde{\gamma}_t^{\xi}(S_t, V_{t-1})$ where $\tilde{\gamma}_t^{\xi}(s,v)$ is l.s.c. and convex in v. Moreover,*

$$\tilde{\gamma}_t^{\xi}(s,v) = \inf_{y\in\mathbf{R}^d}\left(\mathrm{C}_t(s,(0,y^{(2)}-v^{(2)})) + \sup_m \tilde{\gamma}_{t+1}^{\xi}(\alpha_m(s),y)\right).$$

Proof. Under our assumptions, $\theta_t^{\xi}(V_t) = \tilde{\theta}_t^{\xi}(S_t, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ where $\tilde{\theta}_t^{\xi}(s,v) = \sup_m \tilde{\gamma}_{t+1}^{\xi}(\alpha_m(s), v)$ by Proposition 4.7 or Proposition 4.8. As a supremum, $\tilde{\theta}_t^{\xi}(s,v)$ is convex in v if so $\tilde{\gamma}_{t+1}^{\xi}(s,v)$ is. As $\mathrm{C}_t(s,y)$ is also convex in y, we deduce that $D_t^{\xi}(y,v) = \mathrm{C}_t(s,(0,y^{(2)}-v^{(2)}))+\tilde{\theta}_t^{\xi}(s,y)$ is convex in (y,v). By Proposition 5.20 under AEP, $\tilde{\gamma}_t^{\xi}(s,v) = \inf_{y\in\mathbf{R}^d} D_t^{\xi}(y,v) \in \mathbf{R}$ is convex in v hence it is continuous. □

4.2. Computational feasibility under strong AIP no-arbitrage condition

The results of Section 4.1 are not a priori sufficient to compute backwardly θ_{t-1}^{ξ} as we need $\gamma_t^{\xi}(s,v)$ be l.s.c. in s, see Proposition 4.7. This is why, we introduce the following conditions.

Assumption 4. *The payoff function ξ is of the form $\xi = g(S_T)$, where $g \in \mathbf{R}_+^d$ is continuous. Moreover, ξ is hedgeable, i.e. there exists a portfolio process $(V_u^{\xi})_{u=0}^T$ such that $\xi = V_T^{\xi}$.*

Assumption 5. *The conditional support is such that* $\mathrm{supp}_{\mathcal{F}_t} S_{t+1} = \phi_t(S_t)$ *where* ϕ_t *is a set-valued lower hemicontinuous function, see Definition 5.11, with compact values such that* $\phi_t(S_t) \subseteq \bar{B}(0, R_t(S_t))$ *where* R_t *is a continuous function on* $\mathbf{R}^m$*.*

Note that under Assumption 2, $\phi_t(S_t) = \mathrm{cl}\{\alpha_m(S_t) : \ m \geq 1\}$ defines a set-valued lower hemicontinuous function if the functions $(\alpha_m)_{m\geq 1}$ are continuous, see Lemma 5.15.

Definition 4.10. *We say that the condition AIP holds at time* t *if the minimal cost* $c_t(0) = \gamma_t^0(0)$ *of the European zero claim* $\xi = 0$ *is* 0 *at time* $t \leq T$*. We say that AIP holds if AIP holds at any time.*

The condition AIP has been introduced for the first time in the paper [2]. This is a weak no-arbitrage condition which is clearly satisfied in the real financial markets i.e. the price of a non negative payoff is non negative.

Lemma 4.11. *Suppose that the cost functions are either sub-additive or super-additive. Then, AIP implies AEP.*

Proof. We prove it in the case where the cost function is sub-additive, the supper-additive case is similar. Suppose that AIP holds and $\mathrm{C}_t(s, v)$ is sub-additive in v. For any $V_t, \tilde{V}_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we have:

$$\begin{aligned} D_t^0(S_t, V_t, \tilde{V}_t) &= \mathrm{C}_t(S_t, \tilde{V}_t - V_t) + \theta_t^0(S_t, \tilde{V}_t), \\ &\geq \mathrm{C}_t(S_t, \tilde{V}_t) + \theta_t^0(S_t, \tilde{V}_t) - \mathrm{C}_t(S_t, V_t), \\ &= D_t^0(S_t, 0, \tilde{V}_t) - \mathrm{C}_t(S_t, V_t). \end{aligned}$$

Under AIP, $D_t^0(S_t, 0, \tilde{V}_t) \geq 0$ hence $D_t^0(S_t, V_t, \tilde{V}_t) \geq -\mathrm{C}_t(S_t, V_t)$. We deduce that $\gamma_t^0(V_t) = \mathrm{ess\,inf}_{\tilde{V}_t} D_t^0(S_t, V_t, \tilde{V}_t) \geq -\mathrm{C}_t(S_t, V_t) > -\infty$. □

Definition 4.12. *We say that the condition SAIP (Strong AIP condition) holds at time* t *if AIP holds at time* t *and, for any* $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*, we have* $D_t^0(S_t, 0, Z_t) = 0$ *if and only if* $Z_t^{(2)} = 0$ *a.s.. We say that SAIP holds if SAIP holds at any time.*

Recall that $D_t^0(S_t, 0, Z_t)$ is given by (4.4) and it is the minimal cost expressed in cash that is needed at time t to hedge the zero payoff when we start from the initial strategy $V_t = (\theta_t^0(Z_t), Z_t^{(2)})$, initial value of a portfolio process $(V_u)_{t\leq u\leq T}$ such that $V_T = 0$. Therefore, the condition SAIP states that the minimal cost of the zero payoff is 0 at time t and this minimal cost is only attained by the zero strategy $V_t = 0$. This is intuitively clear as soon as any non null transaction implies positive costs.

The following proposition shows that the classical Robust No Arbitrage NA^{r} ([14, Chapter 3]) used to characterize the super hedging prices in the Kabanov model with proportional transaction costs is stronger than the SAIP condition.

Proposition 4.13. *Suppose that* $\mathrm{int}\,\mathbf{G}_{\mathrm{t}}^* \neq \emptyset$ *for any* $t \leq T$*. Then,* NA^{r} *implies SAIP.*

Proof. Recall that NA^{r} is equivalent to the existence of a martingale $(K_s)_{s\leq T}$ such that $K_s \in \mathrm{int}\,\mathbf{G}_{\mathrm{s}}^*$, [14, Theorem 3.2.1]. Consider $Z_{T-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$. As $D_{T-1}(0, Z_{T-1}) = D_{T-1}(0,(0, Z_{T-1}^{(2)}))$, we may suppose that $Z_{T-1} = (0, Z_{T-1}^{(2)})$. By the definition of C_u, there exists $\tilde{g}_u \in L^0(\mathbf{G}_u, \mathcal{F}_u)$, $u = T-1, T$, such that:

$$\mathrm{C}_{T-1}((0, Z_{T-1}^{(2)}))e^1 - g_{T-1} = (0, Z_{T-1}^{(2)})$$
$$\mathrm{C}_{T}((0, -Z_{T-1}^{(2)}))e^1 - \tilde{g}_{T} = (0, -Z_{T-1}^{(2)}).$$

Adding these equalities, we get that $D_{T-1}(0, Z_{T-1})e^1 = g_{T-1} + g_T$ for some $g_T \in L^0(\mathbf{G}_T, \mathcal{F}_T)$, see (4.3). So, we get that $K_T D_{T-1}(0, Z_{T-1})e^1 \geq K_T g_{T-1}$ and, taking the generalized conditional expectation w.r.t $\mathcal{F}_{T-1}$, we deduce that $K_{T-1}D_{T-1}(0, Z_{T-1})e^1 \geq K_{T-1}g_{T-1} \geq 0$. Since $K_{T-1}e^1 = K_{T-1}^1 > 0$, AIP holds at time $T-1$. Moreover, $g_{T-1} \neq 0$ a.s. as soon as $Z_{T-1}^{(2)} \neq 0$. Since $K_{T-1} \in \mathrm{int}\,\mathbf{G}_{T-1}^*$, we finally deduce that

$$K_{T-1}D_{T-1}^0(S_t, 0, Z_{T-1})e^1 \geq K_{T-1}g_{T-1} > 0$$

as soon as $Z_{T-1}^{(2)} \neq 0$, which means that SAIP holds at time $T-1$.

Suppose that we have already shown SAIP for $s \geq t+1$. For a given $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, we consider $g_t \in L^0(\mathbf{G}_t, \mathcal{F}_t)$ such that

$$\mathrm{C}_t((0, Z_t^{(2)}))e^1 - g_t = (0, Z_t^{(2)}). \tag{4.5}$$

Since AIP holds at time $t+1$, by Lemma 4.11, we have $\gamma_{t+1}(Z_t) > -\infty$ under AEP. Since the family $\{D_{t+1}^0(Z_t, Z_{t+1}), Z_{t+1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})\}$ is directed downward, we deduce the existence of a sequence $Z_{t+1}^n \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$, $n \in \mathbb{N}$ such that

$$\gamma_{t+1}^0(Z_t) = \mathrm{ess\,inf}_{Z_{t+1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t+1})} D_{t+1}^0(Z_t, Z_{t+1}) = \inf_n D_{t+1}^0(Z_t, Z_{t+1}^n) > -\infty \text{ a.s.}$$

We deduce that, for any $\epsilon > 0$, there exists $Z^{\epsilon}_{t+1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t+1})$ such that $\gamma^0_{t+1}(Z_t) + \epsilon \geq D^0_{t+1}(Z_t, Z^{\epsilon}_{t+1})$. Proceeding forward with the induction hypothesis, we construct a sequence $g^{\epsilon}_s \in L^0(\mathbf{G}_s, \mathcal{F}_s)$, $s \geq t+1$, such that

$$(D^0_t(0, Z_t) + \epsilon T)e^1 = g_t + \sum_{s=t+1}^{T} g^{\epsilon}_s.$$

Therefore, multiplying by $K_T \in \mathbf{G}^*_T$ and then taking the (generalized) conditional expectation knowing $\mathcal{F}_{T-1}$, we get that

$$K_T(D^0_t(0, Z_t) + \epsilon T)e^1 \geq \; K_T \left(g_t + \sum_{s=t+1}^{T-1} g^{\epsilon}_s \right),$$

$$K_{T-1}(D^0_t(0, Z_t) + \epsilon T)e^1 \geq K_{T-1} \left(g_t + \sum_{s=t+1}^{T-1} g^{\epsilon}_s \right).$$

By successive iterations, we finally get that $K_t(D^0_t(0, Z_t) + \epsilon T)e^1 \geq K_t g_t$. Since g_t does not depend on ϵ, see its definition in (4.5), we deduce as $\epsilon \to 0$, that $K_t D^0_t(0, Z_t)e^1 \geq K_t g_t \geq 0$ and $K_t D^0_t(0, Z_t)e^1 > 0$ if $g_t \neq 0$ when $Z^{(2)}_t \neq 0$. Therefore, SAIP holds at time t and we may conclude. □

The following result is the last main contribution of this section: It states that the minimal cost function γ^{ξ}_t is a l.s.c. function of S_t and V_{t-1}, i.e. γ^{ξ}_t inherits from the lower-semicontinuity of γ^{ξ}_{t+1}, under Assumption 4 and 5, if SAIP holds as we shall see. We introduce the notation

$$S^{d-1}(0,1) = \{z \in \mathbf{R}^d : \; z^1 = 0 \,\text{and}\, |z| = 1\}.$$

Theorem 4.14. *Suppose that* C_t *is positively super* δ*-homogeneous. Suppose that there exists a lower semi-continuous function* $\tilde{\gamma}^{\xi}_{t+1}$ *defined on* $\mathbf{R}^m \times \mathbf{R}^d$ *such that* $\gamma^{\xi}_{t+1}(V_t) = \tilde{\gamma}^{\xi}_{t+1}(S_{t+1}, V_t)$ *for all* $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. *Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost function* $\mathrm{C}_t(s, z)$ *is l.s.c. in* (s, z) *and* C_t *is either super-additive or sub-additive. Then, if* $\inf_{z \in S^{d-1}(0,1)} D^0_t(S_t, 0, z) > 0$, $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(S_t, V_{t-1})$ *where* $\tilde{\gamma}^{\xi}_t(s, v_{t-1})$ *is l.s.c. in* (s, v_{t-1}).

Proof. Since $\tilde{\gamma}^{\xi}_{t+1}(s, v)$ is lower semi-continuous in s, we deduce that $\theta^{\xi}_t(V_t) = \tilde{\theta}^{\xi}_t(S_t, V_t)$ by Proposition 5.6, for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$, where

$$\tilde{\theta}^{\xi}_t(s, v) = \sup_{z \in \phi_t(S_t)} \tilde{\gamma}^{\xi}_{t+1}(z, v).$$

As ϕ_t is lower hemicontinuous by assumption, we deduce by [1, Lemma 17.29] that $\tilde{\theta}_t^\xi(s,v)$ is l.s.c. in (s,v). Therefore, the function

$$D_t^\xi(s,v_{t-1},v_t) = \mathrm{C}_t(s,(0,v_t^{(2)}-v_{t-1}^{(2)})) + \tilde{\theta}_t^\xi(s,v_t)$$

is l.s.c. in (s,v_{t-1},v_t) by assumption on C_t. By Lemma 5.5, we get that $\gamma_t^\xi(V_{t-1}) = \tilde{\gamma}_t^\xi(S_t,V_{t-1})$ where $\tilde{\gamma}_t^\xi(s,v_{t-1}) = \inf_{v_t\in\mathbf{R}^d} D_t^\xi(s,v_{t-1},v_t)$. The next step is to show that $\tilde{\gamma}_t^\xi(s,v_{t-1}) = \inf_{v_t\in\phi_t(s,v_{t-1})} D_t^\xi(s,v_{t-1},v_t)$ where ϕ_t is a set-valued upper hemicontinuous function, see Definition 5.10, with compact values. We then conclude that $\tilde{\gamma}_t^\xi(s,v_{t-1})$ is l.s.c. in (s,v_{t-1}) by Proposition 5.17.

To obtain ϕ_t, first observe that $\gamma_t^\xi(V_{t-1}) \le D_t^\xi(s,v_{t-1},0)$ hence we get that $\gamma_t^\xi(V_{t-1}) = \tilde{\gamma}_t^\xi(S_t,V_{t-1})$ where $\tilde{\gamma}_t^\xi(s,v_{t-1}) = \inf_{v_t\in K_t(s,v_{t-1})} D_t^\xi(s,v_{t-1},v_t)$ and

$$K_t(s,v_{t-1}) = \left\{ v_t \in \mathbf{R}^d : \ D_t^\xi(s,v_{t-1},v_t) \le D_t^\xi(s,v_{t-1},0) \right\}.$$

Since C_T is increasing w.r.t. $\mathbf{R}^d_+$, we deduce that $D_t^\xi(s,v_{t-1},v_t) \ge D_t^0(s,v_{t-1},v_t)$. Moreover,

$$D_t^0(s,v_{t-1},v_t) = \mathrm{C}_t(s,(0,v_t^{(2)}-v_{t-1}^{(2)}))+\theta_t^0(s,v_t) \ge \mathrm{C}_t(s,(0,-v_{t-1}^{(2)}))+D_t^0(s,0,v_t)$$

in the case where C_t is super-additive and, if C_t is sub-additive, we have

$$D_t^0(s,v_{t-1},v_t) = \mathrm{C}_t(s,(0,v_t^{(2)}-v_{t-1}^{(2)}))+\theta_t^0(s,v_t) \ge -\mathrm{C}_t(s,(0,v_{t-1}^{(2)}))+D_t^0(s,0,v_t).$$

As C_t is dominated by a continuous function by hypothesis, we get that $D_t^0(s,v_{t-1},v_t) \ge \tilde{h}_t(s,v_{t-1}) + D_t^0(s,0,v_t)$ where $\tilde{h}_t$ is a continuous function. Moreover, by Lemma 5.21, if $|v_t| \ge 1$,

$$D_t^0(s,0,v_t) \ge \delta(|v_t|)D_t^0(s,0,v_t/|v_t|) \ge \delta(|v_t|) \inf_{z\in S^{d-1}(0,1)} D_t^0(s,0,z). \quad (4.6)$$

By Lemma 5.22, $|D_t^\xi(s,v_{t-1},0)| \le \hat{h}_t^\xi(s,v_{t-1})$ for some continuous function $\hat{h}_t^\xi \ge 0$. Recall that $\inf_{z\in S^{d-1}(0,1)} D_t^0(S_t,0,z) > 0$ a.s. by assumption. It follows that $K_t(s,v_{t-1}) \subseteq \phi_t(s,v_{t-1}) := \bar{B}_t(0, r_t(s,v_{t-1})+1)$ where

$$\begin{aligned} r_t(s,v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(s,v_{t-1}}{i_t(s)}\right), \\ i_t(s) &:= \inf_{z\in S^{d-1}(0,1)} D_t^0(s,0,z), \ \lambda_t(s,v_{t-1}) = |\tilde{h}_t(s,v_{t-1})| + \hat{h}_t^\xi(s,v_{t-1}). \end{aligned}$$

Since λ_t is continuous and i_t is l.s.c. by Proposition 5.17, we deduce that λ_t/i_t is u.s.c. on the open set $\mathcal{O}_t := \{(s, v_{t-1}) \in \mathbf{R}^m \times \mathbf{R}^d : \ i_t(s, v_{t-1}) > 0\}$. As δ^{-1} is continuous and increasing, we finally get that r_t is also u.s.c. in $(s, v_{t-1}) \in \mathcal{O}_t$. By Lemma 5.12, we deduce that the function ϕ_t is upper hemicontinuous in $(s, v_{t-1}) \in \mathcal{O}_t$. Therefore, $\tilde{\gamma}_t^{\xi}(s, v_{t-1}) = \inf_{v_t \in \phi_t(s, v_{t-1})} D_t^{\xi}(s, v_{t-1}, v_t)$ is l.s.c. on $\mathcal{O}_t$ by Proposition 5.17. Observe that $(S_t, z) \in \mathcal{O}_t$ for all $z \in S(0,1)$ a.s. under our hypothesis. By Lemma 5.23, $\gamma_t^{\xi}(S_t, V_{t-1}) \geq \bar{h}_t(S_t, V_{t-1})$ for some continuous function $\bar{h}_t$. Therefore, replace $\tilde{\gamma}_t^{\xi}(s, v_{t-1})$ by $\tilde{\gamma}_t^{\xi}(s, v_{t-1}) \vee \bar{h}_t(s, v_{t-1})$ so that, w.l.o.g., we assume that $\tilde{\gamma}_t^{\xi}(s, v_{t-1}) \geq \bar{h}_t(s, v_{t-1})$. By Lemma 5.18, it is then possible to extend $\tilde{\gamma}_t^{\xi}$ as a l.s.c. function on the whole space $\mathbf{R}^m \times \mathbf{R}^d$. The conclusion follows. $\square$

The following result asserts that the SAIP condition and the condition $\inf_{z \in S^{d-1}(0,1)} D_t^0(S_t, 0, z) > 0$, both with AIP, are actually equivalent.

Theorem 4.15. *Assume that Assumption 4 holds. Suppose that either Assumption 5 holds or the cost functions* $\mathrm{C}_t(s,z)$ *are convex in* z*. Suppose that the cost functions* $\mathrm{C}_t(s,z)$ *are l.s.c. in* (s,z) *and* $\mathrm{C}_t(s,z)$ *are either super-additive or sub-additive, for any* $t \leq T$*. Then, the following statements are equivalent:*

1.) SAIP.

2.) AIP holds and $\inf_{z \in S^{d-1}(0,1)} D_t^0(S_t, 0, z) > 0$ *a.s..*

Proof. Let us show that 1.) implies 2.). Suppose first that Assumption 5 holds. As $\gamma_T^0(Z_T) = \mathrm{C}_T(0, -Z_T^{(2)})$ is l.s.c. in Z_T, we deduce by Proposition 4.1 that $\theta_{T-1}^0(Z_{T-1})$ is l.s.c. in Z_{T-1}. Therefore, $D_{T-1}^0(S_{T-1}, Z_{T-2}, Z_{T-1})$ is l.s.c. in (Z_{T-2}, Z_{T-1}). By lower-semicontinuity on the compact set $S^{d-1}(0,1)$ and by a measurable selection argument, there exists $\hat{Z}_{T-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})$ such that

$$\inf_{z \in S^{d-1}(0,1)} D_{T-1}^0(S_{T-1}, 0, z) = D_{T-1}^0(S_{T-1}, 0, \hat{Z}_{T-1}).$$

Moreover, $D_{T-1}^0(S_{T-1}, 0, \hat{Z}_{T-1}) > 0$, i.e. $\inf_{z \in S^{d-1}(0,1)} D_{T-1}^0(S_{T-1}, 0, z) > 0$ under SAIP. By Theorem 4.14, we deduce that $\gamma_{T-1}^0(S_{T-1}, Z_{T-2})$ is l.s.c. in Z_{T-2}. By Proposition 4.1, we deduce that $\theta_{T-2}^0(Z_{T-2})$ is l.s.c. in Z_{T-2}. Therefore, $D_{T-2}^0(S_{T-2}, Z_{T-3}, Z_{T-2})$ is l.s.c. in (Z_{T-3}, Z_{T-3}) and, as previously, we deduce that $\inf_{z \in S^{d-1}(0,1)} D_{T-2}^0(S_{T-2}, 0, z) > 0$ under SAIP. Then, we may proceed by induction by virtue of Theorem 4.14 and Proposition 4.1.

At last, if the cost functions are convex, recall that AEP holds by Lemma 4.11. Then, it suffices to apply Theorem 4.5 and Proposition 4.1 to deduce that $D_t^0(S_t,0,z)$ is l.s.c. in z so that we may conclude similarly.

Let us show that 2.) implies 1.) Suppose that $D_t^0(S_t,0,Z_t)=0$ for some $Z_t\in L^0(\mathbf{R}^d\setminus\{0\},\mathcal{F}_t)$. By Lemma 5.21,

$$D_t^0(S_t,0,Z_t)\geq \delta(|Z_t|)D_t^0(S_t,0,Z_t/|Z_t|)\geq \delta(|Z_t|)\inf_{z\in S^{d-1}(0,1)}D_t^0(S_t,0,z)>0.$$

This yields a contradiction hence the conclusion follows under Assumption 5. □

We then conclude that, under SAIP, the dynamic programming principle allows to compute $\tilde{\gamma}_t^\xi$ backwardly so that it is possible to deduce the minimal hedging price $c_0(\xi)=\gamma_0^\xi(0)$.

Theorem 4.16. *Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost functions are l.s.c. and either super-additive of sub-additive. Then, under the condition SAIP, there exists l.s.c. functions $\tilde{\gamma}_t^\xi$ defined on $\mathbf{R}^m\times\mathbf{R}^m$ such that, for all $V_{t-1}\in L^0(\mathbf{R}^d,\mathcal{F}_{t-1})$, $\gamma_t^\xi(V_{t-1})=\tilde{\gamma}_t^\xi(S_t,V_{t-1})$. Moreover, the dynamic programming principle 3.2 is computable ω-wise as:*

$$\gamma_t^\xi(S_t,V_{t-1})=\inf_{y\in\mathbf{R}}\left(\mathrm{C}_t(S_t,(0,y^{(2)}-V_{t-1}^{(2)}))+\sup_{s\in\phi_t(S_t)}\gamma_{t+1}^\xi(s,y)\right),$$

where $\phi_t(S_t)=\mathrm{supp}_{\mathcal{F}_t}S_{t+1}$. Also, the infimum hedging cost of ξ at any time t is reached, i.e. $\gamma_t^\xi(V_{t-1})$ is a mimimal cost.

4.3. The case of fixed transaction costs

In the case of fixed costs, the cost functions C_t, $t\leq T$, are not convex in general. Moreover, C_t is a priori positively lower homogeneous, i.e. for any $\lambda\geq 1$, $\mathrm{C}_t(\lambda z)\leq\lambda\mathrm{C}_t(z)$. Then, C_t does not satisfy the assumptions we impose in this paper. Nevertheless, we shall see in this section that we may also implement the dynamic programming principle under a robust SAIP condition imposed on the enlarged market with only proportional transaction costs.

To do so, recall that for a l.s.c. function g, the horizon function (see [26, Section 3.C]) g^∞ of g is defined as:

$$g^\infty(y):=\liminf_{\alpha\to\infty}\frac{g(\alpha y)}{\alpha}.$$

Recall that g^∞ is positively homogeneous and l.s.c. in y. We then define the *horizon* cost function as

$$\hat{\mathrm{C}}_t(s,y) = \mathrm{C}_t^\infty(s,y) = \liminf_{\alpha\to\infty} \frac{\mathrm{C}_t(s,\alpha y)}{\alpha}. \tag{4.7}$$

The liquidation value associated to the cost function $\hat{\mathrm{C}}_t$ is then given by

$$\hat{\text{Ł}}_t(s,y) = \limsup_{\alpha\to\infty} \frac{\text{Ł}_t(s,\alpha y)}{\alpha}.$$

Note that in the case where $\hat{\mathrm{C}}_t(s,y) = \lim_{\alpha\to\infty} \dfrac{\mathrm{C}_t(s,\alpha y)}{\alpha}$, then $\hat{\text{Ł}}_t = \text{Ł}_t^\infty$. Moreover, if $\hat{\mathrm{C}}_t$ is subadditive, we deduce that

$$\hat{\mathbf{G}}_t(\omega) := \{z:\ \hat{\text{Ł}}_t(S_t(\omega), z) \geq 0\}$$

is an $\mathcal{F}_t$-measurable random positive closed cone. We then deduce that the enlarged market defined by the solvency sets $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ corresponds to a model with proportional transaction costs, as defined in [14][Section 3]. The cash invariance property propagates from C_t to $\hat{\mathrm{C}}_t$. In that case, we may verify that $\hat{\text{Ł}}_t(s,z) = \max\{\alpha \in \mathbf{R}:\ z - \alpha e_1 \in \hat{\mathbf{G}}_t\}$ and similarly, we have $\hat{\mathrm{C}}_t(s,z) = \min\{\alpha \in \mathbf{R}:\ \alpha e_1 - z \in \hat{\mathbf{G}}_t\}$. We then deduce the following:

Lemma 4.17. *Suppose that C_t is cash invariant. Then, $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$ if and only if $\hat{\mathrm{C}}_t(S_t, z) \leq \mathrm{C}_t(S_t, z)$ for any z a.s..*

Proof. First suppose that $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$. As $\mathrm{C}_t(S_t,z)e_1 - z \in \mathbf{G}_t$, then we get that $\mathrm{C}_t(S_t,z)e_1 - z \in \hat{\mathbf{G}}_t$. Therefore, we deduce that

$$\hat{\mathrm{C}}_t(s,z) = \min\{\alpha \in \mathbf{R}:\ \alpha e_1 - z \in \hat{\mathbf{G}}_t\} \leq \mathrm{C}_t(S_t, z).$$

Reciprocally, if $\hat{\mathrm{C}}_t \leq \mathrm{C}_t$, then $\hat{\text{Ł}}_t \geq \text{Ł}_t$ hence $\mathbf{G}_t \subseteq \hat{\mathbf{G}}_t$. □

Note that in [19], such an enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ is studied and $\hat{\text{Ł}}_t$ is the liquidation value of the closed conic hull K_t of $\mathbf{G}_t$, i.e. $\hat{\mathbf{G}}_t = \mathrm{K}_t$.

Example 4.18. The market is composed of one bond whose price is $B_t = 1$ and $d-1$ risky assets, $d \geq 2$, whose prices are described by a family of bid and ask prices and fixed costs $S = ((S^{b,i}, S^{a,i}, c^i))_{i=2,\cdots,d}$. In the following, we denote by $s = ((s^{b,i}, s^{a,i}, c^i))_{i=2,\cdots,d}$ any element of $\mathbf{R}^{3(d-1)}$. We consider the fixed costs model defined by the following liquidation process:

$$
\begin{aligned}
Ł_t(s,y) &:= y^1 + \sum_{i=2}^{d} Ł_t^i(s^{b,i}, s^{a,i}, c^i, y^i),\ (s,y) \in \mathbf{R}^{3(d-1)} \times \mathbf{R}^d, \\
Ł_t^i(s^{b,i}, s^{a,i}, c^i, y^i) &:= \left(y^i s^{b,i} - c_t^i\right)^+ 1_{y^i>0} + \left(y^i s^{a,i} - c_t^i\right) 1_{y^i<0}.
\end{aligned}
$$

Note that the $(c^i)_{i=2,\cdots,d}$ are interpreted as fixed costs while $(s^{b,i}, s^{a,i})_{i=2,\cdots,d}$ are bid and ask prices for the risky assets. We may of course generalize this model to an order book with several bid and ask prices for each asset, as in Example 2.1. Recall that by definition $\mathrm{C}_t(s,y) = -Ł_t(s,-y)$ and we may verify that $\mathrm{C}_t(s,y)$ is l.s.c. in every (s,y) such that $(c^i)_{i=2,\cdots,d} \in \mathbf{R}_+^{d-1}$. To see it, it suffices to observe that $Ł_t^i(s,y)$ is continuous at each point (s,y) such that $y \neq 0$. At last, if $y = 0$, $Ł_t(s,y) = 0$ and $\liminf_{r\to s, y\to 0} Ł_t(r,y) \leq 0$ since $c_t^i \geq 0$. Therefore, $Ł_t^i$ is u.s.c. Moreover, $\mathrm{C}_t(s,y)$ subadditive in y. A direct computation yields that $\hat{Ł}_t(s,y) = y^1 + \sum_{i=2}^{d} \hat{Ł}_t^i(s^{b,i}, s^{a,i}, y^i)$ where

$$
\hat{Ł}_t^i(s^{b,i}, s^{a,i}, y^i) = (y^i)^+ s^{b,i} - (y^i)^- s^{a,i}.
$$

Note that $\hat{Ł}_t = Ł_t^\infty$ and we have $\hat{\mathrm{C}}_t(s,y) = y^1 + \sum_{i=2}^{d} \hat{\mathrm{C}}_t^i(s^{b,i}, s^{a,i}, y^i)$ where

$$
\hat{\mathrm{C}}_t^i(s^{b,i}, s^{a,i}, y^i) = (y^i)^+ s^{a,i} - (y^i)^- s^{b,i}.
$$

Observe that $\hat{Ł}_t$ and $\hat{\mathrm{C}}_t$ are continuous in (s,y). Moreover, $\hat{\mathrm{C}}_t \leq \mathrm{C}_t$ and $\hat{\mathrm{C}}_t$ is super δ-homogeneous with $\delta(x) = x$. $\triangle$

In the following, we adapt the notations of Section 3 to the enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$ as follows: We set

$$
\hat{\gamma}_T(S_T, V_{T-1}) = g^1(S_T) + \hat{\mathrm{C}}_T(S_T, (0, g^{(2)}(S_T) - V_{T-1}^{(2)})),
$$

and we define recursively

$$
\begin{aligned}
\hat{\theta}_t^\xi(V_t) &:= \operatorname{ess\,sup}_{\mathcal{F}_t} \hat{\gamma}_{t+1}^\xi(V_t), \\
\hat{D}_t^\xi(S_t, V_{t-1}, V_t) &:= \hat{\mathrm{C}}_t(S_t, (0, V_t^{(2)} - V_{t-1}^{(2)})) + \hat{\theta}_t^\xi(S_t, V_t).
\end{aligned}
$$

Definition 4.19. *We say that the robust no-arbitrage condition RSAIP holds at time t if the SAIP condition holds at time t for the enlarged model $(\hat{\mathbf{G}}_t)_{t\in[0,T]}$. We say that RSAIP holds if it holds at any time.*

Theorem 4.20. *Suppose that the enlarged market satisfies $\hat{\mathrm{C}}_t \le \mathrm{C}_t$, $\hat{\mathrm{C}}$ is super δ-homogeneous and either sub-additive or super-additive. Suppose that there exists a lower semi-continuous function $\tilde{\gamma}^{\xi}_{t+1}$ defined on $\mathbf{R}^m \times \mathbf{R}^d$ such that $\gamma^{\xi}_{t+1}(V_t) = \tilde{\gamma}^{\xi}_{t+1}(S_{t+1}, V_t)$ for all $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$. Assume that Assumption 4 and Assumption 5 hold. Suppose that the cost function $\mathrm{C}_t(s,z)$ is l.s.c. in (s,z) and C_t is either super-additive or sub-additive. Then, if $\inf_{z \in S^{d-1}(0,1)} \hat{D}^0_t(S_t, 0, z) > 0$, $\gamma^{\xi}_t(V_{t-1}) = \tilde{\gamma}^{\xi}_t(S_t, V_{t-1})$ where $\tilde{\gamma}^{\xi}_t(s, v_{t-1})$ is l.s.c. in (s, v_{t-1}).*

Proof. As $\hat{\mathrm{C}}_t(x) \le \mathrm{C}_t(x)$, we deduce by induction that $\hat{D}^0_t(s,0,v_t) \le D^0_t(s,0,v_t)$. We adapt the main arguments of the proof of Theorem 4.14. Recall that $D^0_t(s, v_{t-1}, v_t) \ge \tilde{h}_t(s, v_{t-1}) + D^0_t(s, 0, v_t)$ where $\tilde{h}_t$ is a continuous function. By Lemma 5.21, we have for $|v_t| \ge 1$,

$$D^0_t(s,0,v_t) \ge \hat{D}^0_t(s,0,v_t) \ge \delta(|v_t|)\hat{D}^0_t(s,0,v_t/|v_t|) \ge \delta(|v_t|) \inf_{z \in S^{d-1}(0,1)} \hat{D}^0_t(s,0,z).$$

Therefore, we also get that $\tilde{\gamma}^{\xi}_t(s, v_{t-1}) = \inf_{v_t \in K_t(s, v_{t-1})} D^{\xi}_t(s, v_{t-1}, v_t)$ where $K_t(s, v_{t-1}) \subseteq \phi_t(s, v_{t-1}) := \bar{B}_t(0, r_t(s, v_{t-1}) + 1)$ and

$$\begin{aligned} r_t(s, v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(s, v_{t-1})}{i_t(s)}\right), \\ i_t(s) &:= \inf_{z \in S^{d-1}(0,1)} \hat{D}^0_t(s,0,z),\ \lambda_t(s, v_{t-1}) = |\tilde{h}_t(s, v_{t-1})| + \hat{h}^{\xi}_t(s, v_{t-1}). \end{aligned}$$

Applying Theorem 4.14 by induction to the enlarged market, we deduce that $\hat{D}^0_t(s,0,z)$ is l.s.c. in (s,z), see the proof of Theorem 4.14. We then conclude as in the proof of Theorem 4.14.

□

Remark 4.21. Recall that the condition $\inf_{z \in S^{d-1}(0,1)} \hat{D}^0_t(S_t, 0, z) > 0$ we impose in the theorem above holds under the RSAIP condition by Theorem 4.15. For a fixed costs model, this means that SAIP holds for the enlarged market, a priori without fixed cost. Moreover, the other conditions we impose are also satisfied in the fixed costs model of Example 4.18. △

4.4. Computational feasibility under a weaker SAIP no-arbitrage condition

In this section, we consider a no-arbitrage condition called LAIP, weaker than SAIP, but still sufficient to deduce that the essential infimum in the dynamic programming principle (3.1) is a pointwise infimum so that it can be numerically computed.

Lemma 4.22. *Suppose that* C_t *is sub-additive for any* $t \le T$. *Then, for any payoff* $\xi \in L^0(\mathbf{R}^d, \mathcal{F}_T)$, *the function* D_t^ξ *defined by (4.3) satisfies the following inequality:*

$$D_t^\xi(V_{t-1} + \bar{V}_{t-1}, V_t + \bar{V}_t) \le D_t^\xi(V_{t-1}, V_t) + D_t^0(\bar{V}_{t-1}, \bar{V}_t).$$

Proof. By definition with the sub-additivity of C_T, we have:

$$\begin{aligned}
\gamma_T^\xi(V_{T-1} + \bar{V}_{T-1}) &= \xi^1 + \mathrm{C}_T((0, \xi^{(2)} - V_{T-1}^{(2)} - \bar{V}_{T-1}^{(2)})), \\
&= \xi^1 + \mathrm{C}_T((0, -V_{T-1}^{(2)})) + \mathrm{C}_T((0, -\bar{V}_{T-1}^{(2)})), \\
&\le \gamma_T^\xi(V_{T-1}) + \gamma_T^0(\bar{V}_{T-1}).
\end{aligned}$$

We deduce that $\theta_{T-1}^\xi(V_{T-1} + \bar{V}_{T-1}) \le \theta_{T-1}^\xi(V_{T-1}) + \theta_{T-1}^0(\bar{V}_{T-1})$ and, since $D_{T-1}^\xi(V_{T-2}, V_{T-1}) = \mathrm{C}_{T-1}((0, V_{T-1} - V_{T-2})) + \theta_T^\xi(V_{T-1})$, we get that:

$$D_{T-1}^\xi(V_{T-2} + \bar{V}_{2-1}, V_{T-1} + \bar{V}_{T-1}) \le D_{T-1}^\xi(V_{T-2}, V_{T-1}) + D_{T-1}^0(\bar{V}_{T-2}, \bar{V}_{T-1}).$$

Taking the essential infimum with respect to V_{T-1} and $\bar{V}_{T-1}$, we get that

$$\gamma_{T-1}^\xi(V_{T-2} + \bar{V}_{T-2}) \le \gamma_{T-1}^\xi(V_{T-2}) + \gamma_{T-1}^\xi(\bar{V}_{T-2}).$$

We may pursue by induction and conclude. □

We now introduce the LAIP condition. By Proposition 5.7, we may suppose that the function $D_t^0(y, z)$ defined by (4.3) is l.s.c. in (y, z) and it is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d) \otimes \mathcal{B}(\mathbf{R}^d)$ measurable w.r.t. (ω, y, z). Note that, under AIP, the family of random variables $\mathcal{N}_t := \{Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t), Z_t^1 = 0,\ D_t^0(0, Z_t) = 0\}$ coincides with $\{Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t), Z_t^1 = 0,\ D_t^0(0, Z_t) \le 0\}$. Therefore, by lower semicontinuity, $\mathcal{N}_t$ is a closed subset of $L^0(\mathbf{R}^d, \mathcal{F}_t)$. Moreover, $\mathcal{N}_t$ is $\mathcal{F}_t$- decomposable, see [14, Section 5.4]. Therefore, by [14, Proposition 5.4.3], there exists an $\mathcal{F}_t$-measurable random set N_t such that $\mathcal{N}_t = L^0(\mathrm{N}_t, \mathcal{F}_t)$.

Definition 4.23. *We say that the condition LAIP (Linear AIP condition) holds at time* t *if AIP holds at time* t *and* $\mathcal{N}_t$ *is a linear vector space, or equivalently* N_t *is a.s. a linear subspace of* $\mathbf{R}^d$. *We say that LAIP holds if LAIP holds at any time.*

Note that if $\mathcal{N}_t = \{0\}$, then SAIP, AIP and LAIP are equivalent. In general, SAIP implies LAIP. The following result gives a financial interpretation of LAIP. If LAIP holds, the cost to hedge the zero payoff from an initial risky position $Z_t = V_t^{(2)} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_t)$ is zero if and only if the cost is also zero for the position $-Z_t$. This symmetric property is related to the SRN condition of [17].

Lemma 4.24. *Suppose that* C_t *is sub-additive and is positively super* δ*-homogeneous, for any* $t \leq T$*. The following statements are equivalent:*

1.) LAIP holds.

2.) AIP holds and, if $Z_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*, then* $D_t^0(0, Z_t) = 0$ *if and only if* $D_t^0(0, -Z_t) = 0$*,* $t \leq T$*.*

Proof. The implication 1.) $\implies$ 2.) is immediate. Reciprocally, suppose that 2.) holds. Let us show that $\mathcal{N}_t$ is stable under addition. We consider $Z_t^1, Z_t^2 \in \mathcal{N}_t$. By Proposition 4.22, we get under AIP that

$$0 \leq D_t^0(0, Z_t^1 + Z_t^2) \leq D_t^0(0, Z_t^1) + D_t^0(0, Z_t^2) \leq 0.$$

We deduce that $Z_t^1 + Z_t^2 \in \mathcal{N}_t$. By induction, we then deduce that for any integer n, $n\mathcal{N}_t \subseteq \mathcal{N}_t$. Moreover, by Lemma 5.21, if $\lambda_t \in L^0((0,1], \mathcal{F}_t)$,

$$D_t^0(0, V_t) = D_t^0(0, \lambda_t(\lambda_t)^{-1}V_t) \geq \delta((\lambda_t)^{-1}) D_t^0(0, \lambda_t V_t) \geq 0.$$

So $V_t \in \mathcal{N}_t$ implies that $\lambda_t V_t \in \mathcal{N}_t$ if $\lambda_t \in L^0((0,1], \mathcal{F}_t)$. Finally, as $\mathbb{N}\mathcal{N}_t \subseteq \mathcal{N}_t$, $\lambda_t V_t \in \mathcal{N}_t$ for every $\lambda_t \geq 0$. Moreover, $\mathcal{N}_t$ is symmetric by assumption. The conclusion follows. □

In the following, let us consider $\mathrm{N}_t^{\perp} := \{z \in \mathbf{R}^d : \ zx = 0, \forall x \in \mathrm{N}_t\}$, the random $\mathcal{F}_t$-measurable linear subspace orthogonal to N_t.

Lemma 4.25. *Suppose that* C_t *is sub-additive and LAIP holds. Then, for all* $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_t)$*, there exists* $V_t^2 \in L^0(\mathrm{N}_t^{\perp}, \mathcal{F}_t)$ *such that*

$$D_t^{\xi}(V_{t-1}, V_t) = D_t^{\xi}(V_{t-1}, V_t^2) \ a.s..$$

Proof. By a measurable selection argument, it is possible to decompose any $V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)$ into $V_t = V_t^1 + V_t^2$, where $V_t^1 \in L^0(\mathrm{N}_t, \mathcal{F}_t)$, $V_t^2 \in L^0(\mathrm{N}_t^{\perp}, \mathcal{F}_t)$. By Lemma 4.22, we have

$$D_t^{\xi}(V_{t-1}, V_t) \leq D_t^{\xi}(V_{t-1}, V_t^2) + D_t^0(0, V_t^1) = D_t^{\xi}(V_{t-1}, V_t^2).$$

On the other hand, as $V_t^2 = V_t - V_t^1$ and $-V_t^1 \in \mathcal{N}_t$ under LAIP, we also have

$$D_t^{\xi}(V_{t-1}, V_t^2) \leq D_t^{\xi}(V_{t-1}, V_t) + D_t^0(0, -V_t^1) = D_t^{\xi}(V_{t-1}, V_t).$$

The conclusion follows. □

In the following, we assume the following condition.

Assumption 6. *For any $t \leq T$, $|\mathrm{C}_t((0, x^{(2)}))| < \bar{h}_t(x)$, where $\bar{h}_t$ is a random function $\bar{h}_t : (\omega, x) \in \Omega \times \mathbf{R}^d \mapsto \bar{h}_t(\omega, x) \in \mathbf{R}$ which is $\mathcal{F}_t \otimes \mathcal{B}(\mathbf{R}^d)$-measurable and continuous a.s. in x.*

Note that the condition above holds under our initial hypothesis with $\bar{h}_t(x) = h_t(S_t, x)$ but, here, we dont stress a dependance of C_t on S_t.

Theorem 4.26. *Suppose that there exists a lower semi-continuous function $\tilde{\gamma}_{t+1}^{\xi}$ defined on $\mathbf{R}^d$. Assume that Assumption 6 holds. Suppose that the cost function $\mathrm{C}_t(z)$ is l.s.c. in z and C_t is sub-additive, positively super δ-homogeneous. If LAIP holds, then $\gamma_t^{\xi}(V_{t-1}) = \tilde{\gamma}_t^{\xi}(V_{t-1})$ where $\tilde{\gamma}_t^{\xi}(v_{t-1})$ is l.s.c. in v_{t-1}.*

Proof. By Lemma 4.25, we get that

$$\operatorname*{ess\,inf}_{V_t \in L^0(\mathbf{R}^d, \mathcal{F}_t)}{}_{\mathcal{F}_t} D_t^{\xi}(V_{t-1}, V_t) = \operatorname*{ess\,inf}_{V_t \in L^0(\mathrm{N}_t^{\perp}, \mathcal{F}_t)}{}_{\mathcal{F}_t} D_t^{\xi}(V_{t-1}, V_t).$$

Since $\mathrm{N}_t^{\perp}$ is an $\mathcal{F}_t$-measurable random closed set, by Proposition 5.7 and Lemma 5.5, we have

$$\operatorname*{ess\,inf}_{V_t \in L^0(\mathrm{N}_t^{\perp}, \mathcal{F}_t)}{}_{\mathcal{F}_t} D_t^{\xi}(V_{t-1}, V_t) = \inf_{y \in \mathrm{N}_t^{\perp}} D_t^{\xi}(V_{t-1}, y).$$

On $\left\{\omega : \mathrm{N}_t^{\perp}(\omega) = \{0\}\right\} \in \mathcal{F}_t$, we have $\gamma_t^{\xi}(V_{t-1}) = D_t^{\xi}(V_{t-1}, 0)$. On the complementary set, $\left\{\mathrm{N}_t^{\perp} \neq \{0\}\right\} \in \mathcal{F}_t$, under LAIP, we have $\inf_{z \in M_t} D_t^0(0, z) > 0$, where $M_t = \mathrm{N}_t^{\perp} \cap S^{d-1}(0, 1) \neq \emptyset$. We now adapt the notations and the main arguments in the proof of Theorem 4.14 with $V_t \in \mathrm{N}_t^{\perp}$. In our case, we use Assumption 6 in order to dominate the cost function by a continuous function. By Lemma 5.21, for all $v_t \in \mathrm{N}_t^{\perp}$, we may suppose w.l.o.g. that $v_t^1 = 0$ and we get that

$$D_t^0(0, v_t) \geq \delta(|v_t|) D_t^0(0, v_t/|v_t|) \geq \delta(|v_t|) \inf_{z \in M_t} D_t^0(0, z).$$

Moreover, by Assumption 6, we have:

$$D_t(v_{t-1},0) = \mathrm{C}_t((0,v_{t-1}^{(2)})) + \theta_t^\xi(0) \le \bar{h}_t(v_{t-1}) + \theta_t^\xi(0).$$

Therefore, we deduce that $\tilde{\gamma}_t^\xi(v_{t-1}) = \inf_{v_t\in\phi_t(v_{t-1})} D_t^\xi(v_{t-1},v_t)$ where ϕ is the set-valued mapping $\phi_t(v_{t-1}) := \bar{B}_t(0, r_t(v_{t-1})+1)$ and

$$\begin{aligned} r_t(v_{t-1}) &:= \delta^{-1}\left(\frac{\lambda_t(v_{t-1})}{i_t}\right), \\ i_t &:= \inf_{z\in M_t} D_t^0(0,z),\ \lambda_t(v_{t-1}) = \tilde{h}_t(v_{t-1}) + \bar{h}_t(v_{t-1}) + \theta_t^\xi(0). \end{aligned}$$

By Corollary 5.3, $i_t > 0$ is $\mathcal{F}_t$-measurable while $\lambda_t(\omega, v_{t-1})$ is $\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^d)$-measurable and continuous in v_{t-1}. Therefore, $r_t(\omega, v_{t-1})$ is $\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^d)$-measurable and continuous in v_{t-1}. We deduce that $\bar{B}_t(0, r_t(v_{t-1}))$ is a continuous set-valued mapping by Corollary 5.14. We then conclude by Proposition 5.17.

□

Note that the theorem above states that, under LAIP, $\gamma_t^\xi(V_{t-1})$ is a lower-semicontinuous function of V_{t-1}. Therefore, by Lemma 5.5, $\gamma_t^\xi(V_{t-1})$ may be computed pointwise as $\gamma_t^\xi(V_{t-1}) = \inf_{y\in\mathbf{R}^d}\left(\mathrm{C}_t((0,y^{(2)} - V_{t-1}^{(2)})) + \theta_t^\xi(y)\right)$. Moreover, the infimum is reached so that $\gamma_t^\xi(V_{t-1})$ is a minimal cost.

5. Appendix

5.1. Normal integrands

Definition 5.1. *Let $\mathcal{F}$ be a complete σ-algebra. We say that the function $(\omega,x)\in\Omega\times\mathbf{R}^k\mapsto f(\omega,x)\in\mathbf{R}$ is an $\mathcal{F}$-normal integrand if f is $\mathcal{F}\otimes\mathcal{B}(\mathbf{R}^k)$-measurable and lower semi-continuous in x. If $Z\in L^0(\mathbf{R}^k,\mathcal{F})$, we use the notation $f(Z):\omega\mapsto f(Z(\omega)) = f(\omega,Z(\omega))$. If f is $\mathcal{F}\otimes\mathcal{B}(\mathbf{R}^k)$-measurable then $f(Z)\in L^0(\mathbf{R}^k,\mathcal{F})$.*

By [26, Theorem 14.37], we have:

Proposition 5.2. *If f is an $\mathcal{F}$-normal integrand,* $\inf_{y\in\mathbf{R}^d} f(\omega,y)$ *is $\mathcal{F}$-measurable and $\{(\omega,x)\in\Omega\times\mathbf{R}^d : f(\omega,x) = \inf_{y\in\mathbf{R}^d} f(\omega,y)\}\in\mathcal{F}\otimes\mathcal{B}(\mathbf{R}^d)$ is a measurable closed set.*

Corollary 5.3. *For any $\mathcal{F}$ normal integrand $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ and any $\mathcal{F}$-measurable random set A, let $p(\omega) = \inf_{x\in A} f(\omega, x)$. Then the function $p : \Omega \to \overline{\mathbf{R}}$ is $\mathcal{F}$-measurable.*

Proof. Let us define $\delta_{A(\omega)}(x) = +\infty$ if $x \notin A(\omega)$ and $\delta_{A(\omega)}(x) = 0$ otherwise. Then, the function $g(\omega, x) := f(\omega, x) + \delta_{A(\omega)}(x)$ is an $\mathcal{F}$-normal integrand since A is closed and $\mathcal{F}$-measurable. Moreover, we observe that $p(\omega) = \inf_{x\in A(\omega)} g(\omega, x)$. The conclusion follows from Proposition 5.2. □

Corollary 5.4. *If f is an $\mathcal{F}$-normal integrand, and if K is an $\mathcal{F}$-measurable set-valued compact set, then $\inf_{y\in K(\omega)} f(\omega, y)$ is $\mathcal{F}$-measurable. Moreover, $M(\omega) = \{x \in K(\omega) : f(\omega, x) = \inf_{y\in K(\omega)} f(\omega, y)\} \in \mathcal{F} \otimes \mathcal{B}(\mathbf{R}^d)$ is a non-empty $\mathcal{F}$-measurable closed set. In particular, $\inf_{y\in K(\omega)} f(\omega, y) = f(\omega, y)$ for all $y \in L^0(M, \mathcal{F}) \neq \emptyset$.*

Proof. It suffices to extend the function f to $\mathbf{R}^d$ by setting $f = +\infty$ on $\mathbf{R}^d \setminus K(\omega)$ so that f is still l.s.c. on $\mathbf{R}^d$. Then, we may apply Proposition 5.2. Notice that $M(\omega) \neq \emptyset$ a.s. by compactness argument so that $L^0(M, \mathcal{F}) \neq \emptyset$ by a measurable selection argument. □

In the following, we use the abuse of notation $f(y) = f(\omega, y)$ for any $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$.

Lemma 5.5. *For any $\mathcal{F}$ normal integrand $f : \Omega \times \mathbf{R}^d \to \overline{\mathbf{R}}$ and any non-empty $\mathcal{F}$-measurable closed set A, we have:*

$$\text{ess inf}_{\mathcal{F}} \left\{ f(a),\ a \in L^0(A, \mathcal{F}) \right\} = \inf_{a\in A} f(a) \ \text{a.s.}$$

Proof. We first prove that

$$\text{ess inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\} \le \inf_{a\in A} f(a).$$

Recall that f is $\mathcal{F}$-normal integrand and $\inf_{a\in A} f(a)$ is $\mathcal{F}$-measurable by Corollary 5.3. Therefore, the set

$$\{(\omega, a) : a \in A(\omega), \inf_{x\in A} f(x) \le f(a) < \inf_{x\in A} f(x) + 1/n\}$$

is $\mathcal{F}$-measurable and has non-empty ω sections for each $n \in \mathbb{N}$. By measurable selection argument, we deduce $a^n \in L^0(A, \mathcal{F})$ such that

$$\inf_{a\in A} f(a) \le f(a^n) < \inf_{a\in A} f(a) + 1/n.$$

This implies that $\lim_n f(a^n) = \inf_{a\in A} f(a)$. Therefore,

$$\inf_{a\in A} f(a) = \inf_n f(a^n) \geq \text{ess inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\}.$$

For the reversed inequality, for each $a \in L^0(A, \mathcal{F})$, $f(a) \geq \inf_{a\in A} f(a)$ and, since $\inf_{a\in A} f(a)$ is $\mathcal{F}$-measurable by Corollary 5.3, we deduce by definition of conditional essential infimum that

$$\text{ess inf}_{\mathcal{F}} \left\{ f(a), a \in L^0(A, \mathcal{F}) \right\} \geq \inf_{a\in A} f(a) \text{ a.s..}$$

□

We recall a result from [2] which characterizes a conditional essential supremum as a pointwise supremum on a random set. Let $\mathcal{H}$ and $\mathcal{F}$ be two complete sub-σ-algebras of $\mathcal{F}_T$ such that $\mathcal{H} \subseteq \mathcal{F}$. The conditional support of $X \in L^0(\mathbf{R}^d, \mathcal{F})$ with respect to $\mathcal{H}$ is the smallest $\mathcal{H}$-graph measurable random set $\text{supp}_{\mathcal{H}} X$ containing the singleton $\{X\}$ a.s., see [2].

Proposition 5.6. *Let $h : \Omega \times \mathbf{R}^k \to \mathbf{R}$ be a $\mathcal{H} \otimes \mathcal{B}(\mathbf{R}^k)$-measurable function which is l.s.c. in x. Then, for all $X \in L^0(\mathbf{R}^k, \mathcal{F})$,*

$$\text{ess sup}_{\mathcal{H}}\, h(X) = \sup_{x \in supp_{\mathcal{H}} X} h(x) \quad a.s.$$

Proposition 5.7. *Fix $\xi^1 \in L^0(\mathbf{R}, \mathcal{F})$ and $d \geq 2$. Let us consider a random function $f : \Omega \times \mathbf{R}^d \to \mathbf{R}$ that satisfies $f(z) = z^1 + f(0, z^{(2)})$, for any $z = (z^1, z^{(2)}) \in \mathbf{R}^d$. Suppose that $z \mapsto f(z)$ is l.s.c. a.s.. Then, there exists a $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable random function $F^*_{t-1}(\omega, y)$ such that, for any $Y_{t-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{t-1})$,*

$$F^*_{t-1}(Y_{t-1}) = \text{ess sup}_{\mathcal{F}_{t-1}} \left(\xi^1 + f(0, Y_{t-1}) \right) =: F^{\xi^1, f}_{t-1}(Y_{t-1}), \text{ a.s.}$$

Moreover, $F^*_{t-1}(\omega, y)$ is l.s.c. in y and if, in addition, $y \in \mathbf{R}^{d-1} \mapsto f(0, y)$ is a.s. convex, then $y \mapsto F^*_{t-1}(\omega, y)$ is a.s. convex.

Proof. Consider the family of random variables:

$$\begin{aligned} \Lambda_{t-1} &= \left\{ (x_{t-1}, y_{t-1}) \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1}) : f(-x_{t-1}, y_{t-1}) \leq -\xi^1 \right\} \\ &= \left\{ (x_{t-1}, y_{t-1}) \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1}) : x_{t-1} \geq F^{\xi^1, f}_{t-1}(y_{t-1}) \right\}. \end{aligned}$$

Notice that Λ_{t-1} is closed in L^0 since f is l.s.c.. Moreover, Λ_{t-1} is $\mathcal{F}_{t-1}$-decomposable, i.e. $g^1_{t-1} 1_{A_{t-1}} + g^2_{t-1} 1_{A^c_{t-1}} \in \Lambda_{t-1}$ if g^1_{t-1} and g^2_{t-1} belong to

Λ_{t-1} and $A_{t-1} \in \mathcal{F}_{t-1}$. By [18][Corollary 2.5], there exists an $\mathcal{F}_{t-1}$-measurable random closed set Γ_{t-1} such that $\Lambda_{t-1} = L^0(\Gamma_{t-1}, \mathcal{F}_{t-1})$. Moreover, there is a Castaing representation, i.e. a countable family $(z_{t-1}^n)_{n\geq 1} \in \Lambda_{t-1}$ such that $\Gamma_{t-1}(\omega) = \mathrm{cl}\{z_{t-1}^n(\omega) : n \geq 1\}$, $\omega \in \Omega$. We define

$$F_{t-1}^*(\omega, y) := \inf\{x \in \mathbf{R} : (x, y) \in \Gamma_{t-1}(\omega)\}.$$

We claim that $F_{t-1}^*(\omega, y) = \inf\big\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\big\}$. Indeed, first we have $F_{t-1}^*(\omega, y) \leq \inf\big\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\big\}$. Moreover, in the case where $F_{t-1}^*(\omega, y) > -\infty$, for every $\epsilon > 0$, there exist $x \in \mathbf{R}$ such that $(x, y) \in \Gamma_{t-1}$ and $F_{t-1}^*(\omega, y) + \epsilon \geq x$. Choose $\tilde{x} \in \mathbb{Q} \cap [x, x + \epsilon]$. Observe that $(\tilde{x}, y) \in \Gamma_{t-1}$ as the y-sections of Λ_{t-1} are upper sets. We then have:

$$\begin{aligned}
&F_{t-1}^*(\omega, y) + 2\epsilon \geq x + \epsilon \geq \tilde{x},\\
&F_{t-1}^*(\omega, y) \geq \tilde{x} - 2\epsilon \geq \inf\big\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\big\} - 2\epsilon.
\end{aligned}$$

Since ϵ is arbitrary chosen, we conclude that

$$F_{t-1}^*(\omega, y) = \inf\big\{x \in \mathbb{Q} : (x, y) \in \Gamma_{t-1}(\omega)\big\}.$$

Notice that when $F_{t-1}^*(\omega, y) = -\infty$, then we may choose $x \to -\infty$ so that we also have $\tilde{x} \to -\infty$ and we conclude similary. We then deduce that $F_{t-1}^*(\omega, y)$ is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable. Indeed, for every $c < +\infty$, we have:

$$\big\{(\omega, y) : F_{t-1}^*(\omega, y) \geq c\big\} = \bigcap_{x\in\mathbb{Q}} \big\{(\omega, y) : x 1_{(\omega,x,y)\in \mathrm{Graph}\Gamma_{t-1}} \geq c 1_{(\omega,x,y)\in \mathrm{Graph}\Gamma_{t-1}}\big\}.$$

Since Γ_{t-1} is graph-measurable, $\big\{(\omega, y) : F_{t-1}^*(\omega, y) \geq c\big\} \in \mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$. We then conclude that F_{t-1}^* is $\mathcal{F}_{t-1} \otimes \mathcal{B}(\mathbf{R}^{d-1})$-measurable. Moreover, if f_t is convex, Γ_{t-1} is convex a.s. and we deduce that $F_{t-1}^*(\omega, y)$ is convex in y a.s.

Consider a sequence $y^n \in \mathbf{R}^{d-1}$ which converges to y and let us denote $\beta^n := F_{t-1}^*(\omega, y^n)$. We have $(\beta^n, y^n) \in \Gamma_{t-1}$ if $\beta^n > -\infty$. If $\inf_n \beta^n = -\infty$, then, up to a subsequence, $F_{t-1}^*(\omega, y) - 1 > \beta^n$ for n large enough, hence $(F_{t-1}^*(\omega, y) - 1, y^n) \in \Gamma_{t-1}(\omega)$ since the y^n-sections of Γ_{t-1} are upper sets. As $n \to \infty$, we deduce that $(F_{t-1}^*(\omega, y) - 1, y) \in \Gamma_{t-1}(\omega)$, which contradicts the definition of F_{t-1}^*. Moreover it is trivial that $F_{t-1}^*(\omega, y) \leq \liminf_n \beta^n$ if $\liminf_n \beta^n = \infty$. Otherwise, $\beta^\infty := \liminf_n \beta^n < \infty$ and $(\beta^\infty, y) \in \Gamma_{t-1}$ since Γ_{t-1} is closed. It follows that $F_{t-1}^*(\omega, y) \leq \beta^\infty = \liminf_n \beta^n$ by the definition of F_{t-1}^*. We conclude that $F_{t-1}^*(\omega, x)$ is l.s.c. in x.

We show that $F^{\xi^1,f}_{t-1}(Y_{t-1}) = F^*_{t-1}(Y_{t-1})$ a.s. for all $Y_{t-1} \in L^0(\mathbf{R}^{d-1}, \mathcal{F}_{t-1})$. We first restrict Ω to the $\mathcal{F}_{t-1}$-measurable set $\{\omega : \ \Gamma_{t-1}(\omega) \neq \emptyset\}$. We may then consider a measurable selection $(\tilde{x}_{t-1}, \tilde{y}_{t-1}) \in \Gamma_{t-1} \neq \emptyset$ a.s.. By definition, we have $\tilde{x}_{t-1} \geq F^*_{t-1}(\tilde{y}_{t-1})$. We deduce that $F^*_{t-1}(\tilde{y}_{t-1}) < \infty$ a.s. We define:

$$\widehat{Y}_{t-1} = \tilde{y}_{t-1} 1_{F^*_{t-1}(Y_{t-1})=\infty} + Y_{t-1} 1_{F^*_{t-1}(Y_{t-1})<\infty}.$$

Then:

$$F^*_{t-1}(\widehat{Y}_{t-1}) = F^*_{t-1}(\tilde{y}_{t-1}) 1_{F^*_{t-1}(Y_{t-1})=\infty} + F^*_{t-1}(Y_{t-1}) 1_{F^*_{t-1}(Y_{t-1})<\infty}.$$

Observe that on the set $\{F^*_{t-1}(Y_{t-1}) < \infty\}$, $(F^*_{t-1}(\widehat{Y}_{t-1}), \widehat{Y}_{t-1}) \in \Gamma_{t-1}$ a.s. since Γ_{t-1} is closed. Therefore, $(F^*_{t-1}(\widehat{Y}_{t-1}), \widehat{Y}_{t-1}) \in \Lambda_{t-1} = L^0(\Gamma_{t-1}, \mathcal{F}_{t-1})$ and we deduce that $F^*_{t-1}(\widehat{Y}_{t-1}) \geq F^{\xi^1,f}_{t-1}(\widehat{Y}_{t-1})$ a.s.. We conclude that on the set $\{F^*_{t-1}(Y_{t-1}) < \infty\}$, $F^*_{t-1}(Y_{t-1}) \geq F^{\xi^1,f}_{t-1}(Y_{t-1})$ while the inequality is trivial on the complementary set. On the other hand, let us define

$$\begin{aligned}
\widehat{X}_{t-1} &= F^{\xi^1,f}_{t-1}(Y_{t-1}) 1_{F^{\xi^1,f}_{t-1}(Y_{t-1})<\infty} + F^{\xi^1,f}_{t-1}(\tilde{y}_{t-1}) 1_{F^{\xi^1,f}_{t-1}(Y_{t-1})=\infty},\\
\widehat{Y}_{t-1} &= Y_{t-1} 1_{F^{\xi^1,f}_{t-1}(Y_{t-1})<\infty} + \tilde{y}_{t-1} 1_{F^{\xi^1,f}_{t-1}(Y_{t-1})=\infty}.
\end{aligned}$$

Observe that $(\widehat{X}_{t-1}, \widehat{Y}_{t-1}) \in \Lambda_{t-1}$ hence $F^*_{t-1}(\widehat{Y}_{t-1}) \leq \widehat{X}_{t-1}$ by definition of F^*_{t-1}. Then, $F^*_{t-1}(Y_{t-1}) \leq \widehat{X}_{t-1} = F^{\xi^1,f}_{t-1}(Y_{t-1})$ on $\{F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty\}$. The inequality is trivial on the complementary set so that we may conclude.

On the set $\{\omega : \ \Gamma_{t-1}(\omega) = \emptyset\}$, we have $F^*_{t-1}(Y_{t-1}) = +\infty$. Moreover, if $F^{\xi^1,f}_{t-1}(Y_{t-1}) < \infty$, we deduce that $(F^{\xi^1,f}_{t-1}(Y_{t-1}), Y_{t-1}) \in \Gamma_{t-1} = \emptyset$ since $\xi^1 + f(0, Y_{t-1}) \leq F^{\xi^1,f}_{t-1}(Y_{t-1})$. This is a contradiction hence $F^{\xi^1,f}_{t-1}(Y_{t-1}) = +\infty$ and the conclusion follows. □

Lemma 5.8. *Suppose that Assumption 2 holds and consider an $\mathcal{F}_{t-1}$-normal integrand $\gamma_t : (\omega, s, y) : \Omega \times \mathbf{R}^m \times \mathbf{R}^d \mapsto \gamma_t(\omega, s, y)$. Then, for any $V_{t-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{t-1})$, we have:*

$$\operatorname{ess\,sup}_{\mathcal{F}_{t-1}} \gamma_t(S_t, V_{t-1}) = \sup_{s \in supp_{\mathcal{F}_{t-1}} S_t} \gamma_t(s, V_{t-1}) = \sup_{m \geq 1} \gamma_t(\alpha^m_{t-1}(S_{t-1}), V_{t-1}).$$

Proof. As $(\omega, s) \mapsto \gamma_t(\omega, s, V_{t-1}(\omega))$ is an $\mathcal{F}_{t-1}$-normal integrand under our assumptions, the first equality holds by Theorem 5.6. It remains to observe that, if $s \in \mathrm{supp}_{\mathcal{F}_{t-1}} S_t$, then $s = \lim_m \alpha^m_{t-1}(S_{t-1})$ for a subsequence and, by lower semicontinuity, we deduce that

$$\gamma_t(s, V_{t-1}) \leq \liminf_m \gamma^\xi_t(\alpha^m_{t-1}(S_{t-1})), V_{t-1}) \leq \sup_{m\geq 1} \gamma^\xi_t(\alpha^m_{t-1}(S_{t-1})), V_{t-1}).$$

It follows that $\sup_{s\in \mathrm{supp}_{\mathcal{F}_{t-1}} S_t} \gamma_t(s, V_{t-1}) \leq \sup_{m\geq 1} \gamma_t(\alpha^m_{t-1}(S_{t-1}), V_{t-1})$ and, finally, the equality holds. □

5.2. Continuous set-valued functions

For two topological vector spaces X, Y, consider a set-valued function $\phi : X \twoheadrightarrow Y$. We recall the definition of hemicontinuous set-valued mappings as formulated in [1].

Definition 5.9. *We say that ϕ is **lower hemicontinuous** at x if for every open set $U \subset Y$ such that $\phi(x) \cap U \neq \emptyset$, there exits a neighborhood V of x such that $z \in V$ implies $\phi(x) \cap U \neq \emptyset$.*

Definition 5.10. *We say that ϕ is **upper hemicontinuous** at x if for every open set $U \subset Y$ such that $\phi(x) \subseteq U$, there is a neighborhood V of x such that $z \in V$ implies $\phi(z) \subset U$.*

Definition 5.11. *We say that ϕ is **continuous** at x if it is both upper and lower hemicontinuous at x. It is continuous if it is continuous at any point.*

Lemma 5.12. *Let $f : \mathbf{R}^k \to \mathbf{R}_+$ be an upper semicontinuous function. Then, the mapping $x \mapsto \bar{B}(0, f(x))$ is upper hemicontinuous in the sense of definition 5.10.*

Proof. The upper hemicontinuity is simple to check. Indeed, consider an open set in $U \subseteq \mathbf{R}^k$, such that $\phi(x) = \bar{B}(0, f(x)) \subset U$. We may suppose that U is bounded w.l.o.g. and we deduce $\epsilon > 0$ such that $\bar{B}(0, f(x) + \epsilon) \subset U$. By upper semicontinuity, there exists an open set V containing x such that $z \in V$ implies $f(z) \leq f(x) + \epsilon$ hence $\phi(z) \subseteq U$. □

Lemma 5.13. *Let $f : \mathbf{R}^k \to \mathbf{R}_+$ be a lower semicontinuous function. Then, the mapping $x \mapsto \bar{B}(0, f(x))$ is lower hemicontinuous in the sense of definition 5.9.*

Proof. For any ball $B(y, r) \in \mathbf{R}^k$, we have $\bar{B}(0, f(x)) \cap B(y, r) \neq \emptyset$ if and only if $f(x) + r > |y|$. We also have $f(x) - \epsilon + r > |y|$ for some small $\epsilon > 0$. As f is l.s.c., we deduce that $f(z) \geq f(x) - \epsilon$ for every z in some neighborhood

V of x. This implies that $f(z)+r>|y|$, i.e. $\bar{B}(0,f(x))\cap B(y,r)\neq\emptyset$ for every $z\in V$. The conclusion follows. □

Corollary 5.14. *Let $f:\mathbf{R}^k\to\mathbf{R}_+$ be a continuous function. Then, the mapping $x\mapsto\bar{B}(0,f(x))$ is continuous in the sense of definition 5.11.*

Lemma 5.15. *Consider the set-valued mapping $\alpha:\mathbf{R}^m\twoheadrightarrow\mathbf{R}^m$ defined by $\alpha(s)=\mathrm{cl}\{\alpha^m(s),m\in\mathbb{N}\}$ where $(\alpha^m)_{m\geq 1}$ are continuous functions. Then, α is lower hemicontinuous.*

Proof. Consider $\omega\in\Omega$ and some open set $U\in\mathbf{R}^d$. We have $\alpha_t(\omega,z)\cap U\neq\emptyset$ if and only if there is $m\in\mathbb{N}$ such that $\alpha_t^m(\omega,z)\in U$. Since $\alpha_t^m(\omega,.)$ is continuous, we deduce that there exists an open neighborhood V of z such that $\alpha_t^m(\omega,x)\in U$ for any $x\in V$. The conclusion follows. □

We recall a result from [1][Theorem 17.31].

Proposition 5.16. *Let $\phi:\mathbf{R}^k\twoheadrightarrow\mathbf{R}^m$ be a continuous set-valued mapping with nonempty compact values and suppose that $f:\mathbf{R}^k\times\mathbf{R}^m\to\mathbf{R}$ is continuous. Then, the function $m(x)=\inf_{y\in\phi(x)}f(x,y)$ and the function $M(x)=\sup_{y\in\phi(x)}f(x,y)$ are continuous.*

Proposition 5.17. *Let $\phi:\mathbf{R}^k\twoheadrightarrow\mathbf{R}^m$ be an upper hemicontinous set-valued mapping with nonempty compact values and suppose that $f:\mathbf{R}^k\times\mathbf{R}^m\to\mathbf{R}$ is lower semicontinuous. Then, the function $m(x)=\inf_{y\in\phi(x)}f(x,y)$ is l.s.c.*

Proof. We have $m(x)=-\sup_{y\in\phi(x)}g(x,y)$ where $g=-f$ is upper semicontinuous. By [1][Lemma 17.30], the mapping $x\mapsto\sup_{y\in\phi(x)}g(x,y)$ is upper semicontinuous hence m is l.s.c. □

Lemma 5.18. *Let O be an open subset of $\mathbf{R}^k$, if $\gamma:O\to\mathbf{R}$ is l.s.c. and $\gamma\geq g$ on O for some l.s.c. function $g:\mathbf{R}^k\to\mathbf{R}$. Then, there exists a l.s.c. function $\tilde{\gamma}:\mathbf{R}^k\to\mathbf{R}$ such that $\gamma=\tilde{\gamma}$ on O.*

Proof. It suffices to consider $\tilde{\gamma}=\gamma 1_O+g1_{\Omega\setminus O}$. □

5.3. Auxiliary results

Lemma 5.19. *Suppose that there is a family of $\mathcal{F}_{t-1}$-measurable random variables $(\alpha_{t-1}^m)_{m\geq 1}$ such that $S_t\in\{\alpha_{t-1}^m:\ m\geq 1\}$ a.s. and suppose that $P(S_t=\alpha_{t-1}^m|\mathcal{F}_{t-1})>0$ a.s. for all $m\geq 1$. Then, for any $\mathcal{F}_{t-1}$-measurable* *random function* $f:\Omega\times\mathbf{R}^d\to\mathbf{R}$,

$$\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)=\sup_{m\geq 1} f(\alpha^m_{t-1}).$$

Proof. It is clear that $\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)\leq \sup_{m\geq 1} f(\alpha^m_{t-1})$ a.s. since S_t belongs to $\{\alpha^m_{t-1}:\ m\geq 1\}$ and $\sup_{m\geq 1} f(\alpha^m_{t-1})$ is $\mathcal{F}_{t-1}$-measurable by assumption. On the other hand, consider $\Gamma^m_t:=\{S_t\in\alpha^m_{t-1}\}\in\mathcal{F}_t$. We have:

$$\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)1_{\Gamma^m_t}\geq f(S_t)1_{\Gamma^m_t}\geq f(\alpha^m_{t-1})1_{\Gamma^m_t} \text{ a.s.}$$

Taking the conditional expectation, we get that

$$\begin{aligned}
&E(\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)1_{\Gamma^m_t}|\mathcal{F}_{t-1})\geq E(f(\alpha^m_{t-1})1_{\Gamma^m_t}|\mathcal{F}_{t-1}) \text{ a.s.},\\
&\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)P(\Gamma^m_t|\mathcal{F}_{t-1}))\geq f(\alpha^m_{t-1})P(\Gamma^m_t|\mathcal{F}_{t-1})) \text{ a.s.}
\end{aligned}$$

As $P(\Gamma^m_t|\mathcal{F}_{t-1}))>0$ by assumption, we get that $\text{ess sup}_{\mathcal{F}_{t-1}} f(S_t)\geq f(\alpha^m_{t-1})$ a.s. for any $m\geq 1$ so that the reverse inequality holds. □

Proposition 5.20. *Let D_t be a measurable function on $\Omega\times\mathbf{R}^m\times\mathbf{R}^d\times\mathbf{R}^d$ which is $\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^m)\otimes\mathcal{B}(\mathbf{R}^d)\otimes\mathcal{B}(\mathbf{R}^d)$-measurable. Suppose that the mapping $y\mapsto D_t(\omega,s,v,y)$ is a.s. l.s.c. for all $s,v\in\mathbf{R}^m\times\mathbf{R}^d$. Let us define*

$$\gamma_t(s,v)=\text{ess inf}_{V_t\in L^0(\mathbf{R}^d,\mathcal{F}_t)} D_t(s,v,V_t)=\inf_V D_t(s,v,V),\quad (s,v)\in\mathbf{R}^m,\mathbf{R}^d.$$

Then, there is a $\mathcal{F}_t\otimes\mathcal{B}(\mathbf{R}^m)\otimes\mathcal{B}(\mathbf{R}^d)$-measurable function $\phi_t(\omega,s,v)$ defined on $(\omega,s,v)\in\Omega\times\mathbf{R}^m\times\mathbf{R}^d$ such that for all $S_t\in L^0(\mathbf{R}^m,\mathcal{F}_t)$, $V_t\in L^0(\mathbf{R}^d,\mathcal{F}_t)$,

$$\gamma_t(S_t,V_t)=\phi_t(S_t,V_t) \text{ a.s.}$$

Moreover, if $\gamma_t(s,v)\in\mathbf{R}$, for all $(s,v)\in\mathbf{R}^m\times\mathbf{R}^d$, and if $(v,y)\mapsto D_t(\omega,s,v,y)$ is convex a.s., then the mapping $v\mapsto\phi_t(\omega,s,v)$ is convex for all $s\in\mathbf{R}^m$ a.s..

Proof. Note that by Lemma 5.5, $\gamma_t(s,v)=\inf_V D_t(s,v,V)$. The measurability property is a direct consequence of [26, Theorem 14.37]. For convexity, it suffices to observe that, if $f(x,y)$ is a jointly convex function of $z=(x,y)\in\mathbf{R}^d\times\mathbf{R}^d$, then $g(x)=\inf_{y\in\mathbf{R}^d} f(x,y)$ is a convex function in x as soon as $g(x)\in\mathbf{R}$ for all $x\in\mathbf{R}^d$. □

Lemma 5.21. *Let D^0 given by (4.3) with $\xi=0$. Suppose that* C *is positively super δ-homogeneous. For any $t\leq T$, and any $\lambda_t\in L^0([1,\infty),\mathcal{F}_t)$, we have $D^0_t(\lambda_tV_{t-1},\lambda_tV_t)\geq\delta(\lambda_t)D^0_t(V_{t-1},V_t)$ and $\gamma^0_t(\lambda_tV_{t-1})\geq\delta(\lambda_t)\gamma^0_t(V_{t-1})$ for all $(V_{t-1},V_t)\in L^0(\mathbf{R}^d,\mathcal{F}_t)\times L^0(\mathbf{R}^d,\mathcal{F}_t)$.*

Proof. For $t = T$, we have by assumption:

$$\gamma_T^0(\lambda_T V_{T-1}) = \mathrm{C}_T((0, -\lambda_T V_{T-1}^{(2)}) \geq \delta(\lambda_T)\mathrm{C}_T((0, -V_{T-1}^{(2)}) = \delta(\lambda_T)\gamma_T^0(V_{T-1}).$$

We deduce that

$$\begin{aligned}
\theta_{T-1}^0(\lambda_{T-1}V_{T-1}) &= \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \gamma_T^0(\lambda_{T-1}V_{T-1}), \\
&\geq \delta(\lambda_{T-1}) \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \gamma_T^0(V_{T-1}), \\
&\geq \delta(\lambda_{T-1})\theta_{T-1}^0(V_{T-1}).
\end{aligned}$$

As we also have

$$\mathrm{C}_{T-1}((0, \lambda_{T-1}V_{T-1}^{(2)} - \lambda_{T-1}V_{T-2}^{(2)})) \geq \delta(\lambda_{T-1})\mathrm{C}_{T-1}((0, V_{T-1}^{(2)} - V_{T-2}^{(2)})),$$

we deduce that

$$\begin{aligned}
D_{T-1}(\lambda_{T-1}V_{T-2}, \lambda_{T-1}V_{T-1}) &= \mathrm{C}_{T-1}((0, \lambda_{T-1}V_{T-1}^{(2)} - \lambda_{T-1}V_{T-2}^{(2)})) + \theta_{T-1}^0(\lambda_{T-1}V_{T-1}), \\
&\geq \delta(\lambda_{T-1})\mathrm{C}_{T-1}((0, V_{T-1}^{(2)} - V_{T-2}^{(2)})) + \delta(\lambda_{T-1})\theta_{T-1}^0(V_{T-1}), \\
&\geq \delta(\lambda_{T-1})D_{T-1}(V_{T-2}, V_{T-1}).
\end{aligned}$$

Therefore, as $\lambda_{T-1} \geq 1$,

$$\begin{aligned}
\gamma_{T-1}^0(\lambda_{T-1}V_{T-2}) &= \operatorname{ess\,inf}_{V_{T-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})} D_{T-1}(\lambda_{T-1}V_{T-2}, \lambda_{T-1}V_{T-1}), \\
&\geq \delta(\lambda_{T-1}) \operatorname{ess\,inf}_{V_{T-1} \in L^0(\mathbf{R}^d, \mathcal{F}_{T-1})} D_{T-1}(V_{T-2}, V_{T-1}), \\
&\geq \delta(\lambda_{T-1})\gamma_{T-1}^0(V_{T-2}).
\end{aligned}$$

We then conclude by induction. □

Lemma 5.22. *Suppose that Assumption 4 and Assumption 5 hold. For every $t \leq T$, there exists a continuous function $\hat{h}_t \geq 0$ such that the function D_t^ξ given by (4.4) satisfies $|D_t^\xi(s, v_{t-1}, 0)| \leq \hat{h}_t^\xi(s, v_{t-1})$.*

Proof. Recall that $\gamma_T^\xi(V_T) = g^1(S_T) + \mathrm{C}_T(S_T, (0, g^2(S_T) - V_T^{(2)}))$. By assumption on C_T and g, we deduce that $\gamma_T^\xi(V_T) \leq f_T(S_T, V_T)$ where f_T is continuous. Therefore, by Proposition 5.6,

$$\begin{aligned}
\theta_{T-1}^\xi(V_{T-1}) &= \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} \gamma_T^\xi(V_{T-1}) \leq \operatorname{ess\,sup}_{\mathcal{F}_{T-1}} f_T(S_T, V_{T-1}), \\
&\leq \sup_{z \in \operatorname{supp}_{\mathcal{F}_{T-1}} S_T} f_T(z, V_{T-1}) \leq \sup_{z \in \bar{B}(0, R_{T-1}(S_{T-1}))} f_T(z, V_{T-1}).
\end{aligned}$$

As R_{T-1} is continuous, we deduce by Corollary 5.14 and Proposition 5.16 that $\bar{\theta}^{\xi}_{T-1}(S_{T-1}, V_{T-1}) = \sup_{z \in \bar{B}(0, R_{T-1}(S_{T-1}))} f_T(z, V_{T-1})$ is a continuous function in (S_{T-1}, V_{T-1}). Recall that $\mathrm{C}_{T-1}(S_{T-1}, (0, -V^{(2)}_{T-1}) \leq h_{T-1}(S_{T-1}, V_{T-1})$ where h_{T-1} is continuous. As

$$D^{\xi}_{T-1}(S_{T-1}, V_{T-1}, 0) = \mathrm{C}_{T-1}(S_{T-1}, (0, -V^{(2)}_{T-1}) + \theta^{\xi}_{T-1}(V_{T-1}),$$

we deduce that $D^{\xi}_{T-1}(S_{T-1}, V_{T-1}, 0) \leq \hat{h}^{\xi}_{T-1}(S_{T-1}, V_{T-1})$ where $\hat{h}^{\xi}_{T-1}$ is given by $\hat{h}^{\xi}_{T-1}(S_{T-1}, V_{T-1}) = \bar{\theta}^{\xi}_{T-1}(S_{T-1}, V_{T-1}) + h_{T-1}(S_{T-1}, V_{T-1})$, i.e. $\hat{h}^{\xi}_{T-1}$ is continuous. Since $\gamma^{\xi}_{T-1}(S_{T-1}, V_{T-1}) \leq D^{\xi}_{T-1}(S_{T-1}, V_{T-1}, 0)$, we deduce that $\gamma^{\xi}_{T-1}(S_{T-1}, V_{T-1}) \leq \hat{h}^{\xi}_{T-1}(S_{T-1}, V_{T-1}) = f_{T-1}(S_{T-1}, V_{T-1})$ and we may proceed by induction to conclude. □

Following the same arguments, we also deduce the following:

Lemma 5.23. *Suppose that Assumption 4 and Assumption 5 hold. For every $t \leq T$, there exists a continuous function $\bar{h}_t$ such that $\gamma^{\xi}_t(V_t) \geq \bar{h}_t(S_t, V_t)$.*

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