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# Poisson process and sharp constants in $L^p$ and Schauder estimates for a class of degenerate Kolmogorov operators

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## Abstract

We consider a possibly degenerate Kolmogorov-Ornstein-Uhlenbeck operator of the form  $L = \text{Tr}(BD^2) + \langle Az, D \rangle$ , where  $A, B$  are  $N \times N$  matrices,  $z \in \mathbb{R}^N$ ,  $N \geq 1$ , which satisfy the Kalman condition which is equivalent to the hypoellipticity condition. We prove the following stability result: the Schauder and Sobolev estimates associated with the corresponding parabolic Cauchy problem remain valid, with the same constant, for the parabolic Cauchy problem associated with a second order perturbation of  $L$ , namely for  $L + \text{Tr}(S(t)D^2)$  where  $S(t)$  is a *non-negative*  $N \times N$  matrix depending continuously on  $t \geq 0$ . Our approach relies on the perturbative technique based on the Poisson process introduced in [15].

INDAM

**Keywords:** Degenerate Ornstein-Uhlenbeck operators, multidimensional parabolic equations,  $L^p$  and Sobolev-space estimates, Schauder estimates, Poisson process.

**MSC:** Primary: 35K10, 35K15.

# 1 Introduction

Let us first consider the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t, x, y) = \Delta_x u(t, x, y) + x \cdot \nabla_y u(t, x, y) + f(t, x, y), \\ u(0, x, y) = 0, \end{cases} \quad (1.1)$$

where  $(t, x, y)$  is in  $(0, +\infty) \times \mathbb{R}^{d_0+d_1}$  for two integers  $d_0, d_1 \geq 1$ . The underlying differential operator

$$L^K = \Delta_x + x \cdot \nabla_y = \sum_{i=1}^{d_0} \partial_{x_i x_i}^2 + \sum_{i=1}^{d_0} \sum_{j=1}^{d_1} x_i \partial_{y_j}$$

is the so-called Kolmogorov operator whose fundamental solution was derived in the seminal paper [11]. This particular operator was also mentioned by Hörmander as the starting point for his theory of hypoelliptic operators [9].

Let us write  $z = (x, y) \in \mathbb{R}^{d_0} \times \mathbb{R}^{d_1} = \mathbb{R}^N$  and by  $\partial_{z_j}$  and  $\partial_{z_i z_j}^2$  we denote respectively the first and the second partial derivatives with  $i, j = 1, \dots, N$ .

We are interested in studying the influence of a second order perturbation on Equation (1.1). Precisely, for a time-dependent matrix  $\{S(t) : t \geq 0\}$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$  such that  $t \mapsto S(t)$  is continuous and  $S(t)$  is *symmetric* and *non-negative* for any fixed  $t$ , we consider the *perturbed* Cauchy problem:

$$\begin{cases} \partial_t u_S(t, z) = L^K u_S(t, z) + \sum_{i,j=1}^N S_{ij}(t) \partial_{z_i z_j}^2 u_S(t, z) + f(t, z) =: L^{K,S} u_S(t, z) + f(t, z), \\ u_S(0, z) = 0. \end{cases} \quad (1.2)$$

In particular, we will show that Sobolev (and Schauder) estimates which hold for solutions  $u$  of the Cauchy Problem (1.1) are also true, with the same constants, for solutions  $u_S$  to (1.2). Clearly, the operator  $L^{K,S}$  can be seen as a perturbation of  $L^K$  involving second order partial derivatives with continuous time-dependent coefficients.

For now, let us explain our main results for a special form of Equation (1.1) in the case of  $L^p$ -estimates (or Sobolev estimates). For a statement of our results in the whole generality, we instead refer to Section 2.

For a fixed final time  $T > 0$  and a source  $f$  in  $C_0^\infty((0, T) \times \mathbb{R}^N)$ , it is known from the work of Bramanti *et al.* [2], Theorem 3.1 (see also Section 2.3 below), that Equation (1.1) admits a unique classical bounded solution  $u$  which satisfies for  $p$  in  $(1, +\infty)$  the following estimates:

$$\|\Delta_x u\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)} = C_p \|\partial_t u - L^K u\|_{L^p((0,T) \times \mathbb{R}^N)}. \quad (1.3)$$

Note that  $C_p = C_p(T, d_0, d_1) > 0$ . We will actually manage to prove that the unique classical bounded solution  $u_S$  to (1.2) satisfies the estimate

$$\|\Delta_x u_S\|_{L^p((0,T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)} = C_p \|\partial_t u - L^{K,S} u\|_{L^p((0,T) \times \mathbb{R}^N)}, \quad (1.4)$$

with the **same** previous constant  $C_p$  as in (1.3). This result seems to be new even in dimension  $N = 2$  and even if we only consider  $S(t) = S$ ,  $t \geq 0$ , where  $S$  is a  $2 \times 2$  *symmetric non-negative definite* matrix.

For a uniformly elliptic second order perturbation  $S(t) = S$ ,  $t \geq 0$ , where  $S$  is *positive definite*, we could also have appealed to [3] to derive estimates like in (1.4). For related estimates in the uniformly elliptic case, see also Section 4 in Metafune *et al.* [20]. However, note that from [3] and [20] we could only deduce that the constant  $C_p$  depends on the ellipticity constant of the perturbation (this is the first eigenvalue  $\lambda_S$  of  $S$  if  $0 < \lambda_S \leq 1$ ) and on the maximum eigenvalue of  $S$  (on this respect, see also [14] and [23]).

The remarkable point in (1.4) is that the  $L^p$ -estimates are stable under second order perturbations, which can be possibly degenerate. Namely, the fact that  $S(t)$  might be degenerate for some tin  $(0, T)$ , or even in some non-empty sub-intervals of  $(0, T)$ , does not affect the estimates in (1.4).

To prove (1.4), we combine the results of [2] with a probabilistic *perturbative* approach based on the Poisson process inspired by [15]. There, it was established in particular that the  $L^p$ -estimates for non-degenerate parabolic heat equations with space homogeneous coefficients are valid with constants that are independent of the dimension.

**Remark 1.1.** Importantly, the approach of [15] turns out to be sufficiently robust to handle the estimates in the degenerate directions as well. We recall that the associated maximal  $L^p$  regularity was studied e.g. in [1], [10] or [5]. Fixed  $p$  in  $(1, +\infty)$ , there exists  $\tilde{C}_p > 0$  such that for  $f$  in  $C_0^\infty((0, T) \times \mathbb{R}^N)$  the unique classical bounded solution  $u$  of (1.1) verifies

$$\|(\Delta_y)^{\frac{1}{3}}u\|_{L^p((0,T)\times\mathbb{R}^N)} \leq \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)} = \tilde{C}_p \|\partial_t u - L^K u\|_{L^p((0,T)\times\mathbb{R}^N)}, \quad (1.5)$$

where  $(\Delta_y)^{\frac{1}{3}}$  denotes the fractional Laplacian with respect to the degenerate variables  $y$  in  $\mathbb{R}^{d_1}$ . It turns out that this estimate is also stable for the previously described second order perturbation. Namely, for  $u_S$  solving (1.2),

$$\|(\Delta_y)^{\frac{1}{3}}u_S\|_{L^p((0,T)\times\mathbb{R}^d)} \leq \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^d)} = \tilde{C}_p \|\partial_t u - L^{K,S}u\|_{L^p((0,T)\times\mathbb{R}^N)}, \quad (1.6)$$

where again  $\tilde{C}_p$  is the same as in (1.5). □

**Remark 1.2.** The same type of stability results will also hold for the corresponding Schauder estimates, first established in the framework of anisotropic Hölder spaces for the solution of (1.1) by Lunardi [17].

We point out that our results in Section 3 could also **possibly** be obtained by using the general theorems of Section 4 in [15]. This section in [15] introduces a more general probabilistic approach and provides unexpected regularity results. However checking in

our case all the assumptions given in that section is quite involved. On the other hand, we provide self-contained proofs inspired by Sections 2 and 3 of [15].

It remains a challenging open problem to have a purely analytic proof of the above regularity results.

**$L^p$ -estimates for degenerate Ornstein-Uhlenbeck operators.** Let us now describe the more general framework we are going to consider here. Fixed  $\mathbb{R}^N = \mathbb{R}^{d_0} \times \mathbb{R}^{d_1}$  where  $d_0, d_1$  are two *non-negative* integers such that  $d_0 + d_1 = N$  and  $d_0 \geq 1$ . let us introduce the non-negative, symmetric matrix  $B$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$  given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where  $B_0$  is a symmetric, positive definite matrix in  $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$ .

We will use, as underlying *proxy* operators, the family of degenerate Ornstein-Uhlenbeck generators of the form

$$L^{\text{ou}} f(z) = \text{Tr}(BD^2 f(z)) + \langle Az, Df(z) \rangle, \quad z = (x, y) \in \mathbb{R}^{d_0+d_1} = \mathbb{R}^N, \quad (1.7)$$

for a matrix  $A$  in  $\mathbb{R}^N \otimes \mathbb{R}^N$ , where  $\langle \cdot, \cdot \rangle$  denotes the usual inner product in  $\mathbb{R}^N$ .

Moreover, we assume the Kalman condition:

[**K**] There exists a non-negative integer  $k$ , such that

$$\text{Rank}[B, AB, \dots, A^k B] = N, \quad (1.8)$$

where  $[B, AB, \dots, A^k B]$  is the  $\mathbb{R}^N \otimes \mathbb{R}^{Nk}$  matrix whose blocks are  $B, AB, \dots, A^k B$ .

From the non-degeneracy of  $B_0$ , the above condition amounts to say that the vectors

$$\{e_1, \dots, e_{d_0}, Ae_1, \dots, Ae_{d_0}, \dots, A^k e_1, \dots, A^k e_{d_0}\} \text{ generate } \mathbb{R}^N, \quad (1.9)$$

where  $\{e_i : i \in \{1, \dots, d_0\}\}$  are the first  $d_0$  vectors of the canonical basis for  $\mathbb{R}^N$ .

In the linear framework, assumption [**K**] (which also often appears in control theory; see e.g. [26]) is equivalent to the Hörmander condition on the commutators (c.f. [9]) ensuring the hypoellipticity of the operator  $\partial_t - L^{\text{ou}}$ . In particular, it implies the existence and the smoothness of a distributional solution for the following equation:

$$\begin{cases} \partial_t u(t, z) = L^{\text{ou}} u(t, z) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (1.10)$$

where  $f$  is a function in  $C_0^\infty((0, T) \times \mathbb{R}^N)$ .

Similarly to [15], we will prove below the existence and the uniqueness of bounded regular solutions to (1.10) assuming that the source  $f$  belongs to the space  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , which contains  $C_0^\infty((0, T) \times \mathbb{R}^N)$ , and that can be roughly described as the family of functions which are bounded measurable in time and compactly supported in space uniformly in time (see Section 1.2 for a precise definition). Equation (1.10) will be understood in an integral form.

By Theorem 3 in [3] and exploiting some explicit properties of the underlying heat kernel (see Section 2.3 below), it can be derived that for any fixed  $p$  in  $(1, +\infty)$ , there exists  $C_p = C_p(B_0, d_0, d_1, T)$  such that

$$\|D_x^2 u\|_{L^p((0, T) \times \mathbb{R}^N)} \leq C_p \|u_t - L^{\text{ou}} u\|_{L^p((0, T) \times \mathbb{R}^N)} = C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}, \quad (1.11)$$

where for any  $z$  in  $\mathbb{R}^N$ ,  $D_x^2 u(z)$  stands for the Hessian matrix in  $\mathbb{R}^{d_0} \otimes \mathbb{R}^{d_0}$  with respect to the variable  $x$ . We note, in particular, that (1.11) can be rewritten, up to a modification of  $C_p$  involving the ellipticity constant of  $B_0$ , in the following, equivalent way:

$$\begin{aligned} \|B^{1/2} D^2 u B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)} &= \|B_0^{1/2} D_x^2 u B_0^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)} \\ &\leq C_p \|u_t - L^{\text{ou}} u\|_{L^p((0, T) \times \mathbb{R}^N)} \\ &= C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}, \end{aligned} \quad (1.12)$$

where  $D^2 u = D_z^2 u$  represents instead the full Hessian matrix in  $\mathbb{R}^N \otimes \mathbb{R}^N$  with respect to  $z$ .

Fixed a continuous mapping  $t \mapsto S(t)$  such that  $S(t)$  is a *symmetric* and *non-negative* matrix in  $\mathbb{R}^N \otimes \mathbb{R}^N$ , we consider again the following perturbation of  $L^{\text{ou}}$  :

$$\begin{aligned} L_t^{\text{ou}, S} f(z) &:= \text{Tr}(B D^2 f(z)) + \text{Tr}(S(t) D^2 f(z)) + \langle A z, D f(z) \rangle \\ &= L^{\text{ou}} f(z) + \text{Tr}(S(t) D^2 f(z)), \end{aligned} \quad (1.13)$$

where  $z = (x, y)$  is in  $\mathbb{R}^{d_0+d_1} = \mathbb{R}^N$ . For the solution  $u_s$  of the related Cauchy problem

$$\begin{cases} \partial_t u_S(t, z) = L_t^{\text{ou}, S} u_S(t, z) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u_S(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (1.14)$$

we will prove the following main theorem:

**Theorem 1.1.** *Let us consider (1.14) with  $f \in B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Then, there exists a unique solution  $u_S$  of Cauchy Problem (1.14) which verifies, with the same constant  $C_p$  as in (1.12),*

$$\|B^{1/2} D^2 u_S B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)} \leq C_p \|\partial_t u_S - L_t^{\text{ou}, S} u_S\|_{L^p((0, T) \times \mathbb{R}^N)} = C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}. \quad (1.15)$$

For time-homogeneous non-negative matrices  $S$ , the corresponding elliptic estimate could also be derived from (1.15) following the approach of Corollary 3.5 in [15].

Independently from the constant preservation, we also emphasize that the  $L^p$  estimates in (1.15) for the perturbed operator seem, to the best of our knowledge, to be new and have some interest by their own. We can also refer to the recent work by Fornaro *et al.* [22] for a full description of the spectrum of degenerate OU operators in  $L^p$  spaces.

Let us eventually mention that such stability results could turn out to be useful to investigate the well posedness of some related stochastic differential equations through the corresponding martingale problem.

We could actually derive more general estimates, possibly involving the degenerate directions as well, dependingly on the structure of  $A$ . Some results in that direction are gathered in Section 4. Anyhow, to illustrate our approach we now briefly present the various steps to derive (1.15).

### 1.1 Strategy of the proof for Estimate (1.15).

Fixed a classical bounded solution  $u$  to Cauchy Problem (1.10), let us introduce  $v(t, z) := u(t, e^{-tA}z)$ . This well-known (cf. [7]) transformation precisely allows to get rid of the drift term in the PDE satisfied by  $v$ . Indeed, we have that  $u(t, z) = v(t, e^{tA}z)$  and since  $u$  solves (1.10), it holds for any  $(t, z)$  in  $(0, T) \times \mathbb{R}^N$ , that:

$$\begin{aligned} f(t, z) &= \partial_t u(t, z) - L^{\text{ou}} u(t, z) \\ &= v_t(t, e^{tA}z) + \langle Dv(t, e^{tA}z), Ae^{tA}z \rangle - \text{Tr}(e^{tA} B e^{tA*} D^2 v(t, e^{tA}z)) \\ &\quad - \langle Dv(t, e^{tA}z), Ae^{tA}z \rangle \\ &= v_t(t, e^{tA}z) - \text{Tr}(e^{tA} B e^{tA*} D^2 v(t, e^{tA}z)). \end{aligned} \tag{1.16}$$

Denoting  $\tilde{f}(t, z) := f(t, e^{-tA}z)$ , It now follows that  $v$  satisfies the PDE:

$$\begin{cases} \partial_t v(t, z) = \text{Tr}(e^{tA} B e^{tA*} D^2 v(t, z)) + \tilde{f}(t, z) & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{1.17}$$

In terms of the function  $v$ , the estimates in (1.12) rewrites as:

$$\|B^{1/2} e^{-tA*} D^2 v(t, e^{tA}\cdot) e^{tA} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)} \leq C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0, T) \times \mathbb{R}^N)}, \tag{1.18}$$

where we used the notation  $\|B^{1/2} e^{-tA*} D^2 v(t, e^{tA}\cdot) e^{tA} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)}$  to stress the dependence on  $t$  instead of the more precise formulation  $\|B^{1/2} e^{-\cdot A*} D^2 v(\cdot, e^{-\cdot A}\cdot) e^{-\cdot A} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)}$ .

By changing variable in the integrals, Control (1.18) is equivalent to

$$\|B^{1/2} e^{tA*} D^2 v(t, \cdot) e^{tA} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N, m)} \leq C_p \|\tilde{f}\|_{L^p((0, T) \times \mathbb{R}^N, m)}, \tag{1.19}$$

where  $L^p((0, T) \times \mathbb{R}^N, m)$  denotes the  $L^p$  norms w.r.t. the measure  $m(dt, dx) = \det(e^{-At}) dt dx$ .

Considering now the following, more general equation

$$\begin{cases} \partial_t w(t, z) + \text{Tr}(e^{tA} B e^{tA^*} D^2 w(t, z)) + \text{Tr}(e^{tA} S(t) e^{tA^*} D^2 w(t, z)) = \tilde{f}(t, z); \\ w(0, z) = 0, \end{cases} \quad (1.20)$$

we can establish the well-posedness of the Cauchy problem (1.20), exploiting, for instance, probabilistic arguments and the underlying Gaussian process.

The crucial step consists in adapting some arguments from [15] to derive that the same  $L^p$ -estimates in (1.19) still hold for  $w$ , independently from the non-negative definite, symmetric matrix  $S$ . Precisely,

$$\|B^{1/2} e^{tA^*} D^2 w(t, \cdot) e^{tA} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N, m)} \leq C_p \|\tilde{f}(t, \cdot)\|_{L^p((0, T) \times \mathbb{R}^N, m)}, \quad (1.21)$$

with the *same* constant  $C_p$  appearing in (1.19).

The last step then consists in coming back to the Ornstein-Uhlenbeck operators framework. Namely, we introduce  $\tilde{u}(t, z) := w(t, e^{tA} z)$  which solves, by definition, the following equation:

$$\begin{cases} \partial_t \tilde{u}(t, z) + L_t^{\text{ou}, S} \tilde{u}(t, z) = f(t, z), & (t, z) \in (0, T) \times \mathbb{R}^N, \\ \tilde{u}(0, z) = 0, & z \in \mathbb{R}^N. \end{cases}$$

Noticing that  $D^2 w(t, \cdot) = D^2[\tilde{u}(t, e^{-tA} \cdot)] = e^{-tA^*} D^2 \tilde{u}(t, e^{-tA} \cdot) e^{-tA}$  we thus get from (1.21) that the following estimates hold:

$$\|B^{1/2} D^2 \tilde{u} B^{1/2}\|_{L^p((0, T) \times \mathbb{R}^N)} \leq C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N)}. \quad (1.22)$$

Through the previous steps we have then constructed a solution  $\tilde{u}$  of Cauchy Problem (1.14) which indeed satisfies the estimates in (1.15) with the same  $C_p$ , associated with the *unperturbed* or *proxy* operator. The maximum principle will eventually provide uniqueness for the solution  $\tilde{u}$ .

**Remark 1.3.** i) We point out that we could also consider more general time-dependent Ornstein-Uhlenbeck operators like:

$$M = \text{Tr}(B(t) D^2 f(z)) + \langle Az, Df(z) \rangle.$$

Arguing as before starting from  $L^p$ -estimates (or Schauder estimates) for  $M$  we can derive the same  $L^p$ -estimates (or Schauder estimates) for a perturbation of  $M$  like (1.13).

ii) Moreover, we could extend the  $L^p$ -estimates (or the Schauder estimates) related to  $L^{\text{ou}}$  to more general operators like

$$L_t^{\text{ou}, S} f(z) + \langle b(t), Df(z) \rangle$$



where  $b : \mathbb{R}_+ \rightarrow \mathbb{R}^N$  is continuous and even add a further possibly degenerate non-local perturbation (cf. Section 7 of [15]). For the sake of simplicity in the sequel we will consider  $b(t) = 0$  and we will not deal with non-local perturbations of  $L_t^{\text{ou},S}$ .

**Organization of Paper.** The article is organized as follows. At the end of the current section, we first give some useful notations. In Section 2 we will then focus on driftless second order Cauchy problems associated with a non-negative, possibly degenerate, diffusion matrix. In particular, we will establish through the probabilistic perturbation approach of [15] that if some  $L^p$ -estimates hold for a particular diffusion matrix so does it, with the same associated constant, for a non-negative perturbation of the diffusion matrix as explained before (see Section 3). Finally by the argument of Section 1.1 we will obtain (1.22).

## 1.2 Definition of solution and useful notations

Let us consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t, z) = \text{tr}(Q(t)D^2v(t, z)) + \langle b(t, z), Dv(t, z) \rangle + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0, & \text{on } \mathbb{R}^N; \end{cases} \quad (1.23)$$

where  $Q : [0, T] \rightarrow \mathbb{R}^N \otimes \mathbb{R}^N$  is a *continuous time-dependent, symmetric non-negative matrix* and  $b : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}^N$  is a *continuous function* such that  $|b(t, z)| \leq K_T(1 + |z|)$ ,  $(t, z) \in [0, T] \times \mathbb{R}^N$ , for some constant  $K_T > 0$ .

The function  $f$  belongs to  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , the space of all Borel bounded functions  $\phi : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\phi(t, \cdot)$  is smooth and compactly supported for any  $t$  in  $[0, T]$ , for any  $n$  in  $\mathbb{N}$  the  $C^n(\mathbb{R}^N)$ -norms of  $\phi(t, \cdot)$  are bounded in time and the supports of the functions  $\phi(t, \cdot)$  are contained in the same ball. Moreover, we require that, for any  $z \in \mathbb{R}^N$ , the mapping:

$$t \mapsto \phi(t, z) \quad (1.24)$$

is a *piece-wise continuous* function on  $[0, T]$ , i.e. it is continuous except for a finite number of points.

**Remark 1.4.** Note that to perform the technique used in [15] and based on the Poisson process we need to consider equations like (1.23) with a source  $f$  which is possibly discontinuous in time (cf. the proof in Section 2 of [15] and Section 3.2 below).

We interpret Cauchy Problem (1.23) in an *integral* form:

$$v(t, z) = \int_0^t \left( f(s, z) + \text{Tr}(Q(s)D^2v(s, z)) + \langle b(s, z), Dv(s, z) \rangle \right) ds. \quad (1.25)$$

In particular, we say that a continuous and bounded function  $v : [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  is a solution to Equation (1.23) if  $v(t, \cdot)$  is in  $C^2(\mathbb{R}^N)$ , for any  $t \in [0, T]$ , and (1.25) holds as well.

We finally note that, for any  $z \in \mathbb{R}^N$ , the function  $t \mapsto v(t, z)$  is a  $C^1$ -piece-wise function on  $[0, T]$ .

By Theorem 4.1 in [13] we know that if a solution  $v$  exists then it is unique and the following maximum principle holds:

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t, z)|. \quad (1.26)$$

## 2 Estimates for driftless second order operators and related perturbation

Throughout this section, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t, z) = \text{Tr}(Q(t)D^2v(t, z)) + f(t, z) & \text{on } (0, T) \times \mathbb{R}^N; \\ v(0, z) = 0 & \text{on } \mathbb{R}^N, \end{cases} \quad (2.1)$$

which can be seen as a special case of (1.23) when  $b = 0$ . Moreover, we assume that  $Q$  is not identically zero.

### 2.1 Well-posedness

**Proposition 2.1** (Well-posedness in integral form for the driftless Cauchy problem). *Let  $f$  be in  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Then, there exists a unique solution  $v$  to Cauchy problem (2.1) in an integral sense, i.e. it solves for  $(t, z) \in [0, T] \times \mathbb{R}^N$ :*

$$v(t, z) = \int_0^t \left( f(s, z) + \text{Tr}(Q(s)D^2v(s, z)) \right) ds. \quad (2.2)$$

We will denote in short  $v = PDE(Q, f)$ .

*Proof.* By the maximum principle (cf. Equation (1.26)) uniqueness holds for Cauchy Problem (2.1). We can then focus on proving the existence of a solution.

Let us introduce now

$$v(t, z) := \int_0^t \mathbb{E}[f(s, z + I_{s,t})] ds$$

with the following notation:  $I_{s,u} := \sqrt{2} \int_s^u Q(v)^{1/2} dW_v$ , where  $W$  is an  $N$ -dimensional Brownian motion on some probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  and  $Q(v)^{1/2}$  stands for a square root of  $Q(v)$ , i.e.  $Q(v) = Q(v)^{1/2}(Q(v)^{1/2})^*$ .

Applying the Itô formula in space to  $f(s, z + I_{s,u})_{u \in [s,t]}$ , we get that

$$\mathbb{E}f(s, z + I_{s,t}) = f(s, z) + \mathbb{E}\left[\int_s^t \text{Tr}(Q(u)D^2f(s, z + I_{s,u}))du\right].$$

Hence,

$$v(t, z) = \int_0^t \left( f(s, z) + \mathbb{E}\left[\int_s^t \text{Tr}(Q(u)D^2f(s, z + I_{s,u}))du\right] \right) ds,$$

from which it readily follows that

$$\begin{aligned} \partial_t v(t, z) &= f(t, z) + \int_0^t \mathbb{E}[\text{Tr}(Q(t)D^2f(s, z + I_{s,t}))]ds \\ &= f(t, z) + \text{Tr}\left(Q(t)D^2 \int_0^t \mathbb{E}[f(s, z + I_{s,t})]ds\right) \\ &= f(t, z) + \text{Tr}\left(Q(t)D^2v(t, z)\right). \end{aligned}$$

for almost every  $t \in [0, T]$  and any  $z \in \mathbb{R}^N$ .  $\square$

## 2.2 Relation to the Ornstein-Uhlenbeck dynamics

If now in particular,  $Q(t)$  has the particular form  $Q(t) = e^{tA} B e^{tA^*}$  (cf. Equation (1.17)), we introduce

$$u(t, z) := v(t, e^{tA}z),$$

where  $v$  is the solution to (2.2) (see Proposition 2.1). Since we can differentiate with respect to  $t$  the function  $u(\cdot, z)$  for a.e.  $t \in [0, T]$ , we can perform computations similar to (1.16) and get that  $u(t, z)$  solves in integral form:

$$\begin{cases} \partial_t u(t, z) = L^{\text{ou}} u(t, z) + \bar{f}(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ u(0, z) = 0, & \text{on } \mathbb{R}^N; \end{cases} \quad (2.3)$$

with  $L^{\text{ou}}$  as in (1.7),  $\bar{f}(t, z) = f(t, e^{tA}z)$ . Precisely, for all  $(t, z) \in [0, T] \times \mathbb{R}^N$ ,

$$u(t, z) = \int_0^t \left( \bar{f}(s, z) + L^{\text{ou}}v(s, z) \right) ds. \quad (2.4)$$

We have that  $u$  is a solution to (2.3).

Let us also point out that the well-posedness of (2.3) could also have been obtained directly from Gaussian type calculations, similar to those in the proof of Proposition 2.1, introducing  $u^{\text{ou}}(t, z) := \int_0^t \mathbb{E}[\bar{f}(s, e^{(t-s)A}z + I_{s,t}^{\text{ou}})]ds$  where  $I_{s,t}^{\text{ou}} := \sqrt{2} \int_s^t e^{(u-s)A} B dW_u$ .

### 2.3 About the $L^p$ estimate (1.11) for the OU operator

The aim of this section is to fully justify the estimates in (1.11). This is a consequence of the previous probabilistic representation and Theorem 3 in [3]. For  $u$  solving (1.10) it holds that for all  $(t, z) \in [0, T] \times \mathbb{R}^N$ ,

$$u(t, z) = \int_0^t \mathbb{E}[f(s, e^{A(t-s)}z + I_{s,t}^{\text{ou}})]ds = \int_0^t \int_{\mathbb{R}^N} f(s, z')p^{\text{ou}}(t-s, z, z')dz'ds, \quad (2.5)$$

where for  $v > 0$ ,  $p^{\text{ou}}(v, z, \cdot)$  stands for the density at time  $v$  of the stochastic process

$$X_u^{\text{ou}} := e^{Au}z + \sqrt{2} \int_0^u e^{A(u-w)}BdW_w = z + \int_0^u AX_w^{\text{ou}}dw + BW_u, \quad u \geq 0.$$

We recall from [16] that assumption **[K]** is equivalent to the fact that there exists  $k \in \mathbb{N}$  and positive integers  $(\mathfrak{d}_i)_{i \in \{1, \dots, k\}}$  s.t.  $\sum_{i=1}^k \mathfrak{d}_i = d_1$  and for all  $i \in \{1, \dots, k\}$ , setting  $\mathfrak{d}_0 = d_0$  and  $\sum_{m=0}^{-1} = 0$ , the matrixes

$$\mathcal{A}^i := (A_{j,\ell})_{(j,\ell) \in \{\sum_{m=0}^{i-1} \mathfrak{d}_{m+1}, \dots, \sum_{m=0}^i \mathfrak{d}_m\} \times \{\sum_{m=1}^{i-1} \mathfrak{d}_{m+1}, \dots, \sum_{m=1}^i \mathfrak{d}_m\}},$$

have rank  $\mathfrak{d}_i$ . The matrix  $A$  writes:

$$A = \begin{pmatrix} * & * & \dots & \dots & * \\ \mathcal{A}^1 & * & \ddots & \ddots & \vdots \\ 0_{\mathfrak{d}_2, d_0} & \mathcal{A}^2 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0_{\mathfrak{d}_k, d_0} & \dots & 0_{\mathfrak{d}_k, \mathfrak{d}_{k-1}} & \mathcal{A}^k & * \end{pmatrix}.$$

Following the proof of Lemma 5.5 in [8], where the case  $d_0 = d$ ,  $\mathfrak{d}_i = d, k = n-1$  is addressed, it can be derived that there exists  $C \geq 1$  s.t. for all  $(v, z, z') \in (0, T] \times (\mathbb{R}^N)^2$ ,

$$|D_x^2 p^{\text{ou}}(v, z, z')| \leq \frac{C}{v \sum_{i=0}^{k-1} \mathfrak{d}_i(i+\frac{1}{2})+1} \exp\left(-C^{-1}v|\mathbb{T}_v^{-1}(e^{Av}z - z')|^2\right), \quad (2.6)$$

where  $\mathfrak{d}_0 = d_0$  and

$$\mathbb{T}_v := \text{diag}(vI_{d_0 \times d_0}, v^2I_{d_1 \times d_1}, \dots, v^{k+1}I_{d_k \times d_k}), \quad v \geq 0,$$

reflects the various scales of the system. For a given function  $f \in B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$ , it is then clear from (2.5) and (2.6) that for all  $(t, z) \in (0, T] \times \mathbb{R}^N$ :

$$D_x^2 u(t, z) = \text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s, z') D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds. \quad (2.7)$$

It indeed suffices to observe that:

$$\begin{aligned}
& |\text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s, z') D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds| \\
&= |\text{p.v.} \int_0^t \int_{\mathbb{R}^N} [f(s, z') - f(s, e^{A(t-s)}x)] D_x^2 p^{\text{ou}}(t-s, z, z') dz' ds| \\
&\stackrel{(2.6)}{\leq} \sup_{s \in [0, T]} \|Df(s, \cdot)\|_\infty \\
&\quad \times \int_0^t \int_{\mathbb{R}^N} \frac{C}{(t-s)^{\sum_{i=0}^{k-1} \nu_i(i+\frac{1}{2})+\frac{1}{2}}} \exp\left(-C^{-1}(t-s)|\mathbb{T}_{t-s}^{-1}(e^{A(t-s)}z - z')|^2\right) dz' ds \\
&\leq C \sup_{s \in [0, T]} \|Df(s, \cdot)\|_\infty T^{\frac{1}{2}}.
\end{aligned}$$

The estimates in (1.11) now follows from the proof of Theorem 3 in [3], starting from (2.7) instead of (16) therein. The strategy is clear. It is necessary to introduce a cut-off function which separates the points  $(s, z')$  which do not induce any singularity in (2.7) for the derivatives of the density, namely such that  $t-s \geq c_0$  or  $|e^{A(t-s)}z - z'| \geq c_0$ , for some fixed constant  $c_0 > 0$ , from those who are close to the singularity. For the non-singular part of the integral the expected  $L^p$ -control readily follows from (2.6) and the Young inequality (see also Proposition 5 in [3]), whereas the derivation of the bound for the singular part requires some involved harmonic analysis, see Section 4 on the same reference. We can also refer to Theorem 11 and its proof in [24] for similar issues linked with the corresponding  $L^p$ -estimates for degenerate Ornstein-Uhlenbeck operators in an elliptic setting.

## 2.4 The main result for equation (2.1): perturbation of second order driftless PDE

Let us fix  $p$  in  $(1, +\infty)$  and assume that there exist  $R(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$  depending continuously on  $t \geq 0$  and a constant  $C_p > 0$ , such that for any  $f$  in  $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$ , the unique solution  $v = PDE(Q, f)$  to equation (2.1) satisfies

$$\|R(t)^* D^2 v R(t)\|_{L^p((0, T) \times \mathbb{R}^N, \mathbf{m})} \leq C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N, \mathbf{m})}, \quad (2.8)$$

for some absolutely continuous measure  $\mathbf{m}$  w.r.t. the Lebesgue measure on  $[0, T] \times \mathbb{R}^N$  such that  $\mathbf{m}(dt, dx) = g(t) dt dx$  for some borel bounded function  $g$  (note that in (1.19) we have  $R(t) = e^{tA} B^{1/2}$ ,  $\mathbf{m}(dt, dx) = g(t) dt dx = \det(e^{-At}) dt dx$ ).

We would like to exhibit that a control like (4.10) also holds for the solution  $w$  to the following Cauchy Problem:

$$\begin{cases} \partial_t w(t, z) = \text{tr}(Q(t) D^2 w(t, z)) + \text{tr}(Q'(t) D^2 w(t, z)) + f(t, z), & \text{on } (0, T) \times \mathbb{R}^N; \\ w(0, z) = 0, & \text{on } \mathbb{R}^N, \end{cases} \quad (2.9)$$

Namely we have to prove the following result.

**Theorem 2.2.** *Let us consider equations (2.1) and (2.9) where  $Q(t)$ ,  $Q'(t)$  are two continuous in time, non-negative matrices in  $\mathbb{R}^N \otimes \mathbb{R}^N$  and  $f \in B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . Assume that estimate (4.10) holds as explained above.*

*Then the solution  $w$  to (2.9) verifies*

$$\|R(t)^* D^2 w R(t)\|_{L^p((0, T) \times \mathbb{R}^N, m)} \leq C_p \|f\|_{L^p((0, T) \times \mathbb{R}^N, m)}, \quad (2.10)$$

*$p \in (1, \infty)$  with the same constant  $C_p$  as in (4.10).*

From Theorem 2.2 using the argument of Section 1.1 we can easily derive Theorem 1.1.

### 3 A perturbation argument for proving Theorem 2.2

We aim here at applying the probabilistic perturbative approach considered in [15]. The key idea in that work was, for a well-posed PDE which enjoys some quantitative given estimates, to introduce a *small* random perturbation of the source through a suitable Poisson type process and to investigate the properties of the associated PDE.

When taking the expectation in the associated integral formulation, the contributions associated with the jumps then yield, for an appropriate intensity of the underlying Poisson process, a finite difference operator. For the PDE satisfied by the expectation, involving the finite difference operator, the initial estimates are preserved. Compactness arguments then allow to derive that, the initial estimates still hold at the limit with the finite difference operator replaced by the corresponding differential one of order two.

Below, we start recalling the basic properties on Poisson type processes, and their corresponding stochastic integrals, that are needed for our approach.

#### 3.1 Poisson stochastic integrals

We briefly recall here the very definition of the stochastic integral driven by a Poisson process. We start reminding the construction of such processes.

Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  to be fixed from this point further, we start considering a sequence of independent real-valued random variables  $\{\tau_n\}_{n \in \mathbb{N}}$  on  $\Omega$  whose distribution is exponential of parameter  $\lambda > 0$ :

$$\mathbb{P}(\tau_n > r) = e^{-r\lambda}, \quad r \geq 0.$$

We can then define the partial sums sequence  $\{\sigma_n\}_{n \in \mathbb{N}}$  as follows:

$$\sigma_0 = 0; \quad \sigma_n = \sum_{i=1}^n \tau_i, \quad n = 1, 2, \dots$$

For any fixed  $t \geq 0$ ,  $\pi_t$  now denotes the number of consecutive sums of  $\tau_i$  which lie on  $[0, t]$ , i.e.

$$\pi_t = \sum_{n=0}^{\infty} \mathbf{1}_{\sigma_n \leq t}, \quad (3.1)$$

where  $\mathbf{1}_{\sigma_n \leq t}$  represents the indicator function of the event  $\{\sigma_n \leq t\}$ . The process  $\{\pi_t\}_{t \geq 0}$  we have just constructed is usually known in the literature as a *Poisson process* with intensity  $\lambda$  (see, for instance, [25]).

Now, let  $c : [0, T] \rightarrow \mathbb{R}^N$  be a continuous function. We can define the Poisson stochastic integral  $\int_0^t c(s) d\pi_s = \int_{(0, t]} c(s) d\pi_s$ ,  $t \in (0, T]$ , as

$$b_t := \int_0^t c(s) d\pi_s = \sum_{\sigma_k \leq t, k \geq 1} c(\sigma_k) = \sum_{0 < s \leq t} c(s) (\pi_s - \pi_{s-}) \quad (3.2)$$

$b_0 = 0$  (as usual  $\pi_{s-}(\omega)$  denotes the left limit at  $s$ , for any  $\omega$ ,  $\mathbb{P}$ -a.s.). We now recall the following formula for the expectation of the stochastic integral:

$$\mathbb{E} \left[ \int_0^t c(s) d\pi_s \right] = \lambda \int_0^t c(s) ds. \quad (3.3)$$

(cf. Lemma 2.1 in [15] for a direct proof; see also Theorem 16 in [25] and Theorem 5.3 in [15] for a more general formula involving stochastic integrals with predictable processes against the Poisson process). We also recall the following more general result.

**Lemma 3.1.** *Let  $\{\pi_t\}_{t \geq 0}$  be a Poisson Process of intensity  $\lambda$  on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Let us consider a stochastic process  $(\xi_t)_{t \in [0, T]}$  with values in  $\mathbb{R}$  which has cad-lag paths ( $\mathbb{P}$ -a.s.) and is  $\mathcal{F}_t$ -adapted where  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by the random variables  $\pi_s$ ,  $0 \leq s \leq t$ . Suppose that  $\sup_{\omega \in \Omega, s \in [0, T]} |\xi_s(\omega)| < \infty$ . Then*

$$\mathbb{E} \int_0^t \xi_{s-} d\pi_s = \lambda \int_0^t \mathbb{E} \xi_s ds. \quad (3.4)$$

### 3.2 Proof of Theorem 2.2

According to the notations in Proposition 2.1, let  $v = PDE(Q, f)$  and  $w = PDE(Q+Q', f)$  be the unique solutions of equations (2.1) and (2.9), respectively.

The proof of Theorem 2.2 will be obtained adapting the method developed in [15] (see in particular Section 3, therein). Let  $e_1$  be the first unit vector in  $\mathbb{R}^N$ . We define

$$X_t = \int_0^t \sqrt{Q'(r)} e_1 d\pi_r$$

where  $\sqrt{Q'(t)}$  is the unique  $N \times N$  symmetric non-negative square root of  $Q'(t)$  and  $\{\pi_t\}_{t \geq 0}$  is a Poisson Process of intensity  $\lambda$ . The parameter  $\lambda$  will be chosen appropriately later on.

Recall that the solution  $v$  to (2.1) is given by

$$v(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + z') \mu_{s,t}(dz')] \quad (3.5)$$

where  $\mu_{s,t}$  is the Gaussian law of the stochastic integral  $I_{s,t} := \sqrt{2} \int_s^t Q(v)^{1/2} dW_v$  (see the proof of Proposition 2.1).

Let us fix  $\epsilon > 0$ . We notice that the shifted source  $f_\epsilon(t, z) := f(t, z - \epsilon X_t)$  (which also depends on  $\omega$ ; we have omitted to write such dependence on  $\omega$ ) is again in  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ . This is the reason why we considered such a function space for the source. It precisely allows to take into account the time discontinuities coming from the jumps of the Poisson process.

For any fixed  $\omega$  in  $\Omega$ , Proposition 2.1 readily gives that there exists a unique solution  $v_\epsilon = \text{PDE}(Q, f(t, z - \epsilon X_t))$ , depending also on  $\epsilon$  and  $\omega$  as parameters, such that

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v_\epsilon(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t, z)|. \quad (3.6)$$

Moreover, thanks to the invariance for translations of the  $L^p$ -norms, it follows from (4.10) that

$$\|R(t)^* D^2 v_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})}. \quad (3.7)$$

By Equation (3.5), we know that  $v_\epsilon$  is given by

$$v_\epsilon(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z - \epsilon X_s + z') \mu_{s,t}(dz')].$$

For each  $z \in \mathbb{R}^N$ , the stochastic process  $(v_\epsilon(t, z))_{t \in [0, T]}$  has continuous paths ( $\mathbb{P}$ -a.s.) and is  $\mathcal{F}_t$ -adapted where  $\mathcal{F}_t$  is the  $\sigma$ -algebra generated by the random variables  $\pi_s$ ,  $0 \leq s \leq t$ .

For fixed  $z \in \mathbb{R}^N$ , let us introduce the process  $(v_\epsilon(t, z + \epsilon X_t))_{t \in [0, T]}$  which is given by

$$v_\epsilon(t, z + \epsilon X_t) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + \epsilon X_t - \epsilon X_s + z') \mu_{s,t}(dz')].$$

It is not difficult to check that it is  $\mathcal{F}_t$ -adapted and it has bounded and càdlàg paths.

Applying (2.2) on each interval  $[\sigma_n, \sigma_{n+1} \wedge t)$ ,  $n \in \{0, \dots, \pi_t\}$  on which  $X_s$  is constant, one then derives that:

$$v_\epsilon(t, z + \epsilon X_t) = \int_0^t \left( \text{tr}(Q(s) D_z^2 v_\epsilon(s, z + \epsilon X_s)) + f(s, z) \right) ds + \int_0^t g_\epsilon(s, z) d\pi_s, \quad (3.8)$$

where  $g_\epsilon(s, z) = v_\epsilon(s, z + \epsilon \sqrt{Q'(s)} e_1 + \epsilon X_{s-}) - v_\epsilon(s, z + \epsilon X_{s-})$  is precisely the contribution associated with the jump times. It is clear that  $g_\epsilon(s, z) \neq 0$  if and only if  $\pi_s$  has a jump at time  $s$ . We then have by Lemma 3.1:

$$\mathbb{E} \int_0^t g_\epsilon(s, z) d\pi_s = \lambda \int_0^t \left( \bar{v}_\epsilon(s, z + \epsilon \sqrt{Q'(s)} e_1) - v_\epsilon(s, z) \right) ds, \quad (3.9)$$



where  $\bar{v}_\epsilon(s, z) = \mathbb{E}[v_\epsilon(s, z + \epsilon X_s)]$ . Let us denote

$$l(t) := \sqrt{Q'(t)} e_1.$$

Taking the expectation on both sides of equation (3.8), we find out that  $\bar{v}_\epsilon$  is an integral solution of the following PDE:

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D_z^2 \bar{v}_\epsilon(t, z)) + \lambda (\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + f(t, z), \quad (3.10)$$

with zero initial condition. Moreover by (3.7) we obtain (using also the Jensen inequality and the Fubini theorem)

$$\begin{aligned} \|R(t)^* D^2 \bar{v}_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p &= \int_{(0,T) \times \mathbb{R}^N} |R(t)^* D^2 \bar{v}_\epsilon(t, z) R(t)|^p \mathbf{m}(dt, dz) \\ &= \int_{(0,T) \times \mathbb{R}^N} |\mathbb{E}[R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)]|^p dz g(t) dt \\ &\leq \int_{[0,T] \times \mathbb{R}^N} \mathbb{E}[|R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)|^p] dz g(t) dt \\ &= \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, z + \epsilon X_t) R(t)|^p dz g(t) dt \\ &= \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, \bar{z}) R(t)|^p d\bar{z} g(t) dt \\ &\leq C_p^p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathbf{m})}^p, \end{aligned}$$

using (3.7) for the last inequality ( $L^p$  estimate for the PDE with random source). Choosing  $\lambda = \epsilon^{-2}$  we have from (3.10)

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D_z^2 \bar{v}_\epsilon(t, z)) + \epsilon^{-2} (\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + f(t, z), \quad (3.11)$$

with zero initial condition and moreover

$$\|R(t)^* D^2 \bar{v}_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N)}^p \leq C_p^p \|f\|_{L^p((0,T) \times \mathbb{R}^N)}^p. \quad (3.12)$$

Now, the idea is to apply again the same reasoning above to the equation (3.11) with respect to  $\bar{v}_\epsilon$  and  $f(t, z + \epsilon X_t)$  again with  $\lambda = \epsilon^{-2}$ . We obtain first a solution  $p_\epsilon$  to (3.11) corresponding to  $f(t, z + \epsilon X_t)$  and then derive that

$$w_\epsilon(t, z) = \mathbb{E}[p_\epsilon(t, z - \epsilon X_t)]$$

is the unique bounded continuous (integral) solution  $w_\epsilon$  of the following problem:

$$\begin{cases} \partial_t w_\epsilon(t, z) = \text{tr}(Q(t) D^2 w_\epsilon(t, z)) + \epsilon^{-2} (w_\epsilon(t, z + \epsilon l(t)) - 2w_\epsilon(t, z) + w_\epsilon(t, z - \epsilon l(t))) \\ \quad + f(t, z), \\ w_\epsilon(0, z) = 0. \end{cases} \quad (3.13)$$

Uniqueness follows by the maximum principle as in the proof of Lemma 2.2 in [15].

Moreover, the previous estimates still hold with  $w_\epsilon$  instead of  $v_\epsilon$ , i.e.,

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |w_\epsilon(t,z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |f(t,z)|; \quad (3.14)$$

$$\|R(t)^* D^2 w_\epsilon R(t)\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})} \leq C_p \|f\|_{L^p((0,T) \times \mathbb{R}^N, \mathfrak{m})}. \quad (3.15)$$

We would like now to let  $\epsilon$  goes to zero, possibly passing to a subsequence  $\epsilon_n \rightarrow 0$ , and prove that the associated limit  $w$  solves

$$\begin{cases} \partial_t w(t,z) = \text{tr}(Q(t)D^2 w(t,z)) + \langle D^2 w(t,z) \sqrt{Q'(t)} e_1, \sqrt{Q'(t)} e_1 \rangle + f(t,z), \\ w(0,z) = 0 \end{cases} \quad (3.16)$$

and estimates (3.14) and (3.15) hold with  $w_\epsilon$  replaced by  $w$ .

To do so we will proceed by compactness. Namely, we are going to prove that the family of solutions  $w_\epsilon$  solving (3.13), indexed by the parameter  $\epsilon$ , is equi-Lipschitz on any compact subset of  $[0, T] \times \mathbb{R}^N$  and the same holds for any derivative in space of  $w_\epsilon$ . Indeed, one can apply the finite difference operators with respect to  $z$  at any order in (3.13). We recall that for a *smooth* function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ , the first finite difference  $\delta_{h,i}\phi$ ,  $i \in \{1, \dots, N\}$  of step  $h$  in the direction  $e_i$  ( $i^{\text{th}}$  basis vector) is given by

$$\delta_{h,i}\phi(z) = \frac{\phi(z + h e_i) - \phi(z)}{h}, \quad z \in \mathbb{R}^N.$$

For a given multi-index  $\gamma \in \mathbb{N}^N$ , the  $\gamma$ -th order finite difference operator  $\delta_{h,\gamma}$ , is then defined, for any  $h > 0$ , through composition. Namely,

$$\delta_{h,\gamma}\phi(z) = \delta_{h,1}^{\gamma_1} \delta_{h,2}^{\gamma_2} \dots \delta_{h,N}^{\gamma_N} \phi(z),$$

where  $\delta_{h,i}^{\gamma_i}$  denotes the  $\gamma_i$ -th times composition of  $\delta_{h,i}$  with itself.

Using (3.14) and the fact that any spatial derivative of  $f$  belongs to  $B_b(0, T; C_0^\infty(\mathbb{R}^N))$ , we deduce first that any finite difference of any order of  $w_\epsilon$  is bounded. Consequently,  $w_\epsilon$  is infinitely differentiable in space with bounded derivatives on  $[0, T] \times \mathbb{R}^N$ . Equation (3.13), to be understood in its integral form similarly to (2.2), then gives that those derivatives are themselves Lipschitz continuous in time. This precisely gives the equi-Lipschitz on any compact subset of  $[0, T] \times \mathbb{R}^N$  of the family  $w_\epsilon$  and any spatial derivative.

We can now apply the Arzelà-Ascoli theorem to  $w_\epsilon$  showing the existence of a sub-sequence  $\{w_{\epsilon_n}\}_{n \in \mathbb{N}}$  which converges uniformly on any compact set to a function  $w: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ . Similarly,  $\partial_t w_{\epsilon_n}$  and any derivative in space of  $w_{\epsilon_n}$  tend to the respective derivatives of  $w$ , uniformly on the compact sets.

Passing to the limit for  $n \rightarrow \infty$  along the sequence  $(\epsilon_n)_n$  in Equation (3.13) (written in the integral form), we can then conclude that  $w$  solves (3.16).

Moreover, estimates (3.14) and (3.15) holds with  $w_\epsilon$  replaced by  $w$ .

Iterating the previous argument in  $N$  steps we finally prove that the unique solution  $w$  to

$$\begin{cases} \partial_t w(t, z) = \text{tr}(Q(t)D^2w(t, z)) + \sum_{k=1}^N \langle D^2w(t, z)\sqrt{Q'(t)}e_k, \sqrt{Q'(t)}e_k \rangle + f(t, z), \\ w(0, z) = 0 \end{cases} \quad (3.17)$$

verifies estimates (3.14) and (3.15) with  $w_\epsilon$  replaced by  $w$ . The proof is complete.  $\square$

## 4 Additional stability results in anisotropic Sobolev space and Schauder estimates

In this section we extend the previous approach to derive the stability with respect to a second order perturbation of the general OU in (1.7) for  $L^p$  estimates involving the degenerate components of the operator and some associated Schauder estimates, under the Kalman condition **[K]**.

### 4.1 Anisotropic Sobolev spaces and maximal $L^p$ regularity.

With the notations of Section 2.3 we write  $z \in \mathbb{R}^N$  as  $z = (x, y) = (x, y_1, \dots, y_k)$  with  $x \in \mathbb{R}^{d_0}$ ,  $y_i \in \mathbb{R}^{d_i}$ ,  $i \in \{1, \dots, k\}$ ,  $\sum_{i=1}^k d_i = d_1$ .

Given  $\beta$  in  $(0, 1)$  and  $i$  in  $\llbracket 1, k \rrbracket$ , we want to introduce the  $\beta$ -fractional Laplacian  $\Delta_{y_i}^\beta$  along the *component*  $y_i$ . To do so, we follow [10] by considering the orthogonal projection  $p_i: \mathbb{R}^N \rightarrow \mathbb{R}^{d_i}$  such that  $p_i(z) = p_i((x, y)) = y_i$  and denoting its adjoint by  $E_i: \mathbb{R}^{d_i} \rightarrow \mathbb{R}^N$ . We can now define the  $\beta$ -fractional Laplacian  $\Delta_{y_i}^\beta$  as:

$$\Delta_{y_i}^\beta \phi(z) := \text{p.v.} \int_{\mathbb{R}^{d_i}} [\phi(z + E_i w) - \phi(z)] \frac{dw}{|w|^{d_i + 2\beta}}, \quad z \in \mathbb{R}^N,$$

for any sufficiently regular function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$ .

Fixed  $p$  in  $(1, +\infty)$ , we recall that we have denoted by  $L^p((0, T) \times \mathbb{R}^N)$  the standard  $L^p$ -space with respect to the Lebesgue measure.

We can now define the appropriate anisotropic Sobolev space to state our results. For notational simplicity, let us denote

$$\alpha_i := \frac{1}{1 + 2i}. \quad (4.1)$$

Set now  $\alpha := (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ . The *homogeneous* space  $\dot{W}_\alpha^{2,p}([0, T] \times \mathbb{R}^N)$  is composed by all the functions  $\varphi: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$  in  $L^p([0, T] \times \mathbb{R}^N)$  such that  $(t, z) \in [0, T] \times \mathbb{R}^N \mapsto$

$\Delta_x \varphi(t, z) \in L^p([0, T] \times \mathbb{R}^N)$ , where  $\Delta_x \varphi$  is intended in distributional sense, and for any  $i$  in  $\llbracket 1, k \rrbracket$ ,  $\Delta_{y_i}^{\alpha_i} \varphi(t, z)$  is well defined for almost every  $(t, z)$  and

$$\Delta_{y_i}^{\alpha_i} \varphi(t, z) := \Delta_{y_i}^{\alpha_i} \varphi(t, \cdot)(z) \text{ belongs to } L^p([0, T] \times \mathbb{R}^N).$$

It is endowed with the natural *semi*-norm  $\|\varphi\|_{\dot{W}_\alpha^{2,p}}$  where

$$\|\varphi\|_{\dot{W}_\alpha^{2,p}}^p = \|\Delta_x \varphi\|_{L^p}^p + \sum_{i=1}^k \|\Delta_{y_i}^{\alpha_i} \varphi\|_{L^p}^p. \quad (4.2)$$

The thresholds in (4.1) might seem awkward at first sight. They actually correspond to the indexes needed to get stability of the harmonic functions associated with the principal part of (1.7), that is considering  $A_0$  consisting in the subdiagonal part of  $A$  only, along an associated dilation operator. Namely, setting

$$L_0^{\text{ou}} f(z) = \text{Tr}(BD^2 f(z)) + \langle A_0 z, Df(z) \rangle, \quad z = (x, y) \in \mathbb{R}^{d_0+d_1} = \mathbb{R}^N, \quad (4.3)$$

so that  $A_0, B$  satisfy **[K]**, if  $(\partial_t - L_0^{\text{ou}})u(t, z) = 0$  then for all  $\lambda > 0$   $(\partial_t - \mathcal{L}_0^{\text{ou}})u(\delta_\lambda(t, z)) = 0$  where the dilation operator

$$\delta_\lambda(t, z) = (\lambda^{1/2}t, \lambda x, \lambda^{1/3}y_1, \dots, y_k^{1/(1+2k)}).$$

precisely exhibits the exponents in (4.1) for the degenerate components.

In [10], see also [5] and [21] where time inhomogeneous coefficients are considered as well, it has been proven that when the strictly upper diagonal elements of  $A$  in (1.7) are equal to zero then the following Sobolev estimates hold:

$$\|u\|_{\dot{W}_\alpha^{2,p}} \leq C_p \|f\|_{L^p}, \quad (4.4)$$

where again  $u$  is the unique bounded solution to the corresponding Cauchy problem (1.10). In particular we get also the maximal smoothing effects w.r.t. the degenerate directions. The specific structure assumed on  $A$  is actually due to the fact that for such matrices there is an underlying homogeneous space structure which makes easier to establish maximal regularity estimates (see e.g. [6] in this general setting).

For a general  $A$ , having non zero strictly upper diagonal entries, such that  $A, B$  satisfy **[K]**, We believe that the approach in [3] could extend to show that for (4.4) still hold in this general setting, but such estimates have not been, up to our best knowledge, proven yet.

Setting now, as in Section 1.1,  $u(t, z) = v(t, e^{tA}z)$  and since  $u$  solves (1.10) we have that  $v$  in turn solves (1.17). From the previous computations, setting  $B_I = \begin{pmatrix} I_{d_0, d_0} & 0_{d_0, d_1} \\ 0_{d_1, d_0} & 0_{d_1, d_1} \end{pmatrix}$ , we

derive

$$\begin{aligned}\|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} &= \|B_I^{1/2} e^{tA^*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &\leq C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}.\end{aligned}$$

On the other hand, for all  $i \in \{1, \dots, k\}$  and with  $\alpha_i$  as in (4.1),

$$\begin{aligned}\|\Delta_{y_i}^{\alpha_i} u\|_{L^p((0,T)\times\mathbb{R}^N)}^p &= \int_0^T dt \int_{\mathbb{R}^N} dz \left| \text{p.v.} \int_{\mathbb{R}^{\mathfrak{d}_i}} [u(t, z + E_i w) - u(t, z)] \frac{dw}{|w|^{\mathfrak{d}_i + 2\alpha_i}} \right|^p \\ &= \int_0^T dt \int_{\mathbb{R}^N} dz \left| \text{p.v.} \int_{\mathbb{R}^{\mathfrak{d}_i}} [v(t, e^{tA}(z + E_i w)) - v(t, e^{tA}z)] \frac{dw}{|w|^{\mathfrak{d}_i + 2\alpha_i}} \right|^p \\ &=: \|\Delta^{\alpha_i, i, A} v\|_{L^p((0,T)\times\mathbb{R}^N)}^p.\end{aligned}$$

Hence, setting  $\|\Delta^{\alpha_0, 0, A} v\|_{L^p((0,T)\times\mathbb{R}^N)}^p := \|\text{Tr}(B_I^{1/2} e^{-tA^*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I^{1/2})\|_{L^p((0,T)\times\mathbb{R}^N)}^p$ , we get from the definition (4.2) that the estimate (4.4) rewrites in term of  $v$  as:

$$\|v\|_{\dot{W}_\alpha^{2,p,A}}^p := \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} v\|_{L^p((0,T)\times\mathbb{R}^N)}^p \leq \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}^p \quad (4.5)$$

with  $\tilde{C}_p = C_p^p$ . We now want to prove that for  $w$  solving (1.20), namely

$$\begin{cases} \partial_t w(t, z) + \text{Tr}(e^{tA} B e^{tA^*} D^2 w(t, z)) + \text{Tr}(e^{tA} S(t) e^{tA^*} D^2 w(t, z)) = \tilde{f}(t, z), & (t, z) \in (0, T) \times \mathbb{R}^N, \\ w(0, z) = 0, & z \in \mathbb{R}^N; \end{cases}$$

it also holds that

$$\|w\|_{\dot{W}_\alpha^{2,p,A}}^p := \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} w\|_{L^p((0,T)\times\mathbb{R}^N)}^p \leq \tilde{C}_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}^p, \quad (4.6)$$

with the same constants  $\tilde{C}_p$  as in (4.5). This can be done through the previous perturbative approach of Section 3.2 employed to prove Theorem 2.2, which actually gives the expected control for the the second order derivatives contribution of the semi-norm  $\|\cdot\|_{\dot{W}_\alpha^{2,p,A}}$ .

For the other contributions and with the notations of Section 3.2, with  $Q'(s) = e^{sA} S(s) e^{sA^*}$

and with  $m$  which is the Lebesgue measure on  $[0, T] \times \mathbb{R}^N$ , we would get

$$\begin{aligned}
\|\bar{v}_\epsilon\|_{\dot{W}_\alpha^{2,p,A}}^p &= \sum_{i=0}^k \|\Delta^{\alpha_i, i, A} \bar{v}_\epsilon\|_{L^p((0,T) \times \mathbb{R}^N)}^p = \sum_{i=0}^k \int_{(0,T) \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} \bar{v}_\epsilon(t, z)|^p dz dt \\
&= \sum_{i=0}^k \int_{(0,T) \times \mathbb{R}^N} |\mathbb{E}[\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)]|^p dz dt \\
&\leq \sum_{i=0}^k \int_{[0,T] \times \mathbb{R}^N} \mathbb{E}[|\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)|^p] dz dt \\
&= \sum_{i=0}^k \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} v_\epsilon(t, z + \epsilon X_t)|^p dz dt \\
&= \sum_{i=0}^k \mathbb{E} \int_{[0,T] \times \mathbb{R}^N} |\Delta^{\alpha_i, i, A} v_\epsilon(t, \bar{z})|^p d\bar{z} dt \\
&\leq \tilde{C}_p \|f\|_{L^p((0,T) \times \mathbb{R}^N)}^p,
\end{aligned}$$

using for the last inequality that  $v_\epsilon$  also satisfies (4.5) (similarly to what had been established in (3.7)).

The same previous compactness argument then yields (4.6). Setting eventually  $\tilde{u}(t, z) := w(t, e^{tA}z)$ , which is the unique integral solution, smooth in space, of

$$\begin{cases} \partial_t \tilde{u}(t, z) + L_t^{\text{ou}, S} \tilde{u}(t, z) = f(t, z), & (t, z) \in (0, T) \times \mathbb{R}^N, \\ \tilde{u}(0, z) = 0, & z \in \mathbb{R}^N, \end{cases}$$

where  $L_t^{\text{ou}, S}$  introduced in (1.13) is the Ornstein-Uhlenbeck operator perturbed at second order, we derive that

$$\|\tilde{u}\|_{\dot{W}_\alpha^{2,p}} \leq C_p \|f\|_{L^p}, \quad (4.7)$$

with  $C_p$  as in (4.4). We have thus extended the results of Theorem 1.1 for the anisotropic Sobolev norm in (4.2). The estimate (4.4) is stable for a continuous, non-negative second order perturbation of the underlying degenerate Ornstein-Uhlenbeck operator.

## 4.2 Anisotropic Schauder estimates

Following Krylov [12], for some fixed  $\ell$  in  $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$  and  $\beta$  in  $(0, 1]$ , we introduce for a function  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  the Zygmund-Hölder semi-norm as

$$[\phi]_{C^{\ell+\beta}} := \begin{cases} \sup_{|\vartheta|=\ell} \sup_{x \neq y} \frac{|D^\vartheta \phi(x) - D^\vartheta \phi(y)|}{|x-y|^\beta}, & \text{if } \beta \neq 1; \\ \sup_{|\vartheta|=\ell} \sup_{x \neq y} \frac{|D^\vartheta \phi(x) + D^\vartheta \phi(y) - 2D^\vartheta \phi(\frac{x+y}{2})|}{|x-y|}, & \text{if } \beta = 1 \end{cases}$$

(we are using usual multi-indices  $\vartheta$  for the partial derivatives). Consequently, the Zygmund-Hölder space  $C_b^{\ell+\beta}(\mathbb{R}^N)$  is the family of bounded functions  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $\phi$  and its derivatives up to order  $\ell$  are continuous and the norm

$$\|\phi\|_{C_b^{\ell+\beta}} := \sum_{i=0}^{\ell} \sup_{|\vartheta|=i} \|D^{\vartheta}\phi\|_{\infty} + [\phi]_{C^{\ell+\beta}} \text{ is finite.}$$

We can define now the anisotropic Zygmund-Hölder spaces associated with the current setting and which again reflect the various scales already introduced in (4.1). Fixed  $\gamma \in (0, 3)$ , the space  $C_{b,d}^{\gamma}(\mathbb{R}^N)$  is the family of functions  $\phi: \mathbb{R}^N \rightarrow \mathbb{R}$  such that for any  $i$  in  $\llbracket 0, k \rrbracket$  and any  $z_0$  in  $\mathbb{R}^N$ , the function

$$w \in \mathbb{R}^{\mathfrak{d}_i} \rightarrow \phi(z_0 + E_i(w)) \in \mathbb{R} \text{ belongs to } C_b^{\gamma/(1+2i)}(\mathbb{R}^{\mathfrak{d}_i}),$$

with a norm bounded by a constant independent from  $z_0$ . In the above expression, we recall that the  $(E_i)_{i \in \{1, \dots, k\}}$  have been defined in the previous paragraph,  $\mathfrak{d}_0 = d_0$  and  $E_0$  is the embedding matrix from  $\mathbb{R}^{d_0}$  into  $\mathbb{R}^N$ . It is endowed with the norm

$$\|\phi\|_{C_{b,d}^{\gamma}} := \sup_{z_0 \in \mathbb{R}^N} \|\phi(z_0 + E_0(\cdot))\|_{C_b^{\gamma}(\mathbb{R}^{\mathfrak{d}_0})} + \sum_{i=1}^k \sup_{z_0 \in \mathbb{R}^N} [\phi(z_0 + E_i(\cdot))]_{C^{\gamma/(1+2i)}(\mathbb{R}^{\mathfrak{d}_i})}. \quad (4.8)$$

We denote by  $C_{b,d}^{\gamma}$  this function space because the regularity exponents reflect again the multi-scale features of the system and the norm could equivalently be defined through the corresponding spatial parabolic distance  $d$  defined as follows. For all  $z = (x, y), z' = (x', y') \in \mathbb{R}^N$ :

$$d(z, z') := |x - x'| + \sum_{i=1}^k |y_i - y'_i|^{\frac{1}{1+2i}},$$

where the exponents are again those who appeared in (4.1).

Let as before  $f \in B_b(0, T; C_0^{\infty}(\mathbb{R}^N))$ . We recall that under [K], Lunardi showed in [17] that the following anisotropic Schauder estimates hold for the solution  $u$  of Cauchy Problem (1.10):

$$\|u\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} \leq C_{\beta} \|f\|_{L^{\infty}((0,T), C_{b,d}^{\beta})}, \quad (4.9)$$

for some constant  $C_{\beta}$  independent from  $f$ , i.e.,

$$\sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{C_{b,d}^{2+\beta}} \leq C_{\beta} \sup_{0 \leq t \leq T} \|f(t, \cdot)\|_{C_{b,d}^{\beta}}, \quad (4.10)$$

We again set as in the previous paragraph  $u(t, z) = v(t, e^{tA}z)$  and since  $u$  solves (1.10) we have that  $v$  in turn solves (1.17). Write:

$$\begin{aligned} \|u\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} &= \|v(t, e^{tA}\cdot)\|_{L^{\infty}((0,T), C_{b,d}^{2+\beta})} =: \|v\|_{L^{\infty}((0,T), C_{b,d,A}^{2+\beta})} \\ &\leq C_{\beta} \|f\|_{L^{\infty}((0,T), C_{b,d}^{\beta})} \leq C_{\beta} \|\tilde{f}(t, e^{tA}\cdot)\|_{L^{\infty}((0,T), C_{b,d}^{\beta})} = C_{\beta} \|\tilde{f}\|_{L^{\infty}((0,T), C_{b,d,A}^{\beta})}. \end{aligned} \quad (4.11)$$

We again want to prove as in Section 1.1 that for  $w$  solving (1.20),

$$\|w\|_{L^\infty((0,T),C_{b,d,A}^{2+\beta})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C_{b,d,A}^\beta)} \quad (4.12)$$

with the same constant  $C_\beta$  as in (4.11). We proceed one more time through the previous perturbative approach of Section 3.2. With the notations employed therein, we deduce that there exists a unique solution  $v_\epsilon = \text{PDE}(Q, \tilde{f}(t, z - \epsilon X_t))$ , depending also on  $\epsilon$  and  $\omega$  as parameters such that

$$\sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |v_\epsilon(t, z)| \leq T \sup_{(t,z) \in [0,T] \times \mathbb{R}^N} |\tilde{f}(t, z)|. \quad (4.13)$$

From the translation invariance of the Hölder-norms, it is not difficult to prove that, for any  $\omega$ ,  $\mathbb{P}$ -a.s.,

$$\|\tilde{f}\|_{L^\infty((0,T),C_{b,d,A}^\beta)} = \|\tilde{f}(\cdot, \cdot - \epsilon X_\cdot)\|_{L^\infty((0,T),C_{b,d,A}^\beta)}. \quad (4.14)$$

Thus it also holds from (4.11)

$$\|v_\epsilon\|_{L^\infty((0,T),C_{b,d,A}^{2+\beta})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C_{b,d,A}^\beta)}. \quad (4.15)$$

Recalling now that  $\bar{v}_\epsilon(s, z) = \mathbb{E}[v_\epsilon(s, z + \epsilon X_s)]$ , is an integral solution of

$$\partial_t \bar{v}_\epsilon(t, z) = \text{tr}(Q(t) D_z^2 \bar{v}_\epsilon(t, z)) + \lambda (\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t, z)) + \tilde{f}(t, z),$$

with zero initial condition, we write that for  $i \in \{1, \dots, k\}$ ,  $w, w' \in \mathbb{R}^{d_i}$ ,  $(t, z_0) \in [0, T] \times \mathbb{R}^N$ ,

$$\begin{aligned} & |\bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w))) - \bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w')))| \\ & \leq \mathbb{E}[|v_\epsilon(t, e^{At}(z_0 + E_i(w))) + \epsilon e^{At} e^{-At} X_t - v_\epsilon(t, e^{At}(z_0 + E_i(w'))) + \epsilon e^{At} e^{-At} X_t|] \\ & \leq \mathbb{E}[|v_\epsilon(t, e^{At}(z_0 + E_i(\cdot)))|]_{C^{\frac{2+\beta}{1+2i}}} |w - w'|_{\frac{2+\beta}{1+2i}}. \end{aligned}$$

Hence,

$$[\bar{v}_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))]_{C^{\frac{2+\beta}{1+2i}}} \leq \mathbb{E}[|v_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))|]_{C^{\frac{2+\beta}{1+2i}}}.$$

We would get similarly, for  $w, w' \in \mathbb{R}^{d_0}$ ,

$$[D_x^2 \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))]_{C^\beta} \leq \mathbb{E}[|D_x^2 v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|]_{C^\beta},$$

and for all  $i \in \{0, 1, 2\}$ ,

$$|D_x^i \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|_\infty \leq \mathbb{E}[|D_x^i v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|_\infty].$$



Summing those contributions, we thus derive from (4.8), (4.11) that:

$$\|\bar{v}_\epsilon\|_{L^\infty((0,T),C_{b,d,A}^{2+\beta})} \leq \sup_{0 \leq t \leq T} \mathbb{E}[\|v_\epsilon(t, \cdot)\|_{C_{b,d,A}^{2+\beta}}] \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C_{b,d,A}^\beta)}, \quad (4.16)$$

using (4.15) for the last inequality. The other key property we used above is the triangle inequality, namely we put the absolute value inside the expectation. Iterating the procedure, one would derive from the same previous compactness argument that (4.12) indeed holds.

Going eventually backwards setting  $\tilde{u}(t, z) := w(t, e^{tA}z)$ , which is the unique solution to (1.14) in the function class considered through the document, we derive as in the previous paragraph that

$$\|\tilde{u}\|_{L^\infty((0,T),C_{b,d}^{2+\beta})} \leq C_\beta \|f\|_{L^\infty((0,T),C_{b,d}^\beta)}, \quad (4.17)$$

where  $C_\beta$  is the same constant as in (4.9). Equation (4.17) provides the extension of Theorem 1.1 for the anisotropic Schauder estimates.

Let us mention that for the perturbative analysis to work, few properties were actually needed on the underlying norm. Namely, we used the translation invariance and some kind of commutation between the norm (or a function of the norm in the  $L^p$  case) and the expectation. Hence, this approach could possibly be applied to a much wider class of estimates in other function spaces (like e.g. Besov spaces). This will concern further research.

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