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► **To cite this version:**

Lukas Spiegelhofer, Thomas Stoll. The sum-of-digits function on arithmetic progressions. Moscow Journal of Combinatorics and Number Theory, 2020, Vol. 9 (No. 1), pp.43-49. 10.2140/moscow.2020.9.43 . hal-03283478

HAL Id: hal-03283478

<https://hal.science/hal-03283478>

Submitted on 10 Jul 2021

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THE SUM-OF-DIGITS FUNCTION ON ARITHMETIC PROGRESSIONS

LUKAS SPIEGELHOFER AND THOMAS STOLL

ABSTRACT. Let s_2 be the sum-of-digits function in base 2, which returns the number of non-zero binary digits of a nonnegative integer n . We study s_2 along arithmetic subsequences and show that — up to a shift — the set of m -tuples of integers that appear as an arithmetic subsequence of s_2 has full complexity.

1. RESULTS

The binary sum-of-digits function s_2 is an elementary object studied in number theory. It is defined by the equation

$$s_2(\varepsilon_\nu 2^\nu + \cdots + \varepsilon_0 2^0) = \varepsilon_\nu + \cdots + \varepsilon_0,$$

where $\varepsilon_i \in \{0, 1\}$ for $0 \leq i \leq \nu$. Despite the simplicity of definition, the behaviour of s_2 on arithmetic progressions is not fully understood. Cusick’s conjecture on the sum-of-digits function [3, 9] concerns this area of research: for an integer $t \geq 0$, we define the limit

$$c_t = \lim_{N \rightarrow \infty} \frac{1}{N} |\{n : 0 \leq n < N, s_2(n+t) \geq s_2(n)\}|.$$

(The limit exists, see for example Bésineau [2]. In fact, the set in this definition is periodic with period 2^k for some k .) Cusick’s conjecture states that

$$(1.1) \quad c_t > 1/2$$

for all $t \geq 0$. Drmota, Kauers, and the first author [3] proved that $c_t > 1/2$ for *almost all* t in the sense of asymptotic density; we also wish to note the works by Emme and Prikhod’ko [6] and Emme and Hubert [4, 5], and the recent partial result by the first author [9].

In the current note, motivated by Cusick’s conjecture, we are concerned with the $(m+1)$ -tuple $(s_2(n), s_2(n+t), \dots, s_2(n+mt))$, where $t \geq 0$ and $m \geq 1$ are integers. We aim to understand the set of tuples that can occur, as n and t run. In fact, our theorem states that, up to a shift, all tuples occur.

Theorem 1.1. *Assume that $k_1, \dots, k_m \in \mathbb{Z}$. There exist n and t such that for $1 \leq \ell \leq m$,*

$$k_\ell = s_2(n + \ell t) - s_2(n).$$

2010 *Mathematics Subject Classification.* 11A63, 11B25.

The authors acknowledge support by the project MuDeRa, which is a joint project between the FWF (Austrian Science Fund) and the ANR (Agence Nationale de la Recherche). Moreover, the first author was supported by the FWF project F5502-N26, which is a part of the Special Research Program “Quasi Monte Carlo methods: Theory and Applications”; the second author was supported by the project ANR-18-CE40-0018.

This is a generalization of the statement that the Thue–Morse sequence \mathbf{t} has full *arithmetic complexity*, meaning that every finite word $\omega \in \{0, 1\}^L$ occurs as an arithmetic subsequence of \mathbf{t} . This was first proved in [1] and also follows from Müllner and the first named author [8], and Konieczny [7].

Theorem 1.1 is not hard to prove for $m = 1$. We present three arguments leading to this fact.

- (1) Assume first that $k \geq 0$. Set $n = 2^{k+1}$ and $t = 2^k - 1$. Then $s_2(n+t) = k+1$ and $s_2(n) = 1$, yielding $k = s_2(n+t) - s_2(n)$. If $k < 0$, we set $n = 2^{-k+1} - 1$ and $t = 1$. Then $s_2(n) = -k+1$ and $s_2(n+t) = 1$, which yields $s_2(n+t) - s_2(n) = k$. Alternatively, we may also write, as in the case $m = 2$ presented below, $t = 2^c - 1$ and $n = 2^{c-1}(2^a - 1)$, for positive integers a and c . We obtain $s_2(n+t) = c$ and $s_2(n) = a$, and clearly the difference $c - a$ runs through all integers.
- (2) We have $s_2(n+1) - s_2(n) = 1 - \nu_2(n+1) \leq 1$, where $\nu_2(m) = \max\{k \geq 0 : 2^k \mid m\}$ for $m \geq 1$ is the 2-adic valuation of m . This formula follows by considering the number of 1s with which the binary expansion of n ends. Since $s_2(2^\ell) = 1$ and $s_2(2^{\ell+1} - 1) = \ell + 1$, we obtain the fact that $s_2(n)$ attains all values in $\{1, \dots, \ell + 1\}$ as n varies in $\{2^\ell, \dots, 2^{\ell+1} - 1\}$. Let $k \in \mathbb{Z}$ be given and set $\ell = 2|k|$. Choose $n \in \{2^\ell, \dots, 2^{\ell+1} - 1\}$ such that $s_2(n) = |k| + 1$ and $n' \in \{2^{\ell+1}, \dots, 2^{\ell+2} - 1\}$ such that $s_2(n') = |k| + 1 + k$. Then $s_2(n') - s_2(n) = k$, which implies the statement.
- (3) Consider the densities

$$\delta(k, t) = \lim_{N \rightarrow \infty} \frac{1}{N} |\{n : 0 \leq n < N, s_2(n+t) - s_2(n) = k\}|$$

(as it was the case for c_t , this asymptotic density exists [2]). These quantities satisfy the following recurrence [3]:

$$\begin{aligned} \delta(k, 1) &= \begin{cases} 2^{k-2}, & k \leq 1; \\ 0 & \text{otherwise;} \end{cases} \\ \delta(k, 2t) &= \delta(k, t); \\ \delta(k, 2t+1) &= \frac{1}{2}\delta(k-1, t) + \frac{1}{2}\delta(k+1, t+1). \end{aligned}$$

From this, it is very easy to show that $\delta(k, t) > 0$ for all $k \leq s_2(t)$. For k given, choose t in such a way that $s_2(t) \geq k$; the positivity of the density $\delta(k, t)$ implies that there exists an n such that $s_2(n+t) - s_2(n) = k$.

For $m = 2$, it is also possible to obtain the statement by elementary considerations: consider integers $a, c \geq 1, b, d \geq 0$ and choose the integers n and t in such a way that the binary expansions look as follows:

$$\begin{array}{r} n : \quad \overbrace{1 \cdots 1}^a 1 0 \cdots 0 \quad \overbrace{1 \cdots 1}^b 0 \cdots 0 \\ t : \quad \quad \quad \underbrace{1 1 \cdots 1}_c 0 \cdots 0 \quad \underbrace{0 1 \cdots 1}_d. \end{array}$$

The sums of digits of n , $n+t$ and $n+2t$ respectively are $a+b$, $b+c+d$ and $c+d$ respectively. By varying the variables, we can obtain the statement for all integers k_1

and k_2 such that $k_2 \leq k_1$. For the case $k_1 < k_2$, we use the following configuration of the integers n and t , where $a, d \geq 1$ and $c \geq 0$:

$$\begin{array}{r} n : \quad \quad \quad \overbrace{1 \cdots 110 \cdots 0}^a \\ t : \quad \underbrace{1 \cdots 10 \cdots 0}_{c} 1 \underbrace{1 \cdots 1}_d \end{array}$$

The sums of digits of n , $n+t$ and $n+2t$ are a , d and $c+d$ respectively, and we see that we obtain all pairs $(k_1, k_2) \in \mathbb{Z}^2$ such that $k_1 \leq k_2$.

However, the method quickly experiences difficulties, as multiplication by 3 is not a shift of the binary digits anymore. While we believe that the case $m = 3$ can be made work by some effort, a general principle is not apparent. Therefore we choose a different approach.

We prove Theorem 1.1 by induction on m , the cases $m = 1, 2$ having been discussed above. Assume that $m \geq 3$ and let $k_1, \dots, k_m \in \mathbb{Z}$ be given. By induction hypothesis, there exist t_0 and n_0 such that $k_\ell = s_2(n_0 + \ell t_0) - s_2(n_0 + (\ell - 1)t_0)$ for $1 \leq \ell < m$. Set $k'_m = s_2(n_0 + m t_0) - s_2(n_0 + (m - 1)t_0)$. We are going to show that we may vary k'_m by steps of ± 1 , thus yielding the full statement.

By concatenation of binary expansions, it is sufficient to show the following statement.

$$(1.2) \quad \begin{array}{l} \text{There exist } t_1, n_1 \text{ such that } s_2(n_1 + \ell t_1) - s_2(n_1 + (\ell - 1)t_1) = 0 \text{ for } 1 \leq \ell < m \\ \text{and } s_2(n_1 + m t_1) - s_2(n_1 + (m - 1)t_1) = \pm 1. \end{array}$$

This concatenation is straightforward and summarized in the following lemma, which we will also use again in a moment.

Lemma 1.2. *Let $\ell \geq 1$, $m \geq 1$, n_0, \dots, n_{k-1} and t_0, \dots, t_{k-1} be nonnegative integers. There exist nonnegative integers n and t such that*

$$s_2(n + \ell t) - s_2(n + (\ell - 1)t) = \sum_{0 \leq j < k} (s_2(n_j + \ell t_j) - s_2(n_j + (\ell - 1)t_j))$$

for $1 \leq \ell \leq m$.

Proof. The base case $k = 1$ is trivial; it is sufficient to prove the statement for $k = 2$, the general case following easily from repeated application of this case.

Let N be so large that $n_0 + m t_0 < 2^N$, and set $n = 2^N n_1 + n_0$ and $t = 2^N t_1 + t_0$. Since no carry propagation between the digits below and above N occurs, we can add up the contribution of the two blocks in order to yield the statement. \square

We reduce the problem further, using this block representation again: choose $t_j = 1$ for all $0 \leq j < k$; it is sufficient to find a $k \geq 1$ and nonnegative integers n_j for $0 \leq j < k$ such that

$$(1.3) \quad \sum_{0 \leq j < k} (s_2(n_j + \ell) - s_2(n_j + \ell - 1)) = \begin{cases} 0, & \text{if } 1 \leq \ell < m; \\ \pm 1, & \text{if } \ell = m. \end{cases}$$

In order to show (1.3), we use the telescoping sum

$$\sum_{a \leq j < a+2^L} g(j) = s_2(a + 2^L) - s_2(a) = g(\lfloor a/2^L \rfloor),$$

where $g(j) = s_2(j+1) - s_2(j)$. This representation yields for $1 \leq \ell \leq m$, where L is chosen such that $2^L \leq m < 2^{L+1}$,

$$\begin{aligned} \sum_{2 \cdot 2^L - m + \ell \leq j < 3 \cdot 2^L - m + \ell} g(j) &= g(2 + \lfloor (-m + \ell)/2^L \rfloor) = \begin{cases} g(1) = 0, & \text{if } 1 \leq \ell < m; \\ g(2) = 1, & \text{if } \ell = m; \end{cases} \\ \sum_{2^L - m + \ell \leq j < 2 \cdot 2^L - m + \ell} g(j) &= g(1 + \lfloor (-m + \ell)/2^L \rfloor) = \begin{cases} g(0) = 1, & \text{if } 1 \leq \ell < m; \\ g(1) = 0, & \text{if } \ell = m; \end{cases} \\ \sum_{3 \cdot 2^{L+1} + \ell \leq j < 4 \cdot 2^{L+1} + \ell} g(j) &= g(3) = -1 \text{ for } 1 \leq \ell \leq m. \end{aligned}$$

The first of these three identities yields the “+”-part of (1.3) by choosing $k = 2^L$ and $n_j = 2 \cdot 2^L - m + j$ for $0 \leq j < k$.

The “-”-part is obtained from the second and third identities: by considering the disjoint union $J = [2^L - m, 2 \cdot 2^L - m) \cup [3 \cdot 2^{L+1}, 4 \cdot 2^{L+1})$, we have

$$\sum_{j \in J} (s_2(j + \ell) - s_2(j + \ell - 1)) = \begin{cases} 0, & \text{if } 1 \leq \ell < m; \\ -1, & \text{if } \ell = m. \end{cases}$$

The statement follows by merging the two intervals and choosing n_j accordingly. This finishes the proof of our theorem.

2. POSSIBLE EXTENSIONS

From our proof, it is possible to effectively construct integers n and t such that $s_2(n + \ell t) - s_2(n) = k_\ell$ for $1 \leq \ell \leq m$. In particular, this yields integers n and t such that $\mathbf{t}_{n+\ell t} = \omega_\ell$ for $1 \leq \ell \leq m$, where $(\omega_1, \dots, \omega_m) \in \{0, 1\}^m$ and \mathbf{t} is the Thue–Morse sequence on $\{0, 1\}$. (Note that we also used $\mathbf{t}(2^\lambda + n) = 1 - \mathbf{t}(n)$ for $n < 2^\lambda$.) This gives a constructive result concerning the problem of full arithmetic complexity of the Thue–Morse sequence considered in [1, 7, 8].

As an extension of the presented line of research, we are interested in the proportion of cases in which $s_2(n + \ell t) - s_2(n) = k_\ell$ occurs (for $1 \leq \ell \leq m$). For this, we define more generally

$$\delta(\mathbf{k}, \boldsymbol{\varepsilon}, t) = \text{dens} \{n \in \mathbb{N} : s_2(n + \ell t + \varepsilon_\ell) - s_2(n) = k_\ell \text{ for } 1 \leq \ell \leq m\},$$

where $\mathbf{k} = (k_1, \dots, k_m) \in \mathbb{Z}^m$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m) \in \mathbb{N}^m$. This generalizes the array δ defined before. As in the one-dimensional case, the densities in this definition actually exist, and they satisfy the following recurrence relation:

$$\delta(\mathbf{k}, \boldsymbol{\varepsilon}, 2t) = \frac{1}{2} \delta(\mathbf{k}', \boldsymbol{\varepsilon}', t) + \frac{1}{2} \delta(\mathbf{k}'', \boldsymbol{\varepsilon}'', t),$$

where $k'_\ell = k_\ell - (\varepsilon_\ell \bmod 2)$, $k''_\ell = k_\ell + 1 - ((\varepsilon_\ell + 1) \bmod 2)$, $\varepsilon'_\ell = \lfloor \varepsilon_\ell/2 \rfloor$ and $\varepsilon''_\ell = \lfloor (\varepsilon_\ell + 1)/2 \rfloor$; moreover,

$$\delta(\mathbf{k}, \boldsymbol{\varepsilon}, 2t + 1) = \frac{1}{2} \delta(\mathbf{k}', \boldsymbol{\varepsilon}', t) + \frac{1}{2} \delta(\mathbf{k}'', \boldsymbol{\varepsilon}'', t),$$

where $k'_\ell = k_\ell - ((\varepsilon_\ell + \ell) \bmod 2)$, $k''_\ell = k_\ell + 1 - ((\varepsilon_\ell + \ell + 1) \bmod 2)$, $\varepsilon'_\ell = \lfloor (\varepsilon_\ell + \ell)/2 \rfloor$ and $\varepsilon''_\ell = \lfloor (\varepsilon_\ell + \ell + 1)/2 \rfloor$. This recurrence is the reason for the introduction of $\boldsymbol{\varepsilon}$ in the

definition of $\delta(\mathbf{k}, \boldsymbol{\varepsilon}, t)$. To our knowledge, these identities are new; we give the idea of proof. We have

$$\begin{aligned} \delta(\mathbf{k}, \boldsymbol{\varepsilon}, 2t) &= \frac{1}{2} \text{dens}\{n \in \mathbb{N} : s(2n + 2\ell t + \varepsilon_\ell) - s(2n) = k_\ell \text{ for } 1 \leq \ell \leq m\} \\ &\quad + \frac{1}{2} \text{dens}\{n \in \mathbb{N} : s(2n + 2\ell t + \varepsilon_\ell + 1) - s(2n + 1) = k_\ell \text{ for } 1 \leq \ell \leq m\}, \end{aligned}$$

and using the identities $s(2n) = s(n)$ and $s(2n + 1) = s(n) + 1$, we obtain

$$s(2n + 2\ell t + \varepsilon_\ell) = s(2(n + \ell t + \lfloor \varepsilon_\ell/2 \rfloor) + (\varepsilon_\ell \bmod 2)) = s(n + \ell t + \varepsilon'_\ell) + (\varepsilon_\ell \bmod 2),$$

therefore $s(2n + 2\ell t + \varepsilon_\ell) - s(2n) = k_\ell$ is equivalent to $s(n + \ell t + \varepsilon'_\ell) - s(n) = k'_\ell$. Analogous computations are valid for ε_ℓ replaced by $\varepsilon_\ell + 1$, $\varepsilon_\ell + \ell$ and $\varepsilon_\ell + \ell + 1$ respectively, which yields the claim.

This recurrence can be used to prove statements on the densities $\delta(\mathbf{k}, \boldsymbol{\varepsilon}, t)$. The general intuitive idea is that the differences $s(n + jt) - s(n)$, for $1 \leq j \leq m$, should be *almost independent for most t* ; in the light of this consideration, we consider generalizations of Cusick's conjecture and of Emme and Hubert's result [4]. On the one hand, we may ask for multidimensional generalizations of (1.1), relating the relative sizes of the values $s_2(n), s_2(n + t), \dots, s_2(n + mt)$ to one another. We propose the following conjecture, extending (1.1).

Conjecture 1. Assume that $m \geq 1$ is an integer. For an integer $t \geq 0$, define

$$c_t^{(m)} = \text{dens}\{n \in \mathbb{N} : s(n) \leq s(n + jt) \text{ for } 1 \leq j \leq m\}.$$

Then for all $t \geq 0$,

$$c_t^{(m)} > \frac{1}{2^m}.$$

The statement is wrong for any larger constant in place of $1/2^m$. Also, define

$$C_t^{(m)} = \text{dens}\{n \in \mathbb{N} : s(n) \leq s(n + t) \leq s(n + 2t) \leq \dots \leq s(n + mt)\}.$$

Then for all $t \geq 0$,

$$C_t^{(m)} > \frac{1}{2^m m!}.$$

The constant $1/(2^m m!)$ is maximal.

On the other hand, we could ask for the overall shape of the m -dimensional probability distribution defined by $\delta(\cdot, \boldsymbol{\varepsilon}, t)$.

Problem 1. Prove a multidimensional generalization of the theorem by Emme and Hubert [4]: for most t , the densities $\text{dens}\{n \in \mathbb{N} : s_2(n + \ell t) - s_2(n) = k_\ell \text{ for } 1 \leq \ell \leq m\}$ should define a probability distribution that is close to a multivariate Gaussian law.

We can now understand the intuition behind the constants in Conjecture 1: a bivariate normal distribution with mean $(0, 0)$ has one quarter of its total weight in the quadrant $\{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y\}$, and analogous considerations hold for higher dimensions. Concerning the values $C_t^{(2)}$, a bivariate normal distribution with mean $(0, 0)$ has one *eighth* of its total weight in the octant $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq y\}$, which corresponds to the complex closed region $\{re^{2\pi i x} : r \geq 0, 1/8 \leq x \leq 1/4\}$. Noting the fact that $s(n + 2t) \geq s(n + t)$ if and only if $s(n + 2t) - s(n) \geq s(n + t) - s(n)$, we see

the link between the densities $\delta(\mathbf{k}, \varepsilon, t)$ with \mathbf{k} lying in this octant and the values $C_t^{(2)}$. Concerning higher dimensions, we note that the $m!$ sets

$$\{x \in \mathbb{R}^m : \|m\| \leq 1, 0 \leq x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(m)}\},$$

where σ is a permutation of $\{1, \dots, m\}$, are unions of line segments $[0, z]$, where $\|z\| = 1$; they are pairwise congruent (using a linear transformation that is a permutation matrix in the canonical base); an intersection of two distinct sets of this form has measure zero, and their union comprises a fraction $1/2^m$ of the unit ball in m dimensions.

We implemented the computation of the densities $\delta(\mathbf{k}, \varepsilon, t)$ for two dimensions in the Sage computer algebra system [11], and the resulting Sage worksheet is available from the website of the first named author [10]. We obtain $c_t^{(2)} \geq c_{951}^{(2)} = 94299/262144 = 0.3597\dots > 1/4$ for $t \leq 2^{10}$ and $C_t^{(2)} \geq c_{991}^{(2)} = 43947/262144 = 0.1676\dots > 1/8$ for $t \leq 2^{10}$. The implementation involves nine two-dimensional arrays of rational numbers (corresponding to the nine possibilities $(\varepsilon_1, \varepsilon_2) \in \{0, 1, 2\}^2$) and each of these calculations took about 5 minutes on a standard machine. We note that we did not optimize the Sage code, and certainly this computation can be sped up significantly.

Other conjectures similar to Conjecture 1 are conceivable: what about the other octants and quadrants A in the plane (including the borders)? Is it always true that $1/8$ resp. $1/4$ is a lower bound for the density $\text{dens}\{n \in \mathbb{N} : (s(n+t) - s(n), s(n+2t) - s(n)) \in A\}$? We leave this question open for future discussion, but we note that the analogous problem in one dimension is true for almost all t : we also have

$$\tilde{c}_t = \text{dens}\{n \in \mathbb{N} : s(n+t) \leq s(n)\} > 1/2$$

for t in a set having asymptotic density 1 [3]. In other words, usually the median of the probability distribution defined by $\delta(\cdot, t)$ is very close to the mean value (which is 0). We believe that $\tilde{c}_t \geq 1/2$ is true for all t , which complements Cusick's conjecture (1.1) (note that $\tilde{c}_1 = 1/2$, thus there is no strict inequality).

We expect that nontrivial statements on both Conjecture 1 and Problem 1, at least for small m , can be obtained by extending the study of moments set forward by Emme and Hubert [4]. This is certainly not easy and will introduce technical difficulties that have to be surmounted.

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