



**HAL**  
open science

# Poisson process and sharp constants in $L_p$ and Schauder estimates for a class of degenerate Kolmogorov operators

L. Marino, S. Menozzi, E. Priola

## ► To cite this version:

L. Marino, S. Menozzi, E. Priola. Poisson process and sharp constants in  $L_p$  and Schauder estimates for a class of degenerate Kolmogorov operators. 2021. hal-03284221v2

**HAL Id: hal-03284221**

**<https://hal.science/hal-03284221v2>**

Preprint submitted on 30 Aug 2021 (v2), last revised 26 Sep 2021 (v3)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Poisson process and sharp constants in $L^p$ and Schauder estimates for a class of degenerate Kolmogorov operators

L. Marino<sup>1, 2,\*</sup>, S. Menozzi<sup>1, 3,†</sup>, and E. Priola<sup>2,‡</sup>

<sup>1</sup> Laboratoire de Modélisation Mathématique d'Evry (LaMME), UMR CNRS 8071,  
Université Paris-Saclay, Université d'Evry Val d'Essonne, 23 Boulevard de France 91037  
Evry, France.

<sup>2</sup> Dipartimento di Matematica, Università di Pavia, Via Adolfo Ferrata 5, 27100 Pavia, Italy.

<sup>3</sup> Laboratory of Stochastic Analysis, Higher School of Economics (HSE), Pokrovsky  
Boulevard, 11, Moscow, Russian Federation.

\*lorenzo.marino@univ-evry.fr

†stephane.menozzi@univ-evry.fr

‡enrico.priola@unipv.it

30<sup>th</sup> August, 2021

## Abstract

We consider a possibly degenerate Kolmogorov-Ornstein-Uhlenbeck operator of the form  $L = \text{Tr}(BD^2) + \langle Az, D \rangle$ , where  $A, B$  are  $N \times N$  matrices,  $z \in \mathbb{R}^N$ ,  $N \geq 1$ , which satisfy the Kalman condition which is equivalent to the hypoellipticity condition. We prove the following stability result: the Schauder and Sobolev estimates associated with the corresponding parabolic Cauchy problem remain valid, with the same constant, for the parabolic Cauchy problem associated with a second order perturbation of  $L$ , namely for  $L + \text{Tr}(S(t)D^2)$  where  $S(t)$  is a *non-negative definite*  $N \times N$  matrix depending continuously on  $t \in [0, T]$ . Our approach relies on the perturbative technique based on the Poisson process introduced in [15].

**Keywords:** Degenerate Ornstein-Uhlenbeck operators, multidimensional parabolic equations,  $L^p$  and Sobolev-space estimates, Schauder estimates, Poisson process.

**MSC:** Primary: 35K10, 35K15, 60J76.

# 1 Introduction

Let us first consider the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t, x, y) = \Delta_x u(t, x, y) + x \cdot \nabla_y u(t, x, y) + f(t, x, y), \\ u(0, x, y) = 0, \end{cases} \quad (1.1)$$

where  $(t, x, y)$  is in  $(0, T) \times \mathbb{R}^{d_0+d_1}$ , for two integers  $d_0 \geq d_1 \geq 1$ . The underlying differential operator

$$L^K = \Delta_x + x \cdot \nabla_y = \sum_{i=1}^{d_0} \partial_{x_i x_i}^2 + \sum_{i=1}^{d_0} \sum_{j=1}^{d_1} x_i \partial_{y_j}$$

is the so-called Kolmogorov operator whose fundamental solution was derived in the seminal paper [11]. This particular operator was also mentioned by Hörmander as the starting point for his theory of hypoelliptic operators [9]. Let us write  $z = (x, y) \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} = \mathbb{R}^N$  and by  $\partial_{z_j}$  and  $\partial_{z_i z_j}^2$  we denote respectively the first and the second partial derivatives with  $i, j = 1, \dots, N$ .

We are interested in studying the influence of a second order perturbation on equation (1.1). Precisely, for a time-dependent matrix  $\{S(t) : t \in [0, T]\}$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$  such that  $t \mapsto S(t)$  is continuous and  $S(t)$  is *symmetric* and *non-negative definite* for any fixed  $t$ , we consider the *perturbed* Cauchy problem:

$$\begin{cases} \partial_t u_S(t, z) = L^K u_S(t, z) + \sum_{i,j=1}^N S_{ij}(t) \partial_{z_i z_j}^2 u_S(t, z) + f(t, z) \\ \quad =: L^{K,S} u_S(t, z) + f(t, z), \\ u_S(0, z) = 0, \quad z \in \mathbb{R}^N. \end{cases} \quad (1.2)$$

In particular, we will show that Sobolev (and Schauder) estimates which hold for solutions  $u$  of the Cauchy Problem (1.1) are also true, with the same constants, for solutions  $u_S$  to (1.2). Clearly, the operator  $L^{K,S}$  can be seen as a perturbation of  $L^K$  involving second order partial derivatives with continuous time-dependent coefficients.

For now, let us explain our main results in a special form for equation (1.1) in the case of  $L^p$ -estimates (or Sobolev estimates). For a statement of our results in the whole generality, we instead refer to Section 2.

For a fixed final time  $T > 0$  and a source  $f$  in  $C_0^\infty((0, T) \times \mathbb{R}^N)$ , it is known from the work of Bramanti *et al.* [2], Theorem 3.1 (see also Section 2.3 below), that Equation (1.1) admits a unique classical bounded solution  $u$  which satisfies for  $p$  in  $(1, +\infty)$  the following estimates:

$$\|\Delta_x u\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)} = C_p \|\partial_t u - L^K u\|_{L^p((0,T) \times \mathbb{R}^N)}. \quad (1.3)$$

Note that in this case  $C_p = C_p(d_0, d_1) > 0$ . We will actually manage to prove that the unique classical bounded solution  $u_S$  to (1.2) satisfies the estimate

$$\|\Delta_x u_S\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)} = C_p \|\partial_t u_S - L^{K,S} u_S\|_{L^p((0,T) \times \mathbb{R}^N)}, \quad (1.4)$$

with the **same** previous constant  $C_p$  as in (1.3). This result seems to be new even in dimension  $N = 2$  and even if we only consider  $S(t) = S$ ,  $t \in [0, T]$ , where  $S$  is a  $2 \times 2$  *symmetric non-negative definite* matrix.

For a uniformly elliptic second order perturbation  $S(t) = S$ ,  $t \in [0, T]$ , where  $S$  is *positive definite*, we could also have appealed to [3] to derive estimates like in (1.4). For related estimates in the uniformly elliptic case, see also Section 4 in Metafune *et al.* [20]. However, note that from [3] and [20] we could only deduce that the constant  $C_p$  depends on the ellipticity constant of the perturbation (this is the first eigenvalue  $\lambda_S$  of  $S$ ) and on the maximum eigenvalue of  $S$  (on this respect, see also [14] and [23]).

The remarkable point in (1.4) is that the  $L^p$ -estimates are stable under second order perturbations, which can be possibly degenerate. Namely, the fact that  $S(t)$  might be degenerate for some  $t$  in  $(0, T)$ , or even in some non-empty sub-intervals of  $(0, T)$ , does not affect the estimates in (1.4).

To prove (1.4), we combine the results of [2] with a probabilistic *perturbative* approach based on the Poisson process inspired by [15]. There, it was established in particular that the  $L^p$ -estimates for non-degenerate parabolic heat equations with space homogeneous coefficients are valid with constants that are independent of the dimension.

**Remark 1.1.** Importantly, the approach of [15] turns out to be sufficiently robust to handle the estimates in the degenerate directions as well. We recall that the associated maximal  $L^p$ -regularity was studied e.g. in [1], [10] or [5]. Let  $p$  in  $(1, +\infty)$ , there exists  $\tilde{C}_p > 0$  such that for  $f$  in  $C_0^\infty((0, T) \times \mathbb{R}^N)$  the unique classical bounded solution  $u$  of (1.1) verifies

$$\|(\Delta_y)^{\frac{1}{3}} u\|_{L^p((0, T) \times \mathbb{R}^N)} \leq \tilde{C}_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)} = \tilde{C}_p \|\partial_t u - L^K u\|_{L^p((0, T) \times \mathbb{R}^N)}, \quad (1.5)$$

where  $(\Delta_y)^{\frac{1}{3}}$  denotes the fractional Laplacian with respect to the degenerate variables  $y$  in  $\mathbb{R}^{d_1}$ . It turns out that this estimate is also stable for the previously described second order perturbation. Namely, for  $u_S$  solving (1.2),

$$\|(\Delta_y)^{\frac{1}{3}} u_S\|_{L^p((0, T) \times \mathbb{R}^d)} \leq \tilde{C}_p \|f\|_{L^p((0, T) \times \mathbb{R}^d)} = \tilde{C}_p \|\partial_t u_S - L^{K, S} u_S\|_{L^p((0, T) \times \mathbb{R}^N)}, \quad (1.6)$$

where again  $\tilde{C}_p$  is the same as in (1.5).  $\square$

**Remark 1.2.** The same type of stability results will also hold for the corresponding global Schauder estimates, first established in the framework of anisotropic Hölder spaces for the solution of (1.1) by Lunardi [17] (see also [18], [19] and the references therein). We refer to estimate (4.17).  $\square$

We point out that our results in Section 3 could also possibly be obtained by using the general theorems of Section 4 in [15]. This section in [15] introduces a more general probabilistic approach and provides unexpected regularity results. However checking in our case all the assumptions given in that section

is quite involved. On the other hand, we provide self-contained proofs inspired by Sections 2 and 3 of [15].

It remains a challenging open problem to have a purely analytic proof of our regularity results.

**$L^p$ -estimates for degenerate Ornstein-Uhlenbeck operators.** Let us now describe the more general framework we are going to consider here. Let  $\mathbb{R}^N = \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$  where  $d_0, d_1$  are two *non-negative* integers such that  $d_0 + d_1 = N$  and  $d_0 \geq 1$ . let us introduce the non-negative, symmetric matrix  $B$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$  given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $B_0$  is a symmetric, positive definite matrix in  $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$  such that

$$\nu \sum_{i=1}^{d_0} \xi_i^2 \leq \sum_{i,j=1}^{d_0} (B_0)_{ij} \xi_i \xi_j \leq \frac{1}{\nu} \sum_{i=1}^{d_0} \xi_i^2,$$

for all  $\xi \in \mathbb{R}^{d_0}$ , for some  $\nu > 0$ .

We will use, as underlying *proxy* operators, the family of degenerate Ornstein-Uhlenbeck generators of the form

$$L^{\text{ou}} f(z) = \text{Tr}(BD^2 f(z)) + \langle Az, Df(z) \rangle, \quad z = (x, y) \in \mathbb{R}^{d_0+d_1} = \mathbb{R}^N, \quad (1.7)$$

for a matrix  $A$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $\mathbb{R}^N$ . Moreover, we assume the Kalman condition:

[**K**] There exists a non-negative integer  $k$ , such that

$$\text{Rank}[B, AB, \dots, A^k B] = N, \quad (1.8)$$

where  $[B, AB, \dots, A^k B]$  is the  $\mathbb{R}^N \otimes \mathbb{R}^{N(k+1)}$  matrix whose blocks are  $B, AB, \dots, A^k B$ . From the non-degeneracy of  $B_0$ , the above condition amounts to say that the vectors

$$\{e_1, \dots, e_{d_0}, Ae_1, \dots, Ae_{d_0}, \dots, A^k e_1, \dots, A^k e_{d_0}\} \text{ generate } \mathbb{R}^N, \quad (1.9)$$

where  $\{e_i\}_{i \in \{1, \dots, d_0\}}$  are the first  $d_0$  vectors of the canonical basis for  $\mathbb{R}^N$ .

Assumption [**K**] (which also often appears in control theory; see e.g. [26]) is equivalent to the Hörmander condition on the commutators (c.f. [9]) ensuring the hypoellipticity of the operator  $\partial_t - L^{\text{ou}}$ . In particular, it implies the existence and the smoothness of a distributional solution for the following equation:

$$\begin{cases} \partial_t u(t, z) = L^{\text{ou}} u(t, z) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (1.10)$$

where  $f$  is a function in  $C_0^\infty((0, T) \times \mathbb{R}^N)$ .

Similarly to [15], we will prove below the existence and uniqueness of bounded regular solutions to (1.10) assuming that the source  $f$  belongs to the space  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , which contains  $C_0^\infty((0, T) \times \mathbb{R}^N)$ , and that can be roughly described as the family of functions which are bounded measurable in time and compactly supported in space uniformly in time (see Section 1.2 for a precise definition). Equation (1.10) will be understood in an integral form (cf. formula (1.25)).

By Theorem 3 in [3] and exploiting some explicit properties of the underlying heat kernel (see Section 2.3 below), it can be derived that for any fixed  $p$  in  $(1, +\infty)$ , there exists  $C_p = C_p(\nu, A, d_0, d_1, T)$  such that

$$\|D_x^2 u\|_{L^p((0, T) \times \mathbb{R}^N)} \leq C_p \|u_t - L^{\text{ou}} u\|_{L^p((0, T) \times \mathbb{R}^N)} = C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}, \quad (1.11)$$

where for any  $z$  in  $\mathbb{R}^N$ ,  $t \in [0, T]$ ,  $D_x^2 u(t, z)$  stands for the Hessian matrix in  $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$  with respect to the variable  $x$ . We set

$$B_I = \begin{pmatrix} I_{d_0, d_0} & 0_{d_0, d_1} \\ 0_{d_1, d_0} & 0_{d_1, d_1} \end{pmatrix}$$

and note, in particular, that (1.11) can be rewritten in the following, equivalent way:

$$\begin{aligned} \|B_I D^2 u B_I\|_{L^p((0, T) \times \mathbb{R}^N)} &= \|D_x^2 u\|_{L^p((0, T) \times \mathbb{R}^N)} \\ &\leq C_p \|u_t - L^{\text{ou}} u\|_{L^p((0, T) \times \mathbb{R}^N)} = C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}, \end{aligned} \quad (1.12)$$

where  $D^2 u = D_z^2 u$  represents instead the full Hessian matrix in  $\mathbb{R}^N \otimes \mathbb{R}^N$  with respect to  $z$ .

Fixed a continuous mapping  $t \mapsto S(t)$  such that  $S(t)$  is a *symmetric* and *non-negative definite* matrix in  $\mathbb{R}^N \otimes \mathbb{R}^N$ ,  $t \in [0, T]$ , we consider again the following perturbation of  $L^{\text{ou}}$ :

$$\begin{aligned} L_t^{\text{ou}, S} f(z) &:= \text{Tr}(B D^2 f(z)) + \text{Tr}(S(t) D^2 f(z)) + \langle A z, D f(z) \rangle \\ &= L^{\text{ou}} f(z) + \text{Tr}(S(t) D^2 f(z)), \end{aligned} \quad (1.13)$$

where  $z = (x, y)$  is in  $\mathbb{R}^{d_0+d_1} = \mathbb{R}^N$ . For the solution  $u_S$  of the related Cauchy problem

$$\begin{cases} \partial_t u_S(t, z) = L_t^{\text{ou}, S} u_S(t, z) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u_S(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (1.14)$$

we will prove the following main theorem:

**Theorem 1.1.** *Let us consider (1.14) with  $f \in B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Then, there exists a unique solution  $u_S$  of Cauchy Problem (1.14) which verifies, with the same constant  $C_p$  as in (1.12),*

$$\begin{aligned} \|D_x^2 u_S\|_{L^p((0, T) \times \mathbb{R}^N)} &= \|B_I D^2 u_S B_I\|_{L^p((0, T) \times \mathbb{R}^N)} \\ &\leq C_p \|\partial_t u_S - L_t^{\text{ou}, S} u_S\|_{L^p((0, T) \times \mathbb{R}^N)} = C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}. \end{aligned} \quad (1.15)$$

We point out that for time-homogeneous non-negative definite matrices  $S$ , the corresponding elliptic  $L^p$ -estimates as in formula (5) of [3] (replacing  $\mathcal{A}$  in [3] with  $L^{\text{ou},S} := \text{Tr}(BD^2\cdot) + \text{Tr}(SD^2\cdot) + \langle Az, D\cdot \rangle$ ) with a constant independent of  $S$ , could also be derived from (1.15) using an argument given in [3].

For more information on the OU operator  $L^{\text{ou}}$  we also refer to the recent work by Fornaro *et al.* [22] about full description of the spectrum of degenerate OU operators in  $L^p$ -spaces.

Independently from the constant preservation, we also emphasize that the  $L^p$ -estimates in (1.15) for the perturbed operator seem, to the best of our knowledge, to be new and have some interest by their own.

Let us eventually mention that our stability results could turn out to be useful to investigate the well-posedness of some related stochastic differential equations through the corresponding martingale problem.

We could actually derive more general estimates, possibly involving the degenerate directions as well, dependently on the structure of  $A$ . Some results in that direction are gathered in Section 4. Anyhow, to illustrate our approach we now briefly present the various steps to derive (1.15).

## 1.1 Strategy of the proof for estimate (1.15).

Fixed a classical bounded solution  $u$  to Cauchy Problem (1.10), let us introduce  $v(t, z) := u(t, e^{-tA}z)$ . This well-known transformation (cf. [7]) precisely allows to get rid of the drift term in the PDE satisfied by  $v$ . Indeed, we have that  $u(t, z) = v(t, e^{tA}z)$  and since  $u$  solves (1.10), it holds for any  $(t, z)$  in  $(0, T) \times \mathbb{R}^N$ , that:

$$\begin{aligned} f(t, z) &= \partial_t u(t, z) - L^{\text{ou}}u(t, z) \\ &= v_t(t, e^{tA}z) + \langle Dv(t, e^{tA}z), Ae^{tA}z \rangle - \text{Tr}\left(e^{tA}Be^{tA*}D^2v(t, e^{tA}z)\right) \\ &\quad - \langle Dv(t, e^{tA}z), Ae^{tA}z \rangle \\ &= v_t(t, e^{tA}z) - \text{Tr}\left(e^{tA}Be^{tA*}D^2v(t, e^{tA}z)\right). \end{aligned} \tag{1.16}$$

Denoting  $\tilde{f}(t, z) := f(t, e^{-tA}z)$ , it now follows that  $v$  satisfies the PDE:

$$\begin{cases} \partial_t v(t, z) = \text{Tr}\left(e^{tA}Be^{tA*}D^2v(t, z)\right) + \tilde{f}(t, z) & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{1.17}$$

In terms of the function  $v$ , the known estimates in (1.12) rewrites as:

$$\|B_I e^{tA*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0,T) \times \mathbb{R}^N)}, \tag{1.18}$$

where we used the notation  $\|B_I e^{-tA*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I\|_{L^p((0,T) \times \mathbb{R}^N)}$  to stress the dependence on  $t$  instead of the more precise formulation

$$\|B_I e^{\cdot A*} D^2 v(\cdot, e^{\cdot A}\cdot) e^{\cdot A} B_I\|_{L^p((0,T) \times \mathbb{R}^N)}.$$

By changing variable in the integrals, control (1.18) is equivalent to

$$\|B_I e^{tA^*} D^2 v(t, \cdot) e^{tA} B_I\|_{L^p((0,T) \times \mathbb{R}^N, m)} \leq C_p \|\tilde{f}\|_{L^p((0,T) \times \mathbb{R}^N, m)}, \quad (1.19)$$

where  $L^p((0, T) \times \mathbb{R}^N, m)$  denotes the  $L^p$ -norms w.r.t. the measure  $m(dt, dx) = \det(e^{-At}) dt dx$ .

Considering now the following more general Cauchy problem on  $[0, T] \times \mathbb{R}^N$

$$\begin{cases} \partial_t w(t, z) = \text{Tr}(e^{tA} B e^{tA^*} D^2 w(t, z)) + \text{Tr}(e^{tA} S(t) e^{tA^*} D^2 w(t, z)) + \tilde{f}(t, z); \\ w(0, z) = 0, \end{cases} \quad (1.20)$$

we can establish the well-posedness of the Cauchy problem (1.20), exploiting, for instance, probabilistic arguments, using the underlying Gaussian process.

Now the crucial step consists in adapting some arguments from [15] based on the use of the Poisson process to derive that the same  $L^p$ -estimates in (1.19) still hold for  $w$ , independently from the non-negative definite, symmetric matrices  $S(t)$ . Precisely,

$$\|B_I e^{tA^*} D^2 w(t, \cdot) e^{tA} B_I\|_{L^p((0,T) \times \mathbb{R}^N, m)} \leq C_p \|\tilde{f}(t, \cdot)\|_{L^p((0,T) \times \mathbb{R}^N, m)}, \quad (1.21)$$

with the *same* constant  $C_p$  appearing in (1.19).

The last step then consists in coming back to the Ornstein-Uhlenbeck operators framework. Namely, we introduce  $\tilde{u}(t, z) := w(t, e^{tA} z)$  which solves, by definition, the following equation:

$$\begin{cases} \partial_t \tilde{u}(t, z) = L_t^{\text{ou}, S} \tilde{u}(t, z) + f(t, z), & (t, z) \in (0, T) \times \mathbb{R}^N, \\ \tilde{u}(0, z) = 0, & z \in \mathbb{R}^N. \end{cases}$$

Thus  $\tilde{u} = u_S$ . Noticing that  $D^2 w(t, \cdot) = D^2[\tilde{u}(t, e^{-tA} \cdot)] = e^{-tA^*} D^2 \tilde{u}(t, e^{-tA} \cdot) e^{-tA}$  we thus get from (1.21) that the following estimates hold:

$$\|B_I D^2 \tilde{u} B_I\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)}. \quad (1.22)$$

Through the previous steps we have then constructed a solution  $\tilde{u}$  of Cauchy Problem (1.14) which indeed satisfies the estimates in (1.15) with the same  $C_p$ , associated with the *unperturbed* or *proxy* operator. The maximum principle will eventually provide uniqueness for the solution  $\tilde{u}$ .

**Remark 1.3.** i) We point out that we could also consider more general time-dependent Ornstein-Uhlenbeck operators like:

$$M = \text{Tr}(B(t) D^2 \cdot) + \langle Az, D \cdot \rangle.$$

Arguing as before starting from  $L^p$ -estimates (or Schauder estimates) for  $M$  we can derive the same  $L^p$ -estimates (or Schauder estimates) for a perturbation of  $M$  like (1.13).



ii) We could extend the  $L^p$ -estimates (or the Schauder estimates) related to  $L^{\text{ou}}$  to more general operators like

$$L_t^{\text{ou},S} f(z) + \langle b(t), Df(z) \rangle$$

where  $b : \mathbb{R}_+ \rightarrow \mathbb{R}^N$  is continuous. We can even add to  $L_t$  a possibly degenerate non-local perturbation (cf. Section 7 of [15]). The  $L^p$ -estimates (or Schauder estimates) are still preserved with the same constant. For the sake of simplicity in the sequel we will only consider  $b(t) = 0$  and we will not deal with non-local perturbations of  $L_t^{\text{ou},S}$ .

**Organization of the paper.** The article is organized as follows. At the end of the current section, we first fix some useful notations. In Section 2 we will then focus on driftless second order Cauchy problems associated with a non-negative definite, possibly degenerate, diffusion matrix. We will also consider its relation to the Ornstein-Uhlenbeck dynamics. We will establish through the probabilistic perturbation approach of [15] that if some  $L^p$ -estimate holds for a particular diffusion matrix so does it, with the same associated constant as explained before, for a non-negative perturbation of the diffusion matrix (see Section 3). Finally, by the arguments of Section 1.1 we will obtain (1.22). Stability results in anisotropic Sobolev space and Schauder estimates are given in Section 4.

## 1.2 Definition of solution and useful notations

Let us consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t, z) = \text{tr}(Q(t)D^2v(t, z)) + \langle b(t, z), Dv(t, z) \rangle + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0, & \text{on } \mathbb{R}^N; \end{cases} \quad (1.23)$$

where  $Q : [0, T] \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$  is a *continuous symmetric non-negative definite matrix* and  $b : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a *continuous function* such that  $|b(t, z)| \leq K_T(1 + |z|)$ ,  $(t, z) \in [0, T] \times \mathbb{R}^N$ , for some constant  $K_T > 0$ .

The function  $f$  belongs to  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , the space of all Borel bounded functions  $\phi : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\phi(t, \cdot)$  is smooth and compactly supported for any  $t$  in  $[0, T]$ ; for any  $n$  in  $\mathbb{N}$  the  $C^n(\mathbb{R}^N)$ -norms of  $\phi(t, \cdot)$  are bounded in time and the supports of the functions  $\phi(t, \cdot)$  are contained in the same ball. Moreover, we require that, for any  $z \in \mathbb{R}^N$ , the mapping:

$$t \mapsto \phi(t, z) \quad (1.24)$$

is a *piece-wise continuous* function on  $[0, T]$ , i.e. it is continuous except for a finite number of points.

**Remark 1.4.** Note that to perform the technique used in [15] and based on the Poisson process we need to consider equations like (1.23) with a source  $f$  which is possibly discontinuous in time (cf. the proof in Section 2 of [15] and Section 3.2 below).

We interpret Cauchy Problem (1.23) in an *integral* form:

$$v(t, z) = \int_0^t \left( f(s, z) + \text{Tr}(Q(s)D^2v(s, z)) + \langle b(s, z), Dv(s, z) \rangle \right) ds. \quad (1.25)$$

In particular, we say that a continuous and bounded function  $v : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a solution to equation (1.23) if  $v(t, \cdot)$  belongs to  $C^2(\mathbb{R}^N)$ , for any  $t \in [0, T]$ , and (1.25) holds as well, for any  $(t, z) \in [0, T] \times \mathbb{R}^N$ .

We finally note that, for any  $z \in \mathbb{R}^N$ , the function  $t \mapsto v(t, z)$  is a  $C^1$ -piecewise function on  $[0, T]$ .

By Theorem 4.1 in [13] we deduce in a quite standard way that if a solution  $v$  exists then it is unique and the following maximum principle holds:

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t, z)|. \quad (1.26)$$

About the proof of (1.26) we only make some remarks. By considering  $v$  and  $-v$  we see that it is enough to prove that  $v(t, z) \leq T\|f\|_\infty$ , for all  $(t, z) \in [0, T] \times \mathbb{R}^N$ . Moreover, setting  $\tilde{v} = v - T\|f\|_\infty$ , we note that  $\tilde{v}$  verifies (1.25) with  $f$  replaced by  $f - \|f\|_\infty \leq 0$ . Finally, by considering the equation verified by  $e^{-t}\tilde{v}$ , we can apply Theorem 4.1 in [13] to obtain the result.

## 2 Estimates for driftless second order operators and related perturbation

Throughout this section, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t, z) = \text{Tr}(Q(t)D^2v(t, z)) + f(t, z) & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0 & \text{on } \mathbb{R}^N, \end{cases} \quad (2.1)$$

which can be seen as a special case of (1.23) when  $b = 0$ . Moreover, we assume that  $Q$  is not identically zero.

### 2.1 Well-posedness

**Proposition 2.1** (Well-posedness in integral form for the driftless Cauchy problem). *Let  $f$  be in  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Then, there exists a unique solution  $v$  to Cauchy problem (2.1) in an integral sense, i.e., it solves for  $(t, z) \in [0, T] \times \mathbb{R}^N$ :*

$$v(t, z) = \int_0^t \left( f(s, z) + \text{Tr}(Q(s)D^2v(s, z)) \right) ds. \quad (2.2)$$

We will denote in short  $v = PDE(Q, f)$ .

*Proof.* By the maximum principle (cf. equation (1.26)) uniqueness holds for Cauchy Problem (2.1). We can then focus on proving the existence of a solution. Let us introduce now

$$v(t, z) := \int_0^t \mathbb{E}[f(s, z + I_{s,t})] ds$$

with the following notation:  $I_{s,u} := \sqrt{2} \int_s^u Q(r)^{1/2} dW_r$ , where  $W$  is an  $N$ -dimensional Brownian motion on some probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $Q(r)^{1/2}$  stands for a square root of  $Q(r)$ , i.e.  $Q(r) = Q(r)^{1/2}(Q(r)^{1/2})^*$ .

Applying the Itô formula in space to  $f(s, z + I_{s,u})_{u \in [s,t]}$ , we get that

$$\mathbb{E}f(s, z + I_{s,t}) = f(s, z) + \mathbb{E} \left[ \int_s^t \text{Tr}(Q(u) D^2 f(s, z + I_{s,u})) du \right].$$

Hence,

$$v(t, z) = \int_0^t \left( f(s, z) + \mathbb{E} \left[ \int_s^t \text{Tr}(Q(u) D^2 f(s, z + I_{s,u})) du \right] \right) ds,$$

from which it readily follows that

$$\begin{aligned} \partial_t v(t, z) &= f(t, z) + \int_0^t \mathbb{E}[\text{Tr}(Q(t) D^2 f(s, z + I_{s,t}))] ds \\ &= f(t, z) + \text{Tr} \left( Q(t) D^2 \int_0^t \mathbb{E}[f(s, z + I_{s,t})] ds \right) \\ &= f(t, z) + \text{Tr} \left( Q(t) D^2 v(t, z) \right). \end{aligned}$$

for almost every  $t \in [0, T]$  and any  $z \in \mathbb{R}^N$ . □

## 2.2 Relation to the Ornstein-Uhlenbeck dynamics

If now in particular,  $Q(t)$  has the particular form  $Q(t) = e^{tA} B e^{tA^*}$  (cf. equation (1.17)), we introduce

$$u(t, z) := v(t, e^{tA} z),$$

where  $v$  is the solution to (2.2) (see Proposition 2.1). Since we can differentiate with respect to  $t$  the function  $u(\cdot, z)$  for a.e.  $t \in [0, T]$ , we can perform computations similar to (1.16) and get that  $u(t, z)$  solves in integral form:

$$\begin{cases} \partial_t u(t, z) = L^{\text{ou}} u(t, z) + \bar{f}(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u(0, z) = 0, & \text{on } \mathbb{R}^N; \end{cases} \quad (2.3)$$

with  $L^{\text{ou}}$  as in (1.7),  $\bar{f}(t, z) = f(t, e^{tA} z)$ . Precisely, for all  $(t, z) \in [0, T] \times \mathbb{R}^N$ ,

$$u(t, z) = \int_0^t \left( \bar{f}(s, z) + L^{\text{ou}} v(s, z) \right) ds. \quad (2.4)$$

We have that  $u$  is a solution to (2.3).

Let us also point out that the well-posedness of (2.3) could also have been obtained directly from Gaussian type calculations, similar to those in the proof of Proposition 2.1, introducing  $u^{\text{ou}}(t, z) := \int_0^t \mathbb{E}[f(s, e^{(t-s)A} z + I_{s,t}^{\text{ou}})] ds$  where  $I_{s,u}^{\text{ou}} := \sqrt{2} \int_s^u e^{(u-v)A} B dW_v$ .

### 2.3 About the $L^p$ -estimate (1.11) for the OU operator

The aim of this section is to fully justify the estimates in (1.11). This is a consequence of the previous probabilistic representation and of Theorem 3 in [3]. For  $u$  solving (1.10) it holds that for all  $(t, z) \in [0, T] \times \mathbb{R}^N$ ,

$$u(t, z) = \int_0^t \mathbb{E}[f(s, e^{A(t-s)}z + I_{s,t}^{\text{ou}})]ds = \int_0^t \int_{\mathbb{R}^N} f(s, z')p^{\text{ou}}(t-s, z, z')dz'ds, \quad (2.5)$$

where for  $v > 0$ ,  $p^{\text{ou}}(v, z, \cdot)$  stands for the density at time  $v$  of the stochastic process

$$X_u^{\text{ou}} := e^{Au}z + \sqrt{2} \int_0^u e^{A(u-w)}BdW_w = z + \int_0^u AX_w^{\text{ou}}dw + \sqrt{2}BW_u, \quad u \geq 0.$$

We recall from [16] that assumption **[K]** is equivalent to the fact that there exists  $k \in \mathbb{N}$  and positive integers  $(\mathfrak{d}_i)_{i \in \{1, \dots, k\}}$  s.t.  $\sum_{i=1}^k \mathfrak{d}_i = d_1$  and for all  $i \in \{1, \dots, k\}$ , setting  $\mathfrak{d}_0 = d_0$  and  $\sum_{m=0}^{-1} = 0$ , the matrixes

$$\mathcal{A}^i := (A_{j,\ell})_{(j,\ell) \in \{\sum_{m=0}^{i-1} \mathfrak{d}_{m+1}, \dots, \sum_{m=0}^i \mathfrak{d}_m\} \times \{\sum_{m=1}^{i-1} \mathfrak{d}_{m+1}, \dots, \sum_{m=1}^i \mathfrak{d}_m\}},$$

have rank  $\mathfrak{d}_i$ . The matrix  $A$  writes:

$$A = \begin{pmatrix} * & * & \dots & \dots & * \\ \mathcal{A}^1 & * & \ddots & \ddots & \vdots \\ 0_{\mathfrak{d}_2, d_0} & \mathcal{A}^2 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0_{\mathfrak{d}_k, d_0} & \dots & 0_{\mathfrak{d}_k, \mathfrak{d}_{k-1}} & \mathcal{A}^k & * \end{pmatrix}. \quad (2.6)$$

Following the proof of Lemma 5.5 in [8], where the case  $d_0 = d$ ,  $\mathfrak{d}_i = d, k = n - 1$  is addressed, it can be derived that there exists  $C \geq 1$  s.t. for all  $(v, z, z') \in (0, T] \times (\mathbb{R}^N)^2$ ,

$$|D_x^2 p^{\text{ou}}(v, z, z')| \leq \frac{C}{v \sum_{i=0}^{k-1} \mathfrak{d}_i(i+\frac{1}{2})+1} \exp\left(-C^{-1}v|\mathbb{T}_v^{-1}(e^{Av}z - z')|^2\right), \quad (2.7)$$

where  $\mathfrak{d}_0 = d_0$  and

$$\mathbb{T}_v := \text{diag}(vI_{d_0 \times d_0}, v^2I_{d_1 \times d_1}, \dots, v^{k+1}I_{d_k \times d_k}), \quad v \geq 0,$$

reflects the various scales of the system. For a given function  $f \in B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , it is then clear from (2.5) and (2.7) that for all  $(t, z) \in (0, T] \times \mathbb{R}^N$ :

$$D_x^2 u(t, z) = \text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s, z') D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds. \quad (2.8)$$

It indeed suffices to observe that:

$$\begin{aligned}
& |\text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s, z') D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds| \\
&= |\text{p.v.} \int_0^t \int_{\mathbb{R}^N} [f(s, z') - f(s, e^{A(t-s)}z)] D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds| \\
&\leq \sup_{s \in [0, T]} \|Df(s, \cdot)\|_\infty \\
&\times \int_0^t \int_{\mathbb{R}^N} \frac{C}{(t-s)^{\sum_{i=0}^{k-1} \nu_i(i+\frac{1}{2}) + \frac{1}{2}}} \exp\left(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(e^{A(t-s)}z - z')|^2\right) dz' ds \\
&\leq C \sup_{s \in [0, T]} \|Df(s, \cdot)\|_\infty T^{\frac{1}{2}}.
\end{aligned}$$

The estimates in (1.11) now follows from the proof of Theorem 3 in [3], starting from (2.8) instead of (16) therein. The strategy is clear. It is necessary to introduce a cut-off function which separates the points  $(s, z')$  which do not induce any singularity in (2.8) for the derivatives of the density, namely such that  $t-s \geq c_0$  or  $|e^{A(t-s)}z - z'| \geq c_0$ , for some fixed constant  $c_0 > 0$ , from those who are close to the singularity. For the non-singular part of the integral the expected  $L^p$ -control readily follows from (2.7) and the Young inequality (see also Proposition 5 in [3]), whereas the derivation of the bound for the singular part requires some involved harmonic analysis, see Section 4 on the same reference. We can also refer to Theorem 11 and its proof in [24] for similar issues linked with the corresponding  $L^p$ -estimates for degenerate Ornstein-Uhlenbeck operators in an elliptic setting.

## 2.4 The main result for equation (2.1): perturbation of second order driftless PDE

Let us fix  $p$  in  $(1, +\infty)$  and assume that there exists  $R(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$  depending continuously on  $t \geq 0$  and a constant  $C_p > 0$ , such that for any  $f$  in  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , the unique solution  $v = PDE(Q, f)$  to equation (2.1) satisfies

$$\|R(t)^* D^2 v R(t)\|_{L^p((0, T) \times \mathbb{R}^N, \mathbf{m})} \leq C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N, \mathbf{m})}, \quad (2.9)$$

for some absolutely continuous measure  $\mathbf{m}$  w.r.t. the Lebesgue measure on  $[0, T] \times \mathbb{R}^N$  such that  $\mathbf{m}(dt, dx) = g(t) dt dx$  for some Borel bounded function  $g$  (note that in (1.19) we have  $R(t) = e^{tA} B_I$ ,  $\mathbf{m}(dt, dx) = g(t) dt dx = \det(e^{-At}) dt dx$ ).

We would like to exhibit that a control like (2.9) also holds for the solution  $w$  to the following Cauchy Problem:

$$\begin{cases} \partial_t w(t, z) = \text{tr}(Q(t) D^2 w(t, z)) + \text{tr}(Q'(t) D^2 w(t, z)) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ w(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (2.10)$$

Namely we have to prove the following result.

**Theorem 2.2.** *Let us consider equations (2.1) and (2.10) where  $Q(t)$ ,  $Q'(t)$  are two continuous in time, non-negative definite matrices in  $\mathbb{R}^N \otimes \mathbb{R}^N$  and  $f \in B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Assume that estimate (2.9) holds as explained above. Then the solution  $w$  to (2.10) verifies*

$$\|R(t)^* D^2 w R(t)\|_{L^p((0, T) \times \mathbb{R}^N, m)} \leq C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N, m)}, \quad (2.11)$$

$p \in (1, \infty)$  with the same constant  $C_p$  as in (2.9).

From Theorem 2.2 using the argument of Section 1.1 we can easily derive Theorem 1.1.

### 3 A perturbation argument for proving Theorem 2.2

We aim here at applying the probabilistic perturbative approach considered in [15]. The key idea in that work was, for a well-posed PDE which enjoys some quantitative given estimates, to introduce a *small* random perturbation in the source  $f$  through a suitable Poisson type process and to investigate the properties of the associated PDE involving an unknown function  $v$ . After considering a small random perturbation of  $v$ , we arrive at the useful integral formula (3.8). Taking the expectation the contributions associated with the jumps yield, for an appropriate intensity of the underlying Poisson process, a finite difference operator. For the PDE satisfied by the expectation, involving the finite difference operator, *the initial estimates are preserved*. Repeating the previous argument we can obtain a PDE involving the composition of two finite difference operators.

Compactness arguments then allow to derive that, the initial estimates still hold at the limit with the composition of two finite difference operators replaced by the corresponding differential operator of order two. Iterating this procedure we can obtain the result.

Below, we start recalling basic properties of Poisson type processes and corresponding stochastic integrals, which are needed for our approach.

#### 3.1 Poisson stochastic integrals

We briefly recall here the very definition of the stochastic integral driven by a Poisson process. We start reminding the construction of such processes. Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  to be fixed from this point further, we start considering a sequence of independent real-valued random variables  $\{\tau_n\}_{n \in \mathbb{N}}$  on  $\Omega$  whose distribution is exponential of parameter  $\lambda > 0$ :

$$\mathbb{P}(\tau_n > r) = e^{-r\lambda}, \quad r \geq 0.$$

We can then define the partial sums sequence  $\{\sigma_n\}_{n \in \mathbb{N}}$  as follows:

$$\sigma_0 = 0; \quad \sigma_n = \sum_{i=1}^n \tau_i, \quad n = 1, 2, \dots$$

For any fixed  $t \geq 0$ ,  $\pi_t$  now denotes the number of consecutive sums of  $\tau_i$  which lie on  $[0, t]$ , i.e.

$$\pi_t = \sum_{n=0}^{\infty} \mathbb{1}_{\sigma_n \leq t}, \quad (3.1)$$

where  $\mathbb{1}_{\sigma_n \leq t}$  represents the indicator function of the event  $\{\sigma_n \leq t\}$ . The process  $\{\pi_t\}_{t \geq 0}$  we have just constructed is usually known in the literature as a *Poisson process* with intensity  $\lambda$  (see, for instance, [25]).

Now, let  $c : [0, T] \rightarrow \mathbb{R}^N$  be a continuous function. We can define the Poisson stochastic integral as

$$b_t := \int_0^t c(s) d\pi_s = \sum_{\sigma_k \leq t, k \geq 1} c(\sigma_k) = \sum_{0 < s \leq t} c(s)(\pi_s - \pi_{s-}), \quad t \in [0, T], \quad (3.2)$$

$b_0 = 0$  (as usual  $\pi_{s-}(\omega)$  denotes the left limit at  $s$ , for any  $\omega$ ,  $\mathbb{P}$ -a.s.). We now recall the following formula for the expectation of the stochastic integral:

$$\mathbb{E}\left[\int_0^t c(s) d\pi_s\right] = \lambda \int_0^t c(s) ds \in \mathbb{R}^N. \quad (3.3)$$

(cf. Lemma 2.1 in [15] for a direct proof; see also Theorem 16 in [25] and Theorem 5.3 in [15] for a more general formula involving stochastic integrals of predictable processes against the Poisson process). We also recall the following more general result.

**Lemma 3.1.** *Let  $\{\pi_t\}_{t \geq 0}$  be a Poisson Process of intensity  $\lambda$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let us consider a stochastic process  $(\xi_t)_{t \in [0, T]}$  with values in  $\mathbb{R}$  which has càdlàg paths ( $\mathbb{P}$ -a.s.) and is  $\mathcal{F}_t$ -adapted where  $\mathcal{F}_t$  is the completed  $\sigma$ -algebra generated by the random variables  $\pi_s$ ,  $0 \leq s \leq t$ . Suppose that  $\sup_{\omega \in \Omega, s \in [0, T]} |\xi_s(\omega)| < \infty$ . Then*

$$\mathbb{E} \int_0^t \xi_{s-} d\pi_s = \lambda \int_0^t \mathbb{E} \xi_s ds. \quad (3.4)$$

## 3.2 Proof of Theorem 2.2

According to the notations in Proposition 2.1, let  $v = PDE(Q, f)$  and  $w = PDE(Q + Q', f)$  be the unique solutions of equations (2.1) and (2.10), respectively.

The proof of Theorem 2.2 will be obtained adapting the method developed in [15] (see in particular Section 3, therein). Let  $e_1$  be the first unit vector in  $\mathbb{R}^N$ . We define

$$X_t = \int_0^t \sqrt{Q'(r)} e_1 d\pi_r$$

where  $\sqrt{Q'(t)}$  is the unique  $N \times N$  symmetric non-negative definite square root of  $Q'(t)$  and  $\{\pi_t\}_{t \geq 0}$  is a Poisson Process of intensity  $\lambda$  (cf. (3.2)). The parameter  $\lambda$  will be chosen appropriately later on.

Recall that the solution  $v$  to (2.1) is given by

$$v(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + z') \mu_{s,t}(dz')] \quad (3.5)$$

where  $\mu_{s,t}$  is the Gaussian law of the stochastic integral  $I_{s,t} := \sqrt{2} \int_s^t \sqrt{Q(r)} dW_r$  (see the proof of Proposition 2.1).

Let us fix  $\epsilon > 0$ . We notice that the shifted source  $f_\epsilon(t, z) := f(t, z - \epsilon X_t)$  (which also depends on  $\omega$ ; we have omitted to write such dependence on  $\omega$ ) is again in  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . This is the reason why we have considered such a function space for the source. It precisely allows to take into account the time discontinuities coming from the jumps of the Poisson process.

For any fixed  $\omega$  in  $\Omega$ , Proposition 2.1 readily gives that there exists a unique solution  $v_\epsilon = \text{PDE}(Q, f(t, z - \epsilon X_t))$ , depending on  $\epsilon$  and  $\omega$  as parameters, such that

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v_\epsilon(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t, z)|. \quad (3.6)$$

Moreover, thanks to the invariance for translations of the  $L^p$ -norms, it follows from (2.9) that

$$\|R(t)^* D^2 v_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})} \leq C_p \|f_\epsilon\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})} = C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})}. \quad (3.7)$$

By Equation (3.5), we know that  $v_\epsilon$  is given by

$$v_\epsilon(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z - \epsilon X_s + z') \mu_{s,t}(dz')].$$

For each  $z \in \mathbb{R}^N$ , the stochastic process  $(v_\epsilon(t, z))_{t \in [0, T]}$  has continuous paths ( $\mathbb{P}$ -a.s.) and it is  $\mathcal{F}_t$ -adapted where  $\mathcal{F}_t$  is the completed  $\sigma$ -algebra generated by the random variables  $\pi_s$ ,  $0 \leq s \leq t$ .

For fixed  $z \in \mathbb{R}^N$ ,  $\epsilon > 0$ , let us introduce the process  $(v_\epsilon(t, z + \epsilon X_t))_{t \in [0, T]}$  which is given by

$$v_\epsilon(t, z + \epsilon X_t) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + \epsilon X_t - \epsilon X_s + z') \mu_{s,t}(dz')].$$

It is not difficult to check that it is  $\mathcal{F}_t$ -adapted and it has càdlàg paths.

Applying (2.2) on each interval  $[\sigma_n, \sigma_{n+1} \wedge t)$ ,  $n \in \{0, \dots, \pi_t\}$  on which  $X_s$  is constant, one then derives that:

$$v_\epsilon(t, z + \epsilon X_t) = \int_0^t \left( \text{tr}(Q(s) D^2 v_\epsilon(s, z + \epsilon X_s)) + f(s, z) \right) ds + \int_0^t g_\epsilon(s, z) d\pi_s, \quad (3.8)$$



where  $g_\epsilon(s, z) = v_\epsilon(s, z + \epsilon\sqrt{Q'(s)} e_1 + \epsilon X_{s-}) - v_\epsilon(s, z + \epsilon X_{s-})$  is precisely the contribution associated with the jump times. It is clear that  $g_\epsilon(s, z) \neq 0$  if and only if  $\pi_s$  has a jump at time  $s$ . We then have by Lemma 3.1:

$$\mathbb{E} \int_0^t g_\epsilon(s, z) d\pi_s = \lambda \int_0^t \left( \bar{v}_\epsilon(s, z + \epsilon\sqrt{Q'(s)} e_1) - \bar{v}_\epsilon(s, z) \right) ds, \quad (3.9)$$

where  $\bar{v}_\epsilon(s, z) = \mathbb{E}[v_\epsilon(s, z + \epsilon X_s)]$ . Let us denote

$$l(t) := \sqrt{Q'(t)} e_1.$$

Taking the expectation on both sides of equation (3.8), we find out that  $\bar{v}_\epsilon$  is an integral solution of the following PDE:

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D^2 \bar{v}_\epsilon(t, z)) + \lambda (\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + f(t, z), \quad (3.10)$$

with zero initial condition. Remark that uniqueness of bounded continuous solutions to (3.10) follows by the maximum principle, arguing as in the proof of Lemma 2.2 in [15] (first one considers the case  $\lambda T \leq 1/4$  and then one iterates the procedure by steps of size  $1/(4\lambda)$ ).

Moreover by (3.7) we obtain (using also the Jensen inequality and the Fubini theorem)

$$\begin{aligned} \|R(t)^* D^2 \bar{v}_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p &= \int_{(0,T) \times \mathbb{R}^N} |R(t)^* D^2 \bar{v}_\epsilon(t, z) R(t)|^p \mathbf{m}(dt, dz) \\ &= \int_{(0,T) \times \mathbb{R}^N} |\mathbb{E}[R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)]|^p dz g(t) dt \\ &\leq \int_{[0,T] \times \mathbb{R}^N} \mathbb{E}[|R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)|^p] dz g(t) dt \\ &= \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)|^p dz g(t) dt \\ &= \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, \bar{z}) R(t)|^p d\bar{z} g(t) dt \\ &\leq C_p^p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p, \end{aligned}$$

using (3.7) for the last inequality ( $L^p$ -estimate for the PDE with random source). Choosing  $\lambda = \epsilon^{-2}$  we have from (3.10)

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D^2 \bar{v}_\epsilon(t, z)) + \epsilon^{-2} (\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + f(t, z), \quad (3.11)$$

with zero initial condition and moreover

$$\|R(t)^* D^2 \bar{v}_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p \leq C_p^p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p. \quad (3.12)$$

Now the idea is to apply again the same reasoning above to the equation (3.11) with respect to  $\bar{v}_\epsilon$ , using  $f(t, z + \epsilon X_t)$  again with  $\lambda = \epsilon^{-2}$ . We obtain first a solution  $p_\epsilon$  to (3.11) corresponding to  $f(t, z + \epsilon X_t)$  and then derive that

$$w_\epsilon(t, z) = \mathbb{E}[p_\epsilon(t, z - \epsilon X_t)]$$

is the unique bounded continuous (integral) solution  $w_\epsilon$  of the following problem:

$$\begin{cases} \partial_t w_\epsilon(t, z) = \text{tr}(Q(t)D^2 w_\epsilon(t, z)) \\ + \epsilon^{-2} (w_\epsilon(t, z + \epsilon l(t)) - 2w_\epsilon(t, z) + w_\epsilon(t, z - \epsilon l(t))) + f(t, z), \\ w_\epsilon(0, z) = 0. \end{cases} \quad (3.13)$$

The previous estimates still hold with  $w_\epsilon$  instead of  $v_\epsilon$ , i.e.,

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |w_\epsilon(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t, z)|; \quad (3.14)$$

$$\|R(t)^* D^2 w_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})}. \quad (3.15)$$

We would like now to let  $\epsilon$  goes to zero, possibly passing to a subsequence  $\epsilon_n \rightarrow 0$ , and prove that the associated limit  $w$  solves

$$\begin{cases} \partial_t w(t, z) = \text{tr}(Q(t)D^2 w(t, z)) + \langle D^2 w(t, z) \sqrt{Q'(t)} e_1, \sqrt{Q'(t)} e_1 \rangle + f(t, z), \\ w(0, z) = 0 \end{cases} \quad (3.16)$$

and estimates (3.14) and (3.15) hold with  $w_\epsilon$  replaced by  $w$ .

To do so we will proceed by compactness. Namely, we are going to prove that the family of solutions  $w_\epsilon$  solving (3.13), indexed by the parameter  $\epsilon$ , is equi-Lipschitz on any compact subset of  $[0, T] \times \mathbb{R}^N$  and the same holds for any derivative in space of  $w_\epsilon$ . Indeed, one can apply the finite difference operators with respect to  $z$  at any order in (3.13). We recall that for a *smooth* function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ , the first finite difference  $\delta_{h,i}\phi$ ,  $i \in \{1, \dots, N\}$  of step  $h > 0$  in the direction  $e_i$  ( $i^{\text{th}}$  basis vector) is given by

$$\delta_{h,i}\phi(z) = \frac{\phi(z + h e_i) - \phi(z)}{h}, \quad z \in \mathbb{R}^N.$$

For a given multi-index  $\gamma \in \mathbb{N}^N$ , the  $\gamma$ -th order finite difference operator  $\delta_{h,\gamma}$ , is then defined, for any  $h > 0$ , through composition. Namely,

$$\delta_{h,\gamma}\phi(z) = \delta_{h,1}^{\gamma_1} \delta_{h,2}^{\gamma_2} \dots \delta_{h,N}^{\gamma_N} \phi(z),$$

where  $\delta_{h,i}^{\gamma_i}$  denotes the  $\gamma_i$ -th times composition of  $\delta_{h,i}$  with itself.

Since any spatial derivative of  $f$  belongs to  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , using (3.14) we deduce first that any finite difference of any order of  $w_\epsilon$  is bounded. Consequently,  $w_\epsilon$  is infinitely differentiable in space with bounded derivatives on  $[0, T] \times \mathbb{R}^N$ . Equation (3.13), to be understood in its integral form similarly to (2.2), then gives that those derivatives are themselves Lipschitz continuous in time (uniformly in the space variable). This precisely gives the equi-Lipschitz on any compact subset of  $[0, T] \times \mathbb{R}^N$  of the family  $w_\epsilon$  and of any spatial derivative of  $w_\epsilon$ .

We can now apply the Arzelà-Ascoli theorem to  $w_\epsilon$  showing the existence of a

sub-sequence  $\{w_{\epsilon_n}\}_{n \in \mathbb{N}}$  which converges uniformly on any compact set to a function  $w: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ . Similarly, any derivative in space of  $w_{\epsilon_n}$  tends to the respective derivatives of  $w$ , uniformly on the compact sets.

Passing to the limit as  $n \rightarrow \infty$  along the sequence  $(\epsilon_n)_n$  in equation (3.13) (written in the integral form), we can then conclude that  $w$  solves (3.16).

Moreover, estimates (3.14) and (3.15) holds with  $w_\epsilon$  replaced by  $w$ . Iterating the previous argument in  $N$  steps we finally prove that the unique solution  $w$  to

$$\begin{cases} \partial_t w(t, z) = \text{tr}(Q(t)D^2w(t, z)) + \sum_{k=1}^N \langle D^2w(t, z) \sqrt{Q'(t)}e_k, \sqrt{Q'(t)}e_k \rangle + f(t, z), \\ w(0, z) = 0 \end{cases} \quad (3.17)$$

verifies estimates (3.14) and (3.15) with  $w_\epsilon$  replaced by  $w$ . The proof is complete.  $\square$

## 4 Additional stability results in anisotropic Sobolev space and Schauder estimates

In this section we extend the previous approach to derive the stability with respect to a second order perturbation of the OU operator in (1.7) under the Kalman condition  $[\mathbf{K}]$ . Here we consider also  $L^p$ -estimates involving the degenerate components of the OU operator and some associated Schauder estimates.

### 4.1 Anisotropic Sobolev spaces and maximal $L^p$ regularity

With the notations of Section 2.3 we write  $z \in \mathbb{R}^N$  as  $z = (x, y) = (x, y_1, \dots, y_k)$  with  $x \in \mathbb{R}^{d_0}$ ,  $y_i \in \mathbb{R}^{d_i}$ ,  $i \in \{1, \dots, k\}$ ,  $\sum_{i=1}^k d_i = d_1$ .

Given  $\beta$  in  $(0, 1)$  and  $i$  in  $[[1, k]]$ , we want to introduce the  $\beta$ -fractional Laplacian  $\Delta_{y_i}^\beta$  along the *component*  $y_i$ . To do so, we follow [10] by considering the orthogonal projection  $p_i: \mathbb{R}^N \rightarrow \mathbb{R}^{d_i}$  such that  $p_i(z) = p_i((x, y)) = y_i$  and denoting its adjoint by  $E_i: \mathbb{R}^{d_i} \rightarrow \mathbb{R}^N$ .

We can now define the  $\beta$ -fractional Laplacian  $\Delta_{y_i}^\beta$  as:

$$\Delta_{y_i}^\beta \phi(z) := \text{p.v.} \int_{\mathbb{R}^{d_i}} [\phi(z + E_i w) - \phi(z)] \frac{dw}{|w|^{d_i + 2\beta}}, \quad z \in \mathbb{R}^N,$$

for any sufficiently regular function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ .

Let  $p$  in  $(1, +\infty)$ , we recall that we have denoted by  $L^p((0, T) \times \mathbb{R}^N)$  the standard  $L^p$ -space with respect to the Lebesgue measure.

We can now define the appropriate anisotropic Sobolev space to state our results. For notational simplicity, let us denote

$$\alpha_i := \frac{1}{1 + 2i}. \quad (4.1)$$

Set now  $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ . The *homogeneous* space  $\dot{W}_\alpha^{2,p}([0, T] \times \mathbb{R}^N)$  is composed by all the functions  $\varphi: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  in  $L^p([0, T] \times \mathbb{R}^N)$  such that  $(t, z) \in [0, T] \times \mathbb{R}^N \mapsto \Delta_x \varphi(t, z) \in L^p([0, T] \times \mathbb{R}^N)$ , where  $\Delta_x \varphi$  is intended in distributional sense, and for any  $i$  in  $\llbracket 1, k \rrbracket$ ,  $\Delta_{y_i}^{\alpha_i} \varphi(t, z)$  is well defined for almost every  $(t, z)$  and

$$\Delta_{y_i}^{\alpha_i} \varphi(t, z) := \Delta_{y_i}^{\alpha_i} \varphi(t, \cdot)(z) \text{ belongs to } L^p([0, T] \times \mathbb{R}^N).$$

It is endowed with the natural *semi*-norm  $\|\varphi\|_{\dot{W}_\alpha^{2,p}}$  where

$$\|\varphi\|_{\dot{W}_\alpha^{2,p}}^p = \|\Delta_x \varphi\|_{L^p}^p + \sum_{i=1}^k \|\Delta_{y_i}^{\alpha_i} \varphi\|_{L^p}^p. \quad (4.2)$$

The thresholds in (4.1) might seem awkward at first sight. They actually correspond to the indexes needed to get stability of the harmonic functions associated with the principal part of (1.7), that is considering  $A_0$  consisting in the subdiagonal part of  $A$  only (i.e., considering (2.6) when the diagonal elements and the strictly upper diagonal elements are equal to zero) along an associated dilation operator. Namely, setting

$$L_0^{\text{ou}} f(z) = \text{Tr}(BD^2 f(z)) + \langle A_0 z, Df(z) \rangle, \quad z = (x, y) \in \mathbb{R}^{d_0+d_1} = \mathbb{R}^N, \quad (4.3)$$

so that  $A_0, B$  satisfy **[K]**, if  $(\partial_t - L_0^{\text{ou}})u(t, z) = 0$  then for all  $\lambda > 0$   $(\partial_t - \mathcal{L}_0^{\text{ou}})u(\delta_\lambda(t, z)) = 0$  where the dilation operator

$$\delta_\lambda(t, z) = (\lambda^{1/2}t, \lambda x, \lambda^{1/3}y_1, \dots, \lambda^{1/(1+2k)}y_k).$$

precisely exhibits the exponents in (4.1) for the degenerate components.

In [10], see also [5] and [21] where time inhomogeneous coefficients are considered as well, it has been proven that if  $A, B$  satisfy **[K]** and *the diagonal and the strictly upper diagonal elements of  $A$  in (2.6) are equal to zero* (i.e.,  $A = A_0$ ) then the following Sobolev estimates hold:

$$\|u\|_{\dot{W}_\alpha^{2,p}} \leq C_p \|f\|_{L^p}, \quad (4.4)$$

with  $C_p = C_p(\nu, A, d_0, d_1)$ , where again  $u$  is the unique bounded solution to the corresponding Cauchy problem (1.10). In particular we get also the maximal smoothing effects w.r.t. the degenerate directions. Note that the solution  $u$  to (1.1) verifies (4.4). The specific structure assumed on  $A$  is actually due to the fact that for such matrices there is an underlying homogeneous space structure which makes easier to establish maximal regularity estimates (see e.g. [6] in this general setting).

If  $A, B$  satisfy **[K]** with a general  $A$  as in (2.6), having non zero strictly upper diagonal entries (non zero entries in the diagonal should not create difficulties) we believe that the approach in [3] could extend to show that (4.4) still holds in this general setting. However such estimates have not been, up to our best knowledge, proven yet.

**$L^p$ -estimates for the degenerate directions of special OU operators.**

Setting, as in Section 1.1,  $u(t, z) = v(t, e^{tA}z)$  and since  $u$  solves (1.10) we have that  $v$  in turn solves (1.17). From the previous computations, setting

$B_I = \begin{pmatrix} I_{d_0, d_0} & 0_{d_0, d_1} \\ 0_{d_1, d_0} & 0_{d_1, d_1} \end{pmatrix}$  and considering  $A$  as in [10], with the diagonal and the strictly upper diagonal elements of  $A$  equal to zero in (2.6), we derive

$$\begin{aligned} \|D_x^2 u\|_{L^p((0, T) \times \mathbb{R}^N)} &= \|B_I e^{tA^*} D^2 v(t, e^{tA} \cdot) e^{tA} B_I\|_{L^p((0, T) \times \mathbb{R}^N)} \\ &\leq C_p \|\tilde{f}(t, e^{tA} \cdot)\|_{L^p((0, T) \times \mathbb{R}^N)}. \end{aligned}$$

On the other hand, for all  $i \in \{1, \dots, k\}$  and with  $\alpha_i$  as in (4.1),

$$\begin{aligned} &\|\Delta_{y_i}^{\alpha_i} u\|_{L^p((0, T) \times \mathbb{R}^N)}^p \\ &= \int_0^T dt \int_{\mathbb{R}^N} dz \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [u(t, z + E_i w) - u(t, z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p \\ &= \int_0^T dt \int_{\mathbb{R}^N} dz \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [v(t, e^{tA}(z + E_i w)) - v(t, e^{tA}z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p \\ &= \int_0^T dt \int_{\mathbb{R}^N} dz \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [v(t, z + e^{tA} E_i w) - v(t, z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p \\ &=: \|\Delta^{\alpha_i, i, A} v\|_{L^p((0, T) \times \mathbb{R}^N)}^p, \end{aligned}$$

using that  $\text{Tr}(A) = 0$ . Hence, setting

$$\begin{aligned} \|\Delta^{\alpha_0, 0, A} v\|_{L^p((0, T) \times \mathbb{R}^N)}^p &:= \|\text{Tr}(B_I e^{tA^*} D^2 v(t, e^{tA} \cdot) e^{tA} B_I)\|_{L^p((0, T) \times \mathbb{R}^N)}^p \\ &= \|\text{Tr}(B_I e^{tA^*} D^2 v e^{tA} B_I)\|_{L^p((0, T) \times \mathbb{R}^N)}^p, \end{aligned}$$

we get from the definition (4.2) that the estimate (4.4) rewrites in term of  $v$  as:

$$\|v\|_{\dot{W}_\alpha^{2, p, A}}^p := \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} v\|_{L^p((0, T) \times \mathbb{R}^N)}^p \leq \tilde{C}_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}^p \quad (4.5)$$

with  $\tilde{C}_p = C_p^p$ . We now want to prove that for  $w$  solving (1.20), namely

$$\begin{cases} \partial_t w(t, z) = \text{Tr}(e^{tA} B e^{tA^*} D^2 w(t, z)) + \text{Tr}(e^{tA} S(t) e^{tA^*} D^2 w(t, z)) \\ \quad + \tilde{f}(t, z), \quad (t, z) \in (0, T) \times \mathbb{R}^N, \\ w(0, z) = 0, \quad z \in \mathbb{R}^N, \end{cases}$$

it also holds that

$$\|w\|_{\dot{W}_\alpha^{2, p, A}}^p := \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} w\|_{L^p((0, T) \times \mathbb{R}^N)}^p \leq \tilde{C}_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}^p, \quad (4.6)$$

with the same constants  $\tilde{C}_p$  as in (4.5). This can be done through the previous perturbative approach of Section 3.2 employed to prove Theorem 2.2, which

actually gives the expected control for the second order derivatives contribution of the semi-norm  $\|\cdot\|_{\dot{W}_\alpha^{2,p,A}}$ .

For the other contributions and with the notations of Section 3.2, with  $Q'(s) = e^{sA}S(s)e^{sA^*}$  and with  $m$  which is the Lebesgue measure on  $[0, T] \times \mathbb{R}^N$  (indeed in the present case  $g(t) = \det(e^{-At}) = 1$ , for all  $t$ ) we would get

$$\begin{aligned}
\|\bar{v}_\epsilon\|_{\dot{W}_\alpha^{2,p,A}}^p &= \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} \bar{v}_\epsilon\|_{L^p((0,T) \times \mathbb{R}^N)}^p = \sum_{i=0}^k \int_{(0,T) \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} \bar{v}_\epsilon(t, z)|^p dz dt \\
&= \sum_{i=0}^k \int_{(0,T) \times \mathbb{R}^N} |\mathbb{E}[\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)]|^p dz dt \\
&\leq \sum_{i=0}^k \int_{[0,T] \times \mathbb{R}^N} \mathbb{E}[|\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)|^p] dz dt \\
&= \sum_{i=0}^k \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)|^p dz dt \\
&= \sum_{i=0}^k \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} v_\epsilon(t, \bar{z})|^p d\bar{z} dt \leq \tilde{C}_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)}^p,
\end{aligned}$$

using for the last inequality that  $v_\epsilon$  also satisfies (4.5) (similarly to what had been established in (3.7)).

The same previous procedure and the final compactness argument then yields (4.6). Setting eventually  $\tilde{u}(t, z) := w(t, e^{tA}z)$ , which is the unique integral solution (smooth in space) of

$$\begin{cases} \partial_t u_S(t, z) = L_t^{\text{ou}, S} u_S(t, z) + f(t, z), & (t, z) \in (0, T) \times \mathbb{R}^N, \\ u_S(0, z) = 0, & z \in \mathbb{R}^N, \end{cases}$$

where  $L_t^{\text{ou}, S}$  introduced in (1.13) is the Ornstein-Uhlenbeck operator perturbed at second order, we derive that

$$\|u_S\|_{\dot{W}_\alpha^{2,p}} \leq C_p \|f\|_{L^p}, \quad (4.7)$$

with  $C_p$  as in (4.4). We have thus extended the results of Theorem 1.1 with the anisotropic Sobolev semi-norm in (4.2). The estimate (4.4) is stable for a continuous, non-negative second order perturbation of the underlying degenerate Ornstein-Uhlenbeck operator.

## 4.2 Anisotropic Schauder estimates

Following Krylov [12], for some fixed  $\ell$  in  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  and  $\beta$  in  $(0, 1]$ , we introduce for a function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  the Zygmund-Hölder semi-norm as

$$[\phi]_{C^{\ell+\beta}} := \begin{cases} \sup_{|\vartheta|=\ell} \sup_{x \neq y} \frac{|D^\vartheta \phi(x) - D^\vartheta \phi(y)|}{|x-y|^\beta}, & \text{if } \beta \neq 1; \\ \sup_{|\vartheta|=\ell} \sup_{x \neq y} \frac{|D^\vartheta \phi(x) + D^\vartheta \phi(y) - 2D^\vartheta \phi(\frac{x+y}{2})|}{|x-y|}, & \text{if } \beta = 1 \end{cases}$$

(we are using usual multi-indices  $\vartheta$  for the partial derivatives). Consequently, the Zygmund-Hölder space  $C_b^{\ell+\beta}(\mathbb{R}^N)$  is the family of bounded functions  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\phi$  and its derivatives up to order  $\ell$  are continuous and the norm

$$\|\phi\|_{C_b^{\ell+\beta}} := \sum_{i=0}^{\ell} \sup_{|\vartheta|=i} \|D^{\vartheta}\phi\|_{\infty} + [\phi]_{C_b^{\ell+\beta}} \text{ is finite.}$$

We can now define the anisotropic Zygmund-Hölder spaces associated with the current setting and which again reflect the various scales already introduced in (4.1). Let  $\gamma \in (0, 3)$ , the space  $C_{b,d}^{\gamma}(\mathbb{R}^N)$  is the family of functions  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  such that for any  $i$  in  $\llbracket 0, k \rrbracket$  and any  $z_0$  in  $\mathbb{R}^N$ , the real function

$$w \in \mathbb{R}^{\mathfrak{d}_i} \rightarrow \phi(z_0 + E_i(w)) \text{ belongs to } C_b^{\gamma/(1+2i)}(\mathbb{R}^{\mathfrak{d}_i}),$$

with a norm bounded by a constant independent from  $z_0$ . In the above expression, we recall that the  $(E_i)_{i \in \{1, \dots, k\}}$  have been defined in the previous paragraph,  $\mathfrak{d}_0 = d_0$  and  $E_0$  is the embedding matrix from  $\mathbb{R}^{d_0}$  into  $\mathbb{R}^N$ . It is endowed with the norm

$$\|\phi\|_{C_{b,d}^{\gamma}} := \sup_{z_0 \in \mathbb{R}^N} \|\phi(z_0 + E_0(\cdot))\|_{C_b^{\gamma}(\mathbb{R}^{\mathfrak{d}_0})} + \sum_{i=1}^k \sup_{z_0 \in \mathbb{R}^N} \|\phi(z_0 + E_i(\cdot))\|_{C_b^{\gamma/(1+2i)}(\mathbb{R}^{\mathfrak{d}_i})}. \quad (4.8)$$

We denote by  $C_{b,d}^{\gamma}$  this function space because the regularity exponents reflect again the multi-scale features of the system; the norm could equivalently be defined through the corresponding spatial parabolic distance  $d$  defined as follows. For all  $z = (x, y), z' = (x', y') \in \mathbb{R}^N$ :

$$d(z, z') := |x - x'| + \sum_{i=1}^k |y_i - y'_i|^{\frac{1}{1+2i}},$$

where the exponents are again those who appeared in (4.1).

Let  $f \in B_b(0, T; C_0^{\infty}(\mathbb{R}^N))$ . Under **[K]**, by the results of Lunardi [17] it follows that the unique bounded solution of the Cauchy Problem (1.10) (written in integral form) verifies the following anisotropic Schauder estimates

$$\|u\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} \leq C_{\beta} \|f\|_{L^{\infty}((0,T), C_{b,d}^{\beta})}, \quad (4.9)$$

for some constant  $C_{\beta}$  independent from  $f$ , i.e.,

$$\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{C_{b,d}^{2+\beta}} \leq C_{\beta} \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C_{b,d}^{\beta}}. \quad (4.10)$$

We again set as in the previous paragraph  $u(t, z) = v(t, e^{tA}z)$

$$\begin{aligned} \|u\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} &= \|v(t, e^{tA}\cdot)\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} =: \|v\|_{L^{\infty}((0,T), C_{b,d,A}^{2+\beta})} \\ &\leq C_{\beta} \|f\|_{L^{\infty}((0,T), C_{b,d}^{\beta})} = C_{\beta} \|\tilde{f}(t, e^{tA}\cdot)\|_{L^{\infty}((0,T), C_{b,d}^{\beta})} =: C_{\beta} \|\tilde{f}\|_{L^{\infty}((0,T), C_{b,d,A}^{\beta})}, \end{aligned} \quad (4.11)$$

denoting  $\tilde{f}(t, z) := f(t, e^{-tA}z)$ . We again want to prove as in Section 1.1 that for  $w$  solving (1.20),

$$\|w\|_{L^\infty((0,T), C_{b,d,A}^{2+\beta})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T), C_{b,d,A}^\beta)} \quad (4.12)$$

with the same constant  $C_\beta$  as in (4.11). We proceed through the previous perturbative approach of Section 3.2. With the notations employed therein, we deduce that there exists a unique solution  $v_\epsilon = \text{PDE}(Q, \tilde{f}(t, z - \epsilon X_t))$ , depending also on  $\epsilon$  and  $\omega$  as parameters such that

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v_\epsilon(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |\tilde{f}(t, z)|. \quad (4.13)$$

By the translation invariance of the Hölder-norms, using also that  $X_t = e^{tA}e^{-tA}X_t$ , it is not difficult to prove that, for any  $\omega$ ,  $\mathbb{P}$ -a.s.,

$$\|\tilde{f}\|_{L^\infty((0,T), C_{b,d,A}^\beta)} = \|\tilde{f}(\cdot, \cdot - \epsilon X_\cdot)\|_{L^\infty((0,T), C_{b,d,A}^\beta)}. \quad (4.14)$$

Thus it also holds from (4.11)

$$\|v_\epsilon\|_{L^\infty((0,T), C_{b,d,A}^{2+\beta})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T), C_{b,d,A}^\beta)}. \quad (4.15)$$

Recalling now that  $\bar{v}_\epsilon(s, z) = \mathbb{E}[v_\epsilon(s, z + \epsilon X_s)]$  is an integral solution of

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D_z^2 \bar{v}_\epsilon(t, z)) + \lambda(\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + \tilde{f}(t, z),$$

with zero initial condition, we write that for  $i \in \{1, \dots, k\}$ ,  $w, w' \in \mathbb{R}^{d_i}$ ,  $(t, z_0) \in [0, T] \times \mathbb{R}^N$ ,

$$\begin{aligned} & |\bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w))) - \bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w')))| \\ & \leq \mathbb{E}[|v_\epsilon(t, e^{At}(z_0 + E_i(w)) + \epsilon e^{At}e^{-At}X_t) \\ & \quad - v_\epsilon(t, e^{At}(z_0 + E_i(w')) + \epsilon e^{At}e^{-At}X_t)|] \\ & \leq \mathbb{E}[|v_\epsilon(t, e^{At}(z_0 + E_i(\cdot)))|_{C_{1+2i}^{\frac{2+\beta}{1+2i}}}] |w - w'|_{\mathbb{R}^{d_i}}^{\frac{2+\beta}{1+2i}}. \end{aligned}$$

Hence,

$$|\bar{v}_\epsilon(t, e^{At}(z_0 + E_i(\cdot)))|_{C_{1+2i}^{\frac{2+\beta}{1+2i}}} \leq \mathbb{E}[|v_\epsilon(t, e^{At}(z_0 + E_i(\cdot)))|_{C_{1+2i}^{\frac{2+\beta}{1+2i}}}] .$$

We would get, similarly,

$$|D_x^2 \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|_{C^\beta} \leq \mathbb{E}[|D_x^2 v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|_{C^\beta}],$$

and for all  $k \in \{1, 2\}$ ,

$$\|D_x^k \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))\|_\infty \leq \mathbb{E}[\|D_x^k v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))\|_\infty].$$

Summing those contributions, we thus derive from (4.8), (4.11) that:

$$\|\bar{v}_\epsilon\|_{L^\infty((0,T), C_{b,d,A}^{2+\beta})} \leq \sup_{0 \leq t \leq T} \mathbb{E}[\|v_\epsilon(t, \cdot)\|_{C_{b,d,A}^{2+\beta}}] \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T), C_{b,d,A}^\beta)}, \quad (4.16)$$



using (4.15) for the last inequality. Now, continuing as in Section 3.2, using also a compactness argument, one would derive that (4.12) indeed holds.

Going backwards, setting  $\tilde{u}(t, z) := w(t, e^{tA}z)$ , we find that  $\tilde{u}$  is the unique (integral) solution  $u_S$  to (1.14); we finally derive that

$$\|u_S\|_{L^\infty((0,T),C_{b,d}^{2+\beta})} \leq C_\beta \|f\|_{L^\infty((0,T),C_{b,d}^\beta)}, \quad (4.17)$$

where  $C_\beta$  is the same constant as in (4.9). Estimate (4.17) provides the extension of Theorem 1.1 for the anisotropic Schauder estimates.

**Remark 4.1.** Let us mention that for the perturbative method to work, roughly speaking, few properties were actually needed on the underlying norm. Namely, we used the translation invariance and some kind of commutation between the norm (or a function of the norm in the  $L^p$ -case) and expectation. Hence, this approach could possibly be applied to a much wider class of estimates in other function spaces (like e.g. Besov spaces). This will concern further research.  $\square$

## Acknowledgment

For the first author, the work was supported by a public grant (2018 – 0024H) as part of the FMJH project. For the second author, the article was prepared within the framework of the HSE University Basic Research Program. The third author has been partially supported by GNAMPA of the Istituto Nazionale di Alta Matematica (INdAM).

## References

- [1] F. Bouchut. *Hypoelliptic regularity in kinetic equations*. J. Math. Pures Appl., 81 (2002), 1135–1159.
- [2] M. Bramanti, M. C. Cerutti, and M. Manfredini.  *$L^p$ -estimates for some ultraparabolic operators with discontinuous coefficients*. J. Math. Anal. Appl, 200–2 (1996), 332–354.
- [3] M. Bramanti, G. Cupini, E. Lanconelli, and E. Priola. *Global  $L^p$  estimates for degenerate Ornstein-Uhlenbeck operators*. Math. Z., 266(4) (2010), 789–816.
- [4] P-E. Chaudru de Raynal and S. Menozzi. *Regularization effects of a noise propagating through a chain of differential equations: an almost sharp result*. arXiv:1710.03620. To appear in Trans. American Math. Soc.: <https://doi.org/10.1090/tran/7947> (2017).
- [5] Z.Q. Chen, X. Zhang. *Propagation of regularity in  $L^p$ -spaces for Kolmogorov-type hypoelliptic operators*. J. Evol. Equ., 19(4) (2019), 1041-1069.

- [6] R. Coifman and G. Weiss. *Analyse Harmonique non-commutative sur certains espaces homogènes*, volume 242. Springer, Lecture Notes in Math., 1971.
- [7] G. Da Prato, A. Lunardi. *On the Ornstein-Uhlenbeck operator in spaces of continuous functions*. J. Funct. Anal. 131-1 (1995), 94-114.
- [8] F. Delarue and S. Menozzi. *Density estimates for a random noise propagating through a chain of differential equations*. J. Funct. Anal., 259-6 (2010), 1645–1681.
- [9] L. Hörmander. *Hypoelliptic second order differential equations*. Acta Math. 119 (1967), 147–171, 1967.
- [10] L. Huang, S. Menozzi, and E. Priola.  *$L^p$  estimates for degenerate non-local Kolmogorov operators*. J. Math. Pures Appl. (9) 121 (2019), 162–215.
- [11] A. N. Kolmogorov. *Zufällige Bewegungen (zur Theorie der Brownschen Bewegung)*. Ann. of Math. 2-35 (1934), 116–117.
- [12] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of Graduate Studies in Mathematics. AMS, 1996.
- [13] N. V. Krylov and E. Priola, *Elliptic and parabolic second-order PDEs with growing coefficients*. Comm. Partial Differential Equations, 35 (2010), 1-22.
- [14] N. V. Krylov, *Parabolic equations in  $L^p$ -spaces with mixed norms*. Algebra i Analiz. 14 (2002), 91-106 in Russian; English translation in St. Petersburg Math. J. 14 (2003), 603-614.
- [15] N. V. Krylov and E. Priola. *Poisson stochastic process and basic Schauder and Sobolev estimates in the theory of parabolic equations*. Arch. Ration. Mech. Anal., 225 (2017), 1089–1126.
- [16] E. Lanconelli and S. Polidoro. *On a class of hypoelliptic evolution operators*. Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), 29–63. Partial differential equations, II (Turin, 1993).
- [17] A. Lunardi. *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in  $\mathbb{R}^n$* . Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 24 (1997), 133–164.
- [18] L. Marino. *Schauder estimates for degenerate stable Kolmogorov equations*. Bull. Sci. Math., 162 (2020), 102885, 98 pp.
- [19] L. Marino. *Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators*. Journal of Mathematical Analysis and Applications, 500 (2021), 125168, 37 pp.

- [20] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt. *The domain of the Ornstein-Uhlenbeck operator on an  $L^p$ -space with invariant measure* Ann. Sc. Norm. Super. Pisa Cl. Sci., 1–2 (2002), 471–485.
- [21] S. Menozzi. *Martingale problems for some degenerate Kolmogorov equations.* Stochastic Process. Appl. , 128–3 (2018), 756–802.
- [22] S. Fornaro, G. Metafune, D. Pallara, R. Schnaubelt,  *$L^p$ -spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators* J. Funct. Anal., 280–2 (2021), 108807, 22pp.
- [23] E. Priola,  *$L^p$ -parabolic regularity and non-degenerate Ornstein-Uhlenbeck type operators,* Geometric Methods in PDEs, Citti G. et al. (eds.), Springer INdAM Series, vol. 13 (2015), 121-139.
- [24] E. Priola. *On weak uniqueness for some degenerate SDEs by global  $L^p$  estimates.* Potential Anal. 42 (2015), 247–281.
- [25] P. E. Protter. *Stochastic Integration and Differential Equations.* Springer, 2004.
- [26] J. Zabczyk. *Mathematical control theory: an introduction.* Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.