H^1 regularity of the inviscid total variation and Bingham minimisers for H^1 data

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Abstract

The Bingham model for viscoplastic materials involves the minimization of a non-differentiable functional. The regularity of the associated solution is investigated here. The simplified scalar case is considered first: The total variation minimization problem seeks the unique minimizer $u \in \mathrm{BV}(\Omega)$ of bounded variation of the energy $\frac{1}{2}\|u-f\|^2_{L^2(\Omega)} + |u|_{\mathrm{BV}(\Omega)}$ for data $f \in L^2(\Omega)$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. Our main result proves for a convex domain Ω that $f \in H^1(\Omega)$ implies $u \in H^1(\Omega)$. A modification for homogeneous Dirichlet conditions involves an additional trace term $\|u\|_{L^1(\partial\Omega)}$ and then $f \in H^1_0(\Omega)$ implies $u \in H^1_0(\Omega)$. In the case of the vector Bingham model without viscosity, the boundary conditions are difficult to handle, but we prove the local $H^1_{\mathrm{loc}}(\Omega)^n$ regularity of the solution for a right-hand side $f \in H^1_{\mathrm{loc}}(\Omega)^n$. The proofs rely on several generalizations of a lemma due to H. Brézis and on the approximation with small viscosity. As a consequence, we obtain Euler–Lagrange characterizations of the solution. Homogeneous Dirichlet conditions on the viscous problem lead in the vanishing viscosity limit to relaxed boundary conditions of frictional type.

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1 Introduction and main results

The steady Bingham problem for viscoplastic materials can be written

$$\alpha u - \operatorname{div} \frac{Du}{|Du|} = f \quad \text{in } \Omega, \tag{1.1}$$

where $\alpha > 0$, the unknown u has values in $\mathbb{R}^n$, $Du := (\nabla u + (\nabla u)^t)/2$ and $|A| := \sqrt{\sum_{ij} A^2_{ij}}$ is the Frobenius norm of a matrix A. This problem is the simplest to describe non-Newtonian

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materials. When viscosity is present, this problem has been the object of many works, including the study of numerical approximations, see for example [14, 13]. This problem is very close to the scalar Mosolov problem, or total variation minimization problem. In this case, maximum principle methods are available, but many results do *not* rely on them. The H^2 regularity of the solution to the viscous problem has been established in [6]. The inviscid case is much less studied [5, 3], but nevertheless it is possible to include time dependency [5] and nonlinear transport terms [4]. The solution has to be looked for in the space of functions of bounded variation or bounded deformation (in the Bingham vector case). Then numerical approximations are more difficult to analyse, and optimal estimates as in [9] are difficult to obtain. The main result of this paper is that if the right-hand side f belongs to $H^1(\Omega)$ in a bounded convex domain $\Omega \subset \mathbb{R}^n$, the solution u belongs to $H^1(\Omega)$, proving that sharp discontinuities are indeed *not* possible. A result of the same type was obtained in [16] for a two-dimensional incompressible flow and in [8] for a class of convex minimization problems.
One consequence of the $H^1(\Omega)$ regularity are Euler–Lagrange characterizations of the solution. Homogeneous Dirichlet conditions on the viscous problem lead in the vanishing viscosity limit to relaxed boundary conditions of frictional type.
Our main results are stated in Subsection 1.1 for the (scalar) Mosolov problem and in Subsection 1.2 for the (vector) Bingham problem. The remainder of the paper is devoted to their proofs.

1.1 The total variation minimization problem

The inviscid Mosolov problem is a simplified steady scalar formulation of the vector inviscid Bingham model for viscoplastic fluids. It is also used (in its time evolution version) in image processing for denoising purposes. In this context, it is called the total variation minimization problem. It can be formulated as follows. Given a positive parameter $\alpha > 0$ and a data $f \in L^2(\Omega)$ in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, seek a function u such that

$$\alpha u - \operatorname{div} \frac{\nabla u}{|\nabla u|} = f \quad \text{in } \Omega, \tag{1.2}$$

and

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The problem (1.2)-(1.3) is equivalent (at least formally) to the minimization of the total variation functional

$$J(v) := \frac{\alpha}{2}\|v\|^2_{L^2(\Omega)} + |v|_{\mathrm{BV}(\Omega)} + \beta\,\|v\|_{L^1(\partial\Omega)} - \int_\Omega f v\,dx, \tag{1.4}$$

with $\beta = 1$, over all $v \in V := \mathrm{BV}(\Omega) \cap L^2(\Omega)$ with a finite total variation

$$|v|_{\mathrm{BV}(\Omega)} := \sup\{\int_\Omega v \operatorname{div} \varphi dx : \varphi \in C^1_c(\Omega) \text{ with } |\varphi| \le 1 \text{ in } \Omega\}. \tag{1.5}$$

The latter semi-norm is also equal to $|v|_{\mathrm{BV}(\Omega)} = \int_\Omega |\nabla v| dx$, where ∇v denotes the distributional gradient of v, a measure for $v \in \mathrm{BV}(\Omega)$. The trace theorem for BV functions justifies the existence of the L^1 integral $\|v\|_{L^1(\partial\Omega)}$ on the boundary $\partial\Omega$ of the domain. This term models homogeneous Dirichlet boundary conditions on v. The identity (A.2) justifies that $|v|_{\mathrm{BV}(\Omega)} + \beta\|v\|_{L^1(\partial\Omega)}$ is a lower semi-continuous functional on $L^2(\Omega)$ for $\beta = 1$, which would *not* be true for $\beta > 1$.

The boundary L^1 term is *not* taken for the unconstrained model problem with $\beta = 0$ in (1.4) and this corresponds to (1.2) with the Neumann boundary condition

$$\frac{\nabla u}{|\nabla u|} \cdot \nu = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where ν denotes the unit outward normal to $\partial\Omega$. In this paper, the two cases $\beta = 1$ and $\beta = 0$ will be referred to as Dirichlet and Neumann case, respectively. It is known [7] that the minimum $\min_V J(v) = \min_{L^2(\Omega)} J(v) = J(u)$ is attained at a unique minimizer $u \in V$ characterized by

$$f \in \alpha u + \partial\psi(u) \quad \text{in } L^2(\Omega) \tag{1.7}$$

in terms of the subdifferential $\partial\psi(u)$ of the convex and lower semi-continuous functional $\psi : L^2(\Omega) \to [0, \infty]$ defined by

$$\psi(v) = \begin{cases} |v|_{\mathrm{BV}(\Omega)} + \beta \|v\|_{L^1(\partial\Omega)} & \text{if } v \in L^2(\Omega) \cap \mathrm{BV}(\Omega), \\ +\infty & \text{if } v \in L^2(\Omega) \backslash \mathrm{BV}(\Omega). \end{cases} \tag{1.8}$$

More explicitly, (1.7) means that $u \in V$ and that u satisfies the variational inequality

$$\forall v \in V \quad \int_\Omega f(v-u)dx \le \alpha \int_\Omega u(v-u)dx + \psi(v) - \psi(u). \tag{1.9}$$

1.1.1 Statement of the main results

The two cases $\beta = 1$ and $\beta = 0$ model homogeneous Dirichlet boundary conditions through $\beta = 1$, for which we set $H := H_0^1(\Omega)$, and Neumann boundary conditions through $\beta = 0$, for which we set $H := H^1(\Omega)$.

Theorem 1.1 (H^1 regularity) *If Ω is convex and $f \in H$, then the minimizer u of (1.4) in $\mathrm{BV}(\Omega) \cap L^2(\Omega)$ verifies $u \in H$ and $\alpha \|\nabla u\|_{L^2(\Omega)} \le \|\nabla f\|_{L^2(\Omega)}$.*

The proof in Section 3 utilizes a fundamental lemma of H. Brézis (for $\beta = 1$) in the case of a smooth boundary or a variant (for $\beta = 0$) proved in Subsection 2.1. The case of a nonsmooth convex domain is established in Subsection 2.2 with an approximation argument of functions of bounded variation by smooth functions of compact support provided in the appendix.

The Euler–Lagrange equations (1.2) characterize the minimizer $u \in H$.

Theorem 1.2 (Euler–Lagrange equations) *If Ω is convex and $f \in H$, then the minimizer u of (1.4) in $V := \mathrm{BV}(\Omega) \cap L^2(\Omega)$ is characterized by $u \in H^1(\Omega)$,*

$$\begin{cases} \alpha u - \operatorname{div} \sigma = f \text{ in } \Omega \qquad \text{for some } \sigma \in L^\infty(\Omega)^n, \\ |\sigma| \le 1 \text{ a.e.}, \quad \sigma = \dfrac{\nabla u}{|\nabla u|} \text{ where } \nabla u \ne 0, \quad \text{and} \\ u \in H_0^1(\Omega) \text{ for } \beta = 1 \quad \text{or} \quad \sigma \cdot \nu = 0 \text{ on } \partial\Omega \text{ for } \beta = 0. \end{cases} \tag{1.10}$$

The last statements refer to the Dirichlet or Neumann case and the normal component $\sigma \cdot \nu \in H^{-1/2}(\Omega)$ is well defined for $\sigma \in H(\mathrm{div}, \Omega)$ [17], i.e., the condition $\sigma \cdot \nu = 0$ on $\partial\Omega$ for $\beta = 0$ means that $\int_\Omega v \,\mathrm{div}\, \sigma = -\int_\Omega \nabla v \cdot \sigma$ holds for all $v \in H^1(\Omega)$.

Similar results hold for the time-dependent problem

$$\partial_t u - \mathrm{div}\, \frac{\nabla u}{|\nabla u|} = f \quad \text{in } \Omega \tag{1.11}$$

with either Dirichlet ($\beta = 1$) or Neumann ($\beta = 0$) boundary conditions. It can be formulated as

$$u \in H^1(0, T; L^2(\Omega)) \cap B_T, \tag{1.12}$$

$$f(t) \in \partial_t u(t) + \partial\psi(u(t)) \quad \text{for a.e. } t \in (0, T), \text{ and} \tag{1.13}$$

$$u(0) = u^0. \tag{1.14}$$

The given data are ψ from (1.8), $f \in L^2(0, T; L^2(\Omega))$, $u^0 \in V := L^2(\Omega) \cap \mathrm{BV}(\Omega)$, and the space B_T is defined for $T > 0$ by

$$B_T := \left\{ u \in L^2(0, T; L^2(\Omega)), \quad \int_0^T \psi(u(t))dt < \infty \right\}. \tag{1.15}$$

It has been proved in [5] that (1.12)-(1.14) has a unique solution, and that (1.13) is equivalent to the variational formulation

$$\int_0^T \langle f(t), v(t) - u(t)\rangle_{L^2(\Omega)} dt \le \int_0^T \langle \partial_t u(t), v(t) - u(t)\rangle_{L^2(\Omega)} dt + \int_0^T \big(\psi(v(t)) - \psi(u(t))\big) dt \tag{1.16}$$

for all $v \in B_T$ with the inner product $\langle \cdot, \cdot \rangle_{L^2(\Omega)}$ in $L^2(\Omega)$. Recall the two cases $H = H_0^1(\Omega)$ if $\beta = 1$ and $H = H^1(\Omega)$ if $\beta = 0$. The linear space $C([0, T]; H^1(\Omega)\text{-weak})$ of $H^1(\Omega)$-valued weakly continuous functions on the compact time interval $[0, T]$ consists of bounded functions $f : [0, T] \to H^1(\Omega)$ such that $t_n \to t_\infty$ implies the weak convergence $f(t_n) \rightharpoonup f(t_\infty)$ in $H^1(\Omega)$.

Theorem 1.3 (Time-dependent Mosolov problem) *If Ω is convex, $f \in L^2(0, T; L^2(\Omega)) \cap L^1(0, T; H)$, and $u^0 \in \mathrm{BV}(\Omega) \cap H$, then the solution u to (1.12)-(1.14) verifies $u(t) \in H$ for all $t \in [0, T]$ and*

$$\|\nabla u(t)\|_{L^2(\Omega)} \le \|\nabla u^0\|_{L^2(\Omega)} + \int_0^T \|\nabla f(t)\|_{L^2(\Omega)} dt. \tag{1.17}$$

Moreover, the solution u is characterized by

$$u \in H^1(0, T; L^2(\Omega)) \cap C([0, T]; H^1(\Omega)\text{-weak}), \tag{1.18}$$

$$\begin{cases} \partial_t u - \mathrm{div}\, \sigma = f \text{ in } (0, T) \times \Omega \qquad \text{for some } \sigma \in L^\infty((0, T) \times \Omega)^n, \\ |\sigma| \le 1 \text{ a.e.}, \quad \sigma = \dfrac{\nabla u}{|\nabla u|} \text{ where } \nabla u \ne 0, \\ u(t) \in H_0^1(\Omega) \text{ for all } t \in [0, T] \text{ for } \beta = 1 \text{ and} \\ \sigma \cdot \nu = 0 \text{ on } \partial\Omega \text{ for a.e. } t \in [0, T] \text{ for } \beta = 0, \end{cases} \tag{1.19}$$

$$u(0) = u^0. \tag{1.20}$$

The H^2 regularity of the solution to the steady Mosolov problem with positive viscosity $\mu > 0$ has been proved by H. Brézis [6] in the presence of Dirichlet boundary conditions. In the case of Neumann boundary conditions, we can establish a similar result.

Theorem 1.4 (Steady Mosolov problem with viscosity) *Let Ω be C^3 or convex, $f \in L^2(\Omega)$, $\alpha, \mu > 0$, and let $u \in H$ be the unique minimizer of the functional*

$$\frac{\alpha}{2}\int_\Omega v^2\,dx + \frac{\mu}{2}\int_\Omega |\nabla v|^2\,dx + \int_\Omega |\nabla v|\,dx - \int_\Omega fv\,dx \tag{1.21}$$

amongst all $v \in H$. Then $u \in H^2(\Omega)$ and

$$\|u\|_{H^2(\Omega)} \le C'(\Omega)\left(\|u\|_{L^2(\Omega)} + \frac{1}{\mu}\|f\|_{L^2(\Omega)} + \frac{C(\Omega)}{\mu}\right). \tag{1.22}$$

The constant $C(\Omega)$ stems from Lemma 2.1, 2.2, or 2.4 with $C(\Omega) = 0$ if Ω is convex and the the two constants $C(\Omega)$ and $C'(\Omega)$ solely depend on Ω.

We can state also a local result without the convexity of Ω.

Theorem 1.5 (Local regularity and relaxed Dirichlet condition) *Let Ω be a bounded Lipschitz domain open subset in $\mathbb{R}^n$, $f \in L^2(\Omega) \cap H^1_{\mathrm{loc}}(\Omega)$, $\alpha > 0$, and let u be the unique minimizer of (1.4) in $\mathrm{BV}(\Omega) \cap L^2(\Omega)$. Then u is the weak limit in $L^2(\Omega)$ as $\mu \to 0$ of the minimizer $u_\mu \in H$ of the viscous functional (1.21). Moreover, $u \in H^1_{\mathrm{loc}}(\Omega)$ and, for any $\varphi \in C^\infty_c(\Omega)$ with $\varphi \ge 0$, one has*

$$\|\varphi^2 \nabla u\|_{L^2(\Omega)} \le \frac{C_{\varphi,n}}{\alpha}\Big(\|f\|_{L^2(\Omega)} + \|\varphi^2 \nabla f\|_{L^2(\Omega)} + 1\Big) \tag{1.23}$$

with a constant $C_{\varphi,n}$ that depends only on φ and n. The solution u is characterized by $u \in L^2(\Omega)$, $\nabla u \in L^1(\Omega)^n$,

$$\begin{gathered} \alpha u - \operatorname{div}\sigma = f \text{ in } \Omega \qquad \text{for some } \sigma \in L^\infty(\Omega)^n, \\ |\sigma| \le 1 \text{ a.e.}, \quad \sigma = \frac{\nabla u}{|\nabla u|} \text{ where } \nabla u \ne 0, \\ \sigma \cdot \nu = -\operatorname{sign}(u) \text{ where } u \ne 0 \text{ on } \partial\Omega \text{ for } \beta = 1, \text{ and} \\ \sigma \cdot \nu = 0 \text{ on } \partial\Omega \text{ for } \beta = 0. \end{gathered} \tag{1.24}$$

In the presence of Dirichlet boundary conditions $\beta = 1$, we observe that u does *not* necessarily satisfy $u|_{\partial\Omega} = 0$, but satisfies the weaker relaxed Dirichlet condition

$$\sigma \cdot \nu = -\operatorname{sign}(u) \text{ where } u \ne 0 \text{ on } \partial\Omega. \tag{1.25}$$

This condition can be interpreted as a friction condition on the boundary.
The proofs for the Mosolov problem are given in Section 3, except for Theorem 1.5 for which the full proof of the analogous, but more difficult, Theorem 1.8 is given in Section 4.

1.1.2 Comments

H^1 regularity in convex minimization

The condition $f \in H$ also arises in the H^1_{loc} stress regularity of a class of degenerated convex minimization problems in [8]. The very different assumptions with two-sided p-growth and convexity control in [8] do *not* apply to the example at hand with multiple values for $\operatorname{sign}\nabla u$ in the set where $\nabla u = 0$ vanishes.

H^1 regularity for nonconvex domains

For the H^1 regularity, if Ω is smooth but *not* convex, Theorem 1.1 does *not* apply and the proof does *not* work. The H^2-regularity works however, as stated in Theorem 1.4. This is due to the H^2 term in (2.2) that cannot be absorbed for an H^1 estimate unless $C(\Omega) = 0$. This H^2 norm comes from the estimate of the gradient of u_ε on $\partial\Omega$, see (2.9). Nevertheless, the local result from Theorem 1.5 can be applied and implies the $H^1_{\rm loc}$-regularity, because the local analogue (4.6) of (2.2) involves only an H^1-norm of u_ε in Ω instead of $\partial\Omega$. We do *not* know whether the global H^1 regularity can fail for a non-convex domain.

Inviscid limit and boundary layers

It is important to understand that in Theorem 1.5 and in the presence of Dirichlet boundary conditions $\beta = 1$, it is *not* assumed that $f \in H^1_0(\Omega)$. Indeed if $f \in H^1_0(\Omega)$ (and Ω is convex) then Theorem 1.1 implies $u \in H^1_0(\Omega)$, i.e., u vanishes on $\partial\Omega$. In Theorem 1.5 f can be nonzero on $\partial\Omega$, and so u can be nonzero on $\partial\Omega$, even if u is the limit of u_μ, the solution to the viscous problem, that vanishes on $\partial\Omega$. This means that there can be boundary layers as in the following example.

Example. Take $n = 1$, $\Omega = (0, L)$, and a global constant $f = cst$ in Ω. Then one can directly verify the following. In the Neumann case, the solution reads $u = cst = f/\alpha$ and σ is any constant in $[-1, 1]$. In the Dirichlet case, there are two possibilities: If $|f| \le 2/L$, then the solution $u = 0$ vanishes and $\sigma(x) = \sigma_m - f(x - L/2)$, for any σ_m satisfying $|\sigma_m| \le 1 - |f|L/2$. If $|f| > 2/L$, then the solution reads $u = cst = \mathrm{sign}(f)(|f| - 2/L)/\alpha$ and $\sigma(x) = -\frac{2}{L}\mathrm{sign}(f)(x - L/2)$, which satisfies (1.25). We can compute the viscous solution u_μ. In the Neumann or Dirichlet case with $|f| \le 2/L$, $u_\mu = u$ and $\sigma_\mu = \sigma$. In the Dirichlet case with $|f| > 2/L$, the solution u_μ has a boundary layer given by

$$u_\mu(x) = \frac{\mathrm{sign}(f)}{\alpha}\Big(|f| - \frac{1}{L/2 - \Lambda}\Big), \quad \sigma_\mu(x) = -\frac{\mathrm{sign}(f)}{L/2 - \Lambda}\Big(x - \frac{L}{2}\Big) \quad \text{for } \Lambda \le x \le L - \Lambda; \tag{1.26}$$

$$u_\mu(x) = \frac{f}{\alpha}\left(1 - \frac{\mathrm{ch}\Big(\sqrt{\frac{\alpha}{\mu}}(\Lambda - \min(x, L - x))\Big)}{\mathrm{ch}\Big(\sqrt{\frac{\alpha}{\mu}}\Lambda\Big)}\right), \ \sigma_\mu(x) = -\mathrm{sign}(f)\,\mathrm{sign}(x - L/2) \text{ otherwise}, \tag{1.27}$$

where the width $\Lambda \in (0, L/2)$ of the boundary layer satisfies

$$\mathrm{ch}\Big(\sqrt{\frac{\alpha}{\mu}}\Lambda\Big) = |f|\Big(\frac{L}{2} - \Lambda\Big). \tag{1.28}$$

This is illustrated in Figure 1. We remark that in accordance with Theorem 1.5, when $\mu \to 0$, u_μ is bounded in $H^1_{\rm loc}(\Omega)$ but is *not* bounded in $H^1(\Omega)$.

Optimality of the exponents

In case of viscosity, the solution given by (1.26)-(1.27) satisfies $u_\mu \notin C^2$. In particular $u_\mu \notin H^3$, even if f is smooth (a constant here). In the inviscid case, the construction of the one-dimensional solution given in [5] shows that even for f smooth, in general $u \notin C^1$, whence $u \notin H^2$. It remains as an open problem to investigate the regularity $W^{2,p}$ for the viscous case (respectively, $W^{1,p}$ in the inviscid case) for some $p > 2$.

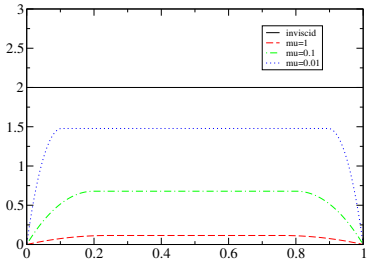

Figure 1: Viscous solution $u_\mu(x)$ given by (1.26), (1.27) for $L = 1$, $\alpha = 1$, $f = 4$, and different values of the viscosity μ. The black line is the inviscid solution.

1.2 The Bingham problem

In the case of the vector Bingham problem with viscosity, we have a local analogue to Theorem 1.4. The steady Bingham problem with viscosity can be written formally as

$$\alpha u - \mu \Delta u - \operatorname{div} \frac{Du}{|Du|} = f \quad \text{in } \Omega, \tag{1.29}$$

where u has values in $\mathbb{R}^n$ and $Du = (\nabla u + (\nabla u)^t)/2$. The rigorous way to define the solution to (1.29) is to minimize the functional

$$\frac{\alpha}{2} \int_\Omega |v|^2 + \frac{\mu}{2} \int_\Omega |\nabla v|^2 + \int_\Omega |Dv| - \int_\Omega f \cdot v \quad \text{for } v \in H, \tag{1.30}$$

where either $H := H_0^1(\Omega)^n$ (Dirichlet boundary conditions) or $H := H^1(\Omega)^n$ (Neumann boundary conditions). This problem has a unique solution and is equivalent to seek $u \in H$ such that

$$\forall v \in H \quad \int_\Omega f \cdot (v - u) \le \alpha \int_\Omega u \cdot (v - u) + \mu \int_\Omega \nabla u : (\nabla v - \nabla u) + \int_\Omega |Dv| - \int_\Omega |Du|, \tag{1.31}$$

with the scalar product $A : B := \operatorname{tr}(A^t B)$ for two matrices A, B.

1.2.1 Statement of the main results

In the case of the Bingham problem, we mostly prove local regularity results and to this purpose we often consider a smooth, compactly supported function $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$. We then use the notation C_φ or $C_{\varphi,n}$ to denote a generic (positive) constant, whose value can change at each occurrence provided it exclusively depends on φ or on φ and n, respectively.

Theorem 1.6 (Bingham problem with viscosity) *Let Ω be a bounded open subset of $\mathbb{R}^n$, $f \in L^2(\Omega)^n$, $\alpha, \mu > 0$, and let u be the unique solution to (1.31) with either $H = H_0^1(\Omega)^n$ or $H = H^1(\Omega)^n$. Then $u \in H^2_{\mathrm{loc}}(\Omega)^n$ and any $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$ satisfies*

$$\|\varphi^2 \nabla^2 u\|_{L^2(\Omega)} \le C_{\varphi,n} \left((\frac{1}{\alpha} + \frac{1}{\mu}) \|f\|_{L^2(\Omega)} + \frac{1}{\sqrt{\alpha\mu}} \right). \tag{1.32}$$

(The constant $C_{\varphi,n}$ exclusively depends on φ and n and $\nabla^2 u$ denotes the Hessian composed of the second derivatives of all components of u.)

Given any compact set $K \subset \Omega$, there exists classically some $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ in $\mathbb{R}^n$ and $\varphi = 1$ in K. Thus, (1.32) implies that $\nabla^2 u$ is square integrable in a neighboorhood of K.
Let us consider the Bingham problem (1.29) in the case without viscosity ($\mu = 0$), i.e., (1.1). We define, for all $v \in L^2(\Omega)^n$,

$$\psi(v) := \sup_{w \in V(\Omega)} \int_\Omega v \cdot w \quad \in [0, \infty], \text{ where} \tag{1.33}$$

$$V(\Omega) := \left\{ w \in L^2(\Omega)^n : \exists \Phi \in C_c^1(\Omega)^{n \times n}, w_i = \frac{1}{2} \sum_{j=1}^n \partial_j (\Phi_{ij} + \Phi_{ji}), \sum_{i,j=1}^n \Phi_{ij}^2 \leq 1 \text{ in } \Omega \right\}. \tag{1.34}$$

Note that the set $V(\Omega)$ does *not* change if we consider only symmetric Φ's ($\Phi_{ij} = \Phi_{ji}$). As shown in [5], the definition (1.33) of $\psi(v)$ is the right definition for $\int_\Omega |Dv|$. Indeed, one has $\psi(v) = \int_\Omega |Dv|$ for all $v \in H^1(\Omega)^n$, and ψ is convex lower semi-continuous on $L^2(\Omega)^n$ with $0 \in \partial\psi(0)$. As for the Mosolov problem, the definition (1.33) models Neumann boundary conditions. In the case of Dirichlet boundary conditions, we define ψ for all $v \in L^2(\Omega)^n$ by

$$\psi(v) = \sup_{w \in V(\mathbb{R}^n)} \int_\Omega v \cdot w \ \in [0, \infty]. \tag{1.35}$$

In other words, it is the functional (1.33) on $\mathbb{R}^n$ applied to the zero-extension $\overline{v}$ of v from Ω to $\mathbb{R}^n$. Then we have $\psi(v) = \int_\Omega |Dv|$ for all $v \in H_0^1(\Omega)^n$, because in this case $\overline{v} \in H^1(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} |D\overline{v}| = \int_\Omega |Dv|$. We consider the solution to (1.1) with Dirichlet or Neumann boundary conditions in the variational sense, which means to minimize the functional

$$\frac{\alpha}{2} \int_\Omega |v|^2 + \psi(v) - \int_\Omega f \cdot v \quad \text{for } v \in L^2(\Omega)^n \tag{1.36}$$

with ψ either defined by (1.35) (Dirichlet) or by (1.33) (Neumann). Since ψ is convex lower semi-continuous in $L^2(\Omega)^n$ with $0 \in \partial\psi(0)$, this problem has a unique solution u. The solution $u \in L^2(\Omega)^n$ is characterized by

$$\forall v \in L^2(\Omega)^n \quad \int_\Omega f \cdot (v - u) \leq \alpha \int_\Omega u \cdot (v - u) + \psi(v) - \psi(u). \tag{1.37}$$

Then $\psi(u) < \infty$ and it is equivalent to consider only test functions v in (1.37) with $\psi(v) < \infty$.

In contrast with Theorem 1.1, the local setting below does *not* assume Ω be convex.

Theorem 1.7 (Inviscid Bingham problem) *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $f \in L^2(\Omega)^n \cap H^1_{\text{loc}}(\Omega)^n$, $\alpha > 0$, and let u be the unique solution to (1.37) with ψ either defined by (1.35) (Dirichlet) or by (1.33) (Neumann). Then u is the weak limit in $L^2(\Omega)^n$ as $\mu \to 0$ of the solution $u_\mu \in H$ to the viscous problem (1.31). Moreover, $u \in H^1_{\text{loc}}(\Omega)^n$ and any $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ satisfies*

$$\|\varphi^2 \nabla u\|_{L^2(\Omega)} \leq \frac{C_{\varphi,n}}{\alpha} \Big(\|f\|_{L^2(\Omega)} + \|\varphi^2 \nabla f\|_{L^2(\Omega)} + 1 \Big). \tag{1.38}$$

(The constant $C_{\varphi,n}$ exclusively depends on φ and n.)

It follows from $u \in L^2(\Omega)^n \cap H^1_{\rm loc}(\Omega)^n$ and $\psi(u) < \infty$ that $Du \in L^1(\Omega)^{n\times n}$ and $\psi(u) = \|Du\|_{L^1(\Omega)^{n\times n}}$. This is a remarkable regularity of the solution u in the space of bounded deformations with only the absolutely continuous part of the symmetrized gradient Du. (Lemma A.2 and $\psi(u) < \infty$ imply that $Du \in L^2_{\rm loc}(\Omega)^{n\times n}$ is a finite measure, cf., e.g., [2] and references quoted therein).
Throughout this paper, the dyadic product $a \otimes b$, a rank-one matrix for vectors $a, b \in \mathbb{R}^n$, allows for the symmetrized dyadic product

$$a \odot b := \operatorname{sym} a \otimes b := \frac{a \otimes b + b \otimes a}{2}. \tag{1.39}$$

Theorem 1.8 (Euler-Lagrange equations for the Bingham problem) *Under the assumptions of Theorem 1.7, u is characterized by $u \in L^2(\Omega)^n$, $Du \in L^1(\Omega)^{n\times n}_{\rm sym}$,*

$$\begin{aligned} \alpha u - \operatorname{div}\sigma = f \ \text{in}\ \Omega \qquad & \text{for some } \sigma \in L^\infty(\Omega)^{n\times n}_{\rm sym},\ \sigma^t = \sigma, \\ |\sigma| \le 1 \ \text{a.e.}, \quad \sigma = \frac{Du}{|Du|} \ & \text{where } Du \neq 0, \\ \sigma_b = -\frac{u_b \odot \nu}{|u_b \odot \nu|} \ & \text{where } u_b \neq 0 \text{ on } \partial\Omega, \ \text{or } \sigma_b \nu = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1.40}$$

The "or" in the last statement (i.e., the third line of (1.40)) refers to the considered Dirichlet or Neumann case and involves the traces u_b of u and σ_b of σ on $\partial\Omega$ defined in Lemma A.2.

We remark that the relaxed Dirichlet condition can be written also in terms of $\sigma_b\nu$, which makes sense as a weak trace

$$\sigma_b\nu = -\frac{u_b + \nu(u_b\cdot\nu)}{\big(2(|u_b|^2 + (u_b\cdot\nu)^2)\big)^{1/2}} \quad \text{where } u_b \neq 0 \text{ on } \partial\Omega. \tag{1.41}$$

Nevertheless, the only value of σ_b that achieves (1.41) and satisfies $|\sigma_b| \le 1$ is the one given in the third line of (1.40). Indeed, since $|u_b \otimes \nu + \nu \otimes u_b|^2 = 2(|u_b|^2 + (u_b\cdot\nu)^2)$, (1.41) implies that

$$\sigma_b : \Big(-\frac{u_b \odot \nu}{|u_b \odot \nu|}\Big) = 1 \quad \text{where } u_b \neq 0 \text{ on } \partial\Omega. \tag{1.42}$$

For the Neumann case, the boundary condition involves the normal trace $\sigma_b\nu = \sigma_b\nu \in H^{-1/2}(\partial\Omega)^n$ of $\sigma \in H(\operatorname{div},\Omega)^n$: The condition $\sigma\nu = 0$ is equivalent to $\int_\Omega v\cdot\operatorname{div}\sigma = -\int_\Omega Dv : \sigma$ for all $v \in H^1(\Omega)^n$.

1.2.2 Comments

Regularity up to the boundary for the Bingham problem

We do *not* have any result of global H^1 regularity for the inviscid Bingham problem, nor of global H^2 regularity for the viscous problem. The difficulty is to obtain appropriate estimates of boundary integral terms, as is possible for the Mosolov problem in the proof of Lemma 2.2 for the terms T_1 and T_2.

Physical viscosity

The viscous Bingham problem (1.29) is stated here with a viscous term of the form $-\mu\Delta u$, while a term of the form $-\operatorname{div}(\mu Du)$ might appear physically more relevant. However, we are *not* able to deal with such a term: Lemma 4.1 does *not* successfully apply in a proof of a possible generalization of Theorem 1.6 or 1.7; an analogue to Lemma 4.1 with $-\operatorname{div}(\mu Du)$ replacing the Laplacian does *not* seem to hold.

The remainder of the paper is devoted to the proofs.

2 Brézis' lemma and variants

2.1 Smooth domain

The regularity of the minimizer u of (1.4) in $V = \mathrm{BV}(\Omega) \cap L^2(\Omega)$ follows from the following lemma from [6].

Lemma 2.1 (Brézis (1971)) *For any open bounded set $\Omega \subset \mathbb{R}^n$ of class C^3, there exists some constant $C(\Omega) \geq 0$, with $C(\Omega) = 0$ if Ω is convex, such that the following holds: Given any $\varepsilon > 0$ and $u \in H^1_0(\Omega)$, the (unique) solution $u_\varepsilon \in H^1_0(\Omega)$ to*

$$u_\varepsilon - \varepsilon\Delta u_\varepsilon = u \quad \text{in } \Omega \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{2.1}$$

satisfies $u_\varepsilon \in H^2(\Omega)$ and

$$\|\nabla u_\varepsilon\|_{L^1(\Omega)} \leq \|\nabla u\|_{L^1(\Omega)} + \varepsilon C(\Omega)\|u_\varepsilon\|_{H^2(\Omega)}. \tag{2.2}$$

The proof of Lemma 2.1 is given in [6, p.118]. We require the following variant for homogeneous Neumann boundary conditions. Recall that ν is the outer unit normal along the boundary $\partial\Omega$.

Lemma 2.2 (variant for Neumann BC) *For any open bounded set $\Omega \subset \mathbb{R}^n$ of class C^3, there exists some constant $C(\Omega) \geq 0$, with $C(\Omega) = 0$ if Ω is convex, such that the following holds: Given any $\varepsilon > 0$ and $u \in H^1(\Omega)$, the (unique) solution $u_\varepsilon \in H^1(\Omega)$ to*

$$u_\varepsilon - \varepsilon\Delta u_\varepsilon = u \quad \text{in } \Omega \quad \text{and} \quad \nabla u_\varepsilon \nu = 0 \quad \text{on } \partial\Omega \tag{2.3}$$

satisfies $u_\varepsilon \in H^2(\Omega)$ and the estimate (2.2).

Proof. Since Ω is of class C^3, the unique solution to (2.3) verifies $u_\varepsilon \in H^3(\Omega)$. In particular, $\nabla^2 u_\varepsilon$ has a trace in $L^2(\partial\Omega)$. As in [6] we consider some regularisation of the Euclidean norm $|\cdot|$. We choose here a C^2 regularisation. Given any $\delta > 0$ and $\xi \in \mathbb{R}^n$, we define

$$j_\delta(\xi) := \begin{cases} \dfrac{3}{4\delta}|\xi|^2 - \dfrac{1}{8\delta^3}|\xi|^4 & \text{if } |\xi| \leq \delta, \\ |\xi| - \dfrac{3\delta}{8} & \text{if } |\xi| \geq \delta. \end{cases} \tag{2.4}$$

Then $j_\delta \in C^2(\mathbb{R}^n)$ is convex and nonnegative, j_δ and its derivative j'_δ are globally Lipschitz continuous, and $|j'_\delta| \leq 1$. The convexity of j_δ leads a.e. in Ω to

$$j_\delta(\nabla u_\varepsilon) - j_\delta(\nabla u) \leq j'_\delta(\nabla u_\varepsilon) \cdot (\nabla u_\varepsilon - \nabla u). \tag{2.5}$$

Since $u_\varepsilon \in H^3(\Omega)$, (2.3) leads to $\nabla u_\varepsilon - \nabla u = \varepsilon \Delta(\nabla u_\varepsilon)$. The integral of (2.5) over Ω reads

$$\begin{aligned} &\int_\Omega j_\delta(\nabla u_\varepsilon)dx - \int_\Omega j_\delta(\nabla u)dx \leq \varepsilon \sum_{m=1}^n \int_\Omega j'_\delta(\nabla u_\varepsilon) \cdot \frac{\partial^2}{\partial x_m^2} \nabla u_\varepsilon dx \qquad (2.6)\\ &= -\varepsilon \sum_{m=1}^n \int_\Omega \Big(\frac{\partial}{\partial x_m} j'_\delta(\nabla u_\varepsilon)\Big) \cdot \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) dx + \varepsilon \sum_{m=1}^n \int_{\partial\Omega} j'_\delta(\nabla u_\varepsilon) \cdot \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) \nu_m ds. \end{aligned}$$

Since j'_δ is C^1 and globally Lipschitz continuous, it follows a.e. in Ω that

$$\frac{\partial}{\partial x_m} \big(j'_\delta(\nabla u_\varepsilon)\big) = j''_\delta(\nabla u_\varepsilon) \frac{\partial}{\partial x_m} \nabla u_\varepsilon. \tag{2.7}$$

The substitution of (2.7) in the first term T_1 on the right-hand side of (2.6) shows that

$$T_1 = -\varepsilon \sum_{m=1}^n \int_\Omega \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) \cdot j''_\delta(\nabla u_\varepsilon) \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) dx \leq 0,$$

since the Hessian $j''_\delta(\nabla u_\varepsilon)$ of the convex function j_δ is symmetric and positive semi-definite. Since $j'_\delta(0) = 0$ as $\delta \to 0$, we have $j'_\delta(\xi) \to \frac{\xi}{|\xi|} \mathbb{1}_{\xi \neq 0}$ for all $\xi \in \mathbb{R}^n$. Thus (2.6), $T_1 \leq 0$, and Lebesgue's theorem lead to

$$\int_\Omega |\nabla u_\varepsilon| dx - \int_\Omega |\nabla u| dx \leq \varepsilon \sum_{m=1}^n \int_{\partial\Omega} \mathbb{1}_{\nabla u_\varepsilon \neq 0} \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \cdot \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) \nu_m ds =: T_2. \tag{2.8}$$

The analysis on the right-hand side T_2 of (2.8) requires an extension of the normal field ν. According to Lemma 2.3 below, the signed distance function $d(x) := \mathrm{dist}(x, \partial\Omega)$ for $x \in \Omega$ and $d(x) := -\mathrm{dist}(x, \partial\Omega)$ for $x \in \mathbb{R}^n \setminus \overline{\Omega}$ allows for $\nu = -\nabla d$ in a neighbourhood of $\partial\Omega$, and $\nu \in C^2$ because Ω is C^3. In this neighbourhood of $\partial\Omega$ intersected with $\overline{\Omega}$, $\nabla\nu = -D^2 d$ and, with the abbreviation $v_\varepsilon := \nabla u_\varepsilon \cdot \nu \in H^2(\Omega)$,

$$\sum_{m=1}^n \Big(\frac{\partial}{\partial x_m} \nabla u_\varepsilon\Big) \nu_m = \nabla v_\varepsilon + D^2 d\, \nabla u_\varepsilon.$$

The substitution of this in T_2 reads

$$T_2 = \varepsilon \int_{\partial\Omega} \frac{\mathbb{1}_{\nabla u_\varepsilon \neq 0}}{|\nabla u_\varepsilon|} \left(\nabla u_\varepsilon \cdot \nabla v_\varepsilon + \nabla u_\varepsilon \cdot D^2 d\, \nabla u_\varepsilon\right) ds.$$

The boundary condition in (2.3) reads $v_\varepsilon = 0$ on $\partial\Omega$, so that the tangential derivatives of v_ε along $\partial\Omega$ vanish as well. Hence $\nabla v_\varepsilon = \nu\, \partial v_\varepsilon / \partial\nu$ on $\partial\Omega$. This and $v_\varepsilon = \nabla u_\varepsilon \cdot \nu = 0$ on $\partial\Omega$ proves $\nabla u_\varepsilon \cdot \nabla v_\varepsilon = (\partial v_\varepsilon / \partial\nu) \nabla u_\varepsilon \cdot \nu = 0$ there. Consequently,

$$T_2 = \varepsilon \int_{\partial\Omega} \frac{\mathbb{1}_{\nabla u_\varepsilon \neq 0}}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \cdot D^2 d\, \nabla u_\varepsilon ds. \tag{2.9}$$

With the maximum $C(\Omega) \geq 0$ of the real eigenvalues of the (symmetric) Hessian $D^2 d$ of the distance function along the compact boundary $\partial\Omega$ (and $C(\Omega) := 0$ if there are no positive eigenvalues), one has $T_2 \leq \varepsilon C(\Omega) \|\nabla u_\varepsilon\|_{L^1(\partial\Omega)}$. This, (2.8), and a trace inequality, lead to (2.2). In the case of a convex domain Ω, Lemma 2.3 below guarantees that the distance function d is concave. Therefore, $\nabla^2 d$ is (symmetric and) negative semidefinite and $C(\Omega) = 0$. □

Lemma 2.3 *Let Ω be an open bounded subset of $\mathbb{R}^n$ of class C^2. Then the function $d(x) := \mathrm{dist}(x, \partial\Omega)$ for $x \in \overline{\Omega}$, $d(x) := -\mathrm{dist}(x, \partial\Omega)$ for $x \in \mathbb{R}^n \backslash \Omega$ is of class C^2 in a neighbourhood of $\partial\Omega$ with gradient $\nabla d(x) = -\nu(z(x))$ for the projection $z(x)$ of x on $\partial\Omega$. Moreover, if Ω is convex, then $-d(x)$ is locally convex in a neighbourhood of $\partial\Omega$.*

Proof. Since Ω is of class C^2, the mapping

$$\partial\Omega \times (-\eta, \eta) \longrightarrow \mathbb{R}^n, \quad (z, t) \longmapsto x = z - t\nu(z) \tag{2.10}$$

is a C^1 diffeomorphism onto an open set W_η containing $\partial\Omega$ for sufficiently small $\eta > 0$. Its inverse can be denoted by $(z(x), t(x))$ for $x \in W_\eta$. Moreover, since $\partial\Omega$ is of class C^2, it has a lower-bounded curvature radius. Therefore, if η is smaller than the minimum curvature radius, for all $x \in W_\eta$, the point $z(x)$ uniquely achieves $|x - z(x)| = \min\limits_{z \in \partial\Omega} |x - z| = \mathrm{dist}(x, \partial\Omega)$, and thus $\mathrm{dist}(x, \partial\Omega) = |t(x)|$. We have that $t(x) > 0$ if and only if $x \in \Omega$, therefore $d(x) = t(x)$ for all $x \in W_\eta$. Differentiating (2.10), we deduce

$$dx = dz - td\nu(z) - \nu(z)dt. \tag{2.11}$$

Since $\nu(z) \cdot dz = 0$ and $\nu(z) \cdot d\nu(z) = 0$, we obtain $\nu(z) \cdot dx = -dt$, which means that

$$\nabla_x t(x) = -\nu(z(x)) \quad \text{for all } x \in W_\eta. \tag{2.12}$$

It follows that $t(x)$ is of class C^2. This concludes the proof of the first part of the lemma.

Assume that Ω is convex in the second part of the proof and define $\overline{d}(x) = \mathrm{dist}(x, \overline{\Omega})$. Then $\overline{d}$ is convex on $\mathbb{R}^n$. Indeed, take $x, y \in \mathbb{R}^n$ and $0 \leq \theta \leq 1$ and define $w = (1-\theta)x + \theta y$. There is some $b_x \in \overline{\Omega}$ such that $\overline{d}(x) = |x - b_x|$ and some $b_y \in \overline{\Omega}$ such that $\overline{d}(y) = |y - b_y|$. Since $\overline{\Omega}$ is convex, we have

$$\overline{d}(w) \leq \big|(1-\theta)x + \theta y - \big((1-\theta)b_x + \theta b_y\big)\big| \leq (1-\theta)|x - b_x| + \theta|y - b_y|. \tag{2.13}$$

This proves the convexity of $\overline{d}$ on $\mathbb{R}^n$. Any $x \in W_\eta \backslash \overline{\Omega}$ satisfies $\overline{d}(x) = \mathrm{dist}(x, \partial\Omega) = -t(x)$. Consequently, $\overline{d}$ is C^2 and convex, whence $D^2 t \leq 0$ in $W_\eta \backslash \overline{\Omega}$. It remains to prove that

$$0 \leq \Big(\nu\big(z(x)\big) - \nu\big(z(y)\big)\Big) \cdot (x - y) \quad \text{for all } x, y \in W_\eta \cap \Omega. \tag{2.14}$$

In fact, (2.12) and (2.14) impy that $D^2 t(x) \leq 0$ in $W_\eta \cap \Omega$. Since $D^2 t(x) \leq 0$ in $W_\eta \backslash \overline{\Omega}$ and since $t(x)$ is C^2 in W_η, this shows $D^2 t(x) \leq 0$ in W_η and that $-t(x)$ is locally convex in W_η.
Thus it remains to prove (2.14), which is equivalent to

$$0 \leq \big(\nu(z_1) - \nu(z_2)\big) \cdot \big(z_1 - t_1\nu(z_1) - z_2 + t_2\nu(z_2)\big) \tag{2.15}$$

for all $z_1, z_2 \in \partial\Omega$, $t_1, t_2 \in (0, \eta)$. To verify this, it is sufficient to prove that

$$0 \le \nu(z_1) \cdot \Big(z_1 - z_2 + t_2\big(\nu(z_2) - \nu(z_1)\big)\Big). \tag{2.16}$$

(Exchanging indices gives $\nu(z_2) \cdot \Big(z_2 - z_1 + t_1\big(\nu(z_1) - \nu(z_2)\big)\Big) \ge 0$ and adding this to (2.16) shows (2.15); note that $\nu(z_1) \cdot (\nu(z_2) - \nu(z_1)) = \nu(z_2) \cdot (\nu(z_1) - \nu(z_2))$.) To prove (2.16), set

$$w := z_2 - t_2\big(\nu(z_2) - \nu(z_1)\big) = y + t_2\nu(z_1) \quad \text{for} \quad y := z_2 - t_2\nu(z_2) \in W_\eta \cap \Omega. \tag{2.17}$$

Then $t_2 = t(y) = \operatorname{dist}(y, \partial\Omega)$, and since $|w - y| = t_2 = \operatorname{dist}(y, \partial\Omega)$, this implies either $w = z_2$ or $w \notin \partial\Omega$. If $w = z_2$, then $\nu(z_2) = \nu(z_1)$ and (2.16) is obvious: $z_2 \in \overline{\Omega}$ is convex and $\nu(z_1)$ is normal to the tangent plane on $\partial\Omega$ at z_1. If $w \notin \partial\Omega$ and $\theta \in (0, 1)$, then

$$w_\theta := (1 - \theta)y + \theta w = y + \theta t_2 \nu(z_1) \tag{2.18}$$

verifies $|y - w_\theta| < t_2 = \operatorname{dist}(y, \partial\Omega)$; whence $w_\theta \notin \partial\Omega$. In summary, $w_\theta \notin \partial\Omega$ holds for all $\theta \in [0, 1]$. Since $y \in \Omega$, this implies $w \in \Omega$. As previously, $\nu(z_1) \cdot (z_1 - w) \ge 0$ follows and results in (2.16). This concludes the proof. □

2.2 General convex domains

Lemma 2.1 and 2.2 also hold for general convex (non-smooth) domains. Recall the notation $H := H^1_0(\Omega)$ (resp. $H := H^1(\Omega)$) for $\beta = 1$ (resp. $\beta = 0$) from Subsection 1.1.

Lemma 2.4 (convex domain) *For any convex bounded open set $\Omega \subset \mathbb{R}^n$, any $\varepsilon > 0$, and any $u \in H$, the solution $u_\varepsilon \in H$ to (2.1) for $\beta = 1$ and to (2.3) for $\beta = 0$ satisfies $\|\nabla u_\varepsilon\|_{L^1(\Omega)} \le \|\nabla u\|_{L^1(\Omega)}$.*

Proof. Given $u \in H$, the solution $u_\varepsilon \in H$ of the assertion satisfies

$$\int_\Omega u_\varepsilon v \, dx + \varepsilon \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dx = \int_\Omega uv \, dx \quad \text{for all } v \in H. \tag{2.19}$$

A convex domain Ω is a Lipschitz domain [15]. Hence there exists a bounded linear extension operator $E : H \to H^1_c(\mathbb{R}^n)$ for $\beta = 0$ and $H = H^1(\Omega)$, while E is the extension by zero for $\beta = 1$ and $H = H^1_0(\Omega)$. Let $\|E\| > 0$ denote the operator norm of E such that

$$\|Ev\|_{H^1(\mathbb{R}^n)} \le \|E\| \, \|v\|_{H^1(\Omega)} \quad \text{for all } v \in H.$$

It is known that the convex domain Ω allows for an outer approximation by a smooth and convex domain $\Omega_\delta \supset \Omega$ for any $\delta > 0$, such that the volume measure $|\Omega_\delta \setminus \Omega|$ of the difference $\Omega_\delta \setminus \Omega$ tends to zero as $\delta \to 0$; apply [1, Corollary 2] to $\overline{\Omega}$ for a proof. The two cases for β and the domain Ω_δ lead to the two cases $H_\delta := H^1_0(\Omega_\delta)$ if $\beta = 1$ and $H_\delta := H^1(\Omega_\delta)$ if $\beta = 0$. We define $u_\delta := (Eu)|_{\Omega_\delta} \in H_\delta$ and consider the solution $u_{\delta\varepsilon} \in H_\delta$ to

$$\int_{\Omega_\delta} u_{\delta\varepsilon} v \, dx + \varepsilon \int_{\Omega_\delta} \nabla u_{\delta\varepsilon} \cdot \nabla v \, dx = \int_{\Omega_\delta} u_\delta v \, dx \quad \text{for all } v \in H_\delta. \tag{2.20}$$

The test with $v = u_{\delta\varepsilon} \in H_\delta$ in (2.20) and a Cauchy–Schwarz inequality show

$$\|u_{\delta\varepsilon}\|^2_{L^2(\Omega_\delta)} + 2\varepsilon\|\nabla u_{\delta\varepsilon}\|^2_{L^2(\Omega_\delta)} \leq \|u_\delta\|^2_{L^2(\Omega_\delta)} \leq \|E\|^2 \|u\|^2_{H^1(\Omega)}. \tag{2.21}$$

For any fixed $\varepsilon > 0$, this proves that $\|u_{\delta\varepsilon}\|_{H^1(\Omega_\delta)}$ is bounded by some $M(\varepsilon)$ independently of $\delta > 0$. Hence, for some sequence $\delta_j \to 0$, there exists a weak limit $u^*_\varepsilon \in H^1(\Omega)$ such that

$$u_{\delta_j\varepsilon}|_\Omega \rightharpoonup u^*_\varepsilon \quad \text{weakly in } H^1(\Omega) \text{ as } j \to \infty. \tag{2.22}$$

The subsequent analysis proves $u^*_\varepsilon \in H$, which is obvious for $\beta = 0$. If $\beta = 1$, then $u_{\delta\varepsilon} \in H^1_0(\Omega_\delta)$ and its extension $\overline{u_{\delta\varepsilon}}$ by zero (outside Ω_δ) is bounded in $H^1_c(\mathbb{R}^n)$. Extracting a subsequence, if necessary and not relabelled, we deduce that $\overline{u_{\delta_j\varepsilon}}$ converges weakly to some limit in $H^1(\mathbb{R}^n)$ as $j \to \infty$. For any open ball $\omega \subset \mathbb{R}^n \setminus \overline{\Omega}$ outside $\overline{\Omega}$, $\overline{u_{\delta\varepsilon}} \in H^1_c(\mathbb{R}^n)$ vanishes outside Ω_δ and the measure of $\Omega_\delta \cap \omega \subset \Omega_\delta \setminus \Omega$ tends to zero as $\delta \to 0$. This, the bound $\|\overline{u_{\delta\varepsilon}}\|_{H^1(\mathbb{R}^n)} \leq M(\varepsilon)$, and a Cauchy–Schwarz inequality prove

$$\|\overline{u_{\delta\varepsilon}}\|_{L^1(\omega)} = \|\overline{u_{\delta\varepsilon}}\|_{L^1(\Omega_\delta\cap\omega)} \leq \|\overline{u_{\delta\varepsilon}}\|_{L^2(\Omega_\delta)}|\Omega_\delta \setminus \Omega| \to 0 \text{ as } \delta \to 0.$$

We conclude that $\overline{u_{\delta_j\varepsilon}} \rightharpoonup \overline{u^*_\varepsilon}$ weakly in $H^1(\mathbb{R}^n)$ as $j \to \infty$, with the extension $\overline{u^*_\varepsilon}$ by zero outside Ω of u^*_ε. Hence, $\overline{u^*_\varepsilon} \in H^1(\mathbb{R}^n)$, and it follows that $u^*_\varepsilon \in H^1_0(\Omega) = H$ for $\beta = 1$.
The proof continues for $\beta = 1$ or $\beta = 0$ knowing that $u^*_\varepsilon \in H$. For any test function $v \in H$ and its extension $v_\delta = (Ev)|_{\Omega_\delta} \in H_\delta$, (2.20) shows that

$$\int_{\Omega_\delta} u_{\delta\varepsilon}(Ev)\,dx + \varepsilon \int_{\Omega_\delta} \nabla u_{\delta\varepsilon} \cdot \nabla(Ev)dx = \int_{\Omega_\delta} (Eu)(Ev)dx \quad \text{for all } v \in H. \tag{2.23}$$

Since $Ev \in H^1_c(\mathbb{R}^n)$, the non-concentration of Lebesgue functions on small sets shows that $\|Ev\|_{L^2(\Omega_\delta\setminus\Omega)} \to 0$ for $|\Omega_\delta \setminus \Omega| \to 0$ as $\delta \to 0$. Similarly $\|\nabla(Ev)\|_{L^2(\Omega_\delta\setminus\Omega)} \to 0$. Consequently, the integrals over Ω_δ in (2.23) can be replaced by the respective integrals over Ω in the limit $\delta \to 0$. This and the weak convergence (2.22) prove in the limit for $\delta_j \to 0$ as $j \to \infty$ that $u^*_\varepsilon \in H$ solves

$$\int_\Omega u^*_\varepsilon v\,dx + \varepsilon \int_\Omega \nabla u^*_\varepsilon \cdot \nabla v\,dx = \int_\Omega uv\,dx \quad \text{for all } v \in H. \tag{2.24}$$

A comparison with (2.19) shows that $u_\varepsilon = u^*_\varepsilon$. Lemma 2.1 or 2.2 apply for the smooth and convex domain Ω_δ for $\delta > 0$ and lead to $\|\nabla u_{\delta\varepsilon}\|_{L^1(\Omega_\delta)} \leq \|\nabla u_\delta\|_{L^1(\Omega_\delta)}$ for the extension $u_\delta := (Eu)|_{\Omega_\delta} \in H_\delta$. This implies that $\|\nabla u_{\delta\varepsilon}\|_{L^1(\Omega)} \leq \|\nabla(Eu)\|_{L^1(\Omega_\delta)}$. The weak convergence (2.22) in $H^1(\Omega)$ and the sequential weak lower semi-continuity of the L^1 norm prove

$$\|\nabla u_\varepsilon\|_{L^1(\Omega)} = \|\nabla u^*_\varepsilon\|_{L^1(\Omega)} \leq \liminf_{j\to\infty} \|\nabla(Eu)\|_{L^1(\Omega_{\delta_j})} = \|\nabla u\|_{L^1(\Omega)}. \quad \square$$

3 Mosolov problem: Proofs

3.1 Proof of Theorem 1.1

Given a convex domain Ω, recall $H := H^1_0(\Omega)$ for $\beta = 1$ and $H := H^1(\Omega)$ for $\beta = 0$. Recall the notation (1.8), i.e., $\psi(v) := |v|_{\mathrm{BV}(\Omega)} + \beta\|v\|_{L^1(\partial\Omega)}$ for $v \in V := \mathrm{BV}(\Omega) \cap L^2(\Omega)$. The minimizer

u of (1.4) in V is equivalently characterized as the solution $u \in V$ to the variational inequality (1.9), i.e.,

$$\int_\Omega (f - \alpha u)(v - u)dx \leq \psi(v) - \psi(u) \quad \text{for all } v \in V. \tag{3.1}$$

The subsequent analysis involves a regularization through viscosity. Given the energy functional J in (1.4), define, for any $\mu > 0$, the regularized functional

$$J_\mu(v) := \frac{\mu}{2}\|\nabla v\|^2_{L^2(\Omega)} + J(v) \quad \text{for all } v \in H.$$

The Mosolov problem with viscosity $\mu > 0$ is the minimization of J_μ in H. The minimal energy $\min_H J_\mu = J_\mu(u_\mu)$ is attained at a unique minimizer $u_\mu \in H$. The latter is equivalently characterized as the solution $u_\mu \in H$ to the variational inequality

$$\int_\Omega (f - \alpha u_\mu)(v - u_\mu)dx \leq \mu \int_\Omega \nabla u_\mu \cdot (\nabla v - \nabla u_\mu)dx + \psi(v) - \psi(u_\mu) \quad \text{for all } v \in H. \tag{3.2}$$

We remark that $\psi(v) = |v|_{\mathrm{BV}(\Omega)} = \|\nabla v\|_{L^1(\Omega)}$ holds for all $v \in H$. Take $\varepsilon > 0$ and consider the solution $u_{\varepsilon\mu} := u_\varepsilon \in H$ to (2.19) for $u := u_\mu \in H$. Then Lemma 2.4 guarantees that $\|\nabla u_{\varepsilon\mu}\|_{L^1(\Omega)} \leq \|\nabla u_\mu\|_{L^1(\Omega)}$. Elementary algebra in the first equation is followed by (2.19) for $v = u_{\varepsilon\mu}$ to verify

$$\|u_{\varepsilon\mu} - u_\mu\|^2_{L^2(\Omega)} + \int_\Omega u_\mu(u_{\varepsilon\mu} - u_\mu)dx = \int_\Omega u_{\varepsilon\mu}(u_{\varepsilon\mu} - u_\mu)dx = -\varepsilon\|\nabla u_{\varepsilon\mu}\|^2_{L^2(\Omega)}.$$

The same algebra for the gradients is followed by (2.19) for $v = u_{\varepsilon\mu} - u_\mu$ to verify

$$|u_{\varepsilon\mu} - u_\mu|^2_{H^1(\Omega)} + \int_\Omega \nabla u_\mu \cdot \nabla(u_{\varepsilon\mu} - u_\mu)dx = \int_\Omega \nabla u_{\varepsilon\mu} \cdot \nabla(u_{\varepsilon\mu} - u_\mu)dx = -\frac{1}{\varepsilon}\|u_{\varepsilon\mu} - u_\mu\|^2_{L^2(\Omega)}.$$

The final argument recalls $\|\nabla u_{\varepsilon\mu}\|_{L^1(\Omega)} \leq \|\nabla u_\mu\|_{L^1(\Omega)}$ and rewrites this as $\psi(u_{\varepsilon\mu}) \leq \psi(u_\mu)$. The two previously displayed identities lead with $v = u_{\varepsilon\mu}$ in (3.2) to

$$\varepsilon\alpha|u_{\varepsilon\mu}|^2_{H^1(\Omega)} + \frac{\mu}{\varepsilon}\|u_{\varepsilon\mu} - u_\mu\|^2_{L^2(\Omega)} \leq \int_\Omega f(u_\mu - u_{\varepsilon\mu})dx. \tag{3.3}$$

Since $f \in H$, one can take $v = f$ in (2.19). This gives

$$\int_\Omega f(u_\mu - u_{\varepsilon\mu})dx = \varepsilon \int_\Omega \nabla f \cdot \nabla u_{\varepsilon\mu}dx \leq \varepsilon|f|_{H^1(\Omega)}|u_{\varepsilon\mu}|_{H^1(\Omega)}. \tag{3.4}$$

This and (3.3) imply the bound $\alpha|u_{\varepsilon\mu}|_{H^1(\Omega)} \leq |f|_{H^1(\Omega)}$, which is independent of ϵ and μ. The estimate (3.3) provides also $\|u_\mu - u_{\varepsilon\mu}\|^2_{L^2(\Omega)} \leq \frac{\epsilon}{\mu}\|f\|_{L^2(\Omega)}\|u_\mu - u_{\varepsilon\mu}\|_{L^2(\Omega)}$. This proves $\lim_{\varepsilon\to 0}\|u_\mu - u_{\varepsilon\mu}\|_{L^2(\Omega)} = 0$. Consequently, for any fixed $\mu > 0$, any weak limit of a subsequence of $u_{\varepsilon\mu}$ as $\varepsilon \to 0$ in H is equal to u_μ. This and the sequential weak lower semi-continuity of the semi-norm in $H^1(\Omega)$ shows in the limit $\varepsilon \to 0$ that

$$\alpha|u_\mu|_{H^1(\Omega)} \leq |f|_{H^1(\Omega)}. \tag{3.5}$$

This bound for the H^1 semi-norm of u_μ is independent of μ. It is standard to test with $v = 0$ and $v = 2u_\mu$ in the variational inequality (3.2) to obtain

$$\alpha\|u_\mu\|^2_{L^2(\Omega)} + \mu|u_\mu|^2_{H^1(\Omega)} + \psi(u_\mu) = \int_\Omega f u_\mu \, dx \leq \frac{\alpha}{2}\|u_\mu\|^2_{L^2(\Omega)} + \frac{1}{2\alpha}\|f\|^2_{L^2(\Omega)}.$$

The result is a uniform bound of the L^2 norm $\|u_\mu\|_{L^2(\Omega)}$ of u_μ. With (3.5), this reads

$$\alpha\|u_\mu\|_{H^1(\Omega)} \leq \|f\|_{H^1(\Omega)} \quad \text{for all } \mu > 0.$$

Thus, there exists a weakly convergent sequence $u_{\mu_j} \rightharpoonup u^* \in H$ with $\mu_j \to 0$ as $j \to \infty$ and $\alpha|u^*|_{H^1(\Omega)} \leq |f|_{H^1(\Omega)}$. The passage to the limit in (3.2) involves the sequential weak lower semi-continuity of ψ and leads to

$$\int_\Omega (f - \alpha u^*)(v - u^*)\, dx \leq \psi(v) - \psi(u^*) \quad \text{for all } v \in H. \tag{3.6}$$

It remains to prove $u = u^*$. Since $u^* \in H \subset \mathrm{BV}(\Omega) \cap L^2(\Omega)$, comparing (3.6) to (3.1) shows that we have to prove that (3.6) holds *not* only for all $v \in H$, but indeed for all $v \in \mathrm{BV}(\Omega) \cap L^2(\Omega)$. This follows from approximation arguments, which are detailed in the proof of Lemma 4.2 for a related assertion and apply here as well; we therefore omit further details. □

3.2 Proof of Theorem 1.2

If $u \in H^1(\Omega)$ satisfies (1.10), then any $v \in C_c^\infty(\Omega)$ satisfies

$$\int_\Omega \alpha u v + \int_\Omega \sigma \cdot \nabla v = \int_\Omega f v. \tag{3.7}$$

A density argument in the Dirichlet case verifies (3.7) for all $v \in H_0^1(\Omega)$. The condition $\sigma \cdot \nu = 0$ on $\partial\Omega$ in the Neumann case implies that (3.7) holds for all $v \in H^1(\Omega)$. Thus in any case, (3.7) holds for all $v \in H$. The test function $v = u$ and $\sigma \cdot \nabla u = |\nabla u|$ a.e. verify

$$\int_\Omega \alpha u^2 + \int_\Omega |\nabla u| = \int_\Omega f u. \tag{3.8}$$

Since (3.7) holds for all $v \in H$ and since $|\sigma| \leq 1$ a.e. in Ω, we have

$$\int_\Omega (f - \alpha u) v \leq \int_\Omega |\nabla v| \quad \text{for all } v \in H. \tag{3.9}$$

The combination of (3.8)-(3.9) reads

$$\int_\Omega (f - \alpha u)(v - u) \leq \int_\Omega |\nabla v| - \int_\Omega |\nabla u| \quad \text{for all } v \in H. \tag{3.10}$$

The approximation argument from the end of the proof of Theorem 1.1 concludes the proof that u is the solution to (3.1). The arguments are displayed in proof of Lemma 4.2 below for a related assertion; we therefore omit further details.

Conversely, let u solve (3.1). Following the proof of Theorem 1.1, u is the weak limit in H as $\mu \to 0$ of u_μ, the solution to the viscous problem (3.2). This u_μ is itself the weak limit in H as $\delta \to 0$ of the solution $u_{\mu\delta} \in H$ to the regularized problem

$$\int_\Omega (f - \alpha u_{\mu\delta})(v - u_{\mu\delta}) \leq \mu \int_\Omega \nabla u_{\mu\delta} \cdot (\nabla v - \nabla u_{\mu\delta}) + \int_\Omega j_\delta(\nabla v) - \int_\Omega j_\delta(\nabla u_{\mu\delta}) \quad \text{for all } v \in H \tag{3.11}$$

with the approximation j_δ of the norm from (2.4). The test with $v = u_{\mu\delta} \pm tw$ for an arbitrary $w \in H$ and $t > 0$, the division by t, and the limit as $t \to 0$ leads to

$$\int_\Omega (f - \alpha u_{\mu\delta}) w = \mu \int_\Omega \nabla u_{\mu\delta} \cdot \nabla w + \int_\Omega j'_\delta(\nabla u_{\mu\delta}) \cdot \nabla w \quad \text{for all } w \in H. \tag{3.12}$$

Recall $|j'_\delta| \leq 1$ so that

$$\sigma_{\mu\delta} := j'_\delta(\nabla u_{\mu\delta}) \tag{3.13}$$

satisfies $|\sigma_{\mu\delta}| \leq 1$. Some sequence $\sigma_{\mu\delta} \rightharpoonup^* \sigma_\mu$ converges in $L^\infty(\Omega)$ weak star as $\delta \to 0$ with $|\sigma_\mu| \leq 1$ a.e. The passage to the limit in (3.12) leads to

$$\int_\Omega (f - \alpha u_\mu) w = \mu \int_\Omega \nabla u_\mu \cdot \nabla w + \int_\Omega \sigma_\mu \cdot \nabla w \quad \text{for all } w \in H. \tag{3.14}$$

The test with $w = u_{\mu\delta}$ in (3.12) will be exploited below. The convexity of j_δ, $|j'_\delta| \leq 1$, and $0 = j_\delta(0)$ lead to $0 \leq j_\delta(\nabla u_{\mu\delta}) \leq j'_\delta(\nabla u_{\mu\delta}) \cdot \nabla u_{\mu\delta} \leq |\nabla u_{\mu\delta}|$ a.e. Since $j'_\delta(\nabla u_{\mu\delta}) \cdot \nabla u_{\mu\delta} = |\nabla u_{\mu\delta}|$ where $|\nabla u_{\mu\delta}| \geq \delta$, it follows that $\big(|\nabla u_{\mu\delta}| - \delta\big)_+ := \max\{0, |\nabla u_{\mu\delta}| - \delta\} \leq j'_\delta(\nabla u_{\mu\delta}) \cdot \nabla u_{\mu\delta}$ a.e. in Ω. Hence, we have

$$\int_\Omega \big(|\nabla u_{\mu\delta}| - \delta\big) \leq \int_\Omega j'_\delta(\nabla u_{\mu\delta}) \cdot \nabla u_{\mu\delta}. \tag{3.15}$$

This and the lower semi-continuity of norms lead for $w = u_{\mu\delta}$ in (3.12) in the limit as $\delta \to 0$ to

$$\mu \int_\Omega |\nabla u_\mu|^2 + \int_\Omega |\nabla u_\mu| \leq \int_\Omega (f - \alpha u_\mu) u_\mu. \tag{3.16}$$

Recall that $u_\mu \rightharpoonup u$ weakly in H as $\mu \to 0$. Some sequence $\sigma_\mu \rightharpoonup^* \sigma$ converges in $L^\infty(\Omega)$ weak star as $\mu \to 0$ and $|\sigma| \leq 1$ a.e. This leads in (3.14) and (3.16) to

$$\int_\Omega (f - \alpha u) w = \int_\Omega \sigma \cdot \nabla w \quad \text{for all } w \in H; \tag{3.17}$$

$$\int_\Omega |\nabla u| \leq \int_\Omega (f - \alpha u) u. \tag{3.18}$$

The test $w = u$ in (3.17) and the comparison with (3.18) show $\int_\Omega |\nabla u| \leq \int_\Omega \sigma \cdot \nabla u \leq \int_\Omega |\nabla u|$ with $|\sigma| \leq 1$ a.e. in the last step. This proves $\sigma = \nabla u / |\nabla u|$ a.e. where $\nabla u \neq 0$. Finally, (3.17) shows the first line of (1.10), and the boundary condition in the Neumann case. □

3.3 Proof of Theorem 1.3

For the time-dependent case, we use the time-discrete approximations that have been proved to converge in [5]. Define the time step $\tau := T/n$ for a positive integer n and

$$f^k := \frac{1}{\tau} \int_{(k-1)\tau}^{k\tau} f(t)\, dt \quad \text{for } k = 1, \ldots, n. \tag{3.19}$$

Define the approximations $u^k \in L^2(\Omega) \cap \mathrm{BV}(\Omega)$ successively by

$$f^k \in \frac{u^k - u^{k-1}}{\tau} + \partial\psi(u^k) \quad \text{for } k = 1, \ldots, n. \tag{3.20}$$

These values define the piecewise constant approximate solution $\widehat{u}_n(t)$ by

$$\widehat{u}_n(0) := u^0 \quad \text{and} \quad \widehat{u}_n(t) := u^k \quad \text{for } (k-1)\tau < t \leq k\tau \text{ and } k = 1, \ldots, n. \tag{3.21}$$

According to [5, Theorem 3.8], the piecewise constant function $\widehat{u}_n(t)$ converges in $L^2(\Omega)$ to the solution $u(t)$ to the problem (1.12)-(1.14) uniformly with respect to $t \in [0,T]$ as $n \to \infty$. Since $f \in L^1(0,T;H)$, one has $f^k \in H$. Notice that (3.20) can be written as

$$f^k + \frac{u^{k-1}}{\tau} \in \frac{u^k}{\tau} + \partial\psi(u^k) \tag{3.22}$$

and so allows an application of Theorem 1.1 with $\alpha = 1/\tau$ and with a right-hand side $g^k = f^k + u^{k-1}/\tau \in H$. Mathematical induction over $k = 1, \ldots, n$ proves that $u^k \in H$ and that

$$\alpha \, \|\nabla u^k\|_{L^2(\Omega)} \leq \|\nabla g^k\|_{L^2(\Omega)} \leq \alpha \|\nabla u^{k-1}\|_{L^2(\Omega)} + \|\nabla f^k\|_{L^2(\Omega)}. \tag{3.23}$$

This and the definition of f^k in (3.19) result in

$$\|\nabla u^k\|_{L^2(\Omega)} - \|\nabla u^0\|_{L^2(\Omega)} \leq \tau \sum_{j=1}^{k} \|\nabla f^j\|_{L^2(\Omega)} \leq \int_0^{k\tau} \|\nabla f(t)\|_{L^2(\Omega)} dt. \tag{3.24}$$

Consequently,

$$\max_{k=1,\ldots,n} \|\nabla u^k\|_{L^2(\Omega)} \leq \|\nabla u^0\|_{L^2(\Omega)} + \int_0^T \|\nabla f(t)\|_{L^2(\Omega)} dt. \tag{3.25}$$

Therefore, $\widehat{u}_n(t)$ from (3.21) belongs to H for all $t \in [0,T]$, with a uniform bound on $\|\nabla \widehat{u}_n(t)\|_{L^2(\Omega)}$ given by the right-hand side of (3.25). Because of the uniform convergence in $L^2(\Omega)$ of $\widehat{u}_n(t)$ to $u(t)$, we conclude that $u(t) \in H$ with the same bound. This concludes the proof of (1.17).

It remains to prove the characterization (1.18)-(1.20). Assume first that u verifies (1.18)-(1.20). Then u clearly satisfies (1.12) and (1.14). Moreover $\partial_t u(t) - \operatorname{div} \sigma(t) = f(t)$ holds at a.e. $t \in (0,T)$ in the sense of distributions in Ω. Thus, at a.e. $t \in (0,T)$, the following holds:

$$\int_\Omega \partial_t u(t) v + \int_\Omega \sigma(t) \cdot \nabla v = \int_\Omega f(t) v \quad \text{for all } v \in H. \tag{3.26}$$

The test function $v = u(t)$ and $\sigma(t) \cdot \nabla u(t) = |\nabla u(t)|$ shows at a.e. $t \in (0,T)$ that

$$\int_\Omega u(t) \partial_t u(t) + \int_\Omega |\nabla u(t)| = \int_\Omega f(t) u(t). \tag{3.27}$$

Recall $|\sigma| \leq 1$ a.e. in $(0,T) \times \Omega$ to deduce from (3.26) at a.e. $t \in (0,T)$ that

$$\int_\Omega (f(t) - \partial_t u(t)) v \leq \int_\Omega |\nabla v| \quad \text{for all } v \in H. \tag{3.28}$$

The combination of (3.27)-(3.28) results at a.e. $t \in (0,T)$ in

$$\int_\Omega (f(t) - \partial_t u(t))(v - u(t)) \leq \int_\Omega |\nabla v| - \int_\Omega |\nabla u(t)| \quad \text{for } v \in H. \tag{3.29}$$

The approximation argument from the end of the proof of Theorem 1.1 concludes the proof of

$$\int_\Omega (f(t)-\partial_t u(t))(v-u(t)) \le \psi(v) - \int_\Omega |\nabla u(t)| \quad \text{for all } v \in \mathrm{BV}(\Omega)\cap L^2(\Omega). \tag{3.30}$$

at a.e. $t \in (0,T)$. Recall that $\psi(u(t)) = \int_\Omega |\nabla u(t)|dx$ for $u(t) \in H$, so that (1.13) follows.

Conversely assume that u solves (1.12)-(1.14). According to (1.17), $u(t)$ is bounded in H. Since $u \in H^1(0,T;L^2(\Omega)) \subset C([0,T];L^2(\Omega))$, it follows that $u \in C([0,T];H\text{-weak})$. Thus u verifies (1.18) and (1.20). It remains to prove (1.19). We know from the above that $u(t)$ is the limit of $\widehat{u}_n(t)$ defined in (3.21) in $L^2(\Omega)$ uniformly in time. Theorem 1.2 applies to (3.22) so that, for all $k = 1,\dots,n$,

$$\int_\Omega \frac{u^k-u^{k-1}}{\tau} v + \int_\Omega \sigma^k \cdot \nabla v = \int_\Omega f^k v \quad \text{for all } v \in H \tag{3.31}$$

holds for some $\sigma^k \in L^\infty(\Omega)^n$, $|\sigma^k| \le 1$, $\sigma^k = \nabla u^k/|\nabla u^k|$ where $\nabla u^k \neq 0$. The latter condition is equivalent to

$$\int_\Omega |\nabla u^k| \le \int_\Omega \sigma^k \cdot \nabla u^k. \tag{3.32}$$

According to [5], the piecewise constant function

$$\delta_t \widehat{u}_n(t) = \frac{u^k-u^{k-1}}{\tau} \quad \text{for } (k-1)\tau < t \le k\tau \text{ and } k = 1,\dots,n \tag{3.33}$$

converges weakly in $L^2(0,T;L^2(\Omega))$ to $\partial_t u$ as $n \to \infty$. We analogously define

$$\sigma_n(t) = \sigma^k \quad \text{for } (k-1)\tau < t \le k\tau \text{ and } k = 1,\dots,n. \tag{3.34}$$

Then $\sigma_n \in L^\infty((0,T)\times\Omega)$, $|\sigma_n| \le 1$, and (3.32) show

$$\int_0^T \int_\Omega |\nabla \widehat{u}_n(t)|\, dt \le \int_0^T \int_\Omega \sigma_n(t) \cdot \nabla \widehat{u}_n(t)\, dt. \tag{3.35}$$

Define f_n similarly as, e.g., (3.34) in terms of f^k. Then it follows from (3.31) that

$$\int_0^T \int_\Omega \delta_t \widehat{u}_n(t) v(t)\, dt + \int_0^T \int_\Omega \sigma_n(t)\cdot\nabla v(t)\, dt = \int_0^T \int_\Omega f_n(t) v(t)\, dt \quad \text{for all } v \in L^2(0,T;H). \tag{3.36}$$

The test function $v = \widehat{u}_n$ and the combination with (3.35) result in

$$\int_0^T \int_\Omega \delta_t \widehat{u}_n(t)\widehat{u}_n(t)dt + \int_0^T \int_\Omega |\nabla \widehat{u}_n(t)|dt \le \int_0^T \int_\Omega f_n(t)\widehat{u}_n(t)dt.$$

Since $\frac{1}{2}((u^k)^2 - (u^{k-1})^2) \le (u^k - u^{k-1})u^k$, this results in

$$\int_\Omega \frac{1}{2}(\widehat{u}_n(T))^2 - \int_\Omega \frac{1}{2}(u^0)^2 + \int_0^T \int_\Omega |\nabla \widehat{u}_n(t)|dt \le \int_0^T \int_\Omega f_n(t)\widehat{u}_n(t)dt. \tag{3.37}$$

At least for some (not relabelled) subsequence, recall that $\widehat{u}_n \to u$ in $L^2(\Omega)$ uniformly in time, $\delta_t \widehat{u}_n \rightharpoonup \partial_t u$ weakly in $L^2(0,T;L^2(\Omega))$, $f_n \to f$ strongly in $L^2(0,T;L^2(\Omega))$, and $\sigma_n \rightharpoonup^* \sigma$ in $L^\infty((0,T)\times\Omega)$ weak star as $n \to \infty$ with $|\sigma| \le 1$ a.e. In the limit of (3.36)-(3.37), we obtain

$$\int_0^T \int_\Omega \partial_t u(t) v(t) dt + \int_0^T \int_\Omega \sigma(t)\cdot\nabla v(t) dt = \int_0^T \int_\Omega f(t) v(t) dt \quad \text{for all } v \in L^2(0,T;H); \tag{3.38}$$

$$\int_\Omega \frac{1}{2}(u(T))^2 - \int_\Omega \frac{1}{2}(u^0)^2 + \int_0^T \int_\Omega |\nabla u(t)| dt \le \int_0^T \int_\Omega f(t)u(t) dt. \tag{3.39}$$

The test function $v = u$ in (3.38) and the combination with (3.39) result in

$$\int_0^T \int_\Omega |\nabla u(t)| dt \le \int_0^T \int_\Omega \sigma(t) \cdot \nabla u(t)\, dt. \tag{3.40}$$

It is a standard argument to rephrase (3.38) equivalently as

$$\int_\Omega \partial_t u(t)\, v + \int_\Omega \sigma(t) \cdot \nabla v = \int_\Omega f(t) v \quad \text{for } v \in H \tag{3.41}$$

at a.e. $t \in (0, T)$. Altogether, (3.40)-(3.41) conclude the proof of (1.19). □

3.4 Proof of Theorem 1.4

The proof of Theorem 1.1 shows that the minimizer $u \in H$ to (1.21) solves the variational inequality (3.2). We proceed as in the proof of Theorem 1.1 and take in (3.2) $v = u_\varepsilon$ from Lemma 2.1, 2.2, or 2.4, that satisfies (2.19) and (2.2). Since $\psi(u_\varepsilon) - \psi(u) \le \varepsilon C(\Omega) \|u_\varepsilon\|_{H^2(\Omega)}$, the analog of (3.3) reads

$$\varepsilon\alpha \int_\Omega |\nabla u_\varepsilon|^2 + \frac{\mu}{\varepsilon} \int_\Omega (u_\varepsilon - u)^2 \le \varepsilon C(\Omega) \|u_\varepsilon\|_{H^2(\Omega)} + \int_\Omega f(u - u_\varepsilon). \tag{3.42}$$

Since $-\varepsilon \Delta u_\varepsilon = u - u_\varepsilon$, this reveals that

$$\begin{aligned} \alpha \int_\Omega |\nabla u_\varepsilon|^2 + \mu \int_\Omega |\Delta u_\varepsilon|^2 &\le C(\Omega) \|u_\varepsilon\|_{H^2(\Omega)} - \int_\Omega f \Delta u_\varepsilon \\ &\le C(\Omega) \|u_\varepsilon\|_{H^2(\Omega)} + \|f\|_{L^2(\Omega)} \|\Delta u_\varepsilon\|_{L^2(\Omega)}. \end{aligned} \tag{3.43}$$

Since Ω is C^3 or convex and u_ε satisfies either Dirichlet or Neumann boundary conditions, the elliptic regularity theory [15] applies to $-\Delta + \mathrm{Id}$ and shows that

$$\|u_\varepsilon\|_{H^2(\Omega)} \le C_1(\Omega) \|u_\varepsilon - \Delta u_\varepsilon\|_{L^2(\Omega)} \le C_1(\Omega) (\|u_\varepsilon\|_{L^2(\Omega)} + \|\Delta u_\varepsilon\|_{L^2(\Omega)}). \tag{3.44}$$

Recall $\|u_\varepsilon\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}$ (from (2.19)) and deduce from (3.43)-(3.44) that

$$\begin{aligned} \frac{\mu}{2} \|\Delta u_\varepsilon\|_{L^2(\Omega)}^2 &\le C(\Omega) C_1(\Omega) \left(\|u\|_{L^2(\Omega)} + \|\Delta u_\varepsilon\|_{L^2(\Omega)} \right) + \frac{1}{2\mu} \|f\|_{L^2(\Omega)}^2 \\ &\le \frac{\mu}{4} \|u\|_{L^2(\Omega)}^2 + 2 \frac{(C(\Omega) C_1(\Omega))^2}{\mu} + \frac{\mu}{4} \|\Delta u_\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{2\mu} \|f\|_{L^2(\Omega)}^2. \end{aligned}$$

Consequently, $2\|\Delta u_\varepsilon\|_{L^2(\Omega)}^2 \le \|u\|_{L^2(\Omega)}^2 + \frac{8\left(C(\Omega)C_1\right)^2}{\mu^2} + \frac{2}{\mu^2} \|f\|_{L^2(\Omega)}^2$. This, $\|u_\varepsilon\|_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)}$, and (3.44) provide an upper bound on $\|u_\varepsilon\|_{H^2(\Omega)}$ that is independent of ε. In the limit $\varepsilon \to 0$, we obtain (1.22) with a constant $C'(\Omega)$ that exclusively depends on $C_1(\Omega)$. □

4 Bingham problem: Proofs

This section is devoted to the proofs related to the Bingham problem.

4.1 Preliminaries

We require a local variation of Lemma 2.2. Recall that C_φ (resp., $C_{\varphi,n}$ and C_n) denote generic constants whose value exclusively depends on φ (resp., on φ, n and on n).

Lemma 4.1 (local estimate) *Given a bounded open set $\Omega \subset \mathbb{R}^n$, $u \in H^1(\Omega)^n$, $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$, and a positive $\varepsilon \leq 2^{-4}\|\varphi\nabla\varphi\|^{-2}_{L^\infty(\Omega)^n}$, there exists a unique solution $u_\varepsilon \in H^1(\Omega)^n$ to*

$$\int_\Omega u_\varepsilon \cdot w + \varepsilon \int_\Omega \varphi^4 \nabla u_\varepsilon : \nabla w = \int_\Omega u \cdot w \quad \text{for all } w \in H^1(\Omega)^n. \tag{4.1}$$

Moreover, $\varphi^2\nabla^2 u_\varepsilon \in L^2(\Omega)^{n\times n\times n}$, $\varphi^4\nabla^3 u_\varepsilon \in L^2(\Omega)^{n\times n\times n\times n}$, and

$$\|u_\varepsilon\|_{L^2(\Omega)^n} \leq \|u\|_{L^2(\Omega)^n}, \tag{4.2}$$

$$\|\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq \frac{1}{1 - 8\varepsilon\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n}} \|\nabla u\|_{L^2(\Omega)^{n\times n}}, \tag{4.3}$$

$$\|\nabla u_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \frac{\varepsilon}{2}\|\varphi^2\Delta u_\varepsilon\|^2_{L^2(\Omega)^n} \leq \int_\Omega \nabla u_\varepsilon : \nabla u + 2^5\varepsilon\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n}\|\nabla u\|^2_{L^2(\Omega)^{n\times n}}, \tag{4.4}$$

$$\|\varphi^2\nabla^2 u_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}} \leq \|\varphi^2\Delta u_\varepsilon\|_{L^2(\Omega)^n} + C_{\varphi,n}\|u\|_{H^1(\Omega)^n}, \tag{4.5}$$

$$\|Du_\varepsilon\|_{L^1(\Omega)^{n\times n}} \leq \|Du\|_{L^1(\Omega)^{n\times n}} + C_{\varphi,n}\varepsilon\left(\|Du\|_{L^1(\Omega)^{n\times n}} + \|\varphi^2\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}}\right). \tag{4.6}$$

Proof. Since the uniqueness is obvious, the proof focuses on the existence of a solution to (4.1). Given any $\gamma > 0$, let $u_{\varepsilon\gamma} \in H^1(\Omega)^n$ solve

$$\int_\Omega u_{\varepsilon\gamma} \cdot w + \varepsilon \int_\Omega (\varphi^4 + \gamma)\nabla u_{\varepsilon\gamma} : \nabla w = \int_\Omega u \cdot w \quad \text{for all } w \in H^1(\Omega)^n. \tag{4.7}$$

The test function $w = u_{\varepsilon\gamma}$ shows

$$\|u_{\varepsilon\gamma}\|_{L^2(\Omega)^n} \leq \|u\|_{L^2(\Omega)^n}. \tag{4.8}$$

Since

$$u_{\varepsilon\gamma} - \varepsilon\,\mathrm{div}((\varphi^4 + \gamma)\nabla u_{\varepsilon\gamma}) = u \quad \text{in } \Omega \tag{4.9}$$

holds in the sense of distributions, we have $\mathrm{div}((\varphi^4 + \gamma)\nabla u_{\varepsilon\gamma}) = (u_{\varepsilon\gamma} - u)/\varepsilon \in L^2(\Omega)^n$, so that $(\varphi^4 + \gamma)\nabla u_{\varepsilon\gamma} \in H(\mathrm{div}, \Omega)^n$. The comparison of (4.7) with (4.9) proves $\nabla u_{\varepsilon\gamma}\nu = 0$ in $H^{-1/2}(\partial\Omega)^n$. Since

$$\mathrm{div}((\varphi^4 + \gamma)\nabla u_{\varepsilon\gamma}) = (\varphi^4 + \gamma)\Delta u_{\varepsilon\gamma} + \nabla u_{\varepsilon\gamma}\nabla(\varphi^4) \tag{4.10}$$

(the Laplacian acts componentwise), we have $(\varphi^4 + \gamma)\Delta u_{\varepsilon\gamma} \in L^2(\Omega)^n$; whence $\Delta u_{\varepsilon\gamma} \in L^2(\Omega)^n$. This and the above homogeneous Neumann boundary conditions prove

$$\int_\Omega \nabla u_{\varepsilon\gamma} : \nabla v = -\int_\Omega v \cdot \Delta u_{\varepsilon\gamma} \quad \text{for all } v \in H^1(\Omega)^n. \tag{4.11}$$

The test function $v = u_{\varepsilon\gamma} - u$ in (4.11) and (4.9)-(4.10) verify

$$\int_\Omega \nabla u_{\varepsilon\gamma} : (\nabla u_{\varepsilon\gamma} - \nabla u) = -\varepsilon \int_\Omega \Delta u_{\varepsilon\gamma} \cdot \Big((\varphi^4 + \gamma)\Delta u_{\varepsilon\gamma} + \nabla u_{\varepsilon\gamma} \nabla(\varphi^4) \Big). \tag{4.12}$$

The crucial point is the observation that

$$|\nabla(\varphi^4)| = 4\varphi^3 |\nabla\varphi| \le 4\|\varphi\nabla\varphi\|_{L^\infty(\Omega)^n}\, \varphi^2. \tag{4.13}$$

This, a Cauchy–Schwarz inequality, and a weighted Young inequality result in

$$\begin{aligned} \big|\textstyle\int_\Omega \Delta u_{\varepsilon\gamma} \cdot (\nabla u_{\varepsilon\gamma} \nabla(\varphi^4))\big| &\le 4\|\varphi\nabla\varphi\|_{L^\infty(\Omega)^n} \|\varphi^2 \Delta u_{\varepsilon\gamma}\|_{L^2(\Omega)^n} \|\nabla u_{\varepsilon\gamma}\|_{L^2(\Omega)^{n\times n}} \\ &\le 8\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n} \|\nabla u_{\varepsilon\gamma}\|^2_{L^2(\Omega)^{n\times n}} + \tfrac{1}{2}\|\varphi^2 \Delta u_{\varepsilon\gamma}\|^2_{L^2(\Omega)^n}. \end{aligned}$$

Its direct consequence

$$\frac{1}{2}\int_\Omega (\varphi^4+\gamma)|\Delta u_{\varepsilon\gamma}|^2 - 8\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n} \|\nabla u_{\varepsilon\gamma}\|^2_{L^2(\Omega)^{n\times n}} \le \int_\Omega \Delta u_{\varepsilon\gamma} \cdot \Big((\varphi^4+\gamma)\Delta u_{\varepsilon\gamma} + \nabla u_{\varepsilon\gamma} \nabla(\varphi^4) \Big)$$

is substituted in (4.12) for the proof of

$$\Big(1 - 8\varepsilon\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n}\Big) \|\nabla u_{\varepsilon\gamma}\|^2_{L^2(\Omega)^{n\times n}} + \frac{\varepsilon}{2}\int_\Omega (\varphi^4+\gamma)|\Delta u_{\varepsilon\gamma}|^2 \le \int_\Omega \nabla u_{\varepsilon\gamma} : \nabla u. \tag{4.14}$$

Recall that $\varepsilon \le 2^{-4}\|\varphi\nabla\varphi\|^{-2}_{L^\infty(\Omega)^n}$ and deduce

$$\|\nabla u_{\varepsilon\gamma}\|_{L^2(\Omega)^{n\times n}} \le \frac{1}{1 - 8\varepsilon\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n}} \|\nabla u\|_{L^2(\Omega)^{n\times n}} \le 2\|\nabla u\|_{L^2(\Omega)^{n\times n}}. \tag{4.15}$$

The combination of (4.14)-(4.15) reveals that

$$\|\nabla u_{\varepsilon\gamma}\|^2_{L^2(\Omega)^{n\times n}} + \frac{\varepsilon}{2}\int_\Omega (\varphi^4+\gamma)|\Delta u_{\varepsilon\gamma}|^2 \le \int_\Omega \nabla u_{\varepsilon\gamma} : \nabla u + 2^5\varepsilon\|\varphi\nabla\varphi\|^2_{L^\infty(\Omega)^n} \|\nabla u\|^2_{L^2(\Omega)^{n\times n}}. \tag{4.16}$$

The bounds (4.15)-(4.16) on $u_{\varepsilon\gamma}$ imply that $u_{\varepsilon\gamma}$ is bounded in $H^1(\Omega)^n$ and $\varphi^2\Delta u_{\varepsilon\gamma}$ is bounded in $L^2(\Omega)^n$ as $\gamma \to 0$. Hence, there exists a sequence $\gamma_j \to 0$ such that $u_{\varepsilon\gamma_j} \rightharpoonup u_\varepsilon$ weakly in $H^1(\Omega)^n$ and $\varphi^2\Delta u_{\varepsilon\gamma_j} \rightharpoonup \varphi^2\Delta u_\varepsilon$ weakly in $L^2(\Omega)^n$. This and the sequential weak lower semi-continuity of norms show that the weak limit $u_\varepsilon \in H^1(\Omega)^n$ solves (4.1) and, owing to (4.8) and (4.15)-(4.16), satisfies the estimates (4.2)-(4.4).

The further investigation of the unique solution $u_\varepsilon \in H^1(\Omega)^n$ to (4.1) rewrites it as

$$u_\varepsilon - \varepsilon\,\mathrm{div}(\varphi^4\nabla u_\varepsilon) = u \quad \text{in } \Omega \tag{4.17}$$

in the sense of distributions. Since $v_\varepsilon = \varphi^2 u_\varepsilon \in H^1(\Omega)^n$ has compact support, its extension by zero $v_\varepsilon \in H^1(\mathbb{R}^n)^n$ satisfies

$$\nabla v_\varepsilon = \varphi^2 \nabla u_\varepsilon + u_\varepsilon \otimes \nabla(\varphi^2) \in L^2(\Omega)^{n\times n} \tag{4.18}$$

and, in the sense of distributions in $\mathbb{R}^n$ for any $i, j = 1, \dots, n$,

$$\partial^2_{ij} v_\varepsilon = \varphi^2 \partial^2_{ij} u_\varepsilon + \partial_i u_\varepsilon \partial_j(\varphi^2) + \partial_i(\varphi^2)\partial_j u_\varepsilon + u_\varepsilon \partial^2_{ij}(\varphi^2). \tag{4.19}$$

Recall that $\varphi^2\Delta u_\varepsilon \in L^2(\Omega)^n$ to deduce from (4.19) that $\Delta v_\varepsilon \in L^2(\Omega)^n$ satisfies

$$\|\Delta v_\varepsilon\|_{L^2(\Omega)^n} \le \|\varphi^2\Delta u_\varepsilon\|_{L^2(\Omega)^n} + C_{\varphi,n}\|u_\varepsilon\|_{H^1(\Omega)^n}. \tag{4.20}$$

Since v_ε has compact support in Ω, $v_\varepsilon \in H^1(\mathbb{R}^n)^n$ satisfies $\Delta v_\varepsilon \in L^2(\mathbb{R}^n)^n$. Invoking a Fourier transformation or the elliptic regularity of the Poisson problem on a large ball that includes Ω [15, Equ. (3,1,2,2)], we deduce that $v_\varepsilon \in H^2(\mathbb{R}^n)^n$ satisfies

$$\|\nabla^2 v_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}} \le \|\Delta v_\varepsilon\|_{L^2(\Omega)^n}. \tag{4.21}$$

Recall (4.2)-(4.3) with $\varepsilon \le 2^{-4}\|\varphi\nabla\varphi\|^{-2}_{L^\infty(\Omega)^n}$ to infer $\|u_\varepsilon\|_{H^1(\Omega)^n} \le 2\|u\|_{H^1(\Omega)^n}$. This and (4.19) show that $\varphi^2\nabla^2 u_\varepsilon \in L^2(\Omega)^{n\times n\times n}$ satisfies

$$\|\varphi^2\nabla^2 u_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}} \le C_{\varphi,n}\|u\|_{H^1(\Omega)^n} + \|\nabla^2 v_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}} \le C_{\varphi,n}\|u\|_{H^1(\Omega)^n} + \|\varphi^2\Delta u_\varepsilon\|_{L^2(\Omega)^n}$$

using (4.20)-(4.21) in the last step. This is (4.5).

Since $\operatorname{div}(\varphi^4\nabla u_\varepsilon) = \varphi^4\Delta u_\varepsilon + \nabla u_\varepsilon\nabla(\varphi^4)$, (4.17) shows $\nabla(\varphi^4\Delta u_\varepsilon) \in L^2(\mathbb{R}^n)^{n\times n}$ with compact support in Ω. This and the above arguments for the proof of (4.20)-(4.21) eventually lead to $\varphi^4\nabla^3 u_\varepsilon \in L^2(\Omega)^{n\times n\times n\times n}$; further details on this are omitted here.

It remains to prove (4.6). Define j_δ as in (2.4) but let the argument ξ be a (symmetric) matrix rather than a vector. Recall the above properties of j_δ hold in the example at hand as well. For instance, the convexity of j_δ implies that

$$j_\delta(Du_\varepsilon) - j_\delta(Du) \le j_\delta'(Du_\varepsilon) : (Du_\varepsilon - Du). \tag{4.22}$$

Moreover, $j_\delta \in C^2$ is convex (its Hessian is positive semi-definite) and j_δ as well as its derivative j_δ' are globally Lipschitz continuous with $|j_\delta'(F)| \le 1$ and $|F|\,|j_\delta''(F)| \le 1$ for any $F \in \mathbb{R}^{n\times n}$. We remark that for a symmetric matrix S and an arbitrary matrix A, one has $S : A = S : A^t = S : (A + A^t)/2$. Consequently,

$$j_\delta(Du_\varepsilon) - j_\delta(Du) \le j_\delta'(Du_\varepsilon) : \nabla(u_\varepsilon - u). \tag{4.23}$$

Recall (4.17) to compute, for any $i, j = 1, \dots, n$, the derivatives in the sense of distributions as follows:

$$\begin{aligned}
\varepsilon^{-1}[\nabla(u_\varepsilon - u)]_{ij} = [\nabla(\operatorname{div}(\varphi^4\nabla u_\varepsilon))]_{ij} &= \sum_{k=1}^n \partial^2_{jk}(\varphi^4\partial_k u^i_\varepsilon) \\
&= \sum_{k=1}^n \partial_k\big(\varphi^4\partial^2_{jk}u^i_\varepsilon + \partial_j(\varphi^4)\partial_k u^i_\varepsilon\big) \\
&= \sum_{k=1}^n \partial_k\big(\varphi^4\partial^2_{jk}u^i_\varepsilon + 2\partial_j(\varphi^4)(Du_\varepsilon)_{ki} - \partial_j(\varphi^4)\partial_i u^k_\varepsilon\big) \\
&= \sum_{k=1}^n \partial_k\Big(\varphi^4\partial^2_{jk}u^i_\varepsilon + 2\partial_j(\varphi^4)(Du_\varepsilon)_{ki} - \delta_{ik}\partial_j(\varphi^4)\operatorname{tr}(Du_\varepsilon) \\
&\qquad\qquad - \partial_j(\varphi^4)\big(\partial_i u^k_\varepsilon - \delta_{ik}\operatorname{tr}(Du_\varepsilon)\big)\Big)
\end{aligned}$$

with the Kronecker delta δ_{ik}. Since $\sum_{k=1}^n \partial_k(\partial_i u^k_\varepsilon - \delta_{ik}\operatorname{tr}(Du_\varepsilon)) = 0$, this is recast as

$$\begin{aligned}
\nabla(u_\varepsilon - u) = \varepsilon\sum_{k=1}^n \partial_k\Big(\varphi^4\partial_k\nabla u_\varepsilon + 2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon)e^k \otimes \nabla(\varphi^4)\Big) \\
-\varepsilon\big((\nabla u_\varepsilon)^t - \operatorname{tr}(\nabla u_\varepsilon)\,\mathrm{Id}\big)\nabla^2(\varphi^4).
\end{aligned} \tag{4.24}$$

The integral over Ω of the combination of (4.23)-(4.24) reads

$$\begin{aligned}
&\int_\Omega j_\delta(Du_\varepsilon) - \int_\Omega j_\delta(Du) \\
&\le \varepsilon \sum_{k=1}^n \int_\Omega j_\delta'(Du_\varepsilon) : \partial_k\Big(\varphi^4 \partial_k \nabla u_\varepsilon + 2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon) e^k \otimes \nabla(\varphi^4)\Big) \\
&\quad - \varepsilon \int_\Omega j_\delta'(Du_\varepsilon) : \big((\nabla u_\varepsilon)^t - \operatorname{tr}(\nabla u_\varepsilon)\,\mathrm{Id}\big)\nabla^2(\varphi^4).
\end{aligned} \tag{4.25}$$

An integration by parts in the first term on the right-hand side of (4.25) shows that

$$\begin{aligned}
&\sum_{k=1}^n \int_\Omega j_\delta'(Du_\varepsilon) : \partial_k(\varphi^4 \partial_k \nabla u_\varepsilon) = -\sum_{k=1}^n \int_\Omega \partial_k\Big(j_\delta'(Du_\varepsilon)\Big) : \varphi^4 \partial_k\big(Du_\varepsilon\big) \\
&= -\sum_{k=1}^n \int_\Omega \varphi^4 j_\delta''(Du_\varepsilon) : \partial_k\big(Du_\varepsilon\big) : \partial_k\big(Du_\varepsilon\big) =: -\sum_{k=1}^n \int_\Omega \varphi^4 j_\delta''(Du_\varepsilon) : \partial_k\big(Du_\varepsilon\big)^{\otimes 2}
\end{aligned} \tag{4.26}$$

with the abbreviation $j_\delta''(Du_\varepsilon) : V^{\otimes 2} = j_\delta''(Du_\varepsilon) : V : V$. An integration by parts in the second term on the right-hand side of (4.25) shows the identity below, which is followed by a Cauchy–Schwarz inequality with respect to the symmetric and positive semi-definite Hessian of j_δ. This proves

$$\begin{aligned}
&\sum_{k=1}^n \int_\Omega j_\delta'(Du_\varepsilon) : \partial_k\big(2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon) e^k \otimes \nabla(\varphi^4)\big) \\
&= -\sum_{k=1}^n \int_\Omega j_\delta''(Du_\varepsilon) : \partial_k\big(Du_\varepsilon\big) : \big(2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon) e^k \otimes \nabla(\varphi^4)\big) \\
&\le \sum_{k=1}^n \int_\Omega \Big(j_\delta''(Du_\varepsilon) : \big(\partial_k\big(Du_\varepsilon\big)\big)^{\otimes 2}\Big)^{1/2} \\
&\qquad\qquad \times \Big(j_\delta''(Du_\varepsilon) : \big(2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon) e^k \otimes \nabla(\varphi^4)\big)^{\otimes 2}\Big)^{1/2} \\
&\le \sum_{k=1}^n \int_\Omega \varphi^4 j_\delta''(Du_\varepsilon) : \big(\partial_k\big(Du_\varepsilon\big)\big)^{\otimes 2} + \frac{C_{\varphi,n}^2}{4} \int_{\operatorname{supp}\varphi} |j_\delta''(Du_\varepsilon)|\,|Du_\varepsilon|^2
\end{aligned}$$

with (4.13) and a Cauchy–Schwarz inequality in $L^2(\Omega)$ in the last step. Since $|Du_\varepsilon||j_\delta''(Du_\varepsilon)| \le 1$ a.e. in Ω, the last term is controlled by $\frac{C_{\varphi,n}^2}{4} \int_{\operatorname{supp}\varphi} |Du_\varepsilon|$, and we have

$$\begin{aligned}
&\sum_{k=1}^n \int_\Omega j_\delta'(Du_\varepsilon) : \partial_k\big(2(Du_\varepsilon)^k \otimes \nabla(\varphi^4) - \operatorname{tr}(Du_\varepsilon) e^k \otimes \nabla(\varphi^4)\big) \\
&\le \sum_{k=1}^n \int_\Omega \varphi^4 j_\delta''(Du_\varepsilon) : \big(\partial_k\big(Du_\varepsilon\big)\big)^{\otimes 2} + \frac{C_{\varphi,n}^2}{4} \int_{\operatorname{supp}\varphi} |Du_\varepsilon|.
\end{aligned} \tag{4.27}$$

Since $|j_\delta'(Du_\varepsilon)| \le 1$ and $|\nabla^2(\varphi^4)| \le C_\varphi \varphi^2$ a.e. in Ω, the last term on the right-hand side of (4.25) is

$$-\int_\Omega j_\delta'(Du_\varepsilon) : \big((\nabla u_\varepsilon)^t - \operatorname{tr}(\nabla u_\varepsilon)\,\mathrm{Id}\big)\nabla^2(\varphi^4) \le C_{\varphi,n} \int_\Omega \varphi^2 |\nabla u_\varepsilon|. \tag{4.28}$$

The combination of (4.25)-(4.28) involves the cancellation of the upper bound (4.26) with the first term on the right-hand side of (4.27). This proves

$$\int_\Omega j_\delta(Du_\varepsilon) - \int_\Omega j_\delta(Du) \le \varepsilon \frac{C_{\varphi,n}^2}{4} \int_{\operatorname{supp}\varphi} |Du_\varepsilon| + \varepsilon C_{\varphi,n} \int_\Omega \varphi^2 |\nabla u_\varepsilon|.$$

The passage to the limit as $\delta \to 0$ leads to

$$\int_\Omega |Du_\varepsilon| - \int_\Omega |Du| \le \varepsilon \frac{C_{\varphi,n}^2}{4} \int_\Omega |Du_\varepsilon| + \varepsilon C_{\varphi,n} \int_\Omega \varphi^2 |\nabla u_\varepsilon|, \tag{4.29}$$

and a Cauchy–Schwarz inequality over the support of φ concludes the proof of (4.6). □

4.2 Proof of Theorem 1.6

The test functions $v = 0$ and $v = 2u$ in (1.31) provide

$$\alpha \int_\Omega |u|^2 + \mu \int_\Omega |\nabla u|^2 + \int_\Omega |Du| = \int_\Omega f \cdot u. \tag{4.30}$$

This leads to the explicit bounds on u in $H^1(\Omega)^n$, namely

$$\|u\|_{L^2(\Omega)^n} \le \frac{\|f\|_{L^2(\Omega)^n}}{\alpha} \quad \text{and} \qquad \|\nabla u\|_{L^2(\Omega)^{n\times n}} \le \frac{\|f\|_{L^2(\Omega)^n}}{\sqrt{\alpha\mu}}. \tag{4.31}$$

Given $\varphi \in C_c^\infty(\Omega)$ with $\varphi \ge 0$ and $\varepsilon > 0$ sufficiently small, Lemma 4.1 provides an approximation u_ε to the solution $u \in H$ to (1.31). Recall that u_ε satisfies (4.17) and φ has compact support, so that u_ε is equal to u outside the support of φ; whence $u_\varepsilon \in H$ (also in case of Dirichlet boundary conditions). Thus, if $u \in H_0^1(\Omega)^n$, we have also $u_\varepsilon \in H_0^1(\Omega)^n$. The test function $v = u_\varepsilon \in H$ in (1.31) plus elementary algebra proves

$$\begin{aligned} &\alpha \|u_\varepsilon - u\|_{L^2(\Omega)^n}^2 + \mu |u_\varepsilon - u|_{H^1(\Omega)^n}^2 + \int_\Omega |Du| - \int_\Omega |Du_\varepsilon| \\ &\le \alpha \int_\Omega u_\varepsilon \cdot (u_\varepsilon - u) + \mu \int_\Omega \nabla u_\varepsilon : (\nabla u_\varepsilon - \nabla u) + \int_\Omega f \cdot (u - u_\varepsilon). \end{aligned} \tag{4.32}$$

The integrals $\int_\Omega u_\varepsilon \cdot (u_\varepsilon - u)$ and $\int_\Omega \nabla u_\varepsilon : (\nabla u_\varepsilon - \nabla u)$ are controlled by (4.1) (with $w = u_\varepsilon$) and (4.4), respectively, while $\|Du_\varepsilon\|_{L^1(\Omega)^{n\times n}} - \|Du\|_{L^1(\Omega)^{n\times n}}$ on the left-hand side of (4.32) is bounded from below by (4.6), and (4.17) rewrites the integral $\int_\Omega f \cdot (u - u_\varepsilon) = -\varepsilon \int_\Omega f \cdot \operatorname{div}(\varphi^4 \nabla u_\varepsilon)$. These arguments and a division by $\varepsilon > 0$ lead in (4.32) to the estimate

$$\begin{aligned} &\frac{\mu}{2} \|\varphi^2 \Delta u_\varepsilon\|_{L^2(\Omega)^n}^2 + \alpha \|\varphi^2 \nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}}^2 + \int_\Omega f \cdot \operatorname{div}(\varphi^4 \nabla u_\varepsilon) \\ &\le C_{\varphi,n} \left(\|Du\|_{L^1(\Omega)^{n\times n}} + \|\varphi^2 \nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \right) + 2^5 \mu \|\varphi \nabla \varphi\|_{L^\infty(\Omega)^n}^2 \|\nabla u\|_{L^2(\Omega)^{n\times n}}^2. \end{aligned} \tag{4.33}$$

Recall the convention on the generic constants $C_{n\varphi}$ and estimates like (4.13) or the estimate $\|\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \le 2\|\nabla u\|_{L^2(\Omega)^{n\times n}}$ from (4.3) (for small ε) to rewrite this as

$$\begin{aligned} \frac{\mu}{C_{n\varphi}} \|\varphi^2 \Delta u_\varepsilon\|_{L^2(\Omega)^n}^2 &\le \|Du\|_{L^1(\Omega)^{n\times n}} + \|\nabla u\|_{L^2(\Omega)^{n\times n}} + \mu \|\nabla u\|_{L^2(\Omega)^{n\times n}}^2 \\ &\quad + \|f\|_{L^2(\Omega)^n} \|\varphi^2 \Delta u_\varepsilon\|_{L^2(\Omega)^n} + \|f\|_{L^2(\Omega)^n} \|\nabla u\|_{L^2(\Omega)^{n\times n}}. \end{aligned} \tag{4.34}$$

Except the term $\|f\|_{L^2(\Omega)^n}\|\varphi^2\Delta u_\varepsilon\|_{L^2(\Omega)^n}$, that shall be absorbed, the a priori bounds (4.30)-(4.31) control the remaining terms on the right-hand side of (4.34). This leads to

$$\frac{\mu}{C_{n\varphi}}\|\varphi^2\Delta u_\varepsilon\|^2_{L^2(\Omega)^n} \leq (\frac{1}{\alpha}+\frac{1}{\mu})\|f\|^2_{L^2(\Omega)^n} + \frac{1}{\sqrt{\alpha\mu}}\|f\|_{L^2(\Omega)^n}. \tag{4.35}$$

The combination with (4.5) bounds $\|\varphi^2\nabla^2 u_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}}$ independently of $\varepsilon>0$. Moreover, the first two terms in (4.32) have been neglected before, but lead to the control of

$$\|\varphi^2\nabla^2 u_\varepsilon\|_{L^2(\Omega)^{n\times n\times n}} + \|u_\varepsilon\|_{H^1(\Omega)^n} + \frac{\alpha}{\varepsilon}\|u-u_\varepsilon\|^2_{L^2(\Omega)^n} + \frac{\mu}{\varepsilon}|u-u_\varepsilon|^2_{H^1(\Omega)^n}$$

by a global upper bound independently of $\varepsilon>0$ as $\varepsilon\to 0$. Hence, there exists a sequence $\varepsilon\to 0$ (not labelled explicitly) such that $u_\varepsilon, \nabla u_\varepsilon, \varphi^2\nabla^2 u_\varepsilon$ are bounded and convergence component-wise weakly in $L^2(\Omega)$ to the weak limit $u, \nabla u, \varphi^2\nabla^2 u$. In particular, $\varphi^2\nabla^2 u \in L^2(\Omega)^{n\times n\times n}$. This and (4.35) lead to the assertion (1.32). □

4.3 Proof of Theorem 1.7

The test functions $v=0$ and $v=2u$ in (1.37) provide

$$\alpha\int_\Omega |u|^2 + \psi(u) = \int_\Omega f\cdot u. \tag{4.36}$$

This leads to the explicit bounds on u in $L^2(\Omega)^n$, namely

$$\|u\|_{L^2(\Omega)^n} \leq \frac{\|f\|_{L^2(\Omega)^n}}{\alpha} \quad\text{and}\quad \psi(u) \leq \frac{\|f\|^2_{L^2(\Omega)^n}}{\alpha}. \tag{4.37}$$

Given any $\mu>0$, let $u_\mu\in H$ solve (1.31) with either $H=H^1_0(\Omega)^n$ (Dirichlet) or $H=H^1(\Omega)^n$ (Neumann). Theorem 1.6 proves that $u_\mu\in H\cap H^2_{\rm loc}(\Omega)^n$ satisfies (1.32) (with u_μ replacing u therein) for any $\varphi\in C^\infty_c(\Omega)$ with $\varphi\geq 0$. Apply Lemma 4.1 as in the proof of Theorem 1.6 and define u_ε as an approximation to u_μ (replace u in the lemma by u_μ) and recall the estimate (4.33) (with u_μ replacing u). Since $f\in L^2(\Omega)^n\cap H^1_{\rm loc}(\Omega)^n$, the first term in the right-hand side of (4.33) reads

$$-\int_\Omega f\cdot{\rm div}(\varphi^4\nabla u_\varepsilon) = \int_\Omega \varphi^4\nabla f:\nabla u_\varepsilon \leq \frac{\alpha}{2}\|\varphi^2\nabla u_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} + \frac{1}{2\alpha}\|\varphi^2\nabla f\|^2_{L^2(\Omega)^{n\times n}}. \tag{4.38}$$

Rewrite the a priori bounds (4.30)-(4.31) as

$$\|Du_\mu\|_{L^1(\Omega)^{n\times n}} + \mu\|\nabla u_\mu\|^2_{L^2(\Omega)^{n\times n}} \leq \frac{\|f\|^2_{L^2(\Omega)^n}}{\alpha}. \tag{4.39}$$

Use (4.38)-(4.39) in (4.33) and follow the arguments in the proof of Theorem 1.6 to deduce

$$\frac{\alpha}{2}\|\varphi^2\nabla u_\varepsilon\|^2_{L^2(\Omega)^{n\times n}} \leq \frac{1}{2\alpha}\|\varphi^2\nabla f\|^2_{L^2(\Omega)^{n\times n}} + C_{n\varphi}\left(\|\varphi^2\nabla u_\varepsilon\|_{L^2(\Omega)^n} + \frac{\|f\|^2_{L^2(\Omega)^n}}{\alpha}\right). \tag{4.40}$$

This and $C_{n\varphi}\|\varphi^2\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq C_{n\varphi}^2/\alpha + \frac{\alpha}{4}\|\varphi^2\nabla u_\varepsilon\|^2_{L^2(\Omega)^{n\times n}}$ provide the ε-independent bound

$$\varepsilon^{-1/2}\|u_\mu - u_\varepsilon\|_{L^2(\Omega)} + \|\varphi^2\nabla u_\varepsilon\|_{L^2(\Omega)^{n\times n}} \leq \frac{C_{n\varphi}}{\alpha}\left(1 + \|f\|_{L^2(\Omega)^n} + \|\varphi^2\nabla f\|_{L^2(\Omega)^{n\times n}}\right). \quad (4.41)$$

The first term in (4.41) stems from the first term in (4.32) and was simply suppressed in the above formulas. Given any fixed $\mu > 0$, there exists a sequence $\varepsilon \to 0$ (not labelled explicitly) such that $u_\varepsilon \to u_\mu$ in $L^2(\Omega)^n$ and $\varphi^2\nabla u_\varepsilon \in L^2(\Omega)^{n\times n}$ is bounded by the μ-independent upper bound in (4.41). Recall $\|u_\mu\|_{L^2(\Omega)^n} \leq \|f\|_{L^2(\Omega)^n}/\alpha$ from (4.31). Hence, there exists a sequence $\mu \to 0$ (not labelled explicitly) such that $u_\mu \rightharpoonup u^* \in L^2(\Omega)^n$ converges weakly in $L^2(\Omega)^n$ with $\|u^*\|_{L^2(\Omega)^n} \leq \|f\|_{L^2(\Omega)^n}/\alpha$ and $\varphi^2\nabla u^* \in L^2(\Omega)^{n\times n}$ bounded by the upper bound in (4.41).
It remains to prove that $u^* = u$ solves (1.37). Recall that $u_\mu \in H$ solves (1.31) and deduce the elementary bound $\mu\int_\Omega \nabla u_\mu : (\nabla v - \nabla u_\mu) \leq \frac{\mu}{4}\int_\Omega |\nabla v|^2$ for any test function $v \in H$. Recall that ψ is defined either by (1.35) (Dirichlet) or by (1.33) (Neumann), that $\psi(v) = ||Dv||_{L^1(\Omega)^{n\times n}}$ for $v \in H$, and that $\alpha||\bullet||_{L^2(\Omega)^n} + \psi(\bullet)$ is sequentially weakly lower semi-continuous. This and the above weak convergence lead in the limit as $\mu \to 0$ in (1.31) to

$$\int_\Omega f\cdot(v-u^*) \leq \alpha\int_\Omega u^*\cdot(v-u^*) + \psi(v) - \psi(u^*) \quad \text{for } v \in H. \quad (4.42)$$

Lemma 4.2 below shows that $u^* = u$ and the uniqueness of the weak limit also implies $u_\mu \rightharpoonup u$ in $L^2(\Omega)^n$ not only for *some* sequence $\mu \to 0$ but for *any*. □

Lemma 4.2 *Any function $u^* \in L^2(\Omega)$ with (4.42) is equal to the solution u to (1.37).*

Proof. In the Dirichlet case, suppose $v \in L^2(\Omega)^n$ satisfies $\psi(v) < \infty$ with ψ defined by (1.35). A modification of Lemma A.1 for BD (rather than BV) functions shows that v can be approximated by a sequence $v_m \in H^1_0(\Omega)^n$ such that $v_m \to v$ in L^2 and $\psi(v_m) \to \psi(v)$ as $m \to \infty$.
In the Neumann case, suppose $v \in L^2(\Omega)^n$ satisfies $\psi(v) < \infty$ with ψ defined by (1.33). The existence of a sequence $v_m \in H^1(\Omega)^n$ such that $v_m \to v$ in $L^2(\Omega)^n$ and $\psi(v_m) \to \psi(v)$ as $m \to \infty$ follows from standard regularization [11].
Take the admissible test function $v = v_m \in H$ in (4.42) for m and then pass to the limit as $m \to \infty$ to deduce that (1.37) holds for the test function $v \in L^2(\Omega)^n$. The uniqueness of the solutions to (1.37) concludes the proof. Note that the assumption $\partial\Omega$ Lipschitz applies only here in each of the two cases. □

Remark 4.3 *The above passage to the limit as $\mu \to 0$ and $u_\mu \rightharpoonup u$ in $L^2(\Omega)^n$ also shows*

$$\lim_{\mu\to 0}\mu\int_\Omega |\nabla u_\mu|^2 = 0 \quad \textit{and} \quad \lim_{\mu\to 0}\psi(u_\mu) = \psi(u).$$

(The proof compares the limit of (4.30) (written in terms of u_μ replacing u therein) with (4.36) and utilizes the sequential weak lower semi-continuity of ψ and $\psi(u_\mu) = \|Du_\mu\|_{L^1(\Omega)^{n\times n}}$.)

4.4 Proof of Theorem 1.8

In the first part of the proof of the characterization, suppose u solves (1.37). Theorem 1.7 asserts that $Du \in L^2_{\rm loc}(\Omega)^{n\times n} \cap L^1(\Omega)^{n\times n}$ and that u is the weak limit in $L^2(\Omega)^n$, $u_\mu \rightharpoonup u$ as $\mu \to 0$, of the solution $u_\mu \in H$ to the viscous problem (1.31). For any $\mu > 0$, the solution $u_\mu \in H \cap H^1_{\rm loc}(\Omega)^n$ satisfies the bound (1.38) and is itself the weak limit in H as $\delta \to 0$ of the solution $u_{\mu\delta} \in H$ to the regularized problem

$$\int_\Omega (f - \alpha u_{\mu\delta}) \cdot (v - u_{\mu\delta}) \le \mu \int_\Omega \nabla u_{\mu\delta} : (\nabla v - \nabla u_{\mu\delta}) + \int_\Omega j_\delta(Dv) - \int_\Omega j_\delta(Du_{\mu\delta}) \quad \text{for all } v \in H, \tag{4.43}$$

with the approximation of the norm j_δ from (2.4) for $\delta > 0$. Recall that j_δ is C^2 and deduce from (4.43) that

$$\int_\Omega (f - \alpha u_{\mu\delta}) \cdot v = \mu \int_\Omega \nabla u_{\mu\delta} : \nabla v + \int_\Omega j'_\delta(Du_{\mu\delta}) : Dv \quad \text{for all } v \in H \tag{4.44}$$

(test with $v = u_{\mu\delta} \pm tv$ for $v \in H$ and $t > 0$, divide by t, and let $t \to 0$). Note that

$$\sigma_{\mu\delta} := j'_\delta(Du_{\mu\delta}) \tag{4.45}$$

is symmetric and satisfies $|\sigma_{\mu\delta}| \le 1$ a.e. The pointwise boundedness leads to a sequence $\delta \to 0$ (not labelled explicitly) such that $\sigma_{\mu\delta} \rightharpoonup^* \sigma_\mu$ weak star in $L^\infty(\Omega)^{n\times n}$ with $|\sigma_\mu| \le 1$ a.e. The passage to the limit in (4.44) leads to

$$\int_\Omega (f - \alpha u_\mu) \cdot v = \mu \int_\Omega \nabla u_\mu : \nabla v + \int_\Omega \sigma_\mu : Dv \quad \text{for all } v \in H. \tag{4.46}$$

The same arguments apply in the passage to the limit as $\mu \to 0$ and the weak convergence $u_\mu \rightharpoonup u$ in $L^2(\Omega)^n$ towards the solution u to (1.37) and some weak star limit σ of σ_μ in $L^\infty(\Omega)^{n\times n}$ with $\sigma = \sigma^t$ and $|\sigma| \le 1$ a.e. This and Remark 4.3 show that

$$\int_\Omega (f - \alpha u) \cdot v = \int_\Omega \sigma : Dv \quad \text{for all } v \in H. \tag{4.47}$$

Consequently, $\sigma \in H(\mathrm{div}, \Omega)^n \cap L^\infty(\Omega)^{n\times n}_{\rm sym}$ with $\mathrm{div}\,\sigma = \alpha u - f$. For the Neumann case $H = H^1(\Omega)^n$, (4.47) also shows the asserted boundary condition $\sigma\nu = 0$ in $H^{-1/2}(\partial\Omega)$; but there is no information on the boundary for test functions in $H = H^1_0(\Omega)^n$ from (4.47).
The traces in BD are considered in [2] and the normal traces in $H(\mathrm{div}, \Omega)$ are well known [17]. For $\sigma \in H(\mathrm{div}, \Omega)^n \cap L^\infty(\Omega)^{n\times n}_{\rm sym}$, the normal trace $\sigma\nu \in H^{-1/2}(\partial\Omega)$ belongs to $L^\infty(\partial\Omega)^n$, as we now show. Since this result can be of independent interest, we state it as a separate lemma.

Lemma 4.4 (traces in BD**)** *If $\sigma \in H(\mathrm{div}, \Omega)^n \cap L^\infty(\Omega)^{n\times n}_{\rm sym}$ satisfies $|\sigma| \le 1$ a.e. in Ω, then the normal trace $\sigma\nu \in H^{-1/2}(\partial\Omega)^n$ satisfies $\sigma\nu \in L^\infty(\partial\Omega)^n$ with $|\sigma\nu| \le 1$ a.e. on the boundary $\partial\Omega$ and the integration by parts formula*

$$\int_\Omega (v \cdot \mathrm{div}\,\sigma + \sigma : Dv) = \int_{\partial\Omega} v \cdot \sigma\nu \quad \textit{holds for all } v \in L^2(\Omega)^n \textit{ with } Dv \in L^1(\Omega)^{n\times n}_{\rm sym}.$$

Recalling the notation $a \odot b$ from (1.39), the trace satisfies $\sigma\nu = (z \odot \nu)\nu$ a.e. on $\partial\Omega$ for a (unique) function $z \in L^\infty(\partial\Omega)^n$ with $|z \odot \nu| \le 1$ a.e. on $\partial\Omega$. The above boundary integral reads $\int_{\partial\Omega} v \cdot \sigma\nu = \int_{\partial\Omega} (z \odot \nu) : (v \odot \nu)$.

Proof of the lemma. The *first step* of the proof recalls the trace $v_b \in L^1(\partial\Omega)^n$ of a function $v \in L^2(\Omega)^n$ with $Dv \in L^1(\Omega)^{n\times n}_{\rm sym}$ [2] that satisfies the integration by parts formula

$$\int_\Omega (v \cdot \operatorname{div} \Phi + \Phi : Dv) = \int_{\partial\Omega} v_b \cdot \Phi\nu \quad \text{for all } \Phi \in C^1(\mathbb{R}^n)^{n\times n}_{\rm sym}. \tag{4.48}$$

Since v_b is unique, we simplify the notation and write v rather than v_b in the Lebesgue integral over the boundary $\partial\Omega$.

The *second step* designs a sequence of smooth approximations $\Phi_j \in C^\infty_c(\mathbb{R}^n)^{n\times n}_{\rm sym}$ with (i) $|\Phi_j| \le 1$ in $\overline{\Omega}$ for all $j \in \mathbb{N}$, (ii) its strong convergence $\Phi_j \to \sigma$ in $H(\operatorname{div},\Omega)^n$, (iii) its weak star convergence $\Phi_j \rightharpoonup^* \sigma$ in $L^\infty(\Omega)^{n\times n}_{\rm sym}$, and (iv) $\Phi_j|_{\partial\Omega} \rightharpoonup^* \sigma_\nu$ (weak star) in $L^\infty(\partial\Omega)^{n\times n}_{\rm sym}$, as $j \to \infty$.

It is a standard argument in the approximation of Sobolev functions by mollification in the Lipschitz domain Ω [12, p. 128-129] to consider a suitable partition of unity $\sum_{i=0}^N \zeta_i = 1$. Then, for $\varepsilon > 0$ sufficiently small, one sets $\Phi_\varepsilon := (\zeta_0\sigma) * \eta_\varepsilon + \sum_{i=1}^N \sigma_i * \eta_\varepsilon$, where the function η_ε is the mollification kernel, $\sigma_i(x) := (\zeta_i\sigma)(x + \varepsilon\tau_i)$ for all $x \in \Omega \cap \omega_i$, and the translation vector τ_i is such that $B(x,\varepsilon) + \varepsilon\tau_i \in \Omega$ for all $x \in \Omega \cap \omega_i$ and all $i \in \{1,\dots,N\}$ with ζ_i supported in ω_i. Since ζ_i is scalar-valued, we have $\operatorname{div}(\zeta_i\sigma) = \sigma\nabla\zeta_i + \zeta_i(\operatorname{div}\sigma)$. The arguments in [12] then show that $\Phi_\varepsilon \in C^\infty_c(\mathbb{R}^n)^{n\times n}$ with strong convergence to σ in $L^2(\Omega)^{n\times n}$ and strong convergence of divergence $\operatorname{div}\Phi_\varepsilon \to \operatorname{div}\sigma$ in $L^2(\Omega)^n$, i.e., strong convergence in $H(\operatorname{div},\Omega)^n$. Moreover, Φ_ε takes symmetric values and $\max|\Phi_\varepsilon| \le 1 + C\varepsilon$ for some constant C, that depends only on the smooth and fixed functions $\zeta_0,\dots,\zeta_N$ in the partition of unity. The pointwise boundedness implies the weak star convergence of a subsequence Φ_{ε_j}.

The *third step* substitutes $\Phi = \Phi_{\varepsilon_j}$ for the sequence $\Phi_{\varepsilon_j} \to \sigma$ in $H(\operatorname{div},\Omega)^n$ from step two for a fixed $v \in L^2(\Omega)^n$ with $Dv \in L^1(\Omega)^{n\times n}_{\rm sym}$ in (4.48). The weak star convergence in $L^\infty(\Omega)^{n\times n}_{\rm sym}$ and in $L^\infty(\partial\Omega)^{n\times n}_{\rm sym}$ leads to the asserted integration by parts formula in the passage to the limit as $j \to \infty$.

The *fourth step* concludes the proof: Given any $\sigma := \sigma(x) \in \mathbb{R}^{n\times n}_{\rm sym}$ and $\nu = \nu(x) \in \mathbb{R}^n$ with $|\sigma| \le 1 = |\nu|$ (at almost every point $x \in \partial\Omega$), the linear functional $\varphi \mapsto \varphi \cdot \sigma\nu = \sigma : (\varphi \odot \nu)$ has a unique Riesz representation $z \in \mathbb{R}^n$ in the Hilbert space $(\mathbb{R}^n, (\!(\bullet,\bullet)\!))$ with respect to the scalar product

$$(\!(a,b)\!) := (a \odot \nu) : (b \odot \nu) \quad \text{for } a,b \in \mathbb{R}^n. \tag{4.49}$$

Then $(\!(z,\varphi)\!) = \sigma : (\varphi \odot \nu)$ for all $\varphi \in \mathbb{R}^n$ implies $(z \odot \nu)\nu = \sigma\nu$ and $|z \odot \nu|^2 = (\!(z,z)\!) = \sigma : (z \odot \nu) \le |\sigma|\,|z \odot \nu| \le |z \odot \nu|$, whence $|z \odot \nu| \le 1$. This pointwise definition of $\sigma(x) \mapsto z(x)$ at a.e. $x \in \partial\Omega$ specifies a measurable function $z \in L^\infty(\partial\Omega)^n$. □

Let us resume the proof of Theorem 1.8. Lemma 4.4 applies to the solution $v = u$ and its trace $u_b \in L^1(\partial\Omega)$. Substitute $\operatorname{div}\sigma = \alpha u - f$ from (4.46) in the resulting identity and compare with (4.36). This proves

$$\psi(u) = \int_\Omega \sigma : Du - \beta \int_{\partial\Omega} (z \odot \nu) : (u_b \odot \nu) \tag{4.50}$$

for $\beta = 1$ in the Dirichlet case and $\beta = 0$ in the Neumann case. (The integration by parts formula (4.47) shows that $\sigma\nu = 0$ in the sense of traces in $H^{-1/2}(\partial\Omega)^n$ if $H = H^1(\Omega)^n$, but traces are involved in the Dirichlet case displayed for $\beta = 1$.) The appendix reports on the traces of BV and BD functions and the weak definition of ψ for the Dirichlet ($\beta = 1$) and Neumann ($\beta = 0$) case. Lemma A.2 clarifies for $u \in L^1(\Omega)^n$ with $Du \in L^1(\Omega)^{n\times n}$ that

$$\psi(u) = \int_\Omega |Du| + \beta \int_{\partial\Omega} |u_b \odot \nu|. \tag{4.51}$$

The equality of (4.50)-(4.51) implies that $\sigma : Du = |Du|$ a.e. in Ω and, if $\beta = 1$, that $(z \odot \nu) : (u_b \odot \nu) = |u_b \odot \nu|$ a.e. on $\partial\Omega$. Since $|\sigma| \leq 1$ a.e. in Ω and $|z \odot \nu| \leq 1$ a.e. in $\partial\Omega$, the preceding identities imply that equality holds in the Cauchy–Schwarz inequality (pointwise a.e. in Ω). Consequently, σ and $z \odot \nu$ are (pointwise a.e. in Ω) a non-negative multiple of Du and $u_b \odot \nu$, respectively. This leads to the final formulas and concludes the proof of (1.40).

The second part of the proof establishes the converse implication: Suppose $u \in L^2(\Omega)^n$ satisfies $Du \in L^1(\Omega)^{n\times n}$ and (1.40). Lemma 4.4 applies to $v = u$ and the arising integrals therein satisfy (from (1.40)) that

$$\sigma : Du = |Du| \quad \text{a.e. in } \Omega \quad \text{and} \quad \sigma_b : (u_b \odot \nu) = -|u_b \odot \nu| \quad \text{a.e. on } \partial\Omega$$

(the second identity holds for $\beta = 1$ only). It follows (for $\beta = 0$ and $\beta = 1$ as above) that

$$-\int_\Omega u \cdot \operatorname{div} \sigma = \int_\Omega |Du| + \beta \int_{\partial\Omega} |u_b \odot \nu|.$$

The combination of this with $\operatorname{div} \sigma = \alpha u - f \in L^2(\Omega)^n$ and (4.51) proves that u satisfies (4.36). Given any $v \in H$, an integration by parts with $\operatorname{div} \sigma = \alpha u - f \in L^2(\Omega)^n$ does not involve any boundary terms ($\sigma\nu = 0$ if $\beta = 0$ and $H = H^1(\Omega)^n$, while $v \in H_0^1(\Omega)^n$ for $\beta = 1$). This and $|\sigma| \leq 1$ a.e and $\psi(v) = \|Dv\|_{L^1(\Omega)^{n\times n}}$ for $v \in H$ verify

$$\int_\Omega (f - \alpha u) \cdot v = \int_\Omega \sigma : Dv \leq \psi(v). \tag{4.52}$$

The comparison of (4.36) and (4.52) proves that $u^* := u$ satisfies (4.42). Thus Lemma 4.2 shows that u solves (1.37) and concludes the proof of Theorem 1.8. □

Appendix: Jump of BV and BD functions across the boundary

The regularity proof for the Mosolov problem requires an approximation result regarding the boundary conditions. It corresponds to $\beta = 1$ in the function ψ in (1.8), thus including the L^1 integral along the boundary. For this, one has to understand the jump of functions in $\mathrm{BV}(\Omega)$ across the boundary $\partial\Omega$ of Ω. A similar analysis is possible in the case of functions of bounded deformation ($v \in \mathrm{BD}(\Omega)$ means $v \in L^2(\Omega)^n$ has a distributional Green strain, the symmetrized gradient, Dv that is a finite measure).

Throughout the remainder of this appendix, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with unit outer normal ν along the boundary $\partial\Omega$, abbreviate $C^1(\mathbb{R}^n)^{n\times n}_{\mathrm{sym}}$ as the set of symmetric-valued functions $\Phi \in C^1(\mathbb{R}^n)^{n\times n}$, and let $\mathcal{M}(\Omega)^{n\times n}$ denote the space of finite (matrix-valued) measures over Ω. Recall the notation $a \odot b$ from (1.39). Given any function $v \in \mathrm{BV}(\Omega)$, its zero extension reads

$$\overline{v}(x) = \begin{cases} v(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^n \backslash \Omega. \end{cases} \tag{A.1}$$

Lemma A.1 (Jump of BV functions across the boundary) *Given any $v \in \mathrm{BV}(\Omega)$, its zero extension $\overline{v}$ is in $\mathrm{BV}(\mathbb{R}^n)$ with*

$$|\overline{v}|_{\mathrm{BV}(\mathbb{R}^n)} = |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)}. \tag{A.2}$$

Moreover, for all $v \in L^p(\Omega)$, $1 \le p < \infty$, and all $\varepsilon > 0$, there exists some $v_\varepsilon \in C_c^\infty(\Omega)$ with

$$|v_\varepsilon|_{\mathrm{BV}(\Omega)} \le |v|_{\mathrm{BV}(\Omega)} + \|v\|_{L^1(\partial\Omega)} \tag{A.3}$$

and $v_\varepsilon \to v$ in $L^p(\Omega)$ as $\varepsilon \to 0$.

Proof. One has $\overline{v} \in \mathrm{BV}(\mathbb{R}^n)$ according to [18, 5.10.4], and the trace of v on $\partial\Omega$ belongs to $L^1(\partial\Omega)$ [18, 5.10.7]. For any $\varphi \in C_c^\infty(\mathbb{R}^n)^n$, the Gauss–Green formula shows that

$$-\int_\Omega v \operatorname{div}\varphi = \int_\Omega \varphi \cdot \nabla v - \int_{\partial\Omega} v\,\varphi\cdot\nu, \tag{A.4}$$

and is an identity on measures in $\mathbb{R}^n$ of the form

$$\nabla\overline{v} = \overline{\nabla v} - (v\nu)\delta_{\partial\Omega}. \tag{A.5}$$

The two measures on the right-hand side are supported at the disjoint sets Ω resp. $\partial\Omega$, the total variation of the sum is the sum of the total variations. This proves (A.2).
The existence of the approximation v_ε is shown here for a strictly star-shaped domain Ω, which means that there is some $x_0 \in \Omega$ such that for any $x \in \Omega$, one has $[x_0, x] \subset \Omega$ (recall that $[x_0, x] := \{tx_0 + (1-t)x \mid t \in [0,1]\}$) and for any $x \in \partial\Omega$, one has $[x_0, x) \subset \Omega$. The case of a convex domain falls into this scope and we refer the reader to [11] for the general case.
Without loss of generality, suppose that Ω is strictly star-shaped with respect to the origin 0. Taking $0 < \varepsilon \le 1/2$, we define $\Omega_\varepsilon := \Omega/(1-\varepsilon) \supset \Omega$ and

$$a_\varepsilon := \frac{1}{4}\operatorname{dist}(\partial\Omega_\varepsilon, \partial\Omega). \tag{A.6}$$

Since Ω is strictly star-shaped with respect to the origin, any half-line from the origin contains a single point belonging to $\partial\Omega$. It follows that $a_\varepsilon > 0$ and $a_\varepsilon \to 0$ as $\varepsilon \to 0$. Define $\rho_\varepsilon(x) = a_\varepsilon^{-n}\rho(x/a_\varepsilon)$, where $\rho \in C_c^\infty(B(0,1);\mathbb{R}^+)$, $\int_{B(0,1)}\rho = 1$, and

$$\widetilde{v}_\varepsilon(x) := \int_\Omega v(y)\rho_\varepsilon(x-y)dy = \int_{\mathbb{R}^n} \overline{v}(y)\rho_\varepsilon(x-y)dy \quad \text{at } x \in \mathbb{R}^n, \tag{A.7}$$

and

$$v_\varepsilon(x) := \widetilde{v}_\varepsilon\left(\frac{x}{1-\varepsilon}\right) \quad \text{at } x \in \mathbb{R}^n. \tag{A.8}$$

Then, $v_\varepsilon \in C_c^\infty(\Omega)$,

$$|v_\varepsilon|_{\mathrm{BV}(\Omega)} = (1-\varepsilon)^{n-1}|\widetilde{v}_\varepsilon|_{\mathrm{BV}(\Omega_\varepsilon)} \le |\overline{v}|_{\mathrm{BV}(\mathbb{R}^n)}, \tag{A.9}$$

and with (A.2), we obtain (A.3). Standard results on the convolution (A.7) show the convergence in $L^p(\Omega)$. □

Lemma A.2 (Jump of BD functions across the boundary) *(i) For all $v \in L^1(\Omega)^n$ with $Dv := (\nabla v + (\nabla v)^t)/2 \in \mathcal{M}(\Omega)^{n\times n}$, there is a unique trace $v_b \in L^1(\partial\Omega)^n$ such that*

$$\int_\Omega \Phi : Dv + \int_\Omega v \cdot \operatorname{div} \Phi = \int_{\partial\Omega} \Phi : v_b \odot \nu \quad \text{for all } \Phi \in C^1(\mathbb{R}^n)^{n\times n}_{\rm sym}. \tag{A.10}$$

(ii) For all $\sigma \in L^\infty(\Omega)^{n\times n}_{\rm sym}$ with $|\sigma| \le 1$ a.e. in Ω and $\operatorname{div}\sigma \in \mathcal{M}(\Omega)^n$, there exists some trace $\sigma_b \in L^\infty(\partial\Omega)^{n\times n}_{\rm sym}$ with $|\sigma_b| \le 1$ a.e. on $\partial\Omega$ and

$$\int_\Omega \varphi \cdot \operatorname{div}\sigma + \int_\Omega \sigma : D\varphi = \int_{\partial\Omega} \sigma_b : \varphi \odot \nu \quad \text{for all } \varphi \in C^1(\mathbb{R}^n)^n. \tag{A.11}$$

(iii) For all $v \in L^2(\Omega)^n$, $\psi_N(v) < \infty$ (with ψ_N defined by the right-hand side of (1.33)) if and only if $Dv \in \mathcal{M}(\Omega)^{n\times n}$. In this case, one has

$$\psi_N(v) = \int_\Omega |Dv|. \tag{A.12}$$

For all $v \in L^2(\Omega)^n$, $\psi_D(v) < \infty$ (with ψ_D defined by the right-hand side of (1.35)) if and only if $Dv \in \mathcal{M}(\Omega)^{n\times n}$. In this case, one has

$$\psi_D(v) = \int_\Omega |Dv| + \int_{\partial\Omega} |v_b \odot \nu|. \tag{A.13}$$

Proof of (i). This is proven in [2]. □

Proof of (ii). Let σ_{ij}, for all $i,j = 1,\dots,n$, denote the components of σ. For all $j = 1,\dots,n$, we consider the vector $\sigma^{(j)} = (\sigma_{j1},\dots,\sigma_{jn})$. Then $\sigma^{(j)} \in L^\infty(\Omega)^n$ and $\operatorname{div}\sigma^{(j)} \in \mathcal{M}(\Omega)$ is a finite measure. It is proven in [10] that there exists a trace $f_j \in L^\infty(\partial\Omega)$ such that

$$\int_\Omega \phi \operatorname{div}\sigma^{(j)} + \int_\Omega \sigma^{(j)} \cdot \nabla\phi = \int_{\partial\Omega} \phi f_j \quad \text{for all } \phi \in C^1(\mathbb{R}^n). \tag{A.14}$$

The substitution of $\phi = \varphi_j$ and the sum over $j = 1,\dots,n$ shows that

$$\int_\Omega \varphi \cdot \operatorname{div}\sigma + \int_\Omega \sigma : D\varphi = \int_{\partial\Omega} \varphi \cdot f \quad \text{for all } \varphi \in C^1(\mathbb{R}^n)^n, \tag{A.15}$$

with $f = (f_1,\dots,f_n)$. If $\sigma \in C^1(\overline{\Omega})^{n\times n}_{\rm sym}$, then $f_j = \sigma^{(j)} \cdot \nu$ and $f = \sigma\nu$ on $\partial\Omega$, whence

$$\varphi \cdot f = \varphi^t \sigma \nu = \sigma : \varphi \odot \nu \quad \text{a.e. on } \partial\Omega. \tag{A.16}$$

Since $|\sigma| := (\sigma:\sigma)^{1/2} \le 1$ a.e. on $\partial\Omega$, it follows that

$$|\varphi \cdot f|^2 \le |\varphi \odot \nu|^2 = \frac{1}{2}(|\varphi|^2|\nu|^2 + (\varphi\cdot\nu)^2) \quad \text{a.e. on } \partial\Omega. \tag{A.17}$$

For an arbitrary (non-smooth) field σ, one can argue as in Lemma 4.4 (even for a measure $\operatorname{div}\sigma$) and deduce that (A.17) remains valid for an arbitrary field σ.

At almost any fixed point x on $\partial\Omega$, with unit vector $\nu := \nu(x)$, we consider the inner product on $\mathbb{R}^n$ such that

$$(\!(a,b)\!) := a \odot \nu : b \odot \nu := (a \odot \nu) : (b \odot \nu) \quad \text{for all } a, b \in \mathbb{R}^n. \tag{A.18}$$

(Since $(\!(a,a)\!) = \frac{1}{2}(|a|^2|\nu|^2 + (a \cdot \nu)^2)$ and $|\nu| = 1$, $(\!(\bullet,\bullet)\!)$ is an inner product on $\mathbb{R}^n$.) Letting $f := f(x) \in \mathbb{R}^n$, the Riesz representation $g \in \mathbb{R}^n$ of the linear form $\varphi \mapsto \varphi \cdot f$ in the Hilbert space $(\mathbb{R}^n, (\!(\bullet,\bullet)\!))$ satisfies

$$\varphi \cdot f = (\!(\varphi, g)\!) = (\varphi \odot \nu) : (g \odot \nu) \quad \text{for all } \varphi \in \mathbb{R}^n. \tag{A.19}$$

Recall the calculation $|\varphi \cdot f|^2 \le (\!(\varphi,\varphi)\!)$ for all $\varphi \in \mathbb{R}^n$ from (A.17). Hence, the functional $\varphi \mapsto \varphi \cdot f$ has norm at most one in the Hilbert space $(\mathbb{R}^n, (\!(\bullet,\bullet)\!))$ and so is the norm of its Riesz representation g bounded from above by one, i.e., we have

$$|g \odot \nu|^2 = (\!(g,g)\!) = \frac{1}{2}(|g|^2|\nu|^2 + (g \cdot \nu)^2) \le 1. \tag{A.20}$$

Therefore, the symmetric tensor $\sigma_b(x) := g \odot \nu$ satisfies $|\sigma_b(x)| \le 1$. This calculation applies to almost any $x \in \partial\Omega$ and leads to a measurable bounded function $\sigma_b \in L^\infty(\partial\Omega)^{n\times n}_{\rm sym}$ with $|\sigma_b| \le 1$ and $\sigma_b : \varphi \odot \nu = \varphi \cdot f$ a.e. on $\partial\Omega$ (from (A.19)). This and (A.15) conclude the proof of (A.11). □

Proof of (iii). Recall the definition of $V(\Omega)$ in (1.34) and consider $v \in L^2(\Omega)^n$ and $w \in V(\Omega)$ with $w = \operatorname{div}\operatorname{sym}\Phi$ for $\Phi \in C^1_c(\Omega)^{n\times n}$. With the duality $\langle \bullet, \bullet \rangle$ in the sense of distributions, the integral in (1.33) reads

$$\begin{aligned} \int_\Omega v \cdot w &= \frac{1}{2}\sum_{i,j=1}^n \int_\Omega v_i \partial_j(\Phi_{ij} + \Phi_{ji}) = -\frac{1}{2}\sum_{i,j=1}^n \langle \partial_j v_i, \Phi_{ij} + \Phi_{ji}\rangle \\ &= -\frac{1}{2}\sum_{i,j=1}^n \langle \partial_j v_i + \partial_i v_j, \Phi_{ij}\rangle = -\langle Dv, \Phi\rangle. \end{aligned} \tag{A.21}$$

The supremum over all $\Phi \in C^1_c(\Omega)^{n\times n}$ with $|\Phi| \le 1$ a.e. in (A.21) leads to the equivalence between $\psi_N(v) < \infty$ and $Dv \in \mathcal{M}(\Omega)^{n\times n}$ and to the formula (A.12).
Since $V(\Omega) \subset V(\mathbb{R}^n)$ and therefore $\psi_N \le \psi_D$, $\psi_D(v) < \infty$ implies $Dv \in \mathcal{M}(\Omega)^{n\times n}$. Hence, the formula (A.10) can be employed for $v \in L^2(\Omega)^n$ with $Dv \in \mathcal{M}(\Omega)^{n\times n}$ and its unique trace $v_b \in L^1(\partial\Omega)^n$ to investigate the supremum (1.35). For any symmetric $\Phi \in C^1_c(\mathbb{R}^n)^{n\times n}_{\rm sym}$ with $w = \operatorname{div}\Phi$, the integral in (1.35) reads

$$\int_\Omega v \cdot w = \int_\Omega v \cdot \operatorname{div}\Phi = -\int_\Omega \Phi : Dv + \int_{\partial\Omega} \Phi : v_b \odot \nu. \tag{A.22}$$

The supremum over all $\Phi \in C^1_c(\mathbb{R}^n)^{n\times n}_{\rm sym}$ with $\max|\Phi| \le 1$ in the right-hand side of (A.22) can be replaced by the supremum over all $\Phi \in C^0_c(\mathbb{R}^n)^{n\times n}_{\rm sym}$ with $\max|\Phi| \le 1$. This means that $\psi(v)$ is the total variation of a measure, which is the sum of two measures corresponding to each term in the right-hand side of (A.22). Since one of the later measures is located in Ω and the other on $\partial\Omega$, the total variation of the sum is the sum of the respective total variations $\int_\Omega |Dv|$ and $\|v_b \odot \nu\|_{L^1(\partial\Omega)}$. This concludes the proof of (A.13). □

Remark A.3 (Uniqueness) *The trace σ_b in (A.11) is not unique since the matrices $\varphi \odot \nu$, for $\varphi \in \mathbb{R}^n$, do not generate the whole space of symmetric matrices. Indeed, we see in (A.11) that only $\sigma_b\,\nu$ is unique; however, σ_b is unique in the above format $g \odot \nu$.*

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