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► **To cite this version:**

| Thomas Ehrhard. Coherent differentiation. 2021. hal-03282799v1

HAL Id: hal-03282799

<https://hal.science/hal-03282799v1>

Preprint submitted on 9 Jul 2021 (v1), last revised 27 Nov 2021 (v4)

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Coherent differentiation

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July 9, 2021

Abstract

The categorical models of the differential lambda-calculus are additive categories because of the Leibniz rule which requires the summation of two expressions. This means that, as far as the differential lambda-calculus and differential linear logic are concerned, these models feature finite non-determinism and indeed these languages are essentially non-deterministic. We introduce a categorical framework for differentiation which does not require additivity and is compatible with deterministic models such as coherence spaces and probabilistic models such as probabilistic coherence spaces. Based on this semantics we sketch the syntax of a deterministic version of the differential lambda-calculus.

Introduction

The differential λ -calculus has been introduced in [ER03], starting from earlier investigations on the semantics of Linear Logic (LL) in models based on various kinds of topological vector spaces [Ehr05, Ehr02]. Later on the same authors proposed in [ER04, Ehr18] an extension of LL featuring differential operations which appear as an additional structure of the exponentials (the resource modalities of LL), offering a perfect duality to the standard rules of dereliction, weakening and contraction. The differential λ -calculus and differential LL are about computing formal derivatives of programs and from this point of view are deeply connected to the kind of formal differentiation of programs used in Machine Learning for propagating gradients (that is, differentials viewed as vectors of partial derivatives) within formal neural networks. As shown by the recent [BMP20, MP21] formal transformations of programs related to the differential λ -calculus can be used for efficiently implementing gradient back-propagation in a purely functional framework. The differential λ -calculus and the differential linear logic are also useful as the foundation for an approach to finite approximations of programs based on the Taylor expansion [ER08, BM20] which provides a precise analysis of the use of resources during the execution of a functional program deeply related with implementations of the λ -calculus in abstract machines such as the Krivine Machine [ER06].

One should insist on the fact that in the differential λ -calculus derivatives are not taken wrt. to a ground type of real numbers as in [BMP20, MP21] but can be computed wrt. elements of all types. For instance it makes sense to compute the derivative of a function $M : (\iota \Rightarrow \iota) \rightarrow \iota$ wrt. its argument which is a function $\iota \rightarrow \iota$ (where ι is the type of integers) thus suggesting the possibility of using this formalism for optimization purposes in a model such as probabilistic coherence spaces [DE11] (PCS) where a program of type $\iota \rightarrow \iota$ is seen as an analytic function transforming probability distributions on the integers. In [Ehr19] it is also shown how such derivatives can be used to compute the expectation of the number of steps in the execution of a program. A major obstacle on the use of these derivatives is the fact that probabilistic coherence spaces are not a model of the differential λ -calculus in spite of the fact that the morphisms, being analytic, are obviously differentiable. The main goal of this paper being to circumvent this obstacle, let us first understand it better.

These formalisms require the possibility of adding terms of the same type. For instance, to define the operational semantics of the differential λ -calculus, given a term t such that $x : A \vdash t : B$ and a term u such that $\Gamma \vdash u : A$ one has to define a term $\frac{\partial t}{\partial x} \cdot u$ such that $\Gamma, x : A \vdash \frac{\partial t}{\partial x} \cdot u : B$ which can be understood as a *linear substitution* of u for x in t and is actually a formal differentiation: x has no reason to occur linearly in t so this operation involves the creation of linear occurrences of x in t and

this is done applying the rules of ordinary differential calculus. The most important case is when t is an application $t = (t_1)t_2$ where $\Gamma, x : A \vdash t_1 : C \Rightarrow B$ and $\Gamma, x : A \vdash t_2 : C$. In that case we set

$$\frac{\partial(t_1)t_2}{\partial x} \cdot u = \left(\frac{\partial t_1}{\partial x} \cdot u\right)t_2 + (Dt_1 \cdot \left(\frac{\partial t_2}{\partial x} \cdot u\right))t_2$$

where we use *differential application* which is a syntactic construct of the language: given $\Gamma \vdash s : C \Rightarrow B$ and $\Gamma \vdash v : C$, we have $\Gamma \vdash Ds \cdot v : C \Rightarrow B$. This crucial definition involves a sum corresponding to the fact that x can appear free in t_1 and in t_2 : this is the essence of the ‘‘Leibniz rule’’ $(fg)' = f'g + fg'$ which has nothing to do with multiplication but everything with the fact that both f and g can have non-zero derivatives wrt. a common variable they share (logically this sharing is implemented by a contraction rule).

For this reason the syntax of the differential λ -calculi and linear logic features an addition operation on terms of the same type and accordingly the categorical models of these formalisms are based on additive categories. Operationally such sums correspond to a form of finite non-determinism: for instance if the language has a ground type of integers ι with constants \underline{n} such that $\Gamma \vdash \underline{n} : \iota$ for each $n \in \mathbb{N}$, we are allowed to consider sums such as $\underline{42} + \underline{57}$ corresponding to the non-deterministic superposition of the two integers (and not at all to their sum $\underline{99}$ in the usual sense!). This can be considered as a weakness of this approach since, even if one has nothing against non-determinism *per se* it is not satisfactory to be obliged to enforce it for allowing differential operations which have nothing to do with it *a priori*. So the fundamental question is:

Does any logical approach to differentiation require non-determinism?

We ground our negative answer to this question on the observation made in [Ehr19] that, in the category of PCS, morphisms of the associated cartesian closed category are analytic functions and therefore admit all iterated derivatives (at least in the ‘‘interior’’ of the domain where they are defined). Consider for instance in this category an analytic $f : 1 \rightarrow 1$ where 1 (the \otimes unit of LL) is the $[0, 1]$ interval, meaning that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ with coefficient $a_n \in \mathbb{R}_{\geq 0}$ such that $\sum_{n=0}^{\infty} a_n x^n \leq 1$. The derivative $f'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n$ has no reason to map $[0, 1]$ to $[0, 1]$ and can even be unbounded on $[0, 1]$ and undefined at $x = 1$ (and there are programs whose interpretation behaves in that way). Though, if $(x, u) \in [0, 1]^2$ satisfy $x + u \in [0, 1]$ then $f(x) + f'(x)u \leq f(x + u) \in [0, 1]$. This is true actually of any analytic morphism f between two PCSs X and Y : we can see the differential of f as mapping a summable pair (x, u) of elements of X to the summable pair $(f(x), f'(x) \cdot u)$ of elements of Y . Seeing the differential as such a pair of functions is central in differential geometry as it allows, thanks to the chain rule, to turn it into a *functor* mapping a smooth map $f : X \rightarrow Y$ (where X and Y are now manifolds) to the function $\mathbb{T}f : \mathbb{T}X \rightarrow \mathbb{T}Y$ which maps (x, u) to $(f(x), f'(x) \cdot u)$ where $\mathbb{T}X$ is the tangent bundle of X , a manifold whose elements are the pairs (x, u) of a point x of X and of a vector u tangent to X at x . The concept of *tangent category* has been introduced in [Ros84, CC14] precisely to describe categorically this construction and its properties. In spite of this formal similarity our central concept of summability cannot be compared with tangent categories in terms of generality, first because when $(x, u) \in \mathbb{T}X$ it makes no sense to add x and u or to consider u alone (independently of x), and second because, given $(x, u_0), (x, u_1) \in \mathbb{T}X$, the local sum $(x, u_0 + u_1) \in \mathbb{T}X$ is always defined in the tangent bundle, whereas in the summability setting, the pair (u_0, u_1) has no reason to be summable.

Content We base our approach on a concept of summable pair that we axiomatize as a general categorical notion in Section 2: a *summable category* is a category \mathcal{L} with 0-morphisms together with a functor $\mathbf{S} : \mathcal{L} \rightarrow \mathcal{L}$ equipped with three natural transformations from $\mathbf{S}X$ to X : two projections and a sum operation. The first projection also exists in the ‘‘tangent bundle’’ functor of a tangent category but the two other morphisms do not. Such a summability structure induces a monad structure on \mathbf{S} (a similar phenomenon occurs in tangent categories). In Section 3 we consider the case where the category is cartesian SMC equipped with a resource comonad $!$ in the sense of LL where we present differentiation as a distributive law between \mathbf{S} and $!$. This allows to lift \mathbf{S} to a strong monad \mathbf{D} on $\mathcal{L}!$ which implements differentiation of non-linear maps. In Section 4 we study the case where the functor \mathbf{S} can be defined using a more basic structure of \mathcal{L} based on the object $1 \& 1$ where $\&$ is the cartesian product and 1 is the unit of \otimes : this is actually what happens in the concrete situations we have in mind. Then the existence of the summability structure becomes a *property* of \mathcal{L} and not an additional structure. We also

study the differential structure in this setting. Last in Section 7 we propose a syntax for a differential λ -calculus corresponding to this semantics.

As a running example along the presentation of our categorical constructions we use the category of coherence spaces, the first model of LL historically [Gir87]. There are many reasons for this choice. It is one of the most popular models of LL and of functional languages, it is a typical example of a model of LL which is not an additive category (in contrast with the relational model or the models of profunctors), *a priori* it does not exhibit the usual features of a model of the differential calculus (no coefficients, no vector spaces etc) and it strongly suggests that our coherent approach to the differential λ -calculus might be applied to programming languages which have nothing to do with probabilities, deep learning or non-determinism.

Related work As already mentioned our approach has strong similarities with tangent categories which have been a major source of inspiration, we explained above the differences. There are also strong connections with differential categories [BCLS20] with the main difference again that differential categories are left-additive which is generally not the case of $\mathcal{L}_!$ in our case, we explained why. There are also interesting similarities with [CLL20] (still in an additive setting): our distributive law ∂_X might play a role similar to the one of the distributive law introduced in the Section 5 of that paper. This needs further investigations.

Recently [KP20] have exhibited a striking connection between Gödel’s Dialectica interpretation and the differential λ -calculus and differential linear logic, with applications to gradient back-propagation in differential programming. One distinctive feature of Pédrot’s approach to Dialectica [Péd15] is to use a “multiset parameterized type” \mathfrak{M} whose purpose seems to provide some control on the summations allowed when performing Pédrot’s analogue of the Leibniz rule (under the Dialectica/differential correspondence of [KP20]) and might therefore play a role similar to our summability functor \mathbf{S} . The precise technical connection is not clear at all but we believe that this analogy will lead to a unified framework for Dialectica interpretation and coherent differentiation of programs and proofs involving denotational semantics, proof theory and differential programming.

The differential λ -calculus that we obtain in Section 7 features strong similarities with the calculus introduced in [BMP20, MP21] for dealing with gradient propagation in a functional setting. Both calculi handle tuples of terms in the spirit of tangent categories which allows to make the chain rule functorial thus allowing to reduce differential terms without creating explicit summations.

Several proofs which are not in the main text can be found in the Appendix.

1 Preliminary notions

1.1 Finite multisets

A finite multiset on a set A is a function $m : A \rightarrow \mathbb{N}$ such that the set $\text{supp}(m) = \{a \in A \mid m(a) \neq 0\}$ is finite, we use $\mathcal{M}_{\text{fin}}(A)$ for the set of all finite multisets of elements of A . The cardinality of m is $\#m = \sum_{a \in A} m(a)$. We use $[\]$ for the empty multiset (so that $\mathbf{D}([\]) = \emptyset$) and if $m_0, m_1 \in \mathcal{M}_{\text{fin}}(A)$ then $m_0 + m_1 \in \mathcal{M}_{\text{fin}}(A)$ is defined by $(m_0 + m_1)(a) = m_0(a) + m_1(a)$. If $a_1, \dots, a_n \in A$ we use $[a_1, \dots, a_n]$ for the $m \in \mathcal{M}_{\text{fin}}(A)$ such that $m(a)$ is the number of $i \in \{1, \dots, n\}$ such that $a_i = a$. If $[a_1, \dots, a_n] \in \mathcal{M}_{\text{fin}}(A)$ and $b_1, \dots, b_p \in \mathcal{M}_{\text{fin}}(B)$ then $m \times p = [(a_i, b_j) \mid i \in \{1, \dots, n\} \text{ and } j \in \{1, \dots, p\}] \in \mathcal{M}_{\text{fin}}(A \times B)$.

1.2 The SMCC of pointed sets

Let \mathbf{Set}_0 be the category of pointed sets. We use 0_X or simply 0 for the distinguished point of the object X . A morphism $f \in \mathbf{Set}_0(X, Y)$ is a function $f : X \rightarrow Y$ such that $f(0_X) = 0_Y$. The terminal object is the singleton $\{0\}$. The cartesian product $X \& Y$ is the ordinary cartesian product, with $0_{X \& Y} = (0_X, 0_Y)$. The tensor product $X \otimes Y$ is defined as

$$X \otimes Y = \{(x, y) \in X \times Y \mid x = 0 \Leftrightarrow y = 0\}$$

with $0_{X \otimes Y} = (0_X, 0_Y)$. The unit of the tensor product is the object $1 = \{0, *\}$ of \mathbf{Set}_0 . This category is enriched over itself, the distinguished point of $\mathbf{Set}_0(X, Y)$ being the constantly 0_Y function. Actually,

it is monoidal closed with $X \multimap Y = \mathbf{Set}_0(X, Y)$ and $0_{X \multimap Y}$ defined by $0_{X \multimap Y}(x) = 0_Y$ for all $x \in X$. A mono in \mathbf{Set}_0 is a morphism of \mathbf{Set}_0 which is injective as a function.

Unless explicitly stipulated, all the categories \mathcal{L} we consider in this paper are enriched over pointed sets, so this assumption will not be mentioned any more. In the case of symmetric monoidal categories, this also means that the tensor product of morphisms is “bilinear” wrt. the pointed structure, that is: if $f \in \mathcal{L}(X_0, Y_0)$ then $f \otimes 0 = 0 \in \mathcal{L}(X_0 \otimes X_1, Y_0 \otimes Y_1)$ and by symmetry we have $0 \otimes f = 0$.

2 Summable categories

Let \mathcal{L} be a category; composition in \mathcal{L} is denoted by simple juxtaposition. We develop a categorical axiomatization of a concept of finite summability in \mathcal{L} which will then be a *partially additive category* [AM80]. The main idea is to equip \mathcal{L} with a functor \mathbf{S} which has the flavor of a monad¹ and intuitively maps an object X to the objects $\mathbf{S}X$ of all pairs (x_0, x_1) of elements of X whose sum $x_0 + x_1$ is well defined. This is another feature of our approach which is to give a crucial role to such pairs, which are the values on which derivatives are computed, very much in the spirit of Clifford’s *dual numbers*. However, contrarily to dual numbers our structures also axiomatize the actual summation of such pairs.

Example 2.1. In order to illustrate the definitions and constructions of the paper we will use the category \mathbf{Coh} of coherence spaces [Gir87] as a running example. An object of this category is a pair $E = (|E|, \supseteq_E)$ where $|E|$ is a set (the web of E) and \supseteq_E is a symmetric and reflexive relation on $|E|$. The set of cliques of a coherence space E is

$$\text{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x \ a \supseteq_E a'\}.$$

Equipped with \subseteq as order relation, $\text{Cl}(E)$ is a cpo. Given coherence spaces E and F , we define the coherence space $E \multimap F$ by $|E \multimap F| = |E| \times |F|$ and

$$(a, b) \supseteq_{E \multimap F} (a', b') \text{ if } a \supseteq_E a' \Rightarrow (b \supseteq_F b' \text{ and } b = b' \Rightarrow a = a').$$

Lemma 2.1. *If $s \in \text{Cl}(E \multimap F)$ and $t \in \text{Cl}(F \multimap G)$ then ts (the compositional relation of t and s) belongs to $\text{Cl}(E \multimap G)$ and the diagonal relation ld_E belongs to $\text{Cl}(E \multimap E)$.*

In that way we have turned the class of coherence spaces into a category enriched over pointed sets, with $0 = \emptyset$. This category is cartesian with E_0 & E_1 given by $|E_0| \& |E_1| = \{0\} \times |E_0| \cup \{1\} \times |E_1|$ and $\text{pr}_i = \{(i, a), a \mid a \in |E|\}$ for $i = 0, 1$ and, given $s_i \in \mathbf{Coh}(F, E_i)$ (for $i = 0, 1$),

$$\langle s_0, s_1 \rangle = \{(b, (i, a)) \mid i \in \{0, 1\} \text{ and } (b, a) \in s_i\}.$$

Given $s \in \mathbf{Coh}(E, F)$ and $x \in \text{Cl}(E)$ one defines $s \cdot x \in \text{Cl}(F)$ by $s \cdot x = \{b \in |F| \mid a \in x \text{ and } (a, b) \in s\}$. Given $x_0, x_1 \in \text{Cl}(E)$ we use $x_0 + x_1$ to denote $x_0 \cup x_1$ if $x_0 \cup x_1 \in \text{Cl}(E)$ and $x_0 \cap x_1 = \emptyset$. With these notations observe that

$$s \cdot \emptyset = \emptyset \text{ and } s \cdot (x_0 + x_1) = s \cdot x_0 + s \cdot x_1$$

explaining somehow the terminology “linear maps” for these morphisms.

Definition 2.1. A *pre-summability structure* on \mathcal{L} is a tuple $(\mathbf{S}, \pi_0, \pi_1, \mathbf{s})$ where $\mathbf{S} : \mathcal{L} \rightarrow \mathcal{L}$ is a functor and π_0, π_1 and \mathbf{s} are natural transformation from \mathbf{S} to the identity functor such that for any two morphisms $f, g \in \mathcal{L}(Y, \mathbf{S}X)$, if $\pi_i f = \pi_i g$ for $i = 0, 1$, then $f = g$. In other words, π_0 and π_1 are jointly monic.

Example 2.2. We give a pre-summability structure on coherence spaces. Given a coherence space E , the coherence space $\mathbf{S}(E)$ is defined by $|\mathbf{S}(E)| = \{0, 1\} \times |E|$ and $(i, a) \supseteq_{\mathbf{S}(E)} (i', a')$ if $i = i'$ and $a \supseteq_E a'$, or $i \neq i'$ and $a \sim_E a'$. Remember that $a \supseteq_E a'$ means that $a \supseteq_E a'$ and $a \neq a'$ (strict coherence relation).

Lemma 2.2. *$\text{Cl}(\mathbf{S}E)$ is isomorphic to the poset of all pairs $(x_0, x_1) \in \text{Cl}(E)^2$ such that $x_0 + x_1 \in \text{Cl}(E)$, equipped with the product order.*

¹And will actually be shown to have a canonical monad structure

Given $s \in \mathbf{Coh}(E, F)$, we define $\mathbf{S}s \subseteq |\mathbf{S}E \rightarrow \mathbf{S}F|$ by

$$\mathbf{S}s = \{(i, a), (i, b) \mid i \in \{0, 1\} \text{ and } (a, b) \in s\}.$$

Then it is easy to check that $\mathbf{S}s \in \mathbf{Coh}(\mathbf{S}E, \mathbf{S}F)$ and that \mathbf{S} is a functor. This is due to the definition of s which entails $s \cdot (x_0 + x_1) = s \cdot x_0 + s \cdot x_1$.

The additional structure is defined as follows:

$$\pi_i = \{((i, a), a) \mid a \in |E|\} \text{ and } \mathbf{s} = \{((i, a), a) \mid i \in \{0, 1\} \text{ and } a \in |E|\}$$

which are easily seen to belong to $\mathbf{Coh}(\mathbf{S}E, E)$. Notice that $\mathbf{s} = \pi_0 + \pi_1$. Of course $\pi_i \cdot (x_0, x_1) = x_i$ and $\mathbf{s} \cdot (x_0, x_1) = x_0 + x_1$.

From now on we assume that we are given such a structure. We say that $f_i \in \mathcal{L}(X, Y)$ (for $i = 0, 1$) are *summable* if there is a morphism $g \in \mathcal{L}(X, \mathbf{S}Y)$ such that

$$\begin{array}{ccc} X & \xrightarrow{g} & \mathbf{S}Y \\ & \searrow f_i & \downarrow \pi_i \\ & & Y \end{array}$$

for $i = 0, 1$. By definition of a pre-summability structure there is only one such g if it exists, we denote it as $\langle f_0, f_1 \rangle_{\mathbf{S}}$. When this is the case we set $f_0 + f_1 = \mathbf{s} \langle f_0, f_1 \rangle_{\mathbf{S}} \in \mathbf{Coh}(X, Y)$. We sometimes call $\langle f_0, f_1 \rangle_{\mathbf{S}}$ the *witness of the summability* of f_0 and f_1 and $f_0 + f_1$ their sum.

Example 2.3. In the case of coherence spaces, saying that $s_0, s_1 \in \mathbf{Coh}(E, F)$ are summable simply means that $s_0 \cap s_1 = \emptyset$ and $s_0 \cup s_1 \in \mathbf{Coh}(E, F)$. This property is equivalent to

$$\forall x \in \text{Cl}(X) \quad (s_0 \cdot x, s_1 \cdot x) \in \text{Cl}(\mathbf{S}E)$$

and in that case the witness is defined exactly in the same way as $\langle s_0, s_1 \rangle \in \mathbf{Coh}(E, F \& F)$.

Lemma 2.3. Assume that $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable and that $g \in \mathcal{L}(U, X)$ and $h \in \mathcal{L}(Y, Z)$. Then $h f_0 g$ and $h f_1 g$ are summable with witness $(\mathbf{S}h) \langle f_0, f_1 \rangle_{\mathbf{S}} g \in \mathcal{L}(U, \mathbf{S}Z)$ and sum $(\mathbf{S}h) (f_0 + f_1) g \in \mathcal{L}(U, Z)$.

The proof boils down to the naturality of π_i and \mathbf{s} . An easy consequence is that the application of \mathbf{S} to a morphism can be written as a witness.

Lemma 2.4. If $f \in \mathcal{L}(X, Y)$ then $f \pi_0, f \pi_1 \in \mathcal{L}(\mathbf{S}X, Y)$ are summable with witness $\mathbf{S}f$ and sum f . That is $\mathbf{S}f = \langle f \pi_0, f \pi_1 \rangle_{\mathbf{S}}$.

Now using this notion of pre-summability structure we start introducing additional conditions to define a summability structure.

Notice that by definition π_0 and π_1 are summable with Id as witness and \mathbf{s} as sum. Here is our first condition:

(S-com) π_1 and π_0 are summable and the witness $\langle \pi_1, \pi_0 \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}X, \mathbf{S}X)$ satisfies $\mathbf{s} \langle \pi_1, \pi_0 \rangle_{\mathbf{S}} = \mathbf{s}$.

Notice that this witness is an involutive iso since $\pi_i \langle \pi_1, \pi_0 \rangle_{\mathbf{S}} \langle \pi_1, \pi_0 \rangle_{\mathbf{S}} = \pi_i$ for $i = 0, 1$.

Lemma 2.5. If $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable then f_1, f_0 are summable with witness $\langle \pi_1, \pi_0 \rangle_{\mathbf{S}} \langle f_0, f_1 \rangle_{\mathbf{S}}$ and we have $f_0 + f_1 = f_1 + f_0$.

(S-zero) For any $f \in \mathcal{L}(X, Y)$, the morphisms f and $0 \in \mathcal{L}(X, Y)$ are summable and their sum is f , that is $\mathbf{s} \langle f, 0 \rangle = f$.

By **(S-com)** this implies that 0 and f are summable with $0 + f = f$.

Notice that we have four morphisms $\pi_0 \pi_0, \pi_1 \pi_1, \pi_0 \pi_1, \pi_1 \pi_0 \in \mathcal{L}(\mathbf{S}^2 X, X)$.

Lemma 2.6. If $f, f' \in \mathcal{L}(X, \mathbf{S}^2 Y)$ satisfy $\pi_i \pi_j f = \pi_i \pi_j f'$ for all $i, j \in \{0, 1\}$ then $f = f'$, that is, the $\pi_i \pi_j$ are jointly monic..

This is an easy consequence of the fact that π_0, π_1 are jointly monic.

(S-witness) Let $f_{00}, f_{01}, f_{10}, f_{11} \in \mathcal{L}(X, Y)$ be morphisms such that (f_{00}, f_{01}) and (f_{10}, f_{11}) are summable, and moreover $(f_{00} + f_{01}, f_{10} + f_{11})$ is summable. Then the witnesses $\langle f_{00}, f_{01} \rangle_{\mathbf{S}}, \langle f_{10}, f_{11} \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}X)$ are summable.

The last axiom requires a little preparation. By Lemma 2.3 the pairs of morphisms $(\pi_0\pi_0, \pi_0\pi_1)$ and $(\pi_1\pi_0, \pi_1\pi_1)$ are summable with sums $\pi_0 \mathbf{s}$ and $\pi_1 \mathbf{s}$ respectively. By the same lemma these two morphisms are summable (with sum $\mathbf{s} \mathbf{s} \in \mathcal{L}(\mathbf{S}^2X, X)$). By Axiom **(S-witness)** it follows that the witnesses $\langle \pi_0\pi_0, \pi_0\pi_1 \rangle_{\mathbf{S}}, \langle \pi_1\pi_0, \pi_1\pi_1 \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}^2X, \mathbf{S}X)$ are summable, let $\mathbf{c} = \langle \langle \pi_0\pi_0, \pi_0\pi_1 \rangle_{\mathbf{S}}, \langle \pi_1\pi_0, \pi_1\pi_1 \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}^2X, \mathbf{S}^2X)$ be the corresponding witness which is easily seen to be an involutive natural iso using Lemma 2.6. Notice that \mathbf{c} (which is similar to the flip of a tangent bundle functor) is characterized by

$$\forall i, j \in \{0, 1\} \quad \pi_i \pi_j \mathbf{c} = \pi_j \pi_i.$$

We can now state our last axiom.

(S-assoc) The following diagram commutes.

$$\begin{array}{ccc} \mathbf{S}^2X & \xrightarrow{\mathbf{c}} & \mathbf{S}^2X \\ & \searrow \mathbf{s}_{\mathbf{S}X} & \downarrow \mathbf{s}_{\mathbf{S}X} \\ & & \mathbf{S}X \end{array}$$

Let us see what this condition has to do with associativity of summation.

Lemma 2.7. *Let $f_{00}, f_{01}, f_{10}, f_{11} \in \mathcal{L}(X, Y)$ be morphisms such that (f_{00}, f_{01}) and (f_{10}, f_{11}) are summable, and moreover $(f_{00} + f_{01}, f_{10} + f_{11})$ is summable. Then (f_{00}, f_{10}) and (f_{01}, f_{11}) are summable, $(f_{00} + f_{10}, f_{01} + f_{11})$ is summable and moreover*

$$(f_{00} + f_{01}) + (f_{10} + f_{11}) = (f_{00} + f_{10}) + (f_{01} + f_{11}).$$

Proof. By Axiom **(S-witness)** we have a ‘‘global witness’’ $g = \langle \langle f_{00}, f_{01} \rangle_{\mathbf{S}}, \langle f_{10}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}^2Y)$. Let $g' = \mathbf{c}g \in \mathcal{L}(X, \mathbf{S}^2Y)$. We have $\pi_0\pi_0 g' = f_{00}$ and $\pi_1\pi_0 g' = f_{10}$ which shows that f_{00} and f_{10} are summable with witness $\langle f_{00}, f_{10} \rangle_{\mathbf{S}} = \pi_0 g' \in \mathcal{L}(X, \mathbf{S}Y)$. Similarly f_{01} and $f_{11} \in \mathcal{L}(X, \mathbf{S}Y)$ are summable with witness $\langle f_{01}, f_{11} \rangle_{\mathbf{S}} = \pi_1 g'$. Since π_0 and π_1 are summable, it results from Lemma 2.3 that $\langle f_{00}, f_{10} \rangle_{\mathbf{S}}$ and $\langle f_{01}, f_{11} \rangle_{\mathbf{S}}$ are summable with witness $\langle \langle f_{00}, f_{10} \rangle_{\mathbf{S}}, \langle f_{01}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} = g'$. We have

$$\begin{aligned} \mathbf{S}_{\mathbf{S}X} \langle \langle f_{00}, f_{10} \rangle_{\mathbf{S}}, \langle f_{01}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} &= \langle \mathbf{s}_X \langle f_{00}, f_{10} \rangle_{\mathbf{S}}, \mathbf{s}_X \langle f_{01}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} \text{ by Lemma 2.3} \\ &= \langle f_{00} + f_{10}, f_{01} + f_{11} \rangle_{\mathbf{S}} \end{aligned}$$

On the other hand, by Axiom **(S-assoc)** and by definition of g' we have

$$\begin{aligned} \mathbf{S}_{\mathbf{S}X} \langle \langle f_{00}, f_{10} \rangle_{\mathbf{S}}, \langle f_{01}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} &= \mathbf{s}_{\mathbf{S}X} \langle \langle f_{00}, f_{01} \rangle_{\mathbf{S}}, \langle f_{10}, f_{11} \rangle_{\mathbf{S}} \rangle_{\mathbf{S}} \\ &= \langle f_{00}, f_{01} \rangle_{\mathbf{S}} + \langle f_{10}, f_{11} \rangle_{\mathbf{S}} \end{aligned}$$

so we have shown that

$$\langle f_{00}, f_{01} \rangle_{\mathbf{S}} + \langle f_{10}, f_{11} \rangle_{\mathbf{S}} = \langle f_{00} + f_{10}, f_{01} + f_{11} \rangle_{\mathbf{S}}$$

that is, the summation of summable pairs is performed component wise. Next we have that $\mathbf{s}_X \langle f_{00} + f_{10}, f_{01} + f_{11} \rangle_{\mathbf{S}} = (f_{00} + f_{10}) + (f_{01} + f_{11})$ and, by Lemma 2.3 we know that $\mathbf{s}_X \langle f_{00}, f_{01} \rangle_{\mathbf{S}} = f_{00} + f_{01}$ and $\mathbf{s}_X \langle f_{10}, f_{11} \rangle_{\mathbf{S}} = f_{10} + f_{11}$ are summable with sum equal to $\mathbf{s}_X (\langle f_{00}, f_{01} \rangle_{\mathbf{S}} + \langle f_{10}, f_{11} \rangle_{\mathbf{S}})$. This shows that $(f_{00} + f_{10}) + (f_{01} + f_{11}) = (f_{00} + f_{01}) + (f_{10} + f_{11})$ as contended. \square

Lemma 2.8. *Let $f_0, f_1, f_2 \in \mathcal{L}(X, Y)$ be such that (f_0, f_1) is summable and $(f_0 + f_1, f_2)$ is summable. Then (f_1, f_2) is summable and $(f_0, f_1 + f_2)$ is summable and we have $(f_0 + f_1) + f_2 = f_0 + (f_1 + f_2)$.*

Proof. It suffices to apply Lemma 2.7 to $f_0, f_1, 0, f_2$, using **(S-zero)** for making sure that $(0, f_2)$ is summable, with sum = f_2 . \square

Example 2.4. All these properties are easy to check in coherence spaces and boil down to the standard algebraic properties of set unions.

Definition 2.2. A summability structure on \mathcal{L} is a pre-summability structure which satisfies axioms **(S-com)**, **(S-zero)**, **(S-witness)** and **(S-assoc)**. We call summable category a tuple $(\mathcal{L}, \mathbf{S}, \pi_0, \pi_1, \mathbf{s})$ consisting of a category \mathcal{L} equipped with a summability structure.

We define a general notion of summable family of morphisms $(f_i)_{i=1}^n$ in $\mathcal{L}(X, Y)$ by induction on n :

- if $n = 0$ then $(f_i)_{i=1}^n$ if summable with sum 0
- if $n = 1$ then $(f_i)_{i=1}^n$ if summable with sum f_1
- and $(f_i)_{i=1}^{n+2}$ is summable if f_{n+1}, f_{n+2} are summable and $(f_1, \dots, f_n, f_{n+1} + f_{n+2})$ is summable, and then $f_1 + \dots + f_{n+2} = (f_1 + \dots + f_n) + (f_{n+1} + f_{n+2})$.

Proposition 2.1. For any $\sigma \in \mathfrak{S}_n$ and any family of morphisms $(f_i)_{i=1}^n$, the family $(f_i)_{i=1}^n$ is summable iff the family $(f_{\sigma(i)})_{i=1}^n$ is summable and then $\sum_{i \in I} f_i = \sum_{i \in I} f_{\sigma_i}$.

So we can define a finite family $(f_i)_{i \in I}$ to be summable if any of its enumerations $(f_{i_1}, \dots, f_{i_n})$ is summable and then $\sum_{i \in I} f_i = \sum_{k=1}^n f_{i_k}$.

Theorem 2.1. A family of morphisms $(f_i)_{i \in I}$ in $\mathcal{L}(X, Y)$ is summable iff for any family of pairwise disjoint sets $(I_j)_{j \in J}$ such that $\cup_{j \in J} I_j = I$:

- for each $j \in J$ the restricted family $(f_i)_{i \in I_j}$ is summable with sum $\sum_{i \in I_j} f_i \in \mathcal{L}(X, Y)$
- the family $(\sum_{i \in I_j} f_i)_{j \in J}$ is summable

and then we have $\sum_{i \in I} f_i = \sum_{j \in J} \sum_{i \in I_j} f_i$.

Proof. By induction on $\#I$. □

Another interesting consequence of **(S-assoc)** is that \mathbf{S} preserves summability.

Theorem 2.2. Let $f_0, f_1 \in \mathcal{L}(X, Y)$ be summable. Then $\mathbf{S}f_0, \mathbf{S}f_1 \in \mathcal{L}(\mathbf{S}X, \mathbf{S}Y)$ are summable, with witness $\langle \mathbf{S}f_0, \mathbf{S}f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}X, \mathbf{S}^2Y)$ given by $\langle \mathbf{S}f_0, \mathbf{S}f_1 \rangle_{\mathbf{S}} = \mathbf{c} \mathbf{S} \langle f_0, f_1 \rangle_{\mathbf{S}}$.

Proof. We must prove that $\pi_i \mathbf{c} \mathbf{S} \langle f_0, f_1 \rangle_{\mathbf{S}} = \mathbf{S}f_i$. For this we use the fact that $\pi_0, \pi_1 \in \mathcal{L}(\mathbf{S}Y, Y)$ are jointly monic. We have

$$\begin{aligned} \pi_j \pi_i \mathbf{c} \mathbf{S} \langle f_0, f_1 \rangle_{\mathbf{S}} &= \pi_i \pi_j \mathbf{S} \langle f_0, f_1 \rangle_{\mathbf{S}} \\ &= \pi_i \langle f_0, f_1 \rangle_{\mathbf{S}} \pi_j \quad \text{by naturality} \\ &= f_i \pi_j = \pi_j \mathbf{S}f_i \quad \text{by naturality.} \end{aligned}$$

□

We will use the notations $\iota_0 = \langle \text{Id}, 0 \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}X)$ and $\iota_1 = \langle 0, \text{Id} \rangle_{\mathbf{S}} \in \mathcal{L}(X, \mathbf{S}X)$. Notice that if \mathcal{L} has products $X \& Y$ and coproducts $X \oplus Y$ then we have

$$X \oplus X \xrightarrow{[\iota_0, \iota_1]} \mathbf{S}X \xrightarrow{\langle \pi_0, \pi_1 \rangle} X \& X$$

locating $\mathbf{S}X$ somewhere in between the coproduct and the product of X with itself. Notice that, in the case of coherence spaces, $\mathbf{S}X$ is neither the product nor the coproduct in general.

In contrast, if \mathcal{L} has biproducts, then we necessarily have $\mathbf{S}X = X \& X = X \oplus X$ with obvious structural morphisms, and \mathcal{L} is additive. Of course this is not the situation we are primarily interested in!

2.1 A monad structure on \mathbf{S}

We already noticed that there is a natural transformation $\iota_0 \in \mathcal{L}(X, \mathbf{S}X)$. As also mentioned the morphisms $\pi_i \pi_j \in \mathcal{L}(\mathbf{S}^2 X, X)$ (for all $i, j \in \{0, 1\}$) are summable, so that the morphisms $\pi_0 \pi_0, \pi_1 \pi_0 + \pi_0 \pi_0 \in \mathcal{L}(\mathbf{S}^2 X, \mathbf{S}X)$ are summable, let $\sigma = \langle \pi_0 \pi_0, \pi_1 \pi_0 + \pi_0 \pi_0 \rangle_{\mathbf{S}} \in \mathcal{L}(\mathbf{S}^2 X, \mathbf{S}X)$ be the witness of this summability.

Theorem 2.3. *The tuple $(\mathbf{S}, \iota_0, \sigma)$ is a monad on \mathcal{L} and we have $\sigma \mathbf{c} = \sigma$.*

Proof. The proof is easy and uses the fact that π_0, π_1 are jointly monic. Let us prove that σ is natural so let $f \in \mathcal{L}(X, Y)$, we have $\pi_0(\mathbf{S}f)\sigma_X = f\pi_0\sigma_X$ by naturality of π_0 and hence $\pi_0(\mathbf{S}f)\sigma_X = f\pi_0\pi_0$, and $\pi_0\sigma_Y(\mathbf{S}^2 f) = \pi_0\pi_0(\mathbf{S}^2 f) = f\pi_0\pi_0$ by naturality of π_0 .

Similarly, using the naturality of π_1 , we have $\pi_1(\mathbf{S}f)\sigma_X = f\pi_1\sigma_X = f(\pi_0\pi_1 + \pi_1\pi_0) = f\pi_0\pi_1 + f\pi_1\pi_0$ and $\pi_1\sigma_Y(\mathbf{S}^2 f) = (\pi_0\pi_1 + \pi_1\pi_0)(\mathbf{S}^2 f) = \pi_0\pi_1\mathbf{S}^2 f + \pi_1\pi_0(\mathbf{S}^2 f) = f\pi_0\pi_1 + f\pi_1\pi_0$.

One proves $\sigma_X\sigma_{\mathbf{S}X} = \sigma_X\mathbf{S}\sigma_X$ by showing in the same manner that $\pi_0\sigma_X\sigma_{\mathbf{S}X} = \pi_0\pi_0\pi_0 = \pi_0\sigma_X\mathbf{S}\sigma_X$ and that $\pi_1\sigma_X\sigma_{\mathbf{S}X} = \pi_0\pi_0\pi_1 + \pi_0\pi_1\pi_0 + \pi_1\pi_0\pi_0 = \pi_1\sigma_X\mathbf{S}\sigma_X$. The commutations involving σ and ι_0 are proved in the same way. The last equation results from $\mathbf{c}\pi_i\pi_j = \pi_j\pi_i$ \square

Example 2.5. In our coherence space running example, we have $\iota_0 \cdot (x, u) = x$ and $\sigma \cdot ((x, u), (y, v)) = (x, u + y)$; notice indeed that since $((x, u), (y, v)) \in \text{Cl}(\mathbf{S}^2 E)$ we have $x + u + y + v \in \text{Cl}(E)$.

Just as in tangent categories, this monad structure will be crucial for expressing that the differential (Jacobian) is a linear morphism.

3 Summability in a monoidal category and differentiation

We assume that our summable category \mathcal{L} is a symmetric monoidal category (SMC), with monoidal product \otimes , unit 1 and isomorphisms $\rho_X \in \mathcal{L}(X \otimes 1, X)$, $\lambda_X \in \mathcal{L}(X \otimes 1, X)$, $\alpha_{X_0, X_1, X_2} \in \mathcal{L}((X_0 \otimes X_1) \otimes X_2, X_0 \otimes (X_1 \otimes X_2))$ and $\gamma_{X_0, X_1} \in \mathcal{L}(X_0 \otimes X_1, X_1 \otimes X_0)$. Most often these isos will be kept implicit to simplify the presentation. Concerning the compatibility of the summability structure with the monoidal structure our axiom stipulates distributivity.

($\mathbf{S} \otimes$ -dist) If (f_{00}, f_{01}) is a summable pair of morphisms in $\mathcal{L}(X_0, Y_0)$ and $f_1 \in \mathcal{L}(X_1, Y_1)$ then $(f_{00} \otimes f_1, f_{01} \otimes f_1)$ is a summable pair of morphisms in $\mathcal{L}(X_0 \otimes X_1, Y_0 \otimes Y_1)$, and moreover

$$f_{00} \otimes f_1 + f_{01} \otimes f_1 = (f_{00} + f_{01}) \otimes f_1$$

As a consequence, using the symmetry of \otimes , if (f_{00}, f_{01}) is summable in $\mathcal{L}(X_0, Y_0)$ and (f_{10}, f_{11}) is summable in $\mathcal{L}(X_1, Y_1)$, the family $(f_{00} \otimes f_{10}, f_{00} \otimes f_{11}, f_{01} \otimes f_{10}, f_{01} \otimes f_{11})$ is summable in $\mathcal{L}(X_0 \otimes X_1, Y_0 \otimes Y_1)$ and we have

$$(f_{00} + f_{01}) \otimes (f_{10} + f_{11}) = f_{00} \otimes f_{10} + f_{00} \otimes f_{11} + f_{01} \otimes f_{10} + f_{01} \otimes f_{11}.$$

We can define a natural transformation $\varphi_{X_0, X_1} \in \mathcal{L}(X_0 \otimes \mathbf{S}X_1, \mathbf{S}(X_0 \otimes X_1))$ by setting $\varphi_{X_0, X_1} = \langle X_0 \otimes \pi_0, X_0 \otimes \pi_1 \rangle_{\mathbf{S}}$ which is well defined by **($\mathbf{S} \otimes$ -dist)**. We use $\varphi_{X_0, X_1}^\gamma \in \mathcal{L}(\mathbf{S}X_0 \otimes X_1, \mathbf{S}(X_0 \otimes X_1))$ for the natural transformation defined from φ using the symmetry isomorphism of the SMC, that is $\varphi_{X_0, X_1}^\gamma = \langle \pi_0 \otimes X_1, \pi_1 \otimes X_1 \rangle_{\mathbf{S}}$.

Lemma 3.1. $\mathbf{s}\varphi_{X_0 \otimes X_1} = X_0 \otimes \mathbf{s}X_1$.

Proof. We have $\mathbf{s}\varphi_{X_0 \otimes X_1} = X_0 \otimes \pi_0 + X_0 \otimes \pi_1 = X_0 \otimes (\pi_0 + \pi_1)$ by **($\mathbf{S} \otimes$ -dist)** and we have $\pi_0 + \pi_1 = \mathbf{s}X_1$. \square

Theorem 3.1. *The natural transformation φ is a strength for the monad $(\mathbf{S}, \iota_0, \sigma)$ and equipped with φ this monad is commutative.*

We set $\mathbf{L}_{X_0, X_1} = \sigma(\mathbf{S}\varphi_{X_0, X_1}^\gamma)\varphi_{\mathbf{S}X_0, X_1} = \sigma(\mathbf{S}\varphi_{X_0, X_1})\varphi_{X_0, \mathbf{S}X_1}^\gamma = \langle \pi_0 \otimes \pi_0, \pi_1 \otimes \pi_0 + \pi_0 \otimes \pi_1 \rangle_{\mathbf{S}}$; it is well known that in such a commutative monad situation, the associated tuple $(\mathbf{S}, \iota_0, \sigma, \mathbf{L})$ is a symmetric monoidal monad on the SMC \mathcal{L} . When the summability structure of the SMC \mathcal{L} satisfies **($\mathbf{S} \otimes$ -dist)** we say that \mathcal{L} is a *summable SMC*.

3.1 Differential structure

We assume that \mathcal{L} is a summable *resource category*, the latter property meaning that:

- \mathcal{L} is a SMC;
- \mathcal{L} is cartesian with terminal object \top and cartesian product of X_0, X_1 denoted $(X_0 \& X_1, \text{pr}_0, \text{pr}_1)$ and pairing of morphisms $(f_i \in \mathcal{L}(Y, X_i))_{i=0,1}$ denoted $\langle f_0, f_1 \rangle \in \mathcal{L}(Y, X_0 \& X_1)$;
- and \mathcal{L} is equipped with a *resource comonad*, that is a tuple $(!_, \text{der}, \text{dig}, \text{m}^0, \text{m}^2)$ where $!_-$ is a functor $\mathcal{L} \rightarrow \mathcal{L}$ which is a comonad with counit der and comultiplication dig , and $\text{m}^0 \in \mathcal{L}(1, !\top)$ and $\text{m}^2 \in \mathcal{L}(!X \otimes !Y, !(X \& Y))$ are the Seely isomorphisms (subject to condition that we do not recall here, see for instance [Mel09]).

Then remember that it is possible to define a contraction morphism $\text{contr}_X \in \mathcal{L}(!X, !X \otimes !X)$ and a weakening morphism $\text{w}_X \in \mathcal{L}(!X, 1)$ turning $!X$ into a commutative comonoid. These morphisms are defined as follows:

$$!X \xrightarrow{!0} !\top \xrightarrow{(\text{m}^0)^{-1}} 1 \quad !X \xrightarrow{!(\text{Id}, \text{Id})} !(X \& X) \xrightarrow{(\text{m}^2)^{-1}} !X \otimes !X$$

In that situation, we need a further assumption on the summability structure.

(S&-pres) The functor \mathbf{S} preserves all finite cartesian products. In other words $0 \in \mathcal{L}(\mathbf{S}\top, \top)$ and $\langle \text{Spr}_0, \text{Spr}_1 \rangle \in \mathcal{L}(\mathbf{S}(X_0 \& X_1), \mathbf{S}X_0 \& \mathbf{S}X_1)$ are isos.

Let $\partial_X \in \mathcal{L}(!\mathbf{S}X, \mathbf{S}!X)$ be a natural transformation. We introduce now a series of conditions that this morphism has to satisfy in order to define a differentiation operation.

$$(\partial\text{-local}) \quad \begin{array}{ccc} !\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \\ & \searrow !\pi_0 & \downarrow \pi_0 \\ & & !X \end{array}$$

$$(\partial\text{-lin}) \quad \begin{array}{ccc} !X & \xrightarrow{!0} & !\mathbf{S}X \\ \downarrow !\iota_0 & \searrow \iota_0 & \downarrow \partial_X \\ !\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \end{array} \quad \begin{array}{ccc} !\mathbf{S}^2X & \xrightarrow{\partial_{\mathbf{S}X}} & \mathbf{S}!\mathbf{S}X \xrightarrow{\mathbf{S}\partial_X} \mathbf{S}^2!X \\ \downarrow !\sigma & & \downarrow \sigma \\ !\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \end{array}$$

This is a first distributive law which allows to lift the functor $!_-$ to the Kleisli category $\mathcal{L}_{\mathbf{S}}$, turning probably $\mathcal{L}_!$ into a “left summable category” (a notion which needs a precise definition). In this category, a morphism $X \rightarrow Y$ can be seen as a pair (f_0, f_1) of two summable morphisms in $\mathcal{L}(X, Y)$, and composition is defined by $g \circ f = (g_0 f_0, g_1 f_0 + g_0 f_1)$, a definition which is very reminiscent of the multiplication of dual numbers.

(∂ -chain) The natural transformation ∂ is a distributive law between \mathbf{S} and $!_-$, that is

$$\begin{array}{ccc} !\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \\ \searrow \text{der}_{\mathbf{S}X} & & \downarrow \mathbf{S} \text{der}_X \\ & & \mathbf{S}X \end{array} \quad \begin{array}{ccc} !\mathbf{S}X & \xrightarrow{\partial_X} & \mathbf{S}!X \\ \downarrow \text{dig}_{\mathbf{S}X} & & \downarrow \mathbf{S} \text{dig}_X \\ !!\mathbf{S}X & \xrightarrow{!\partial_X} & !\mathbf{S}!X \xrightarrow{\partial_{!X}} \mathbf{S}!!X \end{array}$$

This second distributive law allows to lift the functor \mathbf{S} to the Kleisli category $\mathcal{L}_!$. We obtain in that way the functor $\mathbf{D} : \mathcal{L}_! \rightarrow \mathcal{L}_!$ defined as follows: on objects, we set $\mathbf{D}X = \mathbf{S}X$. Next, given $f \in \mathcal{L}_!(X, Y) = \mathcal{L}(!X, Y)$, the morphism $\mathbf{D}f \in \mathcal{L}_!(\mathbf{S}X, \mathbf{S}Y) = \mathcal{L}(!\mathbf{S}X, \mathbf{S}Y)$ is defined by $\mathbf{D}f = (\mathbf{S}f) \partial_X$. The purpose of the two distributive law commutations is precisely to make this operation functorial and this functoriality is a categorical version of the chain rule of calculus, exactly as in tangent categories since, as we shall see, this functor \mathbf{D} essentially computes the derivative of f .

(∂ - $\&$) Compatibility with the cartesian products

$$\begin{array}{ccc} !\mathbf{S}\top & \xrightarrow{\partial_\top} & \mathbf{S}!\top \\ \downarrow !0 & & \downarrow \mathbf{S}(\text{m}^0)^{-1} \\ !\top & \xrightarrow{(\text{m}^0)^{-1}} & 1 \xrightarrow{!0} \mathbf{S}1 \end{array} \quad \begin{array}{ccc} !\mathbf{S}(X_0 \& X_1) & \xrightarrow{\partial_{X_0 \& X_1}} & \mathbf{S}!(X_0 \& X_1) \xrightarrow{\mathbf{S}(\text{m}^2)^{-1}} \mathbf{S}(!X_0 \otimes !X_1) \\ \downarrow \langle \text{Spr}_0, \text{Spr}_1 \rangle & & \downarrow \mathbf{S} \langle \text{spr}_0, \text{spr}_1 \rangle \\ !(\mathbf{S}X_0 \& \mathbf{S}X_1) & \xrightarrow{(\text{m}^2)^{-1}} & !\mathbf{S}X_0 \otimes !\mathbf{S}X_1 \xrightarrow{\partial_{X_0} \otimes \partial_{X_1}} \mathbf{S}!X_0 \otimes \mathbf{S}!X_1 \end{array}$$

Theorem 3.2 (Leibniz rule). *If $(\partial\text{-}\&)$ holds then the following diagrams commute.*

$$\begin{array}{ccc}
!SX & \xrightarrow{\partial_X} & S!X \\
ws_X \downarrow & & \downarrow S w_X \\
1 & \xrightarrow{\iota_0} & S1
\end{array}
\qquad
\begin{array}{ccc}
!SX & \xrightarrow{\partial_X} & S!X \\
\text{contr}_{!SX} \downarrow & & \downarrow S \text{contr}_X \\
!SX \otimes !SX & \xrightarrow{\partial_X \otimes \partial_X} & S!X \otimes S!X \xrightarrow{L_{!X, !X}} S(!X \otimes !X)
\end{array}$$

Proof. This is an easy consequence of the naturality of ∂ and of the definition of w_X and contr_X which is based on the cartesian products and on the Seely isomorphisms. \square

($\partial\text{-Schwarz}$) The following diagram, involving the flip introduced before the statement of (**S-assoc**).

$$\begin{array}{ccc}
!S^2X & \xrightarrow{\partial_{S^2X}} & S!SX \xrightarrow{S\partial_X} S^2!X \\
!c \downarrow & & \downarrow c \\
!S^2X & \xrightarrow{\partial_{S^2X}} & S!SX \xrightarrow{S\partial_X} S^2!X
\end{array}$$

Definition 3.1. A differentiation in summable resource category \mathcal{L} is a natural transformation $\partial_X \in \mathcal{L}(!SX, S!X)$ which satisfies ($\partial\text{-local}$), ($\partial\text{-lin}$), ($\partial\text{-chain}$), ($\partial\text{-}\&$) and ($\partial\text{-Schwarz}$).

3.2 Derivatives and partial derivatives in the Kleisli category

The Kleisli category $\mathcal{L}_!$ of the comonad $(!, \text{der}, \text{dig})$ is well known to be cartesian. In general it is not a differential cartesian category in the sense of [AL20] because it is not required to be additive². Our running example of coherence spaces is an example of such a category which is not a differential category.

There is an inclusion functor $\text{Lin}_! : \mathcal{L} \rightarrow \mathcal{L}_!$ which maps X to X and $f \in \mathcal{L}(X, Y)$ to $f \text{ der}_X \in \mathcal{L}_!(X, Y)$, it is faithful but not full in general and allows to see any morphism of \mathcal{L} as a ‘‘linear morphism’’ of $\mathcal{L}_!$.

We have already mentioned the functor $D : \mathcal{L}_! \rightarrow \mathcal{L}_!$, remember that $DX = SX$ and $Df = (Sf) \partial_X$ when $f \in \mathcal{L}(X, Y)$. Then we have $D \circ \text{Lin}_! = \text{Lin}_! \circ S$ which allows to lift simply the monad structure of S to D by setting $\zeta_X = \text{Lin}_! \iota_0 \in \mathcal{L}_!(X, DX)$ and $\tau_X = \text{Lin}_! \sigma \in \mathcal{L}_!(D^2X, DX)$. Since S preserves cartesian products, we can equip trivially this monad (D, ζ, τ) with a commutative strength $\psi_{X_0, X_1} \in \mathcal{L}_!(X_0 \& DX_1, D(X_0 \& X_1))$ which is the following composition in \mathcal{L}

$$!(X_0 \& SX_1) \xrightarrow{\text{der}} X_0 \& SX_1 \xrightarrow{\iota_0 \& SX_1} SX_0 \& SX_1 \xrightarrow{\theta} S(X_0 \& X_1)$$

where $\theta = \langle \text{Spr}_0, \text{Spr}_1 \rangle^{-1}$ is the canonical iso of (**S&-pres**). It is possible to prove the following commutation in \mathcal{L} , relating the strength of S (wrt. \otimes) with the strength of D (wrt. $\&$) through the Seely isomorphisms

$$\begin{array}{ccc}
!(X_0 \& SX_1) & \xrightarrow{\text{der}} & X_0 \& SX_1 \xrightarrow{\iota_0 \& SX_1} SX_0 \& SX_1 \xrightarrow{\theta} S(X_0 \& X_1) \\
m^2 \uparrow & & \uparrow S \text{der} \\
!X_0 \otimes !SX_1 & \xrightarrow{!X_0 \otimes \tilde{\partial}_{X_1}} & !X_0 \otimes S!X_1 \xrightarrow{\varphi_{!X_0, !X_1}} S(!X_0 \otimes !X_1) \xrightarrow{S m^2} S(!X_0 \& X_1)
\end{array}$$

Given $f \in \mathcal{L}_!(X_0 \& X_1, Y)$, we can define the partial derivatives $D_0f \in \mathcal{L}_!(DX_0 \& X_1, DY)$ and $D_1f \in \mathcal{L}_!(X_0 \& DX_1, DY)$ as $Df \circ \psi'$ and $Df \circ \psi$ where we use ψ' for the strength $DX_0 \& X_1 \rightarrow D(X_0 \& X_1)$ defined from ψ using the symmetry of $\&$.

3.3 Deciphering the diagrams

One should think of the objects of \mathcal{L} as partial commutative monoids (with additional structures depending on the considered category), and SX as the object of pairs (x, u) of elements $x, u \in X$ such that $x + u \in X$ is defined. The morphisms in \mathcal{L} are linear in the sense that they preserve 0 and this partially defined sums whereas the morphisms of $\mathcal{L}_!$ should be thought of as functions which are not

²We postpone the precise axiomatization of this kind of partially additive differential category to further work. Of course it will be based on the concept of summability structure.

linear but admit a “derivative”. More precisely $f \in \mathcal{L}_1(X, Y)$ can be seen as a function $X \rightarrow Y$ and, given $(x, u) \in \mathbf{S}X$ we have

$$Df(x, u) = (f(x), \frac{df(x)}{dx} \cdot u) \in \mathbf{S}Y,$$

where $\frac{df(x)}{dx} \cdot u$ is just a notation for the second component of the pair $Df(x, u)$ which, by construction, is such that the sum $f(x) + \frac{df(x)}{dx} \cdot u$ is a well defined element of Y . Now we assume that this derivative $\frac{df(x)}{dx} \cdot u$ obeys the standard rules of differential calculus and we shall see that the above axioms about ∂ correspond to these rules.

Remark 1. We use the well established notation $\frac{df(x)}{dx} \cdot u$ which must be understood properly: in particular the expression $\frac{df(x)}{dx} \cdot u$ is a function of x (the point where the derivative is computed) and of u (the linear parameter of the derivative). When required we use $\frac{df(x)}{dx}(x_0) \cdot u$ for the evaluation of this derivative at point $x_0 \in X$.

- (∂ -**local**) means that the first component of $Df(x, u)$ is $f(x)$.
- Since $\iota_0 : X \rightarrow \mathbf{S}X$ maps x to $(x, 0)$ and $\sigma : \mathbf{S}^2X \rightarrow \mathbf{S}X$ maps $((x, u), (y, v))$ to $(x, u + y)$. The “second derivative” $D^2f \in \mathcal{L}_1(\mathbf{S}^2X, \mathbf{S}^2Y)$ of $f \in \mathcal{L}_1(X, Y)$ is $(\mathbf{S}^2f)(\mathbf{S}\partial_X) \partial_{\mathbf{S}X}$. Remember that $Df(x, u) = (f(x), \frac{df(x)}{dx} \cdot u)$, therefore applying the standard rules of differential calculus we have

$$\begin{aligned} D^2f((x, u), (x', u')) &= (Df(x, u), \frac{dDf(x, u)}{d(x, u)} \cdot (x', u')) \\ &= ((f(x), \frac{df(x)}{dx} \cdot u), \frac{\partial(f(x), \frac{df(x)}{dx} \cdot u)}{\partial x} \cdot x' + \frac{\partial(f(x), \frac{df(x)}{dx} \cdot u)}{\partial u} \cdot u')) \\ &= ((f(x), \frac{df(x)}{dx} \cdot u), (\frac{df(x)}{dx} \cdot x', \frac{d^2f(x)}{dx^2} \cdot (u, x') + \frac{df(x)}{dx} \cdot u')) \end{aligned}$$

where we have used that fact that $f(x)$ does not depend on u and that $\frac{df(x)}{dx} \cdot u$ is linear in u . Therefore (∂ -**lin**) means that $\frac{df(x)}{dx} \cdot 0 = 0$ and $\frac{df(x)}{dx} \cdot (u + y) = \frac{df(x)}{dx} \cdot u + \frac{df(x)}{dx} \cdot y$, that is $\frac{df(x)}{dx} \cdot u$ is linear in u .

- The first diagram of (∂ -**chain**) means that if $f \in \mathcal{L}_1(X, Y)$ is linear, that is, there is $g \in \mathcal{L}(X, Y)$ such that $f = g \mathbf{der}_X$, then $\frac{df(x)}{dx} \cdot u = f(u)$. Notice that it prevents differentiation from being trivial by setting $\frac{df(x)}{dx} \cdot u = 0$ for all f and all x, u . Consider now $f \in \mathcal{L}_1(X, Y)$ and $g \in \mathcal{L}_1(!Y, Z)$; the second diagram means that $D(g \circ f) = Dg \circ Df$, which amounts to

$$\frac{dg(f(x))}{dx} \cdot u = \frac{dg(y)}{dy}(f(x)) \cdot (\frac{df(x)}{dx} \cdot u)$$

which is exactly the chain rule.

- We have assumed that \mathcal{L} is cartesian and hence \mathcal{L}_1 is also cartesian. Intuitively $X_0 \& X_1$ is the space of pairs (x_0, x_1) with $x_i \in X_i$, and our assumption (**S&-pres**) means that $\mathbf{S}(X_0 \& X_1)$ is the space of pairs $((x_0, x_1), (u_0, u_1))$ such that $(x_i, u_i) \in \mathbf{S}X_i$, and the sum of such a pair is $(x_0 + u_0, x_1 + u_1) \in X_0 \& X_1$. Then, given $f \in \mathcal{L}_1(X_0 \& X_1, Y)$ the second diagram of (∂ -**&**) means that

$$\frac{df(x_0, x_1)}{d(x_0, x_1)} \cdot (u_0, u_1) = \frac{\partial f(x_0, x_1)}{\partial x_0} \cdot u_0 + \frac{\partial f(x_0, x_1)}{\partial x_1} \cdot u_1$$

which can be seen by the following computation of $\pi_1 Df$ using that diagram

$$\begin{aligned} \pi_1 Df &= \pi_1 (\mathbf{S}f) \partial_{X_0 \& X_1} \\ &= f \mathbf{m}^2 \pi_1 \mathbf{L}_{!X_0, !X_1} (\partial_{X_0} \otimes \partial_{X_1}) (\mathbf{m}^2)^{-1} !\langle \mathbf{Spr}_0, \mathbf{Spr}_1 \rangle \\ &= f \mathbf{m}^2 (\pi_1 \otimes \pi_0 + \pi_0 \otimes \pi_1) (\partial_{X_0} \otimes \partial_{X_1}) (\mathbf{m}^2)^{-1} !\langle \mathbf{Spr}_0, \mathbf{Spr}_1 \rangle \\ &= f \mathbf{m}^2 (\pi_1 \partial_{X_0} \otimes !\pi_0) (\mathbf{m}^2)^{-1} !\langle \mathbf{Spr}_0, \mathbf{Spr}_1 \rangle + f \mathbf{m}^2 (!\pi_0 \otimes \pi_1 \partial_{X_0}) (\mathbf{m}^2)^{-1} !\langle \mathbf{Spr}_0, \mathbf{Spr}_1 \rangle \\ &= \pi_1 (D_0 f) !\langle \mathbf{pr}_0 \pi_0, \mathbf{Spr}_1 \rangle + \pi_1 (D_1 f) !\langle \mathbf{Spr}_0, \mathbf{pr}_1 \pi_0 \rangle \end{aligned}$$

the two components of these sums corresponding to the two partial derivatives, see Section 3.2.

Then Theorem 3.2 means that $\frac{df(x,x)}{dx} \cdot u = \frac{\partial f(x_0,x_1)}{\partial x_0}(x,x) \cdot u + \frac{\partial f(x_0,x_1)}{\partial x_1}(x,x) \cdot u$ which is the essence of the Leibniz rule of Calculus.

- The object $\mathbf{S}^2 X$ consists of pairs $((x,u), (x',u'))$ such that x, u, x' and u' are globally summable. Then $c \in \mathcal{L}(\mathbf{S}^2 X, \mathbf{S}^2 X)$ maps $((x,u), (x',u'))$ to $((x,x'), (u,u'))$. Therefore, using the same computation of $\mathbf{D}^2 f((x,u), (x',u'))$ as in the case of $(\partial\text{-lin})$, we see that $(\partial\text{-Schwarz})$ expresses that $\frac{d^2 f(x)}{dx^2} \cdot (u, x') = \frac{d^2 f(x)}{dx^2} \cdot (x', u)$ (upon taking $u' = 0$). So this diagram means that the second derivative (aka. Hessian) is a symmetric bilinear function, a result often called Schwarz Theorem in the literature.

3.4 A differentiation in coherence spaces

Now we exhibit such a differentiation in **Coh**. We define $!E$ as follows: $!E$ is the set of finite multisets³ m of elements of $|E|$ such that $\text{supp}(m) \in \text{Cl}(E)$ (such an m is called a finite multiclique). Given $m_0, m_1 \in !E$, we have $m_0 \supset_{!E} m_1$ if $m_0 + m_1 \in !E$. This operation is a functor $\mathbf{Coh} \rightarrow \mathbf{Coh}$: given $s \in \mathbf{Coh}(E, F)$ one sets

$$!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N}, (a_i, b_i) \in s \text{ for } i = 1, \dots, n \text{ and } [a_1, \dots, a_n] \in !E\}$$

which actually belongs to $\text{Cl}(!E \multimap !F)$ because $s \in \text{Cl}(E \multimap F)$. The comonad structure of this functor and the associated commutative comonoid structure are given by

- $\text{der}_E = \{([a], a) \mid a \in |E|\}$
- $\text{dig}_E = \{(m, [m_1, \dots, m_n]) \in !E \multimap !!E \mid m = m_1 + \dots + m_n\}$
- $\text{w}_E = \{([], *)\}$
- and $\text{contr}_E = \{(m, (m_1, m_2)) \in !E \multimap (!E \otimes !E) \mid m = m_1 + m_2\}$.

Composition in $\mathbf{Coh}_!$ can be described directly as follows: let $s \in \text{Cl}(!E \multimap F)$ and $t \in \text{Cl}(!F \multimap G)$, then $t \circ s \in \mathcal{L}(!E \multimap G)$ is $\{(m_1 + \dots + m_n, c) \mid \exists b_1, \dots, b_n \in |F| ([b_1, \dots, b_n], c) \in t \text{ and } (m_i, b_i) \in s \text{ for } i = 1, \dots, n\}$. A morphism $s \in \mathbf{Coh}_!(E, F)$ induces a function $\widehat{s} : \text{Cl}(E) \rightarrow \text{Cl}(F)$ by $\widehat{s}(x) = \{b \mid \exists m \in \mathcal{M}_{\text{fin}}(x) (m, b) \in s\}$. The functions $f : \text{Cl}(E) \rightarrow \text{Cl}(F)$ definable in that way are exactly the *stable functions*: f is stable if for any $x \in \text{Cl}(E)$ and any $b \in f(x)$ there is exactly one minimal subset x_0 of x such that $b \in f(x_0)$, and moreover this x_0 is finite. When moreover this x_0 is always a singleton f is said *linear* and such linear functions are in bijection with $\mathbf{Coh}(E, F)$ (given $t \in \mathbf{Coh}(E, F)$, the associated linear function $\text{Cl}(E) \rightarrow \text{Cl}(F)$ is the map $x \mapsto t \cdot x$).

Notice that for a given stable function $f : \text{Cl}(E) \rightarrow \text{Cl}(F)$ there can be infinitely many $s \in \mathbf{Coh}_!(E, F)$ such that $f = \widehat{s}$ since the definition of \widehat{s} does not take into account the multiplicities in the multisets m such that $(m, b) \in s$. For instance, if $a \in |E|$ and $b \in |F|$ then $\{([a], b)\}$ and $\{([a, a], b)\}$ define exactly the same stable (actually linear) function.

Up to trivial iso we have $!SE = \{(m_0, m_1) \in !E \mid \text{supp}(m_0) \cap \text{supp}(m_1) = \emptyset \text{ and } m_0 + m_1 \in !E\}$ and $(m_{00}, m_{01}) \supset_{!SE} (m_{10}, m_{11})$ if $m_{00} + m_{01} + m_{10} + m_{11} \in !E$ and $\text{supp}(m_{00} + m_{10}) \cap \text{supp}(m_{01} + m_{11}) = \emptyset$. With this identification we define $\partial_E \subseteq !SE \multimap SE$ as follows:

$$\begin{aligned} \partial_E = & \{((m_0, []), (0, m_0)) \mid m_0 \in !E\} \\ & \cup \{((m_0, [a]), (1, m_0 + [a])) \mid m_0 + [a] \in !E \text{ and } a \notin \text{supp}(m_0)\}. \end{aligned} \quad (1)$$

Let us check that $\partial_E \in \mathbf{Coh}(!SE, SE)$.

Let $((m_{j0}, m_{j1}), (i_j, m_j)) \in \partial_E$ for $j = 0, 1$ and assume that $(m_{00}, m_{01}) \supset_{!SE} (m_{10}, m_{11})$ (2). By symmetry, there are 3 cases to consider.

- If $i_0 = i_1 = 0$ then we have $m_{j1} = []$ and $m_{j0} = m_j$ for $j = 0, 1$. Then we have $(0, m_0) \supset_{SE} (0, m_1)$ by our assumption (2), and if $(0, m_0) = (0, m_1)$ then $(m_{00}, m_{01}) = (m_{10}, m_{11})$.

³There is also a definition using finite sets instead of finite multisets, and this is the one considered by Girard in [Gir87], but it does not seem to be compatible with differentiation, see Remark 2.

- Assume now that $i_0 = i_1 = 1$. We have $m_{j1} = [a_j]$ for $a_j \in |E|$, with $a_j \notin \text{supp}(m_{j0})$ and $m_j = m_{j0} + [a_j]$. Our assumption (2) means that $m_{00} + m_{10} + [a_0, a_1] \in |!E|$ and $\text{supp}(m_{00} + m_{10}) \cap \{a_0, a_1\} = \emptyset$. Therefore $m_0 + m_1 \in |!E|$ and hence $(1, m_0) \supset_{\mathbf{S}!E} (1, m_1)$. Assume moreover that $m_0 = m_1$, that is $m_{00} + [a_0] = m_{10} + [a_1]$. This implies $m_{00} = m_{10}$ and $a_0 = a_1$ since we know that $a_1 \notin \text{supp}(m_{00})$ and $a_0 \notin \text{supp}(m_{10})$.
- Last assume that $i_0 = 1$ and $i_1 = 0$. So we have $m_{01} = [a]$ with $a \notin \text{supp}(m_{00})$ and $m_0 = m_{00} + [a]$; $m_{11} = []$ and $m_1 = m_{10}$. By (2) we know that $\text{supp}(m_0 + m_1) \in \text{Cl}(!E)$. Coming back to the definition of the coherence in \mathbf{SF} (for a coherence space F), we must also prove that $m_0 \neq m_1$: this results from (2) which entails that $a \notin \text{supp}(m_1) = m_{10}$ whereas we know that $a \in \text{supp}(m_0)$.

We postpone the proofs of the other commutations to Section 5 where they will be reduced to slightly simpler diagrams. Given $x \in \text{Cl}(E)$, we can define a coherence space E_x (the local sub-coherence space at x) as follows: $|E_x| = \{a \in |E| \setminus x \mid x \cup \{a\} \in \text{Cl}(X)\}$ and $a_0 \supset_{E_x} a_1$ if $a_0 \supset_E a_1$. Then, given $s \in \mathbf{Coh}_!(E, F)$, we can define the *differential* of s at x as

$$\frac{ds(x)}{dx} = \{(a, b) \in |E_x| \times |F| \mid \exists m \in |!E| (m + [a], b) \in s \text{ and } \text{supp}(m) \subseteq x\} \subseteq |E_x \multimap Y|.$$

Theorem 3.3. *Let $s \in \mathbf{Coh}(E, F)$. Then $Ds \in \mathbf{Coh}_!(\mathbf{SE}, \mathbf{SF})$ satisfies*

$$\forall (x, u) \in \text{Cl}(\mathbf{SE}) \quad \widehat{Ds}(x, u) = (\widehat{s}(x), \frac{ds(x)}{dx} \cdot u)$$

Remark 2. *The definition of Ds depends on s and not only on \widehat{s} : for instance if $s = \{([a], b)\}$ then $Ds = \{((([a], []), (0, b)), (([], [a]), (1, b)))\}$ and if $s' = \{([a, a], b)\}$ then $Ds' = \{((([a, a], []), (0, b)))\}$; in that case the derivative vanishes whereas $\widehat{s} = \widehat{s'}$ are the same function.*

Proof. Let $(x, u) \in \text{Cl}(\mathbf{SE})$ and $(i, b) \in |\mathbf{SF}|$ with $i \in \{0, 1\}$ and $b \in |F|$. We have $(i, b) \in \widehat{Ds}(x, u)$ iff there is $(m_0, m_1) \in |!\mathbf{SE}|$ such that $\text{supp}(m_0) \subseteq x$, $\text{supp}(m_1) \subseteq u$ and $((m_0, m_1), (i, b)) \in Ds = \partial_E \mathbf{S}s$. This latter condition holds iff either $i = 0$, $m_1 = []$, and $(m_0, b) \in s$, or $i = 1$, $m_1 = [a]$ for some $a \in |E| \setminus \text{supp}(m_0)$ such that $m_0 + [a] \in \text{Cl}(E)$, and $(m_0 + [a], b) \in s$.

Assume first that $(i, b) \in \widehat{Ds}(x, u)$ and let (m_0, m_1) be as above. If $i = 0$ we have $(m_0, b) \in s$ and $\text{supp}(m_0) \subseteq x$ and hence $b \in \widehat{s}(x)$, that is $(i, b) \in (\widehat{s}(x), \frac{ds(x)}{dx} \cdot u)$. If $i = 1$ let $a \in |E| \setminus \text{supp}(m_0)$ be such that $m_1 = [a]$, $m_0 + [a] \in |!E|$, $(m_0 + [a], b) \in s$ and $\text{supp}(m_0, [a]) \subseteq (x, u)$ (remember that we consider the elements of $\text{Cl}(\mathbf{SE})$ as pairs of cliques), that is $\text{supp}(m_0) \subseteq x$ and $a \in u$. Then we know that $a \in |E_x|$ since $x \cup u \in \text{Cl}(E)$ and $x \cap u = \emptyset$. Therefore $(i, b) \in (\widehat{s}(x), \frac{ds(x)}{dx} \cdot u)$.

We have proven $\widehat{Ds}(x, u) \subseteq (\widehat{s}(x), \frac{ds(x)}{dx} \cdot u)$, we prove the converse inclusion. Let $(i, b) \in (\widehat{s}(x), \frac{ds(x)}{dx} \cdot u)$. If $i = 0$, we have $b \in \widehat{s}(x)$ and hence there is a uniquely defined $m_0 \in |!E|$ such that $\text{supp}(m_0) \subseteq x$ and $(m_0, b) \in s$. It follows that $((m_0, []), (0, b)) \in \partial_E \mathbf{S}s$ and hence $(i, b) \in \widehat{Ds}(x, u)$. Assume now that $i = 1$ so that $b \in \frac{ds(x)}{dx} \cdot u$ and hence there is $a \in u$ (which implies $a \notin x$) such that $(a, b) \in \frac{ds(x)}{dx}$. So there is $m_0 \in |!E|$ such that $\text{supp}(m_0) \subseteq x$ and $(m_0 + [a], b) \in s$ (notice that $a \notin \text{supp}(m_0)$ since $\text{supp}(m_0) \subseteq x$ and $a \notin x$). It follows that $((m_0, [a]), (1, m_0 + [a])) \in \partial_E$ and hence $((m_0, [a]), (1, b)) \in (\mathbf{S}s) \partial_E$ so that $(1, b) \in \widehat{Ds}(x, u)$. \square

Remark 3. *This shows in particular that $\frac{df(x)}{dx} \in \mathbf{Coh}(E_x, F_{\widehat{s}(x)})$ since $\frac{df(x)}{dx} = \pi_1 \circ g \circ \iota_1$ and also that this derivative is stable wrt. the point x where it is computed and thus differentiation of stable functions can be iterated. However Remark 2 stresses a weakness of this derivative which has as consequence that the morphisms in $\mathbf{Coh}_!$ do not coincide with their Taylor expansion that one can define by iterating this derivative (the expansion of s is s whereas the expansion of s' is \emptyset). This is an effect of the uniformity of the construction $!E$, that is, of the fact that for $m \in \mathcal{M}_{\text{fin}}(|E|)$ to be in $!E$, it is required that $\text{supp}(m)$ be a clique. This can be remedied, without breaking the main feature of our construction, namely that it is compatible with the determinism of the model, by using non-uniform coherence spaces instead, where $!E = \mathcal{M}_{\text{fin}}(E)$ [BE01, Bou11], see Section 5.1.*

4 Canonically summable categories

Let \mathcal{L} be a cartesian⁴ SMC where the object $l = 1 \ \& \ 1$ is exponentiable, that is, the functor $\Delta_l : X \mapsto X \otimes l$ has a right adjoint⁵ $\mathbf{S}_l : X \mapsto (l \multimap X)$. We use $\text{ev} \in \mathcal{L}((l \multimap X) \otimes l, X)$ for the corresponding evaluation morphism. We denote this functor as \mathbf{S}_l , notice that, being a right adjoint, it preserves all limits existing in \mathcal{L} (and in particular the cartesian product).

For $i = 0, 1$ we have a morphism $\overline{\pi}_i^{\&} \in \mathcal{L}(X, X \ \& \ X)$ given by $\overline{\pi}_0^{\&} = \langle \text{Id}, 0 \rangle$ and $\overline{\pi}_1^{\&} = \langle 0, \text{Id} \rangle$. In the sequel we use these morphisms for $X = 1$.

Then we define $\pi_i \in \mathcal{L}(\mathbf{S}_l X, X)$ as the following composition of morphisms

$$(l \multimap X) \xrightarrow{\rho^{-1}} (l \multimap X) \otimes 1 \xrightarrow{\text{Id} \otimes \overline{\pi}_i^{\&}} (l \multimap X) \otimes l \xrightarrow{\text{ev}} X$$

and we define $\mathfrak{s} \in \mathcal{L}(\mathbf{S}_l X, X)$ as

$$(l \multimap X) \xrightarrow{\rho^{-1}} (l \multimap X) \otimes 1 \xrightarrow{\text{Id} \otimes \Delta^{\&}} (l \multimap X) \otimes l \xrightarrow{\text{ev}} X$$

where $\Delta^{\&} = \langle \text{Id}, \text{Id} \rangle$.

Definition 4.1. The category \mathcal{L} is *canonically summable* if $(\mathbf{S}_l, \pi_0, \pi_1, \mathfrak{s})$ is a summability structure.

Remark 4. *Canonical summability is a property of \mathcal{L} and not an additional structure, which is however defined in a rather implicit manner. We show now that three elementary conditions are sufficient for guaranteeing canonical summability. We strongly conjecture that they are also necessary.*

Remember that ρ_X^{-1} is the iso $X \rightarrow X \otimes 1$ provided by the monoidal structure of \mathcal{L} .

Theorem 4.1. *If \mathcal{L} satisfies*

(CS-epi) *For any $X \in \text{Obj}(\mathcal{L})$, the morphisms $(X \otimes \overline{\pi}_0^{\&}) \rho^{-1}, X \otimes \overline{\pi}_1^{\&} \rho^{-1} \in \mathcal{L}(X, X \otimes l)$ are jointly epic, that is: if $f, g \in \mathcal{L}(X \otimes l, Y)$ satisfy $f(X \otimes \overline{\pi}_i^{\&}) = g(X \otimes \overline{\pi}_i^{\&})$ for $i = 0, 1$ then $f = g$,*

then $(\mathbf{S}_l, \pi_0, \pi_1, \mathfrak{s})$ is a pre-summability structure.

Proof. Let $f_0, f_1 \in \mathcal{L}(X, \mathbf{S}_l Y)$ be such that $\pi_i f_0 = \pi_i f_1$ for $i = 0, 1$. We have $\pi_i f_j = \text{ev}((l \multimap Y) \otimes \overline{\pi}_i^{\&}) \rho^{-1} f_j = \text{ev}((l \multimap Y) \otimes \overline{\pi}_i^{\&}) (f_j \otimes 1) \rho^{-1} = \text{ev}(f_j \otimes l) (X \otimes \overline{\pi}_i^{\&}) \rho^{-1}$ and hence $\text{ev}(f_0 \otimes l) = \text{ev}(f_1 \otimes l)$ by **(CS-epi)**, which implies $f_0 = f_1$ since $\text{cur}(\text{ev}(f_0 \otimes l)) = f_j$. We have proven that π_0, π_1 are jointly monic. \square

So if \mathcal{L} satisfies **(CS-epi)** it makes sense to speak of summability in \mathcal{L} (wrt. of course to $(\mathbf{S}_l, \pi_0, \pi_1, \mathfrak{s})$).

Lemma 4.1. *If \mathcal{L} satisfies **(CS-epi)**, two morphisms $f_0, f_1 \in \mathcal{L}(X, Y)$ are summable iff there is $g \in \mathcal{L}(X \otimes l, Y)$ such that $f_i = g(X \otimes \overline{\pi}_i^{\&}) \rho^{-1}$ for $i = 0, 1$ and then $\langle f_0, f_1 \rangle_{\mathfrak{s}} = \text{cur } g$. Moreover $f_0 + f_1 = g(X \otimes \Delta^{\&}) \rho^{-1}$.*

Remark 5. *There is a clear homotopy theoretic intuition: l with its two injections $\overline{\pi}_0^{\&}$ and $\overline{\pi}_1^{\&}$ is similar to a $[0, 1]$ interval object and $\langle f_0, f_1 \rangle_{\mathfrak{s}}$ is similar to a homotopy from f_0 to f_1 . The main specificity here is the assumption that this homotopy is unique which is typically not the case in standard homotopy situations. In spite of this major discrepancy, the analogy clearly suggests a higher dimensional categorification of summability structures which deserves further studies, especially in the canonical case.*

Theorem 4.2. *A cartesian SMC where $l = 1 \ \& \ 1$ is exponentiable is canonically summable as soon as it satisfies **(CS-epi)** as well as*

(CS-sum) *Two morphisms $f_0, f_1 \in \mathcal{L}(X \otimes l, Y)$ are summable as soon as the two morphisms $f_0(X \otimes \Delta^{\&}) \rho^{-1}, f_1(X \otimes \Delta^{\&}) \rho^{-1} \in \mathcal{L}(X, Y)$ are summable.*

⁴Actually we don't need all cartesian products, only all n -ary products of l .

⁵Interestingly this adjunction induces on \mathcal{L} the standard linear state monad associated with l : the functor $X \mapsto (l \multimap X \otimes l)$, a fact which has certainly a computational interpretation related with differentiation.

(**CS-flip**) There is a morphism $\tilde{c} \in \mathcal{L}(l \otimes l, l \otimes l)$ such that $\tilde{c}(\tilde{\pi}_i^{\&} \otimes \tilde{\pi}_j^{\&}) = \tilde{\pi}_j^{\&} \otimes \tilde{\pi}_i^{\&}$ and

$$\begin{array}{ccc} l \otimes l & \xrightarrow{\Delta^{\&} \otimes l} & l \otimes l \\ \gamma \downarrow & & \downarrow \tilde{c} \\ l \otimes l & \xrightarrow{l \otimes \Delta^{\&}} & l \otimes l \end{array}$$

Notice that when these conditions hold, the morphism \tilde{c} is unique by condition (**CS-epi**).

There are cartesian SMC where l is exponentiable and which are not canonically summable. The category \mathbf{Set}_0 provides probably the simplest example of that situation.

Example 4.1. We refer to Section 1.2. We have the functor $\mathbf{S}_l : \mathbf{Set}_0 \rightarrow \mathbf{Set}_0$ defined by $\mathbf{S}_l X = (l \multimap X)$. An element of $\mathbf{S}_l X$ is a function $z : \{0, *\}^2 \rightarrow X$ such that $z(0, 0) = 0$. The projection $\pi_i : \mathbf{S}_l X \rightarrow X$ is characterized by $\pi_0(z) = z(*, 0)$ and $\pi_1(z) = z(0, *)$, so $\langle \pi_0, \pi_1 \rangle$ is not injective since $\langle \pi_0, \pi_1 \rangle(z) = (z(*, 0), z(0, *))$ does not depend on $z(*, *)$. So $(\mathbf{S}_l, \pi_0, \pi_1, s)$ is not even a pre-summability structure in \mathbf{Set}_0 . This failure of injectivity is due to the fact that l lacks an addition which would satisfy $(* , 0) + (0, *) = (*, *)$ and, preserved by z , would enforce injectivity.

4.1 The comonoid structure of l

We assume that \mathcal{L} is a canonically summable cartesian SMC. The morphisms $\tilde{\pi}_0^{\&}, \tilde{\pi}_1^{\&} \in \mathcal{L}(l, l)$ are summable with $\tilde{\pi}_0^{\&} + \tilde{\pi}_1^{\&} = \Delta^{\&}$, with witness $\text{ld} \in \mathcal{L}(l, l)$. As a consequence of (**S \otimes -dist**) the morphisms $(\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$, $(\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&}) \rho^{-1}$ and $(\tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$ are summable in $\mathcal{L}(l, l \otimes l)$. Therefore $(\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$ and $(\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&}) \rho^{-1} + (\tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$ are summable in $\mathcal{L}(l, l \otimes l)$ so there is a uniquely defined $\tilde{L} \in \mathcal{L}(l, l \otimes l)$ such that $\tilde{L} \tilde{\pi}_0^{\&} = (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$ and $\tilde{L} \tilde{\pi}_1^{\&} = (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&}) \rho^{-1} + (\tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1}$.

Theorem 4.3. Equipped with $\text{pr}_0 \in \mathcal{L}(l, l)$ as counit and $\tilde{L} \in \mathcal{L}(l, l \otimes l)$ as comultiplication, l is a commutative comonoid in the SMC \mathcal{L} .

Proof. To prove the required commutations, we use (**CS-epi**). Here are two examples of these computations.

$$\rho(l \otimes \text{pr}_0) \tilde{L} \tilde{\pi}_0^{\&} = \rho(l \otimes \text{pr}_0) (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1} = \rho(\tilde{\pi}_0^{\&} \otimes 1) \rho^{-1} = \tilde{\pi}_0^{\&}$$

and

$$\rho(l \otimes \text{pr}_0) \tilde{L} \tilde{\pi}_1^{\&} = \rho(l \otimes \text{pr}_0) (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&} + \tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1} = \rho(\tilde{\pi}_1^{\&} \otimes 1) \rho^{-1} = \tilde{\pi}_1^{\&}$$

since $\text{pr}_0 \tilde{\pi}_i^{\&}$ is equal to ld_1 if $i = 0$ and to 0 otherwise. Hence $\rho(l \otimes \text{pr}_0) \tilde{L} = l$. Next

$$(l \otimes \tilde{L}) \tilde{L} \tilde{\pi}_0^{\&} = (l \otimes l) (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1} = (\tilde{\pi}_0^{\&} \otimes (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&})) (l \otimes \rho^{-1}) \rho^{-1}$$

and

$$\begin{aligned} (l \otimes \tilde{L}) \tilde{L} \tilde{\pi}_1^{\&} &= (l \otimes l) (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&} + \tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \rho^{-1} \\ &= (\tilde{\pi}_0^{\&} \otimes (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&})) + \tilde{\pi}_0^{\&} \otimes (\tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) + \tilde{\pi}_1^{\&} \otimes (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) (l \otimes \rho^{-1}) \rho^{-1}. \end{aligned}$$

Similar computations show that $(\tilde{L} \otimes l) \tilde{L} \tilde{\pi}_0^{\&} = ((\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \otimes \tilde{\pi}_0^{\&}) (\rho^{-1} \otimes l) \rho^{-1}$ and $(\tilde{L} \otimes l) \tilde{L} \tilde{\pi}_1^{\&} = ((\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_0^{\&}) \otimes \tilde{\pi}_1^{\&} + (\tilde{\pi}_0^{\&} \otimes \tilde{\pi}_1^{\&}) \otimes \tilde{\pi}_0^{\&} + (\tilde{\pi}_1^{\&} \otimes \tilde{\pi}_0^{\&}) \otimes \tilde{\pi}_0^{\&}) (\rho^{-1} \otimes l) \rho^{-1}$. Therefore $\alpha(\tilde{L} \otimes l) \tilde{L} \tilde{\pi}_i^{\&} = (l \otimes \tilde{L}) \tilde{L} \tilde{\pi}_i^{\&}$ for $i = 0, 1$ and \tilde{L} is coassociative. Cocommutativity is proven similarly. \square

4.2 Strong monad structure of \mathbf{S}_l

Therefore Δ_l has a canonical comonad structure given by $\rho(X \otimes \text{pr}_0) \in \mathcal{L}(\Delta_l X, X)$ and $\alpha(X \otimes \tilde{L}) \in \mathcal{L}(\Delta_l X, \Delta_l^2 X)$. Through the adjunction $\Delta_l \dashv \mathbf{S}_l$ the functor \mathbf{S}_l inherits a monad structure which is exactly the same as the monad structure of Section 2.1. This monad structure (ι_0, σ) can be described as the Curry transpose of the following morphisms (the monoidality isos are implicit)

$$X \otimes l \xrightarrow{X \otimes \text{pr}_0} X \quad (l \multimap (l \multimap X)) \otimes l \xrightarrow{\text{ld} \otimes \tilde{L}} (l \multimap (l \multimap X)) \otimes l \otimes l \xrightarrow{\text{ev} \otimes l} (l \multimap X) \otimes l \xrightarrow{\text{ev}} X$$

Similarly the trivial costrength $\alpha \in \mathcal{L}(\Delta_1(X \otimes Y), X \otimes \Delta_1 Y)$ induces the strength $\varphi \in \mathcal{L}(X \otimes \mathbf{S}_1 Y, \mathbf{S}_1(X \otimes Y))$ of \mathbf{S}_1 (the same as the one defined in the general setting of Section 3). We have seen in Section 3 that equipped with this strength \mathbf{S}_1 is a commutative monad and recalled that there is therefore an associated lax monoidality $\mathbf{L}_{X_0, X_1} \in \mathcal{L}(\mathbf{S}_1 X_0 \otimes \mathbf{S}_1 X_1, \mathbf{S}_1(X_0 \otimes X_1))$ which can be seen as arising from \mathbf{L} by transposing the following morphism (again we keep the monoidal isos implicit)

$$(1 \multimap X_0) \otimes (1 \multimap X_1) \otimes 1 \xrightarrow{\text{Id} \otimes \tilde{\mathbf{L}}} (1 \multimap X_0) \otimes (1 \multimap X_1) \otimes 1 \otimes 1 \xrightarrow{\text{ev} \otimes \text{ev}} X_0 \otimes X_1$$

4.3 Canonically summable SMCC

In a SMCC, the conditions of Theorem 4.2 admit a slightly simpler formulation.

Theorem 4.4. *A cartesian SMCC is canonically summable as soon as:*

(CCS-epi) $\overline{\pi}_0^{\&}$ and $\overline{\pi}_1^{\&}$ are jointly epic.

(CCS-sum) Two morphisms $f_0, f_1 \in \mathcal{L}(1, Y)$ are summable as soon as the two morphisms $f_0 \Delta^{\&}, f_1 \Delta^{\&} \in \mathcal{L}(1, Y)$ are summable.

(CS-flip) There is a morphism $\tilde{c} \in \mathcal{L}(1 \otimes 1, 1 \otimes 1)$ such that $\tilde{c}(\overline{\pi}_i^{\&} \otimes \overline{\pi}_j^{\&}) = \overline{\pi}_j^{\&} \otimes \overline{\pi}_i^{\&}$ and

$$\begin{array}{ccc} 1 \otimes 1 & \xrightarrow{\Delta^{\&} \otimes 1} & 1 \otimes 1 \\ \gamma \downarrow & & \downarrow \tilde{c} \\ 1 \otimes 1 & \xrightarrow{1 \otimes \Delta^{\&}} & 1 \otimes 1 \end{array}$$

Example 4.2. The SMCC \mathbf{Coh} is canonically summable, actually the summability structure we have considered on this category is exactly its canonical summability structure. Let us check the three conditions.

The coherence space $1 = 1 \ \& \ 1$ is given by $|| = \{0, 1\}$ with $0 \frown 1$. Then $\overline{\pi}_i^{\&} = \{(*, i)\}$ and $\Delta^{\&} = \{(*, 0), (*, 1)\}$. If $s \in \mathbf{Coh}(1, F)$ then $(i, b) \in s \Leftrightarrow (*, b) \in s \overline{\pi}_i^{\&}$ for $i = 0, 1$ and hence $\overline{\pi}_0^{\&}, \overline{\pi}_1^{\&}$ are jointly epic so \mathbf{Coh} satisfies **(CCS-epi)**.

The functor \mathbf{S}_1 defined by $\mathbf{S}_1 E = (1 \multimap E)$ (and similarly for morphisms) coincides exactly with the functor \mathbf{S} described in Example 2.2. Therefore the associated summability is the one described in Example 2.3.

Let $s_i \in \mathbf{Coh}(1, E)$. Let $t_i = s_i \Delta^{\&} = \{(*, a) \in |1 \multimap E| \mid ((0, a) \in s_i \text{ or } (1, a) \in s_i)\}$. Assume that t_0 and t_1 are summable, that is $t_0 \cap t_1 = \emptyset$ and $t_0 \cup t_1 \in \mathbf{Coh}(1, E)$, we must prove that $s_0 \cap s_1 = \emptyset$ and $s_0 \cup s_1 \in \mathbf{Coh}(1, E)$. Let $(j_i, a_i) \in s_i$ for $i = 0, 1$. We have $(*, a_i) \in t_i$ and hence $a_0 \neq a_1$ from which it follows that $(j_0, a_0) \neq (j_1, a_1)$. Since $j_0 \frown j_1$ and $(j_0, a_0), (j_1, a_1) \in s_0 \cup s_1 \in \mathbf{Coh}(1, E)$, we have $a_0 \supset_E a_1$. Hence s_0 and s_1 are summable.

We define $\tilde{c} = \{((i, j), (j, i)) \mid i, j \in \{0, 1\}\} \in \mathbf{Coh}(1 \otimes 1, 1 \otimes 1)$. Let us check the diagram concerning $\Delta^{\&}$: for $i, j, k \in \{0, 1\}$ we have $((*, i), (j, k)) \in \tilde{c}(\Delta^{\&} \otimes 1) \Leftrightarrow i = k$ and $((*, i), (j, k)) \in \tilde{c}(1 \otimes \Delta^{\&}) \gamma \Leftrightarrow ((i, *), (j, k)) \in \tilde{c}(1 \otimes \Delta^{\&}) \gamma \Leftrightarrow i = k$.

4.4 Differentiation in a canonically summable category

Let \mathcal{L} be a resource category (see the beginning of Section 3.1) which is canonically summable. Doubtlessly the following lemma is a piece of categorical folklore, it relies only on the adjunction $\Delta_1 \dashv \mathbf{S}_1$ and on the functoriality of $!_-$. Let $\eta_X \in \mathcal{L}(X, \mathbf{S}_1 \Delta_1 X)$ and $\varepsilon_X \in \mathcal{L}(\Delta_1 \mathbf{S}_1 X, X)$ be the unit and counit of this adjunction. Let $\varphi_X : \mathcal{L}(!\mathbf{S}_1 X, \mathbf{S}_1!X)$ be a natural transformation, then we define a natural transformation $\varphi_X^- \in \Delta_1!X \rightarrow !\Delta_1 X$ as the following composition of morphisms

$$\Delta_1!X \xrightarrow{\Delta_1! \eta_X} \Delta_1! \mathbf{S}_1 \Delta_1 X \xrightarrow{\Delta_1 \varphi_{\Delta_1 X}} \Delta_1 \mathbf{S}_1! \Delta_1 X \xrightarrow{\varepsilon_1 \Delta_1 X} !\Delta_1 X.$$

Conversely given a natural transformation $\psi_X \in \mathcal{L}(\Delta_1!X, !\Delta_1 X)$ we define a natural transformation $\psi_X^+ \in \mathcal{L}(!\mathbf{S}_1 X, \mathbf{S}_1!X)$ as the following composition of morphisms

$$!\mathbf{S}_1 X \xrightarrow{\eta_1 \mathbf{S}_1 X} \mathbf{S}_1 \Delta_1! \mathbf{S}_1 X \xrightarrow{\mathbf{S}_1 \psi_{\mathbf{S}_1 X}} \mathbf{S}_1! \Delta_1 \mathbf{S}_1 X \xrightarrow{\mathbf{S}_1! \varepsilon_X} \mathbf{S}_1! X.$$

Lemma 4.2. *With the notations above, $\varphi^{-+} = \varphi$ and $\psi^{+-} = \psi$.*

Proof. Simple computation using the basic properties of adjunctions and the naturality of the various morphisms involved. \square

Lemma 4.3. *Let $\tilde{\partial}_X \in \mathcal{L}(\Delta_1!X, !\Delta_1X)$ be a natural transformation. The associated natural transformation $\tilde{\partial}_X^+ \in \mathcal{L}(!\mathbf{S}_1X, \mathbf{S}_1!X)$ satisfies (∂ -chain) iff the two following diagrams commute*

$$\begin{array}{ccc} \Delta_1!X & \xrightarrow{\tilde{\partial}_X} & !\Delta_1X \\ \Delta_1 \text{der}_X \searrow & & \downarrow \text{der}_{\Delta_1X} \\ & & \Delta_1X \end{array} \quad \begin{array}{ccc} \Delta_1!X & \xrightarrow{\tilde{\partial}_X} & !\Delta_1X \\ \Delta_1 \text{dig}_X \downarrow & & \downarrow \text{dig}_{\Delta_1X} \\ \Delta_1!!X & \xrightarrow{\tilde{\partial}_{!X}} & !\Delta_1!X \xrightarrow{! \tilde{\partial}_X} !!\Delta_1X \end{array}$$

in other words $\tilde{\partial}_X$ is a co-distributive law $\Delta_1!X \rightarrow !\Delta_1X$. These conditions will be called ($\mathbf{C}\partial$ -chain).

Let $\tilde{\partial}_X \in \mathcal{L}(!X \otimes !, !(X \otimes !))$ satisfying ($\mathbf{C}\partial$ -chain). We introduce additional conditions.

($\mathbf{C}\partial$ -local)

$$\begin{array}{ccc} !X \otimes ! & \xrightarrow{\tilde{\partial}_X} & !(X \otimes !) \\ !X \otimes \pi_0^{\&} \uparrow & \nearrow & \uparrow !\pi_0^{\&} \\ !X & & \end{array}$$

($\mathbf{C}\partial$ -lin)

$$\begin{array}{ccc} !X \otimes ! & \xrightarrow{\tilde{\partial}_X} & !(X \otimes !) \\ !X \otimes \text{pr}_0 \searrow & & \downarrow !\text{pr}_0 \\ & & !X \end{array} \quad \begin{array}{ccc} !X \otimes ! & \xrightarrow{!X \otimes \tilde{L}} & !X \otimes ! \otimes ! \xrightarrow{\tilde{\partial}_X \otimes !} & !(X \otimes !) \otimes ! \\ \tilde{\partial}_X \downarrow & & \downarrow \tilde{\partial}_X \otimes ! \\ !(X \otimes !) & \xrightarrow{!(X \otimes \tilde{L})} & !(X \otimes ! \otimes !) \end{array}$$

($\mathbf{C}\partial$ -&)

$$\begin{array}{ccc} 1 \otimes ! & \xrightarrow{1 \otimes \text{pr}_0} & 1 \\ m^0 \otimes ! \downarrow & & \downarrow m^0 \\ !\top \otimes ! & \xrightarrow{\tilde{\partial}_{\top}} & !(\top \otimes !) \xrightarrow{!0} & !\top \end{array} \quad \begin{array}{ccc} !X_0 \otimes !X_1 \otimes ! & \xrightarrow{\text{Id} \otimes \tilde{L}} & !X_0 \otimes !X_1 \otimes ! \otimes ! \xrightarrow{\tilde{\partial}_{X_0} \otimes \tilde{\partial}_{X_1}} & !(X_0 \otimes !) \otimes !(X_1 \otimes !) \\ m^2 \otimes ! \downarrow & & \downarrow m^2 \\ !(X_0 \& X_1) \otimes ! & \xrightarrow{\tilde{\partial}_{X_0 \& X_1}} & !((X_0 \& X_1) \otimes !) \xrightarrow{!(\text{pr}_0 \otimes !, \text{pr}_1 \otimes !)} & !((X_0 \otimes !) \& (X_1 \otimes !)) \end{array}$$

($\mathbf{C}\partial$ -Schwarz)

$$\begin{array}{ccc} !X \otimes ! \otimes ! & \xrightarrow{\tilde{\partial}_X \otimes !} & !(X \otimes !) \otimes ! \xrightarrow{\tilde{\partial}_X \otimes !} & !(X \otimes ! \otimes !) \\ !X \otimes \bar{c} \downarrow & & \downarrow !(X \otimes \bar{c}) \\ !X \otimes ! \otimes ! & \xrightarrow{\tilde{\partial}_X \otimes !} & !(X \otimes !) \otimes ! \xrightarrow{\tilde{\partial}_X \otimes !} & !(X \otimes ! \otimes !) \end{array}$$

Theorem 4.5. *Let $\tilde{\partial}_X \in \mathcal{L}(!X \otimes !, !(X \otimes !))$ be a natural transformation satisfying ($\mathbf{C}\partial$ -chain), ($\mathbf{C}\partial$ -local), ($\mathbf{C}\partial$ -lin), ($\mathbf{C}\partial$ -&) and ($\mathbf{C}\partial$ -Schwarz). Then $\tilde{\partial}^+$ is a differentiation in $(\mathcal{L}, \mathbf{S}_1)$ (in the sense of Definition 3.1).*

Proof. Simple categorical computations. \square

Definition 4.2. A differential canonically summable resource category is a canonically summable resource category \mathcal{L} equipped with a natural transformation $\tilde{\partial}_X \in \mathcal{L}(!X \otimes !, !(X \otimes !))$ satisfying ($\mathbf{C}\partial$ -chain), ($\mathbf{C}\partial$ -local), ($\mathbf{C}\partial$ -lin), ($\mathbf{C}\partial$ -&) and ($\mathbf{C}\partial$ -Schwarz). Then we set $\partial = \tilde{\partial}^+$.

5 The differential structure of Coh

Observe first that we can identify⁶ $!|\Delta_1 E|$ with $\{(m_0, m_1) \in !|E| \mid m_0 + m_1 \in !|E|\}$, with the coherence relation $(m_0, m_1) \circ (m'_0, m'_1)$ if $m_0 + m_1 + m'_0 + m'_1 \in !|E|$. Notice that here *it is not required* that $\text{supp}(m_0) \cap \text{supp}(m_1) = \emptyset$ for having $(m_0, m_1) \in !|\Delta_1 E|$. We define $\tilde{\partial}_E \subseteq |\Delta_1!E \multimap !\Delta_1 E|$ by

$$\tilde{\partial}_E = \{((m, 0), (m, [])) \mid m \in !|E|\} \cup \{((m, 1), (m - [a], [a])) \mid m \in !|E| \text{ and } a \in \text{supp}(m)\}.$$

Let $((m_j, i_j), (m_{j0}, m_{j1})) \in \tilde{\partial}_E$ for $j = 0, 1$ and assume that $(m_0, i_0) \circ_{\Delta_1!E} (m_1, i_1)$ which simply means that $m_0 + m_1 \in !|E|$. Since $m_{j0} + m_{j1} = m_j$ for $i = 0, 1$, we have $(m_{00}, m_{01}) \circ_{\Delta_1!E} (m_{10}, m_{11})$ and if moreover we have $(m_{00}, m_{01}) = (m_{10}, m_{11})$ it follows that $m_0 = m_1$ for the same reason. We check the four conditions above.

(C ∂ -local) Let $(m, (m_0, m_1)) \in !|E \multimap !(E \otimes \mathbb{I})|$. Assume first that $(m, (m_0, m_1)) \in \tilde{\partial}_E (!E \otimes \tilde{\pi}_0^{\&})$ so that $((m, 0), (m_0, m_1)) \in \tilde{\partial}_E$, that is $m_1 = []$ and $m = m_0$ which means $(m, (m_0, m_1)) \in !\tilde{\pi}_0^{\&}$. If $(m, (m_0, m_1)) \in !\tilde{\pi}_0^{\&}$ then we have $m = m_0$ and $m_1 = []$ and hence $(m, (m_0, m_1)) \in \tilde{\partial}_E$.

(C ∂ -lin) We prove the second commutation. Let $((m, i), \vec{m}) \in !|E \otimes \mathbb{I} \multimap !(E \otimes \mathbb{I} \otimes \mathbb{I})|$ where $\vec{m} = (m_{00}, m_{01}, m_{10}, m_{11}) \in !|E|^4$. We first make more explicit the condition $((m, i), \vec{m}) \in \tilde{\partial}_{E \otimes \mathbb{I}} (\tilde{\partial}_E \otimes \mathbb{I}) (!E \otimes \tilde{\mathbb{L}})$. It means that there are $i_0, i_1 \in \{0, 1\}$ and $m_0, m_1 \in !|E|$ such that $(i, i_0, i_1) \in \tilde{\mathbb{L}}$, $((m, i_0), (m_0, m_1)) \in \tilde{\partial}_E$ and $((m_0, m_1), i_1), \vec{m}) \in \tilde{\partial}_{E \otimes \mathbb{I}}$. By definition of $\tilde{\mathbb{L}}$ this condition is equivalent to the following trichotomy:

- $i = i_0 = i_1 = 0$ and then we have $m_0 = m$, $m_1 = []$, $(m_{00}, m_{01}) = (m_0, [])$ and $(m_{10}, m_{11}) = []$, that is $m_{00} = m$ and $m_{01} = m_{10} = m_{11} = []$;
- or $i = 1, i_0 = 1, i_1 = 0$ and then there is $a \in \text{supp}(m)$ with $m_0 = m - [a]$, $m_1 = [a]$, $(m_{10}, m_{11}) = (m - [a], [a])$ and $(m_{10}, m_{11}) = []$, that is $m_{00} = m - [a]$, $m_{01} = [a]$, $m_{10} = m_{11} = []$;
- or $i = 1, i_0 = 0, i_1 = 1$ and $m_0 = m$, $m_1 = []$ and there is $(j, a) \in \text{supp}(m_0, m_1)$ such that $(m_{00}, m_{01}) = (m_0, m_1) - [(j, a)]$, $(m_{10}, m_{11}) = [(j, a)]$, so that $j = 0$, $m_{10} = [a]$ and $m_{11} = []$; finally $m_{00} = m - [a]$, $m_{01} = []$, $m_{10} = [a]$ and $m_{11} = []$.

Let us now characterize the condition $((m, i), \vec{m}) \in !(E \otimes \tilde{\mathbb{L}}) \tilde{\partial}_E$. It means that there are $m_0, m_1 \in !|E|$ with $((m, i), (m_0, m_1)) \in \tilde{\partial}_E$ and $((m_0, m_1), \vec{m}) \in !(E \otimes \tilde{\mathbb{L}})$. There two cases.

- If $i = 0$ this condition is equivalent to $m_0 = m$, $m_1 = []$ so that (m_0, m_1) represents actually the multiset $[(a_1, 0), \dots, (a_n, 0)]$ where $m_0 = [a_1, \dots, a_n]$. Since the only pair $(i_0, i_1) \in |\mathbb{I}|^2$ such that $(0, (i_0, i_1)) \in \tilde{\mathbb{L}}$ is $(0, 0)$, the condition $((m, []), \vec{m}) \in !(E \otimes \tilde{\mathbb{L}})$ means that $m_{00} = m$ and $m_{10} = m_{01} = m_{11} = []$.
- If $i = 1$ this condition means first that there is $a \in \text{supp}(m)$ such that $m_0 = m - [a]$ and $m_1 = [a]$. So if $m_0 = [a_2, \dots, a_n]$ then $(m_0, m_1) = [(a, 1), (a_2, 0), \dots, (a_n, 0)]$. Since $(1, (i_0, i_1)) \in \tilde{\mathbb{L}}$ iff $(i_0, i_1) \in \{(1, 0), (0, 1)\}$ the condition $((m_0, m_1), \vec{m}) \in !(E \otimes \tilde{\mathbb{L}})$ means that $\vec{m} = [(a, 01), (a_2, 00), \dots, (a_n, 00)]$ or $\vec{m} = [(a, 10), (a_2, 00), \dots, (a_n, 00)]$, in other words $m_{00} = m - [a]$, $m_{01} = [a]$, $m_{10} = []$ and $m_{11} = []$, or $m_{00} = m - [a]$, $m_{01} = []$, $m_{10} = [a]$ and $m_{11} = []$.

Which is the same trichotomy as above.

(C ∂ -&) We prove the second diagram. Let

$$e = (((m_0, m_1), i), ((p_{00}, p_{01}), (p_{10}, p_{11}))) \in !|E_0 \otimes !E_1 \otimes \mathbb{I} \multimap !(E_0 \otimes \mathbb{I}) \otimes !(E_1 \otimes \mathbb{I})|.$$

We rephrase first the condition $e \in (\tilde{\partial}_{E_0} \otimes \tilde{\partial}_{E_1}) (\text{Id} \otimes \tilde{\mathbb{L}})$. It means that there are $i_0, i_1 \in \{0, 1\}$ such that $(i, (i_0, i_1)) \in \tilde{\mathbb{L}}$ and $((m_j, i_j), (p_{j0}, p_{j1})) \in \tilde{\partial}_{E_j}$ for $j = 0, 1$. This means

- $i = 0, p_{j1} = []$ and $p_{j0} = m_j$ for $j = 0, 1$;
- or $i = 1$ and there is $a \in \text{supp}(m_0)$ with $p_{00} = m_0 - [a]$, $p_{01} = [a]$, $p_{10} = m_1$ and $p_{11} = []$;

⁶Precisely, an element of $!|\Delta_1 E|$ is a multiset $m = [(a_1, i(1)), \dots, (a_n, i(n))]$ with $a_j \in |E|$ and $i(j) \in \{0, 1\}$ for $j = 1, \dots, n$; we write $m = (m_0, m_1)$ with $m_i = [a_j \mid i(j) = i]$ for $i = 0, 1$.

- or $i = 1$ and there is $a \in \text{supp}(m_1)$ with $p_{10} = m_1 - [a]$, $p_{11} = [a]$, $p_{00} = m_0$ and $p_{01} = []$.

One has $e \in (\mathfrak{m}^2)^{-1} \langle \text{pr}_0 \otimes \mathfrak{l}, \text{pr}_1 \otimes \mathfrak{l} \rangle \tilde{\partial}_{X_0 \& X_1} (\mathfrak{m}^2 \otimes \mathfrak{l})$ iff $(([0] \times m_0 + [1] \times m_1, i), (q_0, q_1)) \in \tilde{\partial}_{X_0 \& X_1}$ for some $(q_0, q_1) \in \mathfrak{l} \langle (X_0 \& X_1) \otimes \mathfrak{l} \rangle$ with $((q_0, q_1), ((p_{00}, p_{01}), (p_{10}, p_{11}))) \in (\mathfrak{m}^2)^{-1} \langle \text{pr}_0 \otimes \mathfrak{l}, \text{pr}_1 \otimes \mathfrak{l} \rangle$. This means that

- $i = 0$, $q_0 = [0] \times m_0 + [1] \times m_1$ and $q_1 = []$, $p_{j0} = m_j$ and $p_{j1} = []$ for $j = 0, 1$;
- or $i = 1$ and there is $(0, a) \in \text{supp}([0] \times m_0 + [1] \times m_1)$ (that is $a \in \text{supp}(m_0)$), $q_0 = [0] \times (m_0 - [a]) + [1] \times m_1$, $q_1 = [(0, a)]$, $p_{00} = m_0 - [a]$, $p_{01} = [a]$, $p_{10} = m_1$ and $p_{11} = []$;
- or $i = 1$ and there is $(1, a) \in \text{supp}([0] \times m_0 + [1] \times m_1)$ (that is $a \in \text{supp}(m_1)$), $q_0 = [0] \times m_0 + [1] \times (m_1 - [a])$, $q_1 = [(1, a)]$, $p_{00} = m_0$, $m_{01} = []$, $p_{10} = m_1 - [a]$ and $p_{11} = [a]$

which shows that the commutation holds.

(C ∂ -Schwarz) Let $((m, i_0, i_1), \vec{m}) \in \mathfrak{l} \langle E \otimes \mathfrak{l} \otimes \mathfrak{l} \rangle \dashv \langle E \otimes \mathfrak{l} \otimes \mathfrak{l} \rangle$ with $\vec{m} = (m_{00}, m_{01}, m_{10}, m_{11}) \in \mathfrak{l} \langle E \rangle^4$. The condition $((m, i_0, i_1), \vec{m}) \in \mathfrak{l} \langle E \otimes \tilde{\mathfrak{c}} \rangle \tilde{\partial}_{E \otimes \mathfrak{l}} (\tilde{\partial}_E \otimes \mathfrak{l})$ means that there is (m_0, m_1) such that $((m, i_0), (m_0, m_1)) \in \tilde{\partial}_E$ and $((m_0, m_1), i_1), (m_{00}, m_{10}, m_{01}, m_{11})) \in \tilde{\partial}_{E \otimes \mathfrak{l}}$ (observe that we have applied the flip). That is

- (00) $i_0 = i_1 = 0$, $m_0 = m$, $m_1 = []$, $m_{00} = m$, $m_{01} = m_{10} = m_{11} = []$;
- (10) or $i_0 = 1$ and $i_1 = 0$, there is $a \in \text{supp}(m)$ with $m_{00} = m_0 = m - [a]$, $m_{10} = m_1 = [a]$ and $m_{01} = m_{11} = []$;
- (01) or $i_0 = 0$ and $i_1 = 1$, $m_0 = m$, $m_1 = []$, there is $j \in \{0, 1\}$ and $a \in |E|$ such that $(j, a) \in \text{supp}((m_0, m_1))$ and $(m_{00}, m_{10}) = (m_0, m_1) - [(j, a)]$ and $(m_{01}, m_{11}) = [(j, a)]$. Since $m_1 = []$ we must have $j = 0$ and hence $a \in \text{supp}(m_0)$, $m_{00} = m_0 - [a]$, $m_{10} = []$, $m_{01} = [a]$ and $m_{11} = []$.
- (11) or $i_0 = i_1 = 1$ and there is $a_0 \in \text{supp}(m)$ with $m_0 = m - [a_0]$ and $m_1 = [a_0]$, and there is $(j, a_1) \in (m_0, m_1) = (m - [a_0], [a_0])$ with $(m_{00}, m_{10}) = (m_0, m_1) - [(j, a_1)]$ and $(m_{01}, m_{11}) = [(j, a_1)]$. If $j = 0$ this means $a_1 \in \text{supp}(m - [a_0])$, $m_{00} = m - [a_0, a_1]$, $m_{10} = [a_0]$, $m_{01} = [a_1]$ and $m_{11} = []$. If $j = 1$ this means $a_1 = a_0$, $m_{00} = m - [a_0]$, $m_{10} = []$, $m_{01} = []$ and $m_{11} = [a_1]$.

On the other hand $((m, i_0, i_1), \vec{m}) \in \tilde{\partial}_{E \otimes \mathfrak{l}} (\tilde{\partial}_E \otimes \mathfrak{l}) \langle X \otimes \tilde{\mathfrak{c}} \rangle$ means $((m, i_1, i_0), \vec{m}) \in \tilde{\partial}_{E \otimes \mathfrak{l}} (\tilde{\partial}_E \otimes \mathfrak{l})$ which by the same considerations as above means (now without swapping m_{01} and m_{10})

- (00) $i_0 = i_1 = 0$, $m_0 = m$, $m_1 = []$, $m_{00} = m$, $m_{01} = m_{10} = m_{11} = []$;
- (01) or $i_1 = 1$ and $i_0 = 0$, there is $a \in \text{supp}(m)$ with $m_{00} = m_0 = m - [a]$, $m_{01} = m_1 = [a]$ and $m_{10} = m_{11} = []$;
- (10) or $i_1 = 0$ and $i_0 = 1$, $m_0 = m$, $m_1 = []$, there is $j \in \{0, 1\}$ and $a \in |E|$ such that $(j, a) \in \text{supp}((m_0, m_1))$ and $(m_{00}, m_{01}) = (m_0, m_1) - [(j, a)]$ and $(m_{10}, m_{11}) = [(j, a)]$. Since $m_1 = []$ we must have $j = 0$ and hence $a \in \text{supp}(m_0)$, $m_{00} = m_0 - [a]$, $m_{01} = []$, $m_{10} = [a]$ and $m_{11} = []$.
- (11) or $i_1 = i_0 = 1$ and there is $a_0 \in \text{supp}(m)$ with $m_0 = m - [a_0]$ and $m_1 = [a_0]$, and there is $(j, a_1) \in (m_0, m_1) = (m - [a_0], [a_0])$ with $(m_{00}, m_{01}) = (m_0, m_1) - [(j, a_1)]$ and $(m_{10}, m_{11}) = [(j, a_1)]$. If $j = 0$ this means $a_1 \in \text{supp}(m - [a_0])$, $m_{00} = m - [a_0, a_1]$, $m_{01} = [a_0]$, $m_{10} = [a_1]$ and $m_{11} = []$. If $j = 1$ this means $a_1 = a_0$, $m_{00} = m - [a_0]$, $m_{01} = []$, $m_{10} = []$ and $m_{11} = [a_1]$.

and a simple inspection shows that the conditions (ij) in these two lists are equivalent (for (11) this involves swapping the roles of a_0 and a_1 which are “bound variables”).

5.1 Differentiation in non-uniform coherence spaces

In Remark 3 we have pointed out that the uniform definition of $!E$ in coherence spaces makes our differentials “too thin” in general although they are non trivial and satisfy all the required rules of the differential calculus. We show briefly how this situation can be remedied using non-uniform coherence spaces.

A non-uniform coherence space (NUCS) is a triple $E = (|E|, \wedge_E, \vee_E)$ where $|E|$ is a set and \wedge_E and \vee_E are two *disjoint* binary symmetric relations on $|E|$ called *strict coherence* and *strict incoherence*. The

important point of this definition is not what is written but what is not: contrarily to usual coherence spaces *we do not require* the complement of the union of these two relations to be the diagonal: it can be any (of course symmetric) binary relation on $|E|$ that we call *neutrality* and denote as \equiv_E (warning: it needs not be an equivalence relation!). Then we define coherence as $\circlearrowright_E = \wedge_E \cup \equiv_E$ and incoherence $\asymp_E = \smile_E \cup \equiv_E$ and any pair of relations among these 5 (with suitable relation between them such as $\equiv_E \subseteq \asymp_E$), apart from the trivially complementary ones $(\smile_E, \circlearrowright_E)$ and (\wedge_E, \asymp_E) , are sufficient to define such a structure.

Cliques are defined as usual: $\text{Cl}(E) = \{x \subseteq |E| \mid \forall a, a' \in x \ a \circlearrowright_E a'\}$. Then $(\text{Cl}(E), \subseteq)$ is a cpo (a dI-domain actually) but now there can be some $a \in |E|$ such that $a \smile_E a$, and hence $\{a\} \notin \text{Cl}(E)$ (we show below that this really happens). Given NUCS E and F we define $E \multimap F$ by $|E \multimap F| = |E| \times |F|$ and: $(a_0, b_0) \circlearrowright_{E \multimap F} (a_1, b_1)$ if $a_0 \circlearrowright_E a_1 \Rightarrow (b_0 \circlearrowright_F b_1 \text{ and } b_0 \equiv_F b_1 \Rightarrow a_0 \equiv_E a_1)$ and $(a_0, b_0) \equiv_{E \multimap F} (a_1, b_1)$ if $a_0 \equiv_E a_1$ and $b_0 \equiv_F b_1$. Then we define a category \mathbf{NCoh} by $\mathbf{NCoh}(E, F) = \text{Cl}(E \multimap F)$, taking the diagonal relations as identities and ordinary composition of relations as composition of morphisms.

This is a cartesian SMCC with tensor product given by $|E_0 \otimes E_1| = |E_0| \times |E_1|$ and $(a_{00}, a_{01}) \circlearrowright_{E_0 \otimes E_1} (a_{10}, a_{11})$ if $a_{0j} \circlearrowright_{E_j} a_{1j}$ for $j = 0, 1$, and $\equiv_{E_0 \otimes E_1}$ is defined similarly; the unit is 1 with $|1| = \{*\}$ and $*$ \equiv_1 $*$. The object of linear morphisms from E to F is of course $E \multimap F$ and \mathbf{NCoh} is $*$ -autonomous with 1 as dualizing object. The dual E^\perp is given by $|E^\perp| = |E|$, $\wedge_{E^\perp} = \smile_E$ and $\smile_{E^\perp} = \wedge_E$. The cartesian product $\&_{i \in I} E_i$ of a family $(E_i)_{i \in I}$ of NUCS is given by $|\&_{i \in I} E_i| = \cup_{i \in I} \{i\} \times |E_i|$ with $(i_0, a_0) \equiv_{\&_{i \in I} E_i} (i_1, a_1)$ if $i_0 = i_1 = i$ and $a_0 \equiv_{E_i} a_1$, and $(i_0, a_0) \circlearrowright_{\&_{i \in I} E_i} (i_1, a_1)$ if $i_0 = i_1 = i \Rightarrow a_0 \circlearrowright_{E_i} a_1$. We do not give the definition of the operations on morphisms as they are the most obvious ones (the projections of the product are the relations $\text{pr}_i = \{((i, a), a) \mid i \in I \text{ and } a \in |E_i|\}$). Notice that in the object $\mathbf{Bool} = 1 \oplus 1 = (1 \& 1)^\perp$, the two elements $0, 1$ of the web satisfy $0 \smile_{\mathbf{Bool}} 1$ so that $\{0, 1\} \notin \text{Cl}(\mathbf{Bool})$ which is expected in a model of deterministic computations.

We come to the really interesting feature of this model, which is the definition of the exponential $!E$; we choose here the one of [Bou1]. One sets $!E| = \mathcal{M}_{\text{fin}}(|E|)$ (without any uniformity restrictions), $m_0 \circlearrowright_{!E} m_1$ if $\forall a_0 \in \text{supp}(m_0), a_1 \in \text{supp}(m_1) \ a_0 \circlearrowright_E a_1$, and $m_0 \equiv_{!E} m_1$ if $m_0 \circlearrowright_{!E} m_1$ and $m_j = [a_{j0}, \dots, a_{jn}]$ (for $j = 0, 1$) with $\forall i \in \{1, \dots, n\} \ a_{0i} \equiv_E a_{1i}$ (in particular m_0 and m_1 must have the same size). Observe that $[0, 1] \in !\mathbf{Bool}$ and that $[0, 1] \smile_{!\mathbf{Bool}} [0, 1]$. The action of this functor on morphisms is defined as in the relational model of LL: if $s \in \mathbf{NCoh}(E, F)$ then $!s = \{([a_1, \dots, a_n], [b_1, \dots, b_n]) \mid n \in \mathbb{N} \text{ and } \forall i \ (a_i, b_i) \in s\} \in \mathbf{NCoh}(!E, !F)$.

We obtain in that way a Seely model of LL [Mel09], and hence a resource category, which is easily seen to be canonically summable, exactly as we did in Section 4.3 for ordinary coherence spaces. Next we define $\tilde{\partial}_E \subseteq |\Delta_1!E \multimap !\Delta_1E|$ by

$$\tilde{\partial}_E = \{((m, 0), (m, [])) \mid m \in !E\} \cup \{((m_0 + [a], 1), (m_0, [a])) \mid m_0 \in !E \text{ and } a \in |E|\}.$$

Let us check that $\tilde{\partial}_E \in \mathbf{NCoh}(\Delta_1!E, !\Delta_1E)$ so let $((m_j, i_j), (m_{j0}, m_{j1})) \in \tilde{\partial}_E$ for $j = 0, 1$ and assume that $(m_0, i_0) \circlearrowright_{\Delta_1!E} (m_1, i_1)$, which simply means that $m_0 \circlearrowright_{!E} m_1$ (remember that $\Delta_1 F = F \otimes (1 \& 1)$) so that $(b_0, j_0) \equiv_{\Delta_1 F} (b_1, j_1)$ iff $b_0 \equiv_F b_1$ and $j_0 = j_1$, and $(b_0, j_0) \circlearrowright_{\Delta_1 F} (b_1, j_1)$ iff $b_0 \circlearrowright_F b_1$. Whatever be i_0 and i_1 we have $m_{j0} + m_{j1} = m_j$ for $j = 0, 1$ from which it follows that $(m_{00}, m_{01}) \circlearrowright_{!\Delta_1 E} (m_{10}, m_{11})$ in view of the definition of $\circlearrowright_{!\Delta_1 E}$. Now assume moreover that $(m_{00}, m_{01}) \equiv_{!\Delta_1 E} (m_{10}, m_{11})$; by the definition of $\equiv_{!\Delta_1 E}$ we must have $m_{00} + m_{01} \circlearrowright_{!E} m_{10} + m_{11}$ and $m_{0i} \equiv_{!E} m_{1i}$ for $i = 0, 1$. Therefore m_{01} and m_{11} are either both empty or have both exactly 1 element from which it follows that $j_0 = j_1$ and $m_0 \equiv_{!E} m_1$, that is $(m_0, i_0) \equiv_{\Delta_1!E} (m_1, i_1)$ which ends the proof that $\tilde{\partial}_E$ is an \mathbf{NCoh} morphism. The verification that it fulfills the requirement for turning \mathbf{NCoh} into a differential canonically summable resource category proceeds like in \mathbf{Coh} , see Section 5.

The functor $\mathbf{S}_1 : \mathbf{NCoh} \rightarrow \mathbf{NCoh}$ is then given by $|\mathbf{S}_1 E| = \{0, 1\} \times |E|$ and: $(i_0, a_0) \equiv_{\mathbf{S}_1 E} (i_1, a_1)$ if $i_0 = i_1$ and $a_0 \equiv_E a_1$, and $(i_0, a_0) \circlearrowright_{\mathbf{S}_1 E} (i_1, a_1)$ if $(a_0 \circlearrowright_E a_1 \text{ and } a_0 \equiv_E a_1 \Rightarrow i_0 = i_1)$. Given $s \in \mathcal{L}(E, F)$ we have $\mathbf{S}_1 s = \{((i, a), (i, b)) \mid i \in \{0, 1\} \text{ and } (a, b) \in s\}$. The associated morphism $\partial_E = \tilde{\partial}_E^\perp \in \mathbf{NCoh}(!\mathbf{S}_1 E, \mathbf{S}_1!E)$ is simply $\partial_E = \{((m_0, []), (m_0, 0)) \mid m_0 \in [|E|]\} \cup \{((m_0, [a]), (m_0 + [a], 1)) \mid m_0 \in \mathcal{M}_{\text{fin}}(|E|) \text{ and } a \in |E|\}$ (again we identify $|\mathbf{S}_1 E|$ with $\mathcal{M}_{\text{fin}}(|E|)^2$). This should be compared with Equation (1): we do not have anymore the restriction that a should not belong to $\text{supp}(m_0)$ in the second component of this definition of ∂_E .

6 Summability in a SMCC

Assume now that \mathcal{L} is a summable resource category which is closed wrt. its monoidal product \otimes , so that \mathcal{L}_1 is cartesian closed. We use $X \multimap Y$ for the internal hom object and $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$ for the evaluation morphism. If $f \in \mathcal{L}(Z \otimes X, Y)$ we use $\text{cur } f$ for its transpose $\in \mathcal{L}(Z, X \multimap Y)$.

We can define a natural morphism $\varphi^{-\circ} = \text{cur}((\mathbf{S}\text{ev})\varphi_{X \multimap Y, X}^\gamma) \in \mathcal{L}(\mathbf{S}(X \multimap Y), X \multimap \mathbf{S}Y)$ where $\text{ev} \in \mathcal{L}((X \multimap Y) \otimes X, Y)$.

Lemma 6.1. *We have $(X \multimap \pi_i)\varphi^{-\circ} = \pi_i$ for $i = 0, 1$ and $(X \multimap \mathbf{s}_Y)\varphi^{-\circ} = \mathbf{s}_{X \multimap Y}$.*

Proof. The two first equations come from the fact that $\pi_i\varphi^\gamma = \pi_i \otimes X$. The last one results from Lemma 3.1. \square

Then we introduce a further axiom, required in the case of an SMCC. Its intuitive meaning is that two morphisms f_0, f_1 are summable if they map any element to a pair of summable elements, and that their sum is computed pointwise.

($\mathbf{S}\otimes$ -fun) The morphism $\varphi^{-\circ}$ is an iso.

Lemma 6.2. *If ($\mathbf{S}\otimes$ -fun) holds then $f_0, f_1 \in \mathcal{L}(Z \otimes X, Y)$ are summable iff $\text{cur } f_0$ and $\text{cur } f_1$ are summable. Moreover when this property holds we have $\text{cur}(f_0 + f_1) = \text{cur } f_0 + \text{cur } f_1$.*

Proof. Assume that f_0, f_1 are summable so that we have the witness $\langle f_0, f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(Z \otimes X, \mathbf{S}Y)$ and hence $\text{cur}\langle f_0, f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(Z, X \multimap \mathbf{S}Y)$, so let $h = (\varphi^{-\circ})^{-1} \text{cur}\langle f_0, f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(Z, \mathbf{S}(X \multimap Y))$. By Lemma 6.1 we have $\pi_i h = (1 \multimap \pi_i) \text{cur}\langle f_0, f_1 \rangle_{\mathbf{S}} = \text{cur } f_i$ for $i = 0, 1$. Conversely if $\text{cur } f_0, \text{cur } f_1$ are summable we have the witness $\langle \text{cur } f_0, \text{cur } f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(Z, \mathbf{S}(X \multimap Y))$ and hence $\varphi^{-\circ} \langle \text{cur } f_0, \text{cur } f_1 \rangle_{\mathbf{S}} \in \mathcal{L}(Z, X \multimap \mathbf{S}Y)$ so that $g = \text{ev}((\varphi^{-\circ} \langle \text{cur } f_0, \text{cur } f_1 \rangle_{\mathbf{S}}) \otimes X) \in \mathcal{L}(Z \otimes X, Y)$. Then by naturality of ev and by Lemma 6.1 we get $\pi_i g = f_i$ for $i = 0, 1$ and hence f_0, f_1 are summable.

Assume that these equivalent properties hold so that $\langle \text{cur } f_0, \text{cur } f_1 \rangle_{\mathbf{S}} = (\varphi^{-\circ})^{-1} \text{cur}\langle f_0, f_1 \rangle_{\mathbf{S}}$. So $\text{cur } f_0 + \text{cur } f_1 = \mathbf{s}_{X \multimap Y} \langle \text{cur } f_0, \text{cur } f_1 \rangle_{\mathbf{S}} = (1 \multimap \mathbf{s}_Y) \text{cur}\langle f_0, f_1 \rangle_{\mathbf{S}} = \text{cur}(\mathbf{s}_Y \langle f_0, f_1 \rangle_{\mathbf{S}}) = \text{cur}(f_0 + f_1)$. \square

Theorem 6.1. *If \mathcal{L} is canonically summable then the axiom ($\mathbf{S}\otimes$ -fun) holds.*

Proof. In this case, we know from Section 4.2 that $\varphi^{-\circ}$ is the double transpose of the following morphism of \mathcal{L}

$$(1 \multimap (X \multimap Y)) \otimes X \otimes 1 \xrightarrow{\text{Id} \otimes \gamma} (1 \multimap (X \multimap Y)) \otimes 1 \otimes X \xrightarrow{\text{ev} \otimes X} (X \multimap Y) \otimes X \xrightarrow{\text{ev}} Y$$

and therefore is an iso. \square

We know that \mathcal{L}_1 is a cartesian closed category, with internal hom-object $(X \rightrightarrows Y, \text{Ev})$ (with $X \rightrightarrows Y = (!X \multimap Y)$ and Ev defined using ev). Then if \mathcal{L} is a differential summable resource category which is closed wrt. \otimes and satisfies ($\mathbf{S}\otimes$ -fun), we have a canonical iso between $\mathbf{D}(X \rightrightarrows Y)$ and $X \rightrightarrows \mathbf{D}Y$ and two morphisms $f_0, f_1 \in \mathcal{L}_1(Z \& X, Y)$ are summable (in \mathcal{L}) iff $\text{Cur } f_0, \text{Cur } f_1 \in \mathcal{L}_1(Z, X \rightrightarrows Y)$ are summable and then $\text{Cur } f_0 + \text{Cur } f_1 = \text{Cur}(f_0 + f_1)$.

7 Sketch of a syntax

We outline a tentative syntax corresponding the semantic framework of this paper and strongly inspired by it. Our choice of notations is fully coherent with the notations chosen to describe the model, suggesting a straightforward denotational interpretation. The types are $A, B, \dots := \mathbf{D}^n \iota \mid A \rightrightarrows B$ and then for any type A we define $\mathbf{D}A$ as follows: $\mathbf{D}(\mathbf{D}^n \iota) = \mathbf{D}^{n+1} \iota$ and $\mathbf{D}(A \rightrightarrows B) = (A \rightrightarrows \mathbf{D}B)$. Terms are given by $M, N, \dots := x \mid \lambda x^A M \mid (M)N \mid \mathbf{D}M \mid \pi_i(M) \mid \iota_i(M) \mid \tau(M) \mid 0 \mid M + N$. Given a variable x and a

$$\begin{array}{c}
\frac{}{\Gamma, x : A \vdash x : A} \quad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A M : A \Rightarrow B} \quad \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (M)N : B} \\
\frac{}{\Gamma \vdash 0 : A} \quad \frac{\Gamma \vdash M : DA \quad i \in \{0, 1\}}{\Gamma \vdash \pi_i(M) : A} \quad \frac{\Gamma \vdash M : DA}{\Gamma \vdash \pi_0(M) + \pi_1(M) : A} \quad \frac{\Gamma \vdash M : A \quad i \in \{0, 1\}}{\Gamma \vdash \iota_i(M) : DA} \\
\frac{\Gamma \vdash M : D^2 A}{\Gamma \vdash \tau(M) : DA} \quad \frac{\Gamma \vdash M : A \Rightarrow B}{\Gamma \vdash DM : DA \Rightarrow DB}
\end{array}$$

Figure 1: Typing rules

term N , we define a term $\partial(x, N, M)$ as follows.

$$\begin{array}{ll}
\partial(x, N, y) = \begin{cases} N & \text{if } y = x \\ \iota_0(y) & \text{otherwise} \end{cases} & \partial(x, N, \lambda y^B P) = \lambda y^B \partial(x, N, P) \\
\partial(x, N, DM) = D\partial(x, N, M) & \partial(x, N, (P)Q) = \tau((D\partial(x, N, P))\partial(x, N, Q)) \\
\partial(x, N, 0) = 0 & \partial(x, N, M_0 + M_1) = \partial(x, N, M_0) + \partial(x, N, M_1) \\
\partial(x, N, \pi_i(M)) = \pi_i(\partial(x, N, M)) & \partial(x, N, \tau(M)) = \tau(\partial(x, N, M)) \\
\partial(x, N, \iota_i(M)) = \iota_i(\partial(x, N, M)) &
\end{array}$$

One checks easily that if $\Gamma, x : A \vdash M : B$ and $\Gamma \vdash N : DA$ then $\Gamma \vdash \partial(x, N, M) : DB$. The reduction rules are as follows.

$$\begin{array}{ll}
(\lambda x^A M)N \rightarrow M [N/x] & D(\lambda x^A M) \rightarrow \lambda y^{DA} \partial(x, y, M) \\
D0 \rightarrow 0 & D(M_0 + M_1) \rightarrow DM_0 + DM_1 \\
\lambda x^A 0 \rightarrow 0 & \lambda x^A (M_0 + M_1) \rightarrow \lambda x^A M_0 + \lambda x^A M_1 \\
(0)M \rightarrow 0 & (M_0 + M_1)N \rightarrow (M_0)N + (M_1)N \\
\pi_i(\lambda x^A M) \rightarrow \lambda x^A \pi_i(M) & \pi_i((M)N) \rightarrow (\pi_i(M))N \\
\pi_i(\iota_j(M)) \rightarrow \begin{cases} M & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} & \pi_i(0) \rightarrow 0 \\
\pi_i(M_0 + M_1) \rightarrow \pi_i(M_0) + \pi_i(M_1) & \iota_i(0) \rightarrow 0 \\
\iota_i(M_0 + M_1) \rightarrow \iota_i(M_0) + \iota_i(M_1) & \pi_0(\tau(M)) \rightarrow \pi_0(\pi_0(M)) \\
\pi_1(\tau(M)) \rightarrow \pi_1(\pi_0(M)) + \pi_0(\pi_1(M)) &
\end{array}$$

Semantically, the definition of $\partial(x, N, M)$ and the reduction rules are justified by the fact that when \mathcal{L} is a differential summable resource SMCC, the category $\mathcal{L}!$ is cartesian closed and the functor D acts on it as a strong monad; of course the type DA will be interpreted by DX where X is the interpretation of A . The syntactic construct DM corresponds to the ‘‘internalization’’ $(X \Rightarrow Y) \rightarrow (DX \Rightarrow DY)$ made possible by the strength of D (see Section 3.2). The reduction rules concerning π_i are based on the basic properties of the functor \mathbf{S} and on the definition of the ‘‘multiplication’’ τ of the monad D .

One could add more reduction rules corresponding to the categorical properties of the model, such as for instance $(DM)\iota_0(N) \rightarrow \iota_0((M)N)$. This is probably not necessary if we are mainly interested in reducing terms of types A which are not of shape DB : the reduction rules involving the construction $\pi_i(M)$ seem sufficient for extracting the required information. Of course this requires a proof.

Remark 6. *The only rule introducing sums of terms is the reduction of $\pi_1(\tau(M))$. Since the $\tau(M)$ is created only by the definition of $\partial(x, N, (P)Q)$ we retrieve the fact that, in the differential λ -calculus, sums are introduced by the definition of $\frac{\partial(s)^t}{\partial x} \cdot u$. Therefore the reduction of a term which contains no π_i 's will lead to a sum-free term. It is only when we will want to ‘‘read’’ some information about the differential content of this term that we will apply to it some π_i which will possibly create sums when interacting with the τ 's contained in the term and typically created by the reduction. These τ 's are*

markers of the places where sums will be created. But we can try to be clever and create as few sums as necessary, whereas the differential λ -calculus creates all possible sums immediately in the course of the reduction. This possible parsimony in the creation of sums is very much in the spirit of the effectiveness considerations of [BMP20, MP21].

Remark 7. This is only the purely functional core of a differential programming language where the ground type ι is unspecified. We could extend the language with constants $\underline{n} : \iota$ for $n \in \mathbb{N}$, and with successor, predecessor, and conditional constructs turning it into a type of natural numbers. Since these primitives (as well as many others such as arbitrary recursive types) are easy to interpret in our coherent differential models (such as **Coh**, **NCoh** or **PCS**), we are quite confident that we will be able to integrate them smoothly in the language as well. Notice to finish that, contrarily to what happens in Automatic Differentiation, the operation $+$ on terms is not related to an operation of addition on the ground data type: in **AD**, the ground type is \mathbb{R} (or finite powers thereof) and the $+$ on terms extends the usual $+$ on \mathbb{R} pointwise.

7.1 Recursion

One major feature of the models that we can tackle with the new approach developed in this paper is that they can have fixpoint operators in $\mathcal{L}_1(X \Rightarrow X, X)$ implementing general recursion. This is often impossible in an additive category (typically the categories of topological vector spaces where the differential λ -calculus is usually interpreted): given a closed term M of type A , one can define a term $\lambda x^A (x + M) : A \Rightarrow A$ which cannot have a fixpoint in general if addition is not idempotent.

In contrast consider for instance the category **Pcoh** [DE11]. It is a differential canonically summable resource SMCC where addition is not idempotent and where all least fixpoint operators are available. And accordingly we can extend our language with a construct $\mathbf{Y}M$ typed by $\Gamma \vdash \mathbf{Y}M : A$ if $\Gamma \vdash M : A \Rightarrow A$, with the usual reduction rule $\mathbf{Y}M \rightarrow (M)\mathbf{Y}M$ and so morphisms defined by such fixpoints can also be differentiated. It turns out that we can easily extend the definition of $\partial(x, N, \mathbf{Y}M)$ to the case where $M = \mathbf{Y}P$ with $\Gamma, x : A \vdash P : B \Rightarrow B$ and $\Gamma \vdash N : DA$. The correct definition seems to be

$$\partial(x, N, \mathbf{Y}M) = \mathbf{Y}(\lambda y^{\mathbf{D}B} \tau((\mathbf{D}\partial(x, N, M))y))$$

Conclusion

This coherent setting for the formal differentiation of functional programs should allow to integrate differentiation as an ordinary construct in any functional programming language, without breaking the determinism of its evaluation, contrarily to the original differential λ -calculus, whose operational meaning was unclear essentially for its non-determinism. Moreover the differential construct features commutative monadic structures strongly suggesting to consider it as an effect. The fact that this differentiation is compatible with models such as (non uniform) coherence spaces which have nothing to do with “analytic” differentiation suggests that it could also be used for other operational goals, more internal to the scope of general purpose functional languages, such as *incremental computing*.

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A Appendix

A.1 Proof of Lemma 2.1

Proof. It suffices to prove that if (f_1, \dots, f_n) is summable then the cyclic permutation $(f_n, f_1, \dots, f_{n-1})$ is summable with the same sum. For $n = 0$ and $n = 1$, this is trivial. For $n = 2$ the property results from Lemma 2.5. For $n = 3$ it results from Lemma 2.8 and 2.5. Let us deal with the case $n = 4$, the general case being similar. We know that (f_1, f_2) is summable, (f_3, f_4) is summable and $(f_1 + f_2, f_3 + f_4)$ is summable. It follows that (f_4, f_3) is summable by Lemma 2.5 and hence, by Lemma 2.7 we have that (f_1, f_4) and (f_2, f_3) is summable, so that (f_4, f_1) is summable so that finally (f_4, f_1, f_2, f_3) is summable. The identity of summations results from the corresponding identities in the lemmas we have used. \square

A.2 Proof of Theorem 3.1

Proof. The fact that φ is a strength corresponds to the following two commutations:

$$\begin{array}{ccc} X_0 \otimes X_1 & & X_0 \otimes \mathbf{S}^2 X_1 \xrightarrow{\varphi} \mathbf{S}(X_0 \otimes \mathbf{S} X_1) \xrightarrow{\mathbf{S}\varphi} \mathbf{S}^2(X_0 \otimes X_1) \\ X_0 \otimes \iota_0 \downarrow & \searrow \iota_0 & \downarrow X_0 \otimes \sigma \\ X_0 \otimes \mathbf{S} X_1 \xrightarrow{\varphi} \mathbf{S}(X_0 \otimes X_1) & & (X_0 \otimes \mathbf{S} X_1) \xrightarrow{\varphi} \mathbf{S}(X_0 \otimes X_1) \end{array}$$

Let us prove for instance the second one. We have

$$\begin{aligned} \sigma(\mathbf{S}\varphi)\varphi &= \langle \pi_0 \pi_0, \pi_1 \pi_0 + \pi_0 \pi_1 \rangle_{\mathbf{S}} \langle \varphi \pi_0, \varphi \pi_1 \rangle_{\mathbf{S}} \varphi \\ &= \langle \pi_0 \varphi \pi_0, \pi_1 \varphi \pi_0 + \pi_0 \varphi \pi_1 \rangle_{\mathbf{S}} \varphi \\ &= \langle (X_0 \otimes \pi_0) \pi_0 \varphi, (X_0 \otimes \pi_1) \pi_0 \varphi + (X_0 \otimes \pi_0) \pi_1 \varphi \rangle_{\mathbf{S}} \\ &= \langle (X_0 \otimes \pi_0) (X_0 \otimes \pi_0), (X_0 \otimes \pi_1) (X_0 \otimes \pi_0) + (X_0 \otimes \pi_0) (X_0 \otimes \pi_1) \rangle_{\mathbf{S}} = X_0 \otimes \sigma. \end{aligned}$$

The fact that $(\mathbf{S}, \iota_0, \sigma, \varphi)$ is a commutative monad means that, moreover, the following diagram commutes:

$$\begin{array}{ccc} \mathbf{S} X_0 \otimes \mathbf{S} X_1 \xrightarrow{\varphi_{\mathbf{S} X_0, X_1}} \mathbf{S}(\mathbf{S} X_0 \otimes X_1) \xrightarrow{\mathbf{S}\varphi_{X_0, X_1}^\gamma} \mathbf{S}^2(X_0 \otimes X_1) & & \\ \varphi_{X_0, \mathbf{S} X_1}^\gamma \downarrow & & \downarrow \sigma \\ \mathbf{S}(X_0 \otimes \mathbf{S} X_1) \xrightarrow{\mathbf{S}\varphi_{X_0, X_1}} \mathbf{S}^2(X_0 \otimes X_1) \xrightarrow{\sigma} \mathbf{S}(X_0 \otimes X_1) & & \end{array}$$

which results from the commutation of

$$\begin{array}{ccc} \mathbf{S} X_0 \otimes \mathbf{S} X_1 \xrightarrow{\varphi_{\mathbf{S} X_0, X_1}} \mathbf{S}(\mathbf{S} X_0 \otimes X_1) & & \\ \varphi_{X_0, \mathbf{S} X_1}^\gamma \downarrow & & \downarrow \mathbf{S}\varphi_{X_0, X_1}^\gamma \\ \mathbf{S}(X_0 \otimes \mathbf{S} X_1) \xrightarrow{\mathbf{S}\varphi_{X_0, X_1}} \mathbf{S}^2(X_0 \otimes X_1) \xrightarrow{c_{X_0 \otimes X_1}} \mathbf{S}^2(X_0 \otimes X_1) & & \end{array}$$

and from Theorem 2.3. The last commutation is proved as follows: $\pi_i \pi_j (\mathbf{S}\varphi_{X_0, X_1}^\gamma) \varphi_{\mathbf{S} X_0, X_1} = \pi_i \varphi_{X_0, X_1}^\gamma \pi_j \varphi_{\mathbf{S} X_0, X_1} = (\pi_i \otimes X_1) (\mathbf{S} X_0 \otimes \pi_j) = \pi_i \otimes \pi_j$ and similarly $\pi_i \pi_j c(\mathbf{S}\varphi_{X_0, X_1}) \varphi_{X_0, \mathbf{S} X_1}^\gamma = \pi_j \pi_i (\mathbf{S}\varphi_{X_0, X_1}) \varphi_{X_0, \mathbf{S} X_1}^\gamma = \pi_i \otimes \pi_j$. \square

A.3 Proof of Lemma 4.1

Proof. Assume that f_0, f_1 are summable so that we have a uniquely defined $\langle f_0, f_1 \rangle_{\mathbf{S}_1} \in \mathcal{L}(X, \mathbb{1} \multimap Y)$ with $\pi_i \langle f_0, f_1 \rangle_{\mathbf{S}_1} = f_i$ for $i = 0, 1$. We have

$$\begin{aligned} f_i &= \pi_i \langle f_0, f_1 \rangle_{\mathbf{S}_1} = \text{ev}((\mathbb{1} \multimap Y) \otimes \bar{\pi}_i^{\&}) \rho_{\mathbb{1} \multimap Y}^{-1} \langle f_0, f_1 \rangle_{\mathbf{S}_1} \\ &= \text{ev}((\mathbb{1} \multimap Y) \otimes \bar{\pi}_i^{\&}) (\langle f_0, f_1 \rangle_{\mathbf{S}_1} \otimes \mathbb{1}) \rho_X^{-1} \\ &= \text{ev}(\langle f_0, f_1 \rangle_{\mathbf{S}_1} \otimes \mathbb{1}) (X \otimes \bar{\pi}_i^{\&}) \rho^{-1} \end{aligned}$$

So $g = \text{ev}(\langle f_0, f_1 \rangle_{\mathbf{S}_1} \otimes \mathbb{1})$ satisfies the announced condition. Moreover

$$\begin{aligned} f_0 + f_1 &= \mathbf{s} \langle f_0, f_1 \rangle_{\mathbf{S}_1} = \text{ev}((\mathbb{1} \multimap Y) \otimes \Delta^{\&}) \rho^{-1} \langle f_0, f_1 \rangle_{\mathbf{S}_1} \\ &= g(X \otimes \Delta^{\&}) \rho^{-1} \end{aligned}$$

Assume conversely that we have such a $g \in \mathcal{L}(X \otimes \mathbb{1}, Y)$. Then

$$\begin{aligned} \pi_i \text{cur } g &= \text{ev}(\text{cur } g \otimes \mathbb{1}) (X \otimes \bar{\pi}_i^{\&}) \rho^{-1} \\ &= g(X \otimes \bar{\pi}_i^{\&}) \rho^{-1} = f_i \end{aligned}$$

hence f_0, f_1 are summable and $\text{cur } g = \langle f_0, f_1 \rangle_{\mathbf{S}_1}$. □

A.4 Proof of Theorem 4.2

Proof. Let us prove that **(S-com)** holds. Let $f = \text{cur } g \in \mathcal{L}(\mathbf{S}_1 X, \mathbf{S}_1 X)$ where g is the following composition of morphisms

$$(\mathbb{1} \multimap X) \otimes \mathbb{1} \xrightarrow{\text{Id} \otimes (\text{pr}_1, \text{pr}_0)} (\mathbb{1} \multimap X) \otimes \mathbb{1} \xrightarrow{\text{ev}} X$$

then by the same computation as in the proof above we have

$$\begin{aligned} \pi_i f &= \text{ev}(f \otimes \mathbb{1}) ((\mathbb{1} \multimap X) \otimes \bar{\pi}_i^{\&}) \rho^{-1} \\ &= g((\mathbb{1} \multimap X) \otimes \bar{\pi}_i^{\&}) \rho^{-1} \\ &= \text{ev}((\mathbb{1} \multimap X) \otimes \bar{\pi}_{1-i}^{\&}) \rho^{-1} \\ &= \pi_{1-i}. \end{aligned}$$

Moreover by the same kind of computation we get

$$\mathbf{s} f = g((\mathbb{1} \multimap X) \otimes \Delta^{\&}) \rho^{-1} = \text{ev}((\mathbb{1} \multimap X) \otimes \Delta^{\&}) \rho^{-1} = \mathbf{s}.$$

Now we check **(S-zero)** so let $f \in \mathcal{L}(X, Y)$. Let $g \in \mathcal{L}(X \otimes \mathbb{1}, Y)$ be defined by $g = f \rho(X \otimes \text{pr}_0)$ (where pr_0 is the first projection $I \rightarrow \mathbb{1}$). We have $g(X \otimes \bar{\pi}_i^{\&}) \rho^{-1} = f \rho(X \otimes (\text{pr}_0 \bar{\pi}_i^{\&})) \rho^{-1}$ so that $g(X \otimes \bar{\pi}_i^{\&}) \rho^{-1} = f$ is $i = 0$ and $g(X \otimes \bar{\pi}_i^{\&}) \rho^{-1} = 0$ is $i = 1$. By Lemma 4.1 this shows that $f, 0$ are summable with sum $f + 0 = g(X \otimes \Delta^{\&}) \rho^{-1} = f \rho(X \otimes (\text{pr}_0 \Delta^{\&})) \rho^{-1} = f$.

Let us prove that condition **(S-witness)** holds. Let $f_{ij} \in \mathcal{L}(X, Y)$ be such that the pairs (f_{00}, f_{01}) and (f_{10}, f_{11}) are summable and assume also that $(f_{00} + f_{01}, f_{10} + f_{11})$ is summable. Let $g_i = \text{ev}(\langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1} \otimes \mathbb{1}) \in \mathcal{L}(X \otimes \mathbb{1}, Y)$. We have $g_i(X \otimes \Delta^{\&}) \rho^{-1} = \text{ev}(X \otimes \Delta^{\&}) \langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1} = \mathbf{s} \langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1} = f_{i0} + f_{i1}$ and hence by **(CS-sum)** g_0 and g_1 are summable. Let $h \in \mathcal{L}(X \otimes \mathbb{1} \otimes \mathbb{1}, Y)$ be such that $h(X \otimes \mathbb{1} \otimes \bar{\pi}_i^{\&}) \rho_{X \otimes \mathbb{1}}^{-1} = g_i$ for $i = 0, 1$. Let $h' = h(X \otimes \bar{\mathbb{c}}) \in \mathcal{L}(X \otimes \mathbb{1} \otimes \mathbb{1}, Y)$, we have $\text{cur}(h') \in \mathcal{L}(X, \mathbb{1} \multimap (\mathbb{1} \multimap X))$ and $\pi_i \text{cur}(h') = \langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1}$ as we prove now. Let $j \in \{0, 1\}$, we have

$$\begin{aligned} \pi_j \pi_i \text{cur}(h') &= \pi_j \text{ev}(\text{cur}(h') \otimes \mathbb{1}) (X \otimes \bar{\pi}_i^{\&}) \rho_X^{-1} = \pi_j (\text{cur } h') (X \otimes \bar{\pi}_i^{\&}) \rho_X^{-1} \\ &= \pi_j \text{cur}(h' (X \otimes \bar{\pi}_i^{\&} \otimes \mathbb{1})) \rho_X^{-1} \\ &= \text{ev}(\text{cur}(h' (X \otimes \bar{\pi}_i^{\&} \otimes \mathbb{1})) \otimes \mathbb{1}) (X \otimes \mathbb{1} \otimes \bar{\pi}_j^{\&}) \rho_{X \otimes \mathbb{1}}^{-1} \rho_X^{-1} \\ &= h(X \otimes \bar{\mathbb{c}}) (X \otimes \bar{\pi}_i^{\&} \otimes \bar{\pi}_j^{\&}) \rho_{X \otimes \mathbb{1}}^{-1} \rho_X^{-1} \\ &= h(X \otimes \bar{\pi}_j^{\&} \otimes \bar{\pi}_i^{\&}) \rho_{X \otimes \mathbb{1}}^{-1} \rho_X^{-1} \quad \text{by (CS-flip)} \\ &= h(X \otimes \mathbb{1} \otimes \bar{\pi}_i^{\&}) \rho_{X \otimes \mathbb{1}}^{-1} (X \otimes \bar{\pi}_j^{\&}) \rho_X^{-1} = g_i(X \otimes \bar{\pi}_j^{\&}) \rho_X^{-1} \\ &= \text{ev}(\langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1} \otimes \mathbb{1}) (X \otimes \bar{\pi}_j^{\&}) \rho_X^{-1} \\ &= \pi_j \langle f_{i0}, f_{i1} \rangle_{\mathbf{S}_1} = f_{ij} \end{aligned}$$

proving our contention.

To prove **(S-assoc)** we define $c_X \in \mathcal{L}(\mathbf{S}_1^2 X, \mathbf{S}_1^2 X)$ by $c_X = \text{cur}(\text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \tilde{c})))$ where the transposed morphism is typed as follows.

$$\mathbf{S}_1^2 X \otimes \text{l} \otimes \text{l} \xrightarrow{\text{Id} \otimes \tilde{c}} \mathbf{S}_1^2 X \otimes \text{l} \otimes \text{l} \xrightarrow{\text{ev} \otimes \text{l}} \mathbf{S}_1 X \otimes \text{l} \xrightarrow{\text{ev}} X$$

A computation similar to the previous ones shows that $\pi_i \pi_j c = \pi_j \pi_i$ as required. We have moreover

$$\begin{aligned} \mathbf{s}_{\mathbf{S}_1 X} c_X &= \text{ev}(\mathbf{S}_1^2 X \otimes \Delta^{\&}) \rho^{-1} c \quad \text{by def. of } \mathbf{s}_{\mathbf{S}_1 X} \\ &= \text{ev}(c \otimes \text{l}) ((\mathbf{S}_1^2 X) \otimes \Delta^{\&}) \rho^{-1} \\ &= \text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \tilde{c})) ((\mathbf{S}_1^2 X) \otimes \Delta^{\&}) \rho^{-1} \quad \text{by def. of } c \\ &= \text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \tilde{c})(\mathbf{S}_1^2 X \otimes \Delta^{\&} \otimes \text{l})) \rho^{-1} \\ &= \text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes (\tilde{c}(\Delta^{\&} \otimes \text{l})))) \rho^{-1} \\ &= \text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes ((\text{l} \otimes \Delta^{\&}) \gamma))) \rho^{-1} \quad \text{by (CS-flip)} \\ &= \text{cur}(\text{ev}(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \text{l} \otimes \Delta^{\&})(\mathbf{S}_1^2 X \otimes \gamma)) \rho^{-1} \\ &= \text{cur}(\text{ev}(\mathbf{S}_1 X \otimes \Delta^{\&})(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \gamma)) \rho^{-1} \\ &= \text{cur}(\mathbf{s}_X \rho(\text{ev} \otimes \text{l})(\mathbf{S}_1^2 X \otimes \gamma)) \rho^{-1} \quad \text{by def. of } \mathbf{s}_X \\ &= \text{cur}(\mathbf{s}_X \text{ev}(\rho \otimes \text{l})) \rho^{-1} \quad \text{by nat. of } \rho \\ &= \text{cur}(\mathbf{s}_X \text{ev}) = \mathbf{S} \mathbf{1} \mathbf{s}. \end{aligned}$$

Let us prove **(S⊗-dist)** so let (f_{00}, f_{01}) be a summable pair of morphisms in $\mathcal{L}(X_0, Y_0)$ and let $f_1 \in \mathcal{L}(X_1, Y_1)$. Let $g \in \mathcal{L}(X_0 \otimes \text{l}, Y_0)$ be such that $g(X \otimes \tilde{\pi}_i^{\&}) \rho^{-1} = f_{0i}$ for $i = 0, 1$ (that is $g = \text{ev}(\langle f_{00}, f_{01} \rangle_{\mathbf{S}_1} \otimes \text{l})$). Let $h \in \mathcal{L}(X_0 \otimes X_1 \otimes \text{l}, Y_0 \otimes Y_1)$ be defined as the following composition of morphisms:

$$X_0 \otimes X_1 \otimes \text{l} \xrightarrow{X_0 \otimes \gamma} X_0 \otimes \text{l} \otimes X_1 \xrightarrow{g \otimes f_1} Y_0 \otimes Y_1$$

then we have

$$\begin{aligned} h(X_0 \otimes X_1 \otimes \tilde{\pi}_i^{\&}) \rho_{X_0 \otimes X_1}^{-1} &= (g \otimes f_1)(X_0 \otimes \tilde{\pi}_i^{\&} \otimes X_1)(X_0 \otimes \gamma) \rho_{X_0 \otimes X_1}^{-1} \\ &= ((f_{0i} \rho_{X_0}) \otimes f_1)(X_0 \otimes \gamma) \rho_{X_0 \otimes X_1}^{-1} \\ &= (f_{0i} \otimes f_1)(\rho_{X_0} \otimes X_1)(X_0 \otimes \gamma) \rho_{X_0 \otimes X_1}^{-1} \\ &= f_{0i} \otimes f_1. \end{aligned}$$

which shows that $(f_{00} \otimes f_1, f_{01} \otimes f_1)$ is summable. Moreover a completely similar computation (with $\Delta^{\&}$ instead of $\tilde{\pi}_i^{\&}$) shows that $f_{00} \otimes f_1, f_{01} \otimes f_1 = (f_{00} + f_{01}) \otimes f_1$. \square

A.5 Proof of Lemma 4.3

Proof. Consists of computations using naturality and adjunction properties. As an example, assume the second commutation and let us prove the second diagram of **(∂-chain)**:

$$\begin{array}{ccc} \mathbf{!S}_1 X & \xrightarrow{\tilde{\partial}_X^+} & \mathbf{S}_1 \mathbf{!} X \\ \text{dig}_{\mathbf{S}_1 X} \downarrow & & \downarrow \mathbf{S}_1 \text{dig}_X \\ \mathbf{!!S}_1 X & \xrightarrow{\mathbf{!}\tilde{\partial}_X^+} \mathbf{!S}_1 \mathbf{!} X & \xrightarrow{\tilde{\partial}_{\mathbf{!} X}^+} \mathbf{S}_1 \mathbf{!!} X \end{array}$$

We have

$$\begin{aligned}
(\mathbf{S}_1 \text{dig}_X) \tilde{\partial}_X^+ &= (\mathbf{S}_1 \text{dig}_X) (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 \text{dig}_{\Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 ! \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \Delta_1 \text{dig}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 ! \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 ! \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 ! \varepsilon_{\Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 ! \Delta_1 \eta_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 ! \varepsilon_{\Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 ! \Delta_1 \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 \Delta_1 ! \eta_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_{\mathbf{S}_1, X}) (\mathbf{S}_1 ! \Delta_1 \mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 ! \Delta_1 \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X}) \eta_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X} ! \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_{\mathbf{S}_1, X}) (\mathbf{S}_1 ! \Delta_1 \mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 \Delta_1 ! \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X} ! \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_{\mathbf{S}_1, X}) (\mathbf{S}_1 ! \Delta_1 \mathbf{S}_1 ! \varepsilon_X) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X}) (\mathbf{S}_1 \Delta_1 ! \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X} ! \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) (\mathbf{S}_1 \Delta_1 ! \mathbf{S}_1 ! \varepsilon_X) \eta_{\mathbf{S}_1, \Delta_1, \mathbf{S}_1, X} (! \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) ! \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= (\mathbf{S}_1 ! \varepsilon_{\mathbf{S}_1, X}) (\mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) \eta_{\mathbf{S}_1, X} (! \mathbf{S}_1 ! \varepsilon_X) (! \mathbf{S}_1 \tilde{\partial}_{\mathbf{S}_1, X}) ! \eta_{\mathbf{S}_1, X} \text{dig}_{\mathbf{S}_1, X} \\
&= \tilde{\partial}_{\mathbf{S}_1, X}^+ ! \tilde{\partial}_{\mathbf{S}_1, X}^+ \text{dig}_{\mathbf{S}_1, X}
\end{aligned}$$

The other computations are similar. □