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## Chirality defined The end of the even/odd paradigm

Michel Petitjean

Université de Paris, BFA, CNRS UMR 8251, INSERM ERL U1133
http:/ / petitjeanmichel.free.fr/itoweb.petitjean.html
(i) 0000-0002-1745-5402

## MIRROR SYMMETRY AND CHIRALITY

Intuitive definition, commonly encountered:
An object is chiral if its mirror image cannot be superimposed onto it
This is known since centuries, if not since millenaries, in arts, architecture, ...

## SOME FAMOUS NAMES:

- Leonardo da Vinci (second half of $15^{\text {th }}$ century): mirror writing
- Immanuel Kant (1768): differentiation of regions in space, Kant's hands
- Louis Pasteur (1848): dissymmetry ${ }^{*}$, tartrate crystals
*Not to be confused with asymmetry, which means "no symmetry at all".

The term dissymmetry is now deprecated: it is replaced by the term chirality.

## HISTORICAL DEFINITION OF CHIRALITY

Lord Kelvin, 1894: I call any geometrical figure, or group of points, 'chiral', and say that it has chirality if its image in a plane mirror, ideally realized, cannot be brought to coincide with itself.
(Thomson, W. The Molecular Tactics of a Crystal; Clarendon Press: Oxford, UK, 1894; chapter 22, p. 27)

- This definition is still used for teaching, but it is NOT mathematical.
- It was thought for Euclidean spaces, not for the spacetime (relativity appeared later).

Eddington, 1946: reintroduction in physics of the word chirality. He wrote that the term was introduced by Kelvin.
(Eddington, A.S. Fundamental Theory; Cambridge University Press: Cambridge, UK, 1946; p. 111)
In the same page, Eddington used the terms right-handed and left-handed.
That seems to be at the basis of the use of these two terms in physics (Quantum Field Theory and related fields).

## CHIRALITY IN PHYSICS

Looking for a definition of chirality in physics literature:

- Euclidean case: Kelvin's definition suffices*.
${ }^{*}$ The case of flexible objects, as encountered in chemistry, is not considered here.
- Special relativity: the readers are directed to the so-called $\gamma^{5}$ matrix.

In fact, the full story of the $\gamma^{5}$ matrix goes back to Dirac (1928).
A technical presentation is out of scope.
(see: Petitjean, M. Chirality of Dirac spinors revisited. Symmetry 2020, 12(4), 616; DOI 10.3390/sym12040616)
Rough summary:
In the Weyl basis, the matrices $\left[\mathbf{I}-\gamma^{5}\right] / 2$ and $\left[\mathbf{I}+\gamma^{5}\right] / 2$ project Dirac bispinors to get respectively their left-handed and right-handed parts.

This appears to be the main explanation why $\gamma^{5}$ is called the chirality operator.

But that cannot be used to build a general definition of chirality !

## SYMMETRY, CHIRALITY, GROUPS

In 1952, Weyl explained in his book what is symmetry, its importance in nature, arts and science, and its strong relation with groups.
(Weyl, H. Symmetry; Princeton University Press: Princeton, New Jersey, USA, 1952)

Weyl did not use the word chirality: he dealt with proper and improper rotations.

Weyl's book is sometimes cited as a reference for a definition of symmetry, but no unifying mathematical definition of symmetry is given there.

Therefore, a general mathematical definition of chirality was lacking, too.

## OTHER USES OF THE WORLD CHIRAL IN THE LITERATURE

Many algebraic structures were qualified chiral in the mathematical literature.
Among them:
chiral bracket, chiral chain complex, chiral algebra, chiral homology, chiral Lie algebroid, chiral module, chiral monoid, chiral operation, chiral pairings, chiral product, chiral structure.

Most of them are defined in the book of Beilinson and Drinfeld.
(Beilinson, A.; Drinfeld, V. Chiral Algebras; AMS Colloquium Publ.: Providence, Rhode Island, USA, 2004; chap. 3)

They are defined without reference to the common meaning of chirality. It seems that chiral algebra and associated concepts are related to antisymmetry rather than to chirality.

## Remark:

It exists a wide class of functions which are both antisymmetric and achiral*. (Petitjean, M. Symmetry, antisymmetry and chirality: use and misuse of terminology. Symmetry 2021, 13, 603; DOI 10.3390/sym13040603)
*The general difinition of the terms chiral and achiral is given further.

## THE EVEN/ODD PARADIGM

In Euclidean spaces:

- The composition of an EVEN number of reflections is a DIRECT symmetry
- The composition of an ODD number of reflections is an INDIRECT symmetry

The classification of symmetries as DIRECT or INDIRECT is at the basis of the definition of chirality:

Definition:
An object* is ACHIRAL when it is invariant under an INDIRECT symmetry. If it is not achiral, it is chiral.

* The definition of an object is given further
*** We retain the definition of the terms chiral and achiral ${ }^{* * *}$
... but NOT the even/odd paradigm : WHY ?


## INADEQUATION OF THE EVEN/ODD PARADIGM

In Euclidean spaces, the matrix representation of orthogonal transformations is considered: the determinant of the matrix of an indirect orthogonal transformation is -1 , then we see that the even/odd paradigm works.

This is used for space orientation.

But modern physics is based on spaces which are not Euclidean.
In special relativity, the Minkowski spacetime is considered.

The product PT of parity inversion P by time reversal T , is an involution.*
${ }^{*}$ It is known that $\mathrm{PT}=\mathrm{TP}$, so that $(\mathbf{P T})^{2}=\mathrm{PTTP}=$ identity.
P performs a reflection, T performs a reflection.

> Classifying PT as a direct isometry, discarding which metric is considered, would negate the fact that symmetry operators MUST be referred to a metric*

> There would be no difference with the Euclidean space !!

[^0]
## DEFINING SYMMETRY

Before defining chirality, we need to define symmetry. Everybody knows what is symmetry!

But we are looking for an unifying mathematical definition, not ...many mathematical definitions.

## Definitions*:

*Petitjean, M. A definition of symmetry. Symmetry Cult. Sci. 2007, 18(2-3), 99-119;
https://hal.archives-ouvertes.fr/hal-01552499

- An object is a function having its input argument in a metric space. (the space of the returned values may be any space)
- An object is symmetric when it is invariant under an isometry which is not the identity.
... works for geometric figures (with or without colors, as encountered in arts), works for functions, probability distributions, graphs, matrices, strings, etc.
... works also when the metric is a true one, thus, suitable for fields (physics).


## DEFINING CHIRALITY

Symetries are usually classified via classification of isometry groups. Do the same with direct/indirect isometries: classify them with groups.

Summarizing mathematical requirements for a safe chirality definition:

- Should work in any space for any metric (true ones or not true ones).
- Classify isometries as direct or indirect with the help of isometry groups.


## YES SUCH DEFINITION EXISTS !

An isometry is direct when it can be expressed as a product of squared isometries. An isometry which is not direct is an indirect isometry.

Petitjean, M. Chirality in metric spaces. In memoriam Michel Deza.
Optim. Letters 2020, 14, 329-338. DOI 10.1007/s11590-017-1189-7.
(original version: Petitjean, M. Chirality in metric spaces. Symmetry Cult. Sci. 2010, 21 (1-3), 27-36)
It recovers Kelvin's definition in Euclidean spaces.
CARE: when the space is not Euclidean, the composition of two indirect
symmetries may be a direct symmetry ...or not !!

## NO MORE EVEN/ODD PARADIGM

## We do not consider anymore matrix determinants.

## Theorem*:

In a quadratic space, a product of reflections is classified as direct or indirect as follows:
(a) When the product contains an odd number of reflections, it is an indirect isometry.
When the product contains an even number of reflections:
(b) It is a direct isometry when an even number of the squares of the supporting vectors have a negative sign, and the squares of the other supporting vectors have a positive sign.
(c) It is an indirect isometry when an odd number of the squares of the supporting vectors have a negative sign, and the squares of the other supporting vectors have a positive sign.
*Petitjean, M. Chirality in geometric algebra. Mathematics 2021, 9(13), 1521. DOI 10.3390/math9131521.

## CONSEQUENCES IN THE MINKOWSKI SPACETIME

In special relativity, the signature of Minkowski spacetime is $(+,-,-,-)$.

$$
\text { (may be }(+,+,+,-) \text {, as in general relativity) }
$$

- Boosts are direct symmetry operators.
(Petitjean, M. About chirality in Minkowski spacetime. Symmetry 2019, 11(10), 1320. DOI 10.3390/sym11050662)
- PT is an indirect symmetry; it is called "full reflection" in the literature.
(Varlamov, V.V. Universal coverings of the orthogonal groups, Adv. Appl. Cliff. Alg. 2004, 14(1), 81-168)

|  | I | T | P | PT |
| :---: | :---: | :---: | :---: | :---: |
| I | I | T | P | PT |
| T | T | I | PT | P |
| P | P | PT | I | T |
| PT | PT | P | T | I |


|  | I | R | G | B |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbf{I}$ | $\mathbf{I}$ | $\mathbf{R}$ | $\mathbf{G}$ | $\mathbf{B}$ |
| $\mathbf{R}$ | $\mathbf{R}$ | $\mathbf{I}$ | $\mathbf{B}$ | $\mathbf{G}$ |
| $\mathbf{G}$ | $\mathbf{G}$ | $\mathbf{B}$ | $\mathbf{I}$ | $\mathbf{R}$ |
| $\mathbf{B}$ | $\mathbf{B}$ | $\mathbf{G}$ | $\mathbf{R}$ | $\mathbf{I}$ |

The discrete subgroup $\{\mathbf{I}, \mathrm{T}, \mathrm{P}, \mathrm{PT}\}$ of the full Lorentz group is isomorphic to the Klein 4 -group $\{\mathbf{I}, \mathbf{R}, \mathrm{G}, \mathrm{B}\}$, in which the three colors play equivalent roles.
... there is no reason to classify one of them with the identity I.
It is more natural to classify P, T and PT all together, as indirect isometries, rather than to classify PT with I , as a direct isometry.

## THE LEFT/RIGHT CLASSIFICATION IS MEANINGLESS



A planar set of 4 points is continuously deformated until it is congruent to its mirror image: the sets (A) and (D) are mirror images through the first bisector. But, when going from (A) to (D), all encountered sets are chiral, so (A) and (D) must fall in the same class !

From (A) to (B): $\mathrm{x}_{1}=(0,0), \mathrm{x}_{2}=(2,0), \mathrm{x}_{3}=(0,2+2 \alpha), \mathrm{x}_{4}=(-1,0)$, with $\alpha \in\{0,1\}$. From (B) to (C): $\mathrm{x}_{1}=(\mathbf{0}, \mathbf{0}), \mathrm{x}_{2}=(2,0), \mathrm{x}_{3}=(0,4), \mathrm{x}_{4}=(-1+\beta,-\beta)$, with $\beta \in\{0,1\}$. From (C) to (D): $\mathrm{x}_{1}=(0,0), \mathrm{x}_{2}=(2,0), \mathrm{x}_{3}=(0,4-2 \gamma), \mathrm{x}_{4}=(0,-1)$, with $\gamma \in\{0,1\}$. (Petitjean, M. Symmetry, antisymmetry and chirality: use and misuse of terminology. Symmetry 2021, 13, 603; DOI 10.3390/sym13040603)

This comes from a general result established by Mezey.
(Mezey, P.G. Rules on chiral and cchiral molecular transformations. J. Math. Chem. 1995, 17(2) 185-202 and 1995, 18 (2), 133-139)

## CONCLUSION

## Chirality received a mathematical definition

This definition works in very general spaces
It recovers Kelvin's definition

It is expected that it helps to avoid conceptual ambiguities in the literature about chirality.


[^0]:    ${ }^{*}$ The metric is not necessarily a true one

