HAL open science

Poynting vector and beam shape coefficients: On new families of symmetries (non-dark axisymmetric beams of the second kind and dark axisymmetric beams)

Gérard Gouesbet, Luiz Felipe Machado Votto, Leonardo André Ambrosio, Jiajie Wang

To cite this version:

Gérard Gouesbet, Luiz Felipe Machado Votto, Leonardo André Ambrosio, Jiajie Wang. Poynting vector and beam shape coefficients: On new families of symmetries (non-dark axisymmetric beams of the second kind and dark axisymmetric beams). Journal of Quantitative Spectroscopy and Radiative Transfer, 2021, 271, pp.107745. 10.1016/j.jqsrt.2021.107745 . hal-03282454

HAL Id: hal-03282454

https://hal.science/hal-03282454v1

Submitted on 13 Jun 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Version of Record: https://www.sciencedirect.com/science/article/pii/S0022407321002387
Manuscript_04169cb8ca780fa4bd732333e6f695d5

Poynting vector and beam shape coefficients: on new families of symmetries (non-dark axisymmetric beams of the second kind and dark axisymmetric beams).

Gérard Gouesbet[1] and Luiz Felipe Machado Votto[2]
and Leonardo André Ambrosio[2]
and Jiajie Wang[3].

1. CORIA-UMR 6614- Normandie Université
CNRS-Université et INSA de Rouen
Campus Universitaire du Madrillet
76800, Saint-Etienne-du Rouvray, France
2. Department of Electrical and Computer Engineering
São Carlos School of Engineering, University of São Paulo
400 Trabalhador são-carlense Ave., São Paulo, SP 13566-590, Brazil.
3. School of Physics and Optoelectronic Engineering,
Xidian University, Xian, 710071, China.
Corresponding author: gouesbet@coria.fr

May 12, 2021

Abstract

On the one hand, some electromagnetic symmetries of laser beams are reflected in symmetry properties of the Poynting vector. On the other hand, these symmetries are in turn embedded in symmetries relating the beam shape coefficients which encode the structure of laser beams. The present paper extends the beam shape coefficient symmetries of laser beams, already uncovered, to new families. Examples and counter-examples are furthermore provided to illustrate these symmetries and formal laboratories are proposed as a way to discover still more symmetries.

Keywords: Generalized Lorenz-Mie theories; beam shape coefficients; Poynting vector; non-dark axisymmetric beams; dark axisymmetric beams; T-matrix.

© 2021 published by Elsevier. This manuscript is made available under the CC BY NC user license
https://creativecommons.org/licenses/by-nc/4.0/

1 Introduction.

Laser beams may be encoded in a set of beam shape coefficients (BSCs) usually denoted as $g^m_{n,TM}$ and $g^m_{n,TE}$, with TM and TE standing for "Transverse Magnetic" and "Transverse Electric" respectively, and n ranging from 1 to infinity with $-n < m < +n$. These BSCs have been introduced in the framework of generalized Lorenz-Mie theories (GLMTs), e.g. [1], [2], [3] with [4], [5] for recent reviews. These GLMTs describe the interaction between "arbitrarily shaped beams" and scattering particles which possess enough degrees of symmetries for allowing the use of the method of separation of variables. BSCs may also be useful to deal with the extended boundary condition method (EBCM) [6], [7] which describes scattering phenomena when arbitrary shaped particles are illuminated by structured beams [8].

An issue of interest is then to investigate the symmetries of BSCs related to symmetries of the fields. In a scattering problem, symmetries of the illuminating fields are indeed reflected in symmetries of the scattered fields, and of the fields internal to the scatterer, these symmetries being in turn related to BSCs symmetries.

To discuss this issue, we must begin with the introduction of a few notations to describe the configurations in hand. As usual in the GLMTs for spherical particles (in particular homogeneous particles described by the GLMT *stricto sensu*), we consider two parallel Cartesian coordinate systems, one Cartesian coordinate system O_{Buvw} attached to the beam and in which the beam is described, and another Cartesian coordinate system O_{Pxyz} attached to the scatterer and in which the scattering fields and associated quantities are described, with axis O_Bu parallel to the axis O_Px, and similarly for O_Bv and O_Py, and again for O_Bw and O_{Pz}. The coordinates of O_B in the system O_{Pxyz} are denoted by (x_0, y_0, z_0). It is assumed that the beam propagates along the w-direction with a propagation term of the form $\exp(ikw)$, or $\exp(ikw\cos\alpha)$ for beams exhibiting an axicon angle α such as Bessel beams to be considered later in the sequel. It is usually assumed in GLMT that the time-dependence of the beam is of the form $\exp(i\omega t)$, although some authors may have preferred to use a time-dependence of the form $\exp(-i\omega t)$. BSC relationships between the first convention – named P-convention with P standing for positive – and the second convention – named N-convention with N standing for negative – will be discussed later in the sequel. Conventions concerning associated Legendre functions will be discussed as well later when appropriate.

As an example, let us consider a Gaussian beam in the fundamental mode TEM_{00} which possesses an axis of symmetry and let us choose this axis as being the axis O_Bw. The description of the beam in the coordinate system O_{Pxyz} is then generally called an off-axis description. However, if $x_0 = y_0 = 0$, with z_0 still different from 0, we say that the off-axis description has become an on-axis description. From this example, although the introduction of a scatterer has been convenient to introduce the different coordinate systems O_{Buvw} and O_{Pxyz}, the existence of a scatterer is not required to distinguish between off-axis

and on-axis configurations.

A first attack on the search for symmetry relations has been published in 1994 in the general framework of off-axis configurations [9], with BSCs evaluated using a technique called quadrature technique [10], [11]. It was stated that symmetry relations may be used to simplify or check some analytical works, or to speed up the numerical evaluations of BSCs by avoiding useless repetitive calculations. For instance, among other relations, it has been demonstrated that:

$$g_{n,TM}^{-m}(-x_0, y_0, z_0) = (-1)^{m-1} g_{n,TM}^{m}(x_0, y_0, z_0) \tag{1}$$

$$g_{n,TM}^{-m}(x_0, -y_0, z_0) = g_{n,TM}^{m}(x_0, y_0, z_0) \tag{2}$$

$$g_{n,TM}^{-m*}(x_0, y_0, -z_0) = (-1)^{m-1} g_{n,TM}^{m}(x_0, y_0, z_0) \tag{3}$$

and that such relationships imply symmetry relationships for quantities associated with a scattering process, such as:

$$C_{ext}(x_0, y_0, z_0) = C_{ext}(-x_0, y_0, z_0) = C_{ext}(x_0, -y_0, z_0) = C_{ext}(x_0, y_0, -z_0) \tag{4}$$

$$C_{sca}(x_0, y_0, z_0) = C_{sca}(-x_0, y_0, z_0) = C_{sca}(x_0, -y_0, z_0) = C_{sca}(x_0, y_0, -z_0) \tag{5}$$

in which C_{ext} and C_{sca} are the extinction and scattering cross-sections respectively, then showing that these cross-sections are symmetrical with respect to x_0, y_0 and z_0 respectively.

These symmetries, however, are related to the configurations used to describe the beam. They are not related to the beam itself. The search for symmetries of BSCs related to the symmetries of the beam started in 1996 [12]. This paper introduced what we call axisymmetric beams defined as beams for which the component of the Poynting vector in the direction of propagation does not depend on the azimuthal angle in a suitably chosen coordinate system. In practice, the suitably chosen coordinate system is a coordinate system O_{Pxyz} in which O_Pz is the axis of the beam, and the component of the Poynting vector in the direction of propagation is the component S_z which does not depend on φ, in which (r, θ, φ) are the usual spherical coordinates attached to the Cartesian coordinates (x, y, z). We then exhibited a class of symmetries of

BSCs, characterizing what we shall from now on call an on-axis axisymmetric beam of the first kind, reading as:

$$g_{n,TM}^{m} = g_{n,TE}^{m} = 0,\ m \neq \pm 1 \tag{6}$$

$$g_n/2 = g_{n,TM}^{1} = g_{n,TM}^{-1}/K = -i\varepsilon g_{n,TE}^{1} = i\varepsilon g_{n,TE}^{-1}/K \tag{7}$$

in which K is related to the polarization of the beam and $\varepsilon = \pm 1$ depends on whether the beam propagates toward positive or negative z's. A particular interest of such beams is that the bi-index BSCs $g_{n,X}^{m}$ (with X being TM or TE) reduces to uni-index BSCs g_n. Such a reduction occurs in particular for Gaussian beams and the uni-index BSCs g_n actually appeared at the very beginning of the development of GLMT [13] and, when it occurs, it allows one to dramatically simplify the description of the beam under consideration as well as the formulation of GLMT, in particular allowing one to easily demonstrate that LMT is indeed a special case of GLMT, e.g. Chapter 6 in [3]. The existence of on-axis axisymmetric beams of the first kind has been revisited in 2017 [14] in an enlarged context, including a discussion of vortex and non vortex beams, of dark beams, and of BSCs symmetries in spheroidal and cylindrical coordinates. In particular, dark beams (more specifically on-axis dark beams, i.e. dark beams in an on-axis configuration in the present paper) are beams whose intensity S_z is equal to 0 on the axis, i.e. for $\theta = 0$. According to a darkness theorem [15], these beams satisfy, whether they are axisymmetric or not [14]:

$$g_{n,TM}^{\pm 1} = g_{n,TE}^{\pm 1} = 0 \tag{8}$$

Therefore, on-axis axisymmetric beams of the first kind are not dark. However, non-dark axisymmetric beams of the first kind do not exhaust all the kinds of non-dark axisymmetric beams, as it can been seen from a comment between Eqs.(65) and (66) of [14]. Indeed, between the case of non-dark axisymmetric beams of the first kind in which all BSCs of the form $g_{n,X}^{\pm 1}$ are different from 0, as shown in Eq.7, and the case when all of them are 0 in the case of dark beams, as shown in Eq.8, there is some room for a possible case when some of them, but not all of them, are 0. The aim of the present paper is to study the possibility for such cases, and indeed to exhibit a class of such beams that we shall name axisymmetric beams of the second kind (which are non-dark). As a by-product, Eq.8 which does not tell whether the concerned beams are axisymmetric or not will be completed by two equations which ensure an axisymmetry. Therefore, beside non-dark axisymmetric beams of the second kind, a class of axisymmetric dark beams will be added to the list of axisymmetric beams.

The present study is motivated by a recent work devoted to optical forces in the Rayleigh limit of the generalized Lorenz-Mie theory [16], [17] and its relationship with the dipole theory of forces [18], [19], [20], [21]. Specific studies have been devoted to the cases of on-axis Bessel beams [22] and to the more general case of on-axis axisymmetric beams of the first kind in a Part I-paper [23]. As a propaedeutical step toward a follow-up of this Part I-paper, the present paper is therefore devoted to non-dark axisymmetric beams of the second kind and to axisymmetric dark beams.

The paper is organized as follows. Section 2 expounds preliminaries useful in the sequel, recalling basic expressions for the longitudinal component of the Poynting vector and the different ways to be used to evaluate BSCs. Section 3 demonstrates new expressions for the BSCs of axisymmetric beams. Section 4 deals with examples and Section 5 with counter-examples. Section 6 exemplifies the fact that a class of beams named frozen waves provides laboratories to discover new symmetries in the BSCs. Section 7 is a conclusion.

2 Preliminaries.

2.1 Longitudinal component of the Poynting vector.

The longitudinal component S_z of the Poynting vector, with the normalization condition in which $E_0 H_0^*/2 = 1$ (E_0 and H_0 are field strengths in a lossless medium in which the beam propagates), reads as [12], [14]:

$$\begin{aligned} S_z &= \frac{-1}{r^2} \mathrm{Re} \sum_{n=1}^{\infty} \sum_{m=-n}^{+n} \sum_{p=1}^{\infty} \sum_{q=-p}^{+p} i c_n^{pw} c_p^{pw*} e^{i(m-q)\varphi} \\ & (\sin\theta S_{np}^{mq} + \cos\theta C_{np}^{mq}) \end{aligned} \tag{9}$$

in which:

$$\begin{aligned} S_{np}^{mq} &= kr[-g_{n,TM}^{m} g_{p,TM}^{q*} \psi_p (\psi_n + \psi_n^{''}) P_n^{|m|} \tau_p^{|q|} \\ & + g_{n,TE}^{m} g_{p,TE}^{q*} \psi_n (\psi_p + \psi_p^{''}) P_p^{|q|} \tau_n^{|m|} \\ & + q g_{n,TM}^{m} g_{p,TE}^{q*} \psi_p^{'} (\psi_n + \psi_n^{''}) P_n^{|m|} \pi_p^{|q|} \\ & + m g_{n,TM}^{m} g_{p,TE}^{q*} \psi_n^{'} (\psi_p + \psi_p^{''}) P_p^{|q|} \pi_n^{|m|}] \end{aligned} \tag{10}$$

$$
\begin{aligned}
C_{np}^{mq} \quad = \quad & -g_{n,TM}^{m} g_{p,TM}^{q*} \psi_p \psi_n^{'} (\tau_n^{|m|} \tau_p^{|q|} + mq \pi_n^{|m|} \pi_p^{|q|}) \\
& + g_{n,TM}^{m} g_{p,TE}^{q*} \psi_n^{'} \psi_p^{'} (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\
& - g_{n,TE}^{m} g_{p,TM}^{q*} \psi_p \psi_n (m \pi_n^{|m|} \tau_p^{|q|} + q \pi_p^{|q|} \tau_n^{|m|}) \\
& + g_{n,TE}^{m} g_{p,TE}^{q*} \psi_n \psi_p^{'} (mq \pi_n^{|m|} \pi_p^{|q|} + \tau_n^{|m|} \tau_p^{|q|})
\end{aligned} \tag{11}
$$

in which ψ_n are Riccati-Bessel functions with the argument kr (in which k is the wave-number) omitted for convenience, a prime (a double prime) denotes a derivative (a double derivative) with respect to the argument, τ_n^m and π_n^m are generalized Legendre functions (with the argument $\cos\theta$ omitted as well for convenience), defined according to:

$$
\tau_n^m(\cos\theta) = \frac{dP_n^m(\cos\theta)}{d\theta} \tag{12}
$$

$$
\pi_n^m(\cos\theta) = \frac{P_n^m(\sin\theta)}{\sin\theta} \tag{13}
$$

in which $P_n^m(\cos\theta)$ are associated Legendre functions defined according to Hobson's convention [24]:

$$
P_n^{|m|}(\cos\theta) = (-1)^{|m|} (\sin\theta)^{|m|} \frac{d^{|m|} P_n(\cos\theta)}{(d\cos\theta)^{|m|}} \tag{14}
$$

in which P_n's are usual Legendre polynomials and, furthermore, the coefficients c_n^{pw} (with "pw" standing for "plane waves") are coefficients which originally appeared in the Bromwich approach to the generalized Lorenz-Mie theory [13] and have been defined in such a way that the uni-index BSCs g_n are phase-factors (usually taken equal to 1) which do not depend on the value of the index n, in the case of plane waves, propagating along the z-direction, but see [25] for the case of oblique illumination. They read as:

$$
c_n^{pw} = \frac{1}{ik} (-i)^n \frac{2n+1}{n(n+1)} \tag{15}
$$

2.2 Evaluating BSCs.

BSCs may be evaluated using quadrature techniques which have been the first ones to be developed, see again [10], [11]. These techniques will not be used in the present paper, except when they allow one to obtain closed form expressions (examples will be provided). Because they are computationally inefficient when the quadratures, instead of being carried out analytically, have to be carried out numerically, there has been a search for alternative methods, beginning with the finite series (FS) technique, see original papers in [26], [27] and a recent modified FS-method in [28]. Up to recently, the most efficient and famous technique has however been the localized approximation approach, which exists under several variants, in particular the integral localized approximation (ILA) technique [29] which may be viewed as a hybrid method between the quadrature techniques and the original localization technique. Localized approximations have been reviewed in [30], to be complemented with [31], [32]. These approximation techniques have been widely used because (i) they are formally very easy to implement and (ii) they are more computationally efficient than the quadrature techniques, possibly with a gain of computational times by three or four orders of magnitude.

However, it has recently been demonstrated that the accuracy of these approximation techniques may be in question when used for laser beams whose descriptions exhibit an axicon angle and/or a topological charge (when the axicon angle is large, and/or when the topological charge is not small enough), e.g. [33], [34], with applications to Bessel beams [35], [36], frozen waves which are built from discrete or continuous superpositions of Bessel beams [37], Mathieu beams [38], Laguerre-Gauss beams freely propagating or focused by a lens [39], [40], and Bessel-Gauss beams [41]. As a result, the FS-techniques which had been overlooked for a long time was born again, with applications to Laguerre-Gauss beams freely propagating [42], [43], or focused by a lens [44], [45], and Bessel-Gauss beams [41].

When the beam is Maxwellian (that is to say when its description satisfies Maxwell's equations), the quadrature and FS-techniques are mathematically exact and then provide the same values for the BSCs (notwithstanding residual numerical inaccuracies). However, the localization techniques rely on approximations which may be very close (or not, e.g. beams with axicon angles and/or topological charges) to the intended exact values. If the beam is not Maxwellian however, all techniques provide a remodelling of the description, i.e. they change a non-Maxwellian description to a Maxwellian description. The process of remodelling depends however on the method used and, although we may claim that remodelled solutions describe Maxwellian beams in their own right, they may depart more or less from the intended beam. For the sake of completeness, let us mention as well the use of the angular spectrum decomposition (ASD) which is reviewed in Section 2.6 of [5]. The interest of this subsection will be made obvious when we deal with examples and counter-examples.

3 New families demonstrated.

Without pretending to exhaustivity, we are going to look for a class of non-dark axisymmetric beams in which all BSCs are zero but those associated with values of $m = M$ or Q. To begin with, let us proceed without specifying any value of M and Q except that $M \neq Q$. We then look for a class of beams in which all BSCs are zero except those which read g_n^M and g_n^Q. We then consider Eq.9, restrict the values of m and q to M and Q and force the result denoted as S_z^{MQ} to be independent of φ. Therefore, we demand that:

$$\begin{aligned} S_z^{MQ} &= \frac{-1}{r^2}\,\mathrm{Re}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} i c_n^{pw} c_p^{pw*} \\ &\quad \{e^{i(M-Q)\varphi}(\sin\theta S_{np}^{MQ} + \cos\theta C_{np}^{MQ}) + e^{i(Q-M)\varphi}(\sin\theta S_{np}^{QM} + \cos\theta C_{np}^{QM})\} \\ &= 0 \end{aligned} \tag{16}$$

This equation shows that, if $M = Q$, then the beam is axisymmetric. This is a somewhat trivial special case which is therefore excluded from the present paper. Using $\mathrm{Re}(z) = \mathrm{Re}(z^*)$, Eq.16 may be rewritten as:

$$\begin{aligned} S_z^{MQ} &= \frac{-1}{r^2}\,\mathrm{Re}\{i e^{i(M-Q)\varphi}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \\ &\quad [c_n^{pw} c_p^{pw*}(\sin\theta S_{np}^{MQ} + \cos\theta C_{np}^{MQ}) \\ &\quad - c_n^{pw*} c_p^{pw}(\sin\theta S_{np}^{QM*} + \cos\theta C_{np}^{QM*})]\} \\ &= 0 \end{aligned} \tag{17}$$

Invoking Eq.15, Eq.17 may be rewritten as:

$$\begin{aligned} S_z^{MQ} &= \frac{-1}{k^2 r^2}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{np}\,\mathrm{Re}\{i e^{i(M-Q)\varphi} \\ &\quad [(-i)^n i^p(\sin\theta S_{np}^{MQ} + \cos\theta C_{np}^{MQ}) \\ &\quad - i^n(-i)^p(\sin\theta S_{np}^{QM*} + \cos\theta C_{np}^{QM*})]\} \\ &= 0 \end{aligned} \tag{18}$$

in which:

$$A_{np} = A_{pn} = \frac{2n+1}{n(n+1)} \frac{2p+1}{p(p+1)} \tag{19}$$

In Eq.18, the $\sin\theta$- and $\cos\theta$-terms must be separately equal to 0, leading to:

$$S_z^{MQ,\sin} = \frac{-\sin\theta}{k^2 r^2} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{np} \operatorname{Re}\{ie^{i(M-Q)\varphi}[(-i)^n i^p S_{np}^{MQ} - i^n (-i)^p S_{np}^{QM*}]\} = 0 \tag{20}$$

$$S_z^{MQ,\cos} = \frac{-\cos\theta}{k^2 r^2} \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} A_{np} \operatorname{Re}\{ie^{i(M-Q)\varphi}[(-i)^n i^p C_{np}^{MQ} - i^n (-i)^p C_{np}^{QM*}]\} = 0 \tag{21}$$

Let us consider Eq.20. Examining the functional dependences in Eq.10, we notice that terms with subscripts np may have the same functional dependences than terms with subscripts np interchanged. For instance, let us consider the two first lines of Eq.10. The first line contains $\psi_p(\psi_n + \psi_n^{''})$ becoming $\psi_n(\psi_p + \psi_p^{''})$ of the second line. The same happens for $P_n^{|m|}\tau_p^{|q|}$ of the first line becoming $P_p^{|m|}\tau_n^{|q|}$ of the second line if $|M| = |Q|$. Isolating such terms, we then must have:

$$\operatorname{Re}\{ie^{i(M-Q)\varphi}[(-i)^n i^p S_{np}^{MQ} - i^n (-i)^p S_{np}^{QM*} + (-i)^p i^n S_{pn}^{MQ} - i^p (-i)^n S_{pn}^{QM*}]\} = 0 \tag{22}$$

This equation has to be worked out (i) for the case when n and p have the same parity and (ii) for the case when they have different parities. Both cases leading to the same results, let us be content with the first case. Then Eq.22 becomes:

$$\operatorname{Re}\{ie^{i(M-Q)\varphi}[S_{np}^{MQ} - S_{np}^{QM*} + S_{pn}^{MQ} - S_{pn}^{QM*}]\} = 0 \tag{23}$$

Using $\operatorname{Re}(z_1 z_2) = \operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_1)\operatorname{Im}(z_2)$, we then obtain:

$$\operatorname{Re} ie^{i(M-Q)\varphi} \operatorname{Re}(z_2) - \operatorname{Im} ie^{i(M-Q)\varphi} \operatorname{Im}(z_2) = 0 \tag{24}$$

in which:

$$z_2 = S_{np}^{MQ} - S_{np}^{QM*} + S_{pn}^{MQ} - S_{pn}^{QM*} \tag{25}$$

implying that both $\mathrm{Re}(z_2)$ and $\mathrm{Im}(z_2) = 0$, that is to say:

$$(S_{np}^{MQ} - S_{np}^{QM*} + S_{pn}^{MQ} - S_{pn}^{QM*})/(kr) = 0 \tag{26}$$

Similarly, we may start from Eq.11 instead of from Eq.10, and consider again separately the case when n and p have the same parities, and the case when they have different parities. Both cases leading again to the same results, we shall only consider the first case, leading to:

$$C_{np}^{MQ} - C_{np}^{QM*} + C_{pn}^{MQ} - C_{pn}^{QM*} = 0 \tag{27}$$

From Eq.10, the coefficients occurring in Eq.26 read as:

$$\begin{aligned} S_{np}^{MQ}/kr &= -g_{n,TM}^{M} g_{p,TM}^{Q*} \psi_p (\psi_n + \psi_n^{''}) P_n^{|M|} \tau_p^{|Q|} \\ &+ g_{n,TE}^{M} g_{p,TE}^{Q*} \psi_n (\psi_p + \psi_p^{''}) P_p^{|Q|} \tau_n^{|M|} \\ &+ Q g_{n,TM}^{M} g_{p,TE}^{Q*} \psi_p^{'} (\psi_n + \psi_n^{''}) P_n^{|M|} \pi_p^{|Q|} \\ &+ M g_{n,TM}^{M} g_{p,TE}^{Q*} \psi_n^{'} (\psi_p + \psi_p^{''}) P_p^{|Q|} \pi_n^{|M|} \end{aligned} \tag{28}$$

$$\begin{aligned} -S_{np}^{QM*}/kr &= +g_{n,TM}^{Q*} g_{p,TM}^{M} \psi_p (\psi_n + \psi_n^{''}) P_n^{|Q|} \tau_p^{|M|} \\ &- g_{n,TE}^{Q*} g_{p,TE}^{M} \psi_n (\psi_p + \psi_p^{''}) P_p^{|M|} \tau_n^{|Q|} \\ &- M g_{n,TM}^{Q*} g_{p,TE}^{M} \psi_p^{'} (\psi_n + \psi_n^{''}) P_n^{|Q|} \pi_p^{|M|} \\ &- Q g_{n,TM}^{Q*} g_{p,TE}^{M} \psi_n^{'} (\psi_p + \psi_p^{''}) P_p^{|M|} \pi_n^{|Q|} \end{aligned} \tag{29}$$

$$\begin{aligned} S_{pn}^{MQ}/kr &= -g_{p,TM}^{M} g_{n,TM}^{Q*} \psi_n (\psi_p + \psi_p^{''}) P_p^{|M|} \tau_n^{|Q|} \\ &+ g_{p,TE}^{M} g_{n,TE}^{Q*} \psi_p (\psi_n + \psi_n^{''}) P_n^{|Q|} \tau_p^{|M|} \\ &+ Q g_{p,TM}^{M} g_{n,TE}^{Q*} \psi_n^{'} (\psi_p + \psi_p^{''}) P_n^{|M|} \pi_p^{|Q|} \\ &+ M g_{p,TM}^{M} g_{n,TE}^{Q*} \psi_p^{'} (\psi_n + \psi_n^{''}) P_n^{|Q|} \pi_p^{|M|} \end{aligned} \tag{30}$$

$$
\begin{aligned}
-S_{pn}^{QM*}/kr \quad = \quad & +g_{p,TM}^{Q*} g_{n,TM}^{M} \psi_n (\psi_p + \psi_p^{''}) P_p^{|Q|} \tau_n^{|M|} \\
& -g_{p,TE}^{Q*} g_{n,TE}^{M} \psi_p (\psi_n + \psi_n^{''}) P_n^{|M|} \tau_p^{|Q|} \\
& -M g_{p,TM}^{Q*} g_{n,TE}^{M} \psi_n^{'} (\psi_p + \psi_p^{''}) P_p^{|Q|} \pi_n^{|M|} \\
& -Q g_{p,TM}^{Q*} g_{n,TE}^{M} \psi_p^{'} (\psi_n + \psi_n^{''}) P_n^{|M|} \pi_p^{|Q|}
\end{aligned} \tag{31}
$$

Eq.26 then leads to eight equations reading as:

$$\psi_p (\psi_n + \psi_n^{''}) P_n^{|M|} \tau_p^{|Q|} [g_{n,TM}^{M} g_{p,TM}^{Q*} + g_{p,TE}^{Q*} g_{n,TE}^{M}] = 0 \tag{32}$$

$$\psi_n (\psi_p + \psi_p^{''}) P_q^{|Q|} \tau_n^{|M|} [g_{n,TE}^{M} g_{p,TE}^{Q*} + g_{p,TM}^{Q*} g_{n,TM}^{M}] = 0 \tag{33}$$

$$\psi_p' (\psi_n + \psi_n^{''}) P_n^{|M|} \pi_p^{|Q|} [g_{n,TM}^{M} g_{p,TE}^{Q*} - g_{p,TM}^{Q*} g_{n,TE}^{M}] = 0 \tag{34}$$

$$\psi_n' (\psi_p + \psi_p^{''}) P_p^{|Q|} \pi_n^{|M|} [g_{n,TM}^{M} g_{p,TE}^{Q*} - g_{p,TM}^{Q*} g_{n,TE}^{M}] = 0 \tag{35}$$

$$\psi_p (\psi_n + \psi_n^{''}) P_n^{|Q|} \tau_p^{|M|} [g_{n,TM}^{Q*} g_{p,TM}^{M} + g_{p,TE}^{M} g_{n,TE}^{Q*}] = 0 \tag{36}$$

$$\psi_n (\psi_p + \psi_p^{''}) P_p^{|M|} \tau_n^{|Q|} [g_{n,TE}^{Q*} g_{p,TE}^{M} + g_{p,TM}^{M} g_{n,TM}^{Q*}] = 0 \tag{37}$$

$$\psi_p' (\psi_n + \psi_n^{''}) P_n^{|Q|} \pi_p^{|M|} [g_{n,TM}^{Q*} g_{p,TE}^{M} - g_{p,TM}^{M} g_{n,TE}^{Q*}] = 0 \tag{38}$$

$$\psi_n' (\psi_p + \psi_p^{''}) P_p^{|M|} \pi_n^{|Q|} [g_{n,TM}^{Q*} g_{p,TE}^{M} - g_{p,TM}^{M} g_{n,TE}^{Q*}] = 0 \tag{39}$$

After elimination of redundancies, these eight equations lead to two equations reading as:

$$g_{n,TM}^{M} g_{p,TM}^{Q*} + g_{p,TE}^{Q*} g_{n,TE}^{M} = 0 \tag{40}$$

$$g^{M}_{n,TM}g^{Q*}_{p,TE} - g^{Q*}_{p,TM}g^{M}_{n,TE} = 0 \tag{41}$$

With a similar procedure, it is then found that Eq.27 leads again to these Eqs.40 and 41. We then must distinguish between two cases as follows. First, M and Q are different from ± 1. Then, according to the darkness theorem, we are facing a dark beam, e.g. Eq.8, which however is now axisymmetric (recall that Eq.8 alone does not tell whether the beam is axisymmetric or not). Second, one of the indices M or Q is equal to ± 1. Then the beam is non-dark and axisymmetric. This class of non-dark beams, complementing the non-dark beams of Eqs.6 and 7 which were specified as being of the first kind, then encompasses what we shall from now on call (non-dark) axisymmetric beams of the second kind.

It is furthermore worthwhile to remark that there is no confusion or orverlap between axisymmetric beams of the first kind (all $g^{\pm 1}_{n,X}$ are different from 0) and axisymmetric beams of the second kind (not all $g^{\pm 1}_{n,X}$ are different from 0). To illustrate this fact, let us rewrite Eqs.40-41 for $M = Q = 1$, leading to:

$$g^{1}_{n,TM}g^{1*}_{p,TM} + g^{1*}_{p,TE}g^{1}_{n,TE} = 0 \tag{42}$$

$$g^{1}_{n,TM}g^{1*}_{p,TE} - g^{1*}_{p,TM}g^{1}_{n,TE} = 0 \tag{43}$$

But, from Eq.7, we have $g^{1}_{n,TE} = i\varepsilon g^{1}_{n,TM}$. Recalling that $\varepsilon^2 = 1$, Eqs.42 and 43 become:

$$g^{1}_{n,TM}g^{1*}_{p,TM} + g^{1*}_{p,TE}g^{1}_{n,TE} = 2g^{1}_{n,TM}g^{1*}_{p,TM} = 0 \tag{44}$$

$$g^{1}_{n,TM}g^{1*}_{p,TE} - g^{1*}_{p,TM}g^{1}_{n,TE} = -2i\varepsilon g^{1}_{n,TM}g^{1*}_{p,TM} = 0 \tag{45}$$

The first kind conditions imply that these equations cannot be satisfied, illustrating the fact that axisymmetric beams of the first kind and of the second kind are indeed of a different "nature".

4 Examples.

4.1 (1,0) x-polarized circularly symmetric Bessel beams

The introduction of Bessel beams is due to Durnin and co-workers [46], [47]. They are praised due their appealing property of being self-healing and nondiffracting, meaning that they possess an invariance property, namely the fact that the intensity of the beam is constant along the direction of propagation. They also possess a not often mentioned property that the speed of propagation of these beams may be much smaller than the speed of light, being equal to $c\cos\alpha_0$ [48], [49], in which c is the speed of light and α_0 an angle, named axicon angle, or half-cone angle, which occurs in the propagation term reading $\exp(\pm ikz\cos\alpha_0)$ instead of being of the form $\exp(\pm ikz)$ which is somehow more usual. There actually exists an infinity of kinds of Bessel beams (i) with different polarizations and (ii) depending on the form given to a certain function $g(\alpha_0)$, see [48], [50], [51], [52]. In particular, when $g(\alpha_0) = (1+\cos\alpha_0)/4$, the beam reduces to a Davis circularly symmetric beam, e.g. [48], [53] while, when $g(\alpha_0) = 1/2$ it reduces to another kind of beams discussed in [54], [55], [56].

Also, to avoid any confusion, it is now important to comment on the definitions of axisymmetric beams and of circularly symmetric Bessel beams. In the present paper, we follow a definition which is about 25 years old according to which axisymmetric light beams are beams for which the component S_z of the Poynting vector does not depend on the azimuthal angle φ (at least in a suitably chosen coordinate system), e.g. [12]. With this definition, circularly symmetric beams are axisymmetric beams for which, in the same suitably chosen coordinate system, the transverse component $S_t = (S_x^2+S_y^2)^{1/2}$ of the Poynting vector as well does not depend on φ. Outside of the context of the present paper, in order to avoid any confusion, axisymmetric beams might better be called longitudinal axisymmetric beams. The relationship in terms of symmetries of BSCs between the BSCs of axisymmetric beams and the BSCs of circularly symmetric beams has not been worked out, but it is sufficient in this paper to have in mind that circularly symmetric beams are (longitudinal) axisymmetric beams, although the converse is not (necessarily) true.

Using a quadrature technique and solving quadratures analytically, BSCs of circularly symmetric beams have been obtained in closed forms in an off-axis configuration in [52] and conveniently rewritten in [16]. In the case of on-axis configurations, they simplify to [22]:

$$g_{n,TM}^{l+1} = -g(\alpha_0)\exp(ik_z z_0)A_n^{l+1} \tag{46}$$

$$g_{n,TM}^{l-1} = -g(\alpha_0)\exp(ik_z z_0)B_n^{l-1} \tag{47}$$

$$g_{n,TE}^{l+1} = ig(\alpha_0)\exp(ik_z z_0)A_n^{l+1} \tag{48}$$

$$g_{n,TE}^{l-1} = -ig(\alpha_0)\exp(ik_z z_0)B_n^{l-1} \tag{49}$$

in which l denotes the order of the beam, $k_z = k\cos\alpha_0$ is the longitudinal wave number, and:

$$A_n^{l+1} = (-1)^{(l+1-|l+1|)/2}\frac{(n-l-1)!}{(n+|l+1|)!}[\tau_n^{l+1}(\cos\alpha_0) + (l+1)\pi_n^{l+1}(\cos\alpha_0)] \tag{50}$$

$$B_n^{l-1} = (-1)^{(l-1-|l-1|)/2}\frac{(n-l+1)!}{(n+|l-1|)!}[\tau_n^{l-1}(\cos\alpha_0) - (l-1)\pi_n^{l-1}(\cos\alpha_0)] \tag{51}$$

We then observe that the only nonzero BSCs are those for which $m = l \pm 1$. For $l = 0$ ($m = \pm 1$), we readily find that Eqs.6-7 are satisfied with $K = B_n^{-1}/A_n^1$ and $\varepsilon = -1$, so that we are facing non-dark axisymmetric beams of the first kind. For $l = +2$ ($m = 1$ and 3) and $l = -2$ ($m = -1$ and -3), it is readily found that Eqs.40-41 are satisfied, so that we are facing non-dark axisymmetric beams of the second kind. Next, let us assume l different from 0, $+2$, and (-2) already considered just above, i.e. m different from ± 1, so that we are facing dark beams. We readily find:

$$g_{n,TM}^{l+1}g_{p,TM}^{l-1*} + g_{p,TE}^{l-1*}g_{n,TE}^{l+1} = 0 \tag{52}$$

$$g_{n,TM}^{l+1}g_{p,TE}^{l-1*} - g_{p,TM}^{l-1*}g_{n,TE}^{l+1} = 0 \tag{53}$$

showing that Eqs.40-41 are satisfied, i.e. these dark beams are axisymmetric. The fact that all these Bessel beams considered in this section are axisymmetric are however not surprising insofar as they are already circularly symmetric.

4.2 Bessel beams. Other polarizations.

Other polarizations, which are circularly symmetric as well [50], have been discussed in [51]. BSCs for an off-axis configuration are given in [57]. For on-axis configurations, they read as:

$$\begin{aligned} g_{n,TM}^{m} &= ig(\alpha_0)(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\exp(ik_z z_0)\{ \\ &[ip_x\tau_n^m(\cos\alpha_0)+p_y m\pi_n^m(\cos\alpha_0)] \\ &\times[i^{l-m+1}e^{i(l-m+1)\phi_0}J_{l-m+1}(0)+i^{l-m-1}e^{i(l-m-1)\phi_0}J_{l-m-1}(0)] \\ &+[ip_y\tau_n^m(\cos\alpha_0)-p_x m\pi_n^m(\cos\alpha_0)] \\ &\times[-i.i^{l-m+1}e^{i(l-m+1)\phi_0}J_{l-m+1}(0)+i.i^{l-m-1}e^{i(l-m-1)\phi_0}J_{l-m-1}(0)]\} \end{aligned} \tag{54}$$

$$\begin{aligned} g_{n,TE}^{m} &= g(\alpha_0)(-1)^{(m-|m|)/2}\frac{(n-m)!}{(n+|m|)!}\exp(ik_z z_0)\{ \\ &[ip_x m\pi_n^m(\cos\alpha_0)+p_y\tau_n^m(\cos\alpha_0)] \\ &\times[i^{l-m+1}e^{i(l-m+1)\phi_0}J_{l-m+1}(0)+i^{l-m-1}e^{i(l-m-1)\phi_0}J_{l-m-1}(0)] \\ &+[-p_x\tau_n^m(\cos\alpha_0)+ip_y m\pi_n^m(\cos\alpha_0)] \\ &\times[-i.i^{l-m+1}e^{i(l-m+1)\phi_0}J_{l-m+1}(0)+i.i^{l-m-1}e^{i(l-m-1)\phi_0}J_{l-m-1}(0)]\} \end{aligned} \tag{55}$$

in which ϕ_0 is an angle used to specify the rotational location of the off-axis coordinate system, $J_k(.)$ are cylindrical Bessel function of the first kind of order k, and p_x, p_y define the polarization of the beam in the framework of an ASD according to $(p_x, p_y) = (1, 0)$ for x-polarization, $(0, 1)$ for y-polarization, $(1, i)$ for left circular polarization, $(1, -i)$ for right circular polarization, $(\cos\beta, \sin\beta)$ for radial polarization, and $(-\sin\beta, \cos\beta)$ for azimuthal polarization in which β is an azimuthal angle [58], [59].

Because $J_k(0) = \delta_{k0}$, Eqs.54-55 show that the only nonzero BSCs are those for which $m = l \pm 1$, according to:

$$\begin{aligned} g_{n,TM}^{l+1} &= ig(\alpha_0)(-1)^{(l+1-|l+1|)/2}\frac{(n-l-1)!}{(n+|l+1|)!}\exp(ik_z z_0) \\ &\times(ip_x+p_y)[\tau_n^{l+1}(\cos\alpha_0)+(l+1)\pi_n^{l+1}(\cos\alpha_0)] \end{aligned} \tag{56}$$

$$\begin{aligned} g_{n,TM}^{l-1} &= ig(\alpha_0)(-1)^{(l-1-|l-1|)/2}\frac{(n-l+1)!}{(n+|l-1|)!}\exp(ik_z z_0) \\ &\times(ip_x-p_y)[\tau_n^{l-1}(\cos\alpha_0)-(l-1)\pi_n^{l-1}(\cos\alpha_0)] \end{aligned} \tag{57}$$

$$g_{n,TE}^{l+1} = g(\alpha_0)(-1)^{(l+1-|l+1|)/2}\frac{(n-l-1)!}{(n+|l+1|)!}\exp(ik_z z_0) \times(ip_x+p_y)[\tau_n^{l+1}(\cos\alpha_0)+(l+1)\pi_n^{l+1}(\cos\alpha_0)] \quad (58)$$

$$g_{n,TE}^{l-1} = -g(\alpha_0)(-1)^{(l-1-|l-1|)/2}\frac{(n-l+1)!}{(n+|l-1|)!}\exp(ik_z z_0) \times(ip_x-p_y)[\tau_n^{l-1}(\cos\alpha_0)-(l-1)\pi_n^{l-1}(\cos\alpha_0)] \quad (59)$$

which satisfies $g_{nTM}^{l+1} = ig_{n,TE}^{l+1}$ and $g_{nTM}^{l-1} = -ig_{n,TE}^{l+1}$. For $l = 0$, we then find again that Eqs.6-7 are satisfied with $\varepsilon = -1$, and:

$$K = \frac{(n+1)!}{(n-1)!}\frac{(p_y-ip_x)[\tau_n^{-1}(\cos\alpha_0)+\pi_n^{-1}(\cos\alpha_0)]}{(ip_x+p_y)[\tau_n^{1}(\cos\alpha_0)+\pi_n^{1}(\cos\alpha_0)]} \quad (60)$$

This equation can be simplified by complementing Eq. 14 with another equation useful to deal with negative superscripts according to [60]:

$$P_n^{-m}(\cos\alpha_0) = (-1)^m\frac{(n-m)!}{(n+m)!}P_n^m(\cos\alpha_0) \quad (61)$$

leading to:

$$K = \frac{ip_x-p_y}{ip_x+p_y} \quad (62)$$

We are then facing again non-dark axisymmetric beams of the first kind. For $l = +2$ ($m = 1$ and 3) and $l = -2$ ($m = -1$ and -3), it is readily found again that Eqs.40-41 are satisfied, so that we are facing non-dark axisymmetric beams of the second kind. For dark beams, i.e. for $l \neq 0, \pm 2$, we find as well that Eqs.40-41 are satisfied, i.e. these dark beams are axisymmetric.

4.3 Lommel beams.

Lommel beams are constituted by a combination of Bessel beams of various orders (i.e. topological charges) as exemplified by the expression of the basic electric modes in cylindrical coordinates (r, φ, z) reading as, e.g. Eq.(1) in [61]:

$$E_l = \exp(ik_z z \sum_{p=0}^{\infty} (-1)^p c^{2p} \exp[i(l+2p)\varphi] J_{\nu+2p}(k_t r) \tag{63}$$

in which E_l is the x- or y-component of the electric field which is obtained from a solution of the scalar wave equation (meaning that the beam is not Maxwellian), $k_z = k\cos\alpha_0$ and $k_t = k\sin\alpha_0$ are the longitudinal and transverse wavenumbers respectively. The beam being not Maxwellian, its expression in terms of BSCs will therefore provide a remodelling of the beam, turning it from a non-Maxwellian beam to a Maxwellian beam. In [61], these BSCs are evaluated using the ILA which is fairly satisfactory for small enough axicon angles, as discussed in subsection 2.2 (although it is likely that quadratures of the quadrature techniques could be performed analytically), leading to localized BSCs for localized Lommel beams. These BSCs for x- and y-polarizations in an off-axis configuration are provided in Eqs.(7) and (9). Relying on Fig.1 in [61] and remembering that we are currently providing examples of axisymmetric beams, we set $c = 0$ and specify Eqs.(7) and (9) for on-axis configurations. Once more, it is then found that the nonzero BSCs are for $m = l \pm 1$.

For $l = 0$, and for x- and y-polarizations, the BSCs then read as:

$$g^1_{n,TM} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z^1_n}{2} J_0(w_n) \exp(ik_z z_0) \tag{64}$$

$$g^{-1}_{n,TM} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z^{-1}_n}{2} J_0(w_n) \exp(ik_z z_0) \tag{65}$$

$$g^1_{n,TE} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z^1_n}{2} J_0(w_n) \exp(ik_z z_0) \tag{66}$$

$$g^{-1}_{n,TE} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z^{-1}_n}{2} J_0(w_n) \exp(ik_z z_0) \tag{67}$$

in which:

$$w_n = (n + \frac{1}{2}) \sin \alpha_0 \tag{68}$$

and [29]:

$$Z_n^m = (\frac{-2i}{2n+1})^{|m|-1} \text{ for } m \neq 0 \tag{69}$$

$$Z_n^0 = \frac{2n(n+1)}{(2n+1)} \tag{70}$$

and, in particular, for $m = \pm 1$, we have $Z_n^1 = Z_n^{-1} = 1$. Eqs.64-67 then satisfy Eqs.6-7 for both polarizations, with $K = \pm 1$ and $\varepsilon = -1$, so that we are facing non-dark axisymmetric beams of the first kind. The BSCs for $\nu = +2$ ($m = 1$ and 3), and x- and y-polarizations, are found to read as:

$$g_{n,TM}^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z_n^1}{2} J_2(w_n) \exp(ik_z z_0) \tag{71}$$

$$g_{n,TM}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z_n^3}{2} J_2(w_n) \exp(ik_z z_0) \tag{72}$$

$$g_{n,TE}^1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z_n^1}{2} J_2(w_n) \exp(ik_z z_0) \tag{73}$$

$$g_{n,TE}^3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z_n^3}{2} J_2(w_n) \exp(ik_z z_0) \tag{74}$$

while, for $\nu = -2$ ($m = -1$ and -3), we have:

$$g_{n,TM}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z_n^{-1}}{2} J_{-2}(w_n) \exp(ik_z z_0) \tag{75}$$

$$g_{n,TM}^{-3} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z_n^{-3}}{2} J_{-2}(w_n) \exp(ik_z z_0) \tag{76}$$

$$g_{n,TE}^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z_n^{-1}}{2} J_{-2}(w_n) \exp(ik_z z_0) \tag{77}$$

$$g_{n,TE}^{-3}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z_n^{-3}}{2} J_{-2}(w_n)\exp(ik_z z_0) \tag{78}$$

It is then readily checked that these BSCs satisfy Eqs.40-41 so that we are facing non-dark axisymmetric beams of the second kind. As a last example of Lommel beams, we shall consider the case $v = 4$ ($m = 3$ and 5), because it is displayed (for the $x-$polarization) in Fig.1a (with the parameter $c = 0$). This figure vividly suggests that we are facing an axisymmetric dark beam. Indeed, the BSCs are found to be:

$$g_{n,TM}^{3}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix} \frac{Z_n^{3}}{2} J_{4}(w_n)\exp(ik_z z_0) \tag{79}$$

$$g_{n,TM}^{5}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix} \frac{Z_n^{5}}{2} J_{4}(w_n)\exp(ik_z z_0) \tag{80}$$

$$g_{n,TE}^{3}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} i \\ -1 \end{pmatrix} \frac{Z_n^{3}}{2} J_{4}(w_n)\exp(ik_z z_0) \tag{81}$$

$$g_{n,TE}^{5}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -i \\ -1 \end{pmatrix} \frac{Z_n^{5}}{2} J_{4}(w_n)\exp(ik_z z_0) \tag{82}$$

which satisfy Eqs.40-41.

4.4 Bessel-Gauss beams using finite series.

Bessel beams possess the defect of carrying an infinite amount of energy so that they are not physical. Bessel-Gauss beams result from an apodization of Bessel beams by a Gaussian beam which truncates them in order to obtain a finite energy, physically feasible, beam [62], [63]. The beam description is based on a paraxial approximation in which the x-component of the electric field is given in cylindrical coordinates (ρ, φ, z) by the scalar field of Eq.(1) in [41], according to:

$$E_x = \frac{E_0}{\mu}\exp(\frac{-q^2\rho^2}{\mu})J_l(\frac{\rho k_t}{\mu})\exp(\frac{ik_t^2 z}{2k\mu})e^{il\varphi}e^{-ikz} \tag{83}$$

in which $k_t = k\sin\alpha_0$ is the transverse wave number, q is a parameter which regulates the transverse intensity width of the Gaussian apodization, and

$\mu = 1 - 2iks^2 z$, where s is the beam confinement factor of the beam, according to a usual definition for Gaussian beams, namely $s = 1/(kw_0)$ in which w_0 is the beam waist radius, and, as usual in this paper, l denotes the order of the beam. It is worthwhile to insist on the fact that the term $\exp(ikz)$ in Eq.83 is correct although the reader might have expected to read $\exp(-ik\cos\alpha_0 z) = \exp(-ik_z z)$. Actually the longitudinal wavenumber k_z is embedded in the exponential factor whose exponent is proportional to k_t^2. This fact may be retrieved from Eqs.(15) and (20) in [64] in which k_t and k_z are denoted η_n and ζ_n respectively, from which we may establish that $k_t^2/(2k) = (k - k_z)$.

Faraday's law is afterward used to determine the y-component of the magnetic field according to $(\bigtriangledown \times \mathbf{E})_y = \partial_z E_x - \partial_x E_z \approx \partial_z E_x = -i\omega\mu_0 H_y$ in which we assumed a highly paraxial situation and neglected the x-derivative of E_z. The z-derivative is lengthy to perform but it is sufficient to remark that it does not alter the topological term $e^{il\varphi}$ of Eq.83, so that H_y is proportional to it. As a result, the Poynting vector, reduced to its z-component, does not depend on φ. Therefore, we are facing a non-Maxwellian (paraxial) beam which is axisymmetric. This implies that the ILA- and the FS-technique will proceed by changing a non-Maxwellian beam to a Maxwellian beam. Furthermore, when $s \to 0$, the Gaussian envelope used for apodization becomes larger and larger, and the Bessel-Gauss beams returns to a Bessel beam. For $\alpha_0 = 0$, it returns to a paraxial Gaussian beam. For $s = \alpha_0 = 0$, it becomes a plane wave.

This subsection is devoted to the FS-technique. The use of the ILA-technique will provide an interesting counter-example in subsection 5.1. BSCs obtained by using the FS-technique are given by Eqs.(19)-(20) and (30)-(31) in [41] for the TM-coefficients. These BSCs may be conveniently rewritten as:

$$g_{n,TM}^{l+1} = \frac{i^{n+1}}{2^{l+1}} \frac{\Gamma(\frac{n-l+1}{2})}{\Gamma(\frac{n+l+2}{2})} \mathcal{A}_n^l, \ (n-m) \text{ even} \tag{84}$$

$$g_{n,TM}^{l-1} = \frac{-i^{n+1}}{2^{l-1}} \frac{\Gamma(\frac{n-l+3}{2})}{\Gamma(\frac{n+l}{2})} \mathcal{A}_n^l, \ (n-m) \text{ even} \tag{85}$$

$$g_{n,TM}^{l+1} = \frac{i^{n}}{2^{l+1}} \frac{\Gamma(\frac{n-l}{2})}{\Gamma(\frac{n+l+3}{2})} \mathcal{B}_n^l, \ (n-m) \text{ odd} \tag{86}$$

$$g_{n,TM}^{l-1} = \frac{-i^{n}}{2^{l-1}} \frac{\Gamma(\frac{n-l+2}{2})}{\Gamma(\frac{n+l+1}{2})} \mathcal{B}_n^l, \ (n-m) \text{ odd} \tag{87}$$

which shows that the nonzero BSCs are those for which $m = l \pm 1$. The fact that we have to distinguish between the cases $(n-m)$ even and $(n-m)$ odd is typical of the FS-technique. Furthermore, $\mathcal{A}_n^l$ and $\mathcal{B}_n^l$ are finite series whose expressions are fairly complicated but that we do not need to provide

explicitly. Furthermore, in [41], rather than applying the FS-technique to obtain TE-coefficients (i.e. by working on the magnetic field rather than working on the electric field), it has been preferred to force them by using:

$$g_{n,TM}^{l\pm 1} = \pm i g_{n,TE}^{l\pm 1} \tag{88}$$

It is then an exercise, using Eqs.84-88, to check that, for $l = 0$ (i.e. $m = \pm 1$), Eqs.6-7 are satisfied with $\varepsilon = -1$ and, for both $(n - m)$ even and $(n - m)$ odd:

$$K = -4\frac{\Gamma(\frac{n+2}{2})\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+1}{2}\Gamma(\frac{n}{2})} = -n(n+1) \tag{89}$$

so that we are facing non-dark axisymmetric beams of the first kind. For $l \neq 0$, we afterward may simply check, using *only* Eq.88 that Eqs.40-41 are satisfied. Therefore, for $l = \pm 2$, we are facing non-dark axisymmetric beams of the second kind while, otherwise, for $l \neq 0, \pm 2$, we are facing axisymmetric dark beams. Hence, the FS-technique, as used above, remodelled non-Maxwellian axisymmetric beams to Maxwellian axisymmetric beams. We shall see in subsection 5.1 that the situation will be different for the ILA-remodelling.

4.5 Laguerre-Gauss beams freely propagating.

As a last example, we now consider Laguerre-Gauss beams freely propagating which provide again a case of paraxial non-Maxwellian beams. The electric field is obtained from the following expression [42], [43]:

$$\mathbf{E} = (E_x, E_y, E_z) = (E_0 \mathcal{E} e^{-ikz}, 0, 0) \tag{90}$$

in which $\mathcal{E}$ is extracted from the radial component E_r of the electric field given by Siegman [65], [66], and reads as:

$$\begin{aligned} \mathcal{E}(r,\theta,\varphi) &= \frac{1}{kw(r\cos\theta)}\left(\frac{r\sin\theta\sqrt{2}}{w(r\cos\theta)}\right)^l L_\mu^l\left(\frac{2r^2\sin^2\theta}{w^2(r\cos\theta)}\right) \\ &\quad \exp\left[-\frac{ikr^2\sin^2\theta}{2\widetilde{q}(r\cos\theta)}\right]\exp[i(2\mu+l+1)\psi(r\cos\theta)]\exp(il\varphi) \end{aligned} \tag{91}$$

in which:

$$w(r\cos\theta) = w_0\sqrt{1+\left(\frac{2r\cos\theta}{kw_0^2}\right)^2} \tag{92}$$

$$\widetilde{q}(r\cos\theta) = r\cos\theta + \frac{ikw_0^2}{2} \tag{93}$$

$$\psi(r\cos\theta) = \tan^{-1}\frac{2r\cos\theta}{kw_0^2} \tag{94}$$

in which w_0 is the beam waist radius of the beam, while μ and l define the type of Laguerre-Gauss beam under study. Also, L_μ^l denotes associated Laguerre polynomials defined as follows [67], [68]:

$$L_\mu^l(x) = \frac{1}{\mu!}\sum_{i=0}^{\mu}\frac{\mu!}{i!}\left(\begin{array}{c} l+\mu \\ \mu-i \end{array}\right)(-x)^i \tag{95}$$

in which () denotes a binomial coefficient. The "usual" Laguerre polynomials $L_\mu(x)$ are $L_\mu^0(x)$. The associated Laguerre polynomials can be evaluated recursively from the first two polynomials as follows:

$$L_0^\alpha(x) = 1 \tag{96}$$

$$L_1^\alpha(x) = 1+\alpha-x \tag{97}$$

$$L_{k+1}^\alpha(x) = \frac{(2k+1+\alpha-x)L_k^\alpha(x)-(k+\alpha)L_{k-1}^\alpha(x)}{k+1} \tag{98}$$

Faraday's law may then be used to evaluate the magnetic field according to:

$$\mathbf{H} = \frac{1}{-i\omega\mu}\,\mathrm{curl}\,\mathbf{E} = \frac{1}{-i\omega\mu}(0,\frac{\partial E_x}{\partial z},0) \tag{99}$$

in which μ is the permeability. From Eqs.90, 91 and 99, and removing artefacts which include $\partial\mathcal{E}/\partial z$, we obtain:

$$\mathbf{H} = (0, H_0 \mathcal{E} e^{-ikz}, 0) \tag{100}$$

in which we have used $E_0/H_0 = \omega\mu/k$. The removal of the artefacts aimed to symmetrize expressions for the electric and magnetic field components. Considering that $\mathcal{E}$ is proportional to $e^{il\varphi}$, S_z does not depend on φ, and we therefore are facing a non-Maxwellian axisymmetric beam. The remodelling will therefore turns a non-Maxwellian axisymmetric beam to a Maxwellian beam whose axisymmetry is to be tested.

The remodelling is carried out using the FS-technique and it is found, as usual, that the only nonzero BSCs are those for which $m = l \pm 1$. For the sake of conciseness, we shall not consider dark beams in this section. We then begin with the case when the order $l = 0$ ($m = \pm 1$). We recall that the FS-technique distinguishes the case when $(n - m)$ is even and the one when it is odd. For $(n-m)$ even, the expressions for the BSCs $g_{n,TM}^{\pm 1}$ are given in Eqs.(68) and (70) of [42] while they are given by Eqs.(101) and (103) for the case $(n - m)$ odd. In all cases, we readily obtain $g_{n,TM}^{+1} = g_{n,TM}^{-1}$. For the TE-BSCs, as the result of the symmetry forcing explained above, we have, from Eqs.(104) and (106) in [42]: $g_{n,TM}^{+1} = i g_{n,TE}^{+1}$ and $g_{n,TM}^{-1} = -i g_{n,TE}^{-1}$. These BSCs therefore satisfy Eqs.6-7. Hence Laguerre-Gauss beams freely propagating of order 0 remodeled by the FS-technique are on-axis axisymmetric beams of the first kind.

We now consider the case when the order is equal to 2 (the case when the order is -2 is not considered because [42] does not deal with negative orders). From Eqs.(68)-(69) in [42], we may write, with μ denoting the case of L_μ^l-Laguerre-Gauss polynomial in discussion, for $(n - m)$ even:

$$[g_{n,TM}^{3}]_\mu^{even} = \frac{1}{16} \frac{(-1)^{\frac{n+3}{2}} (\frac{n-3}{2})!}{\Gamma(\frac{n}{2} + 2)} S_{TM}^{even} \tag{101}$$

$$[g_{n,TM}^{1}]_\mu^{even} = \frac{1}{4} \frac{(-1)^{\frac{n+1}{2}} (\frac{n-1}{2})!}{\Gamma(\frac{n}{2} + 1)} S_{TM}^{even} \tag{102}$$

in which n is odd, $n \succeq 3$ in Eq.101 and $n \succeq 1$ in Eq.102, and in which:

$$S_{TM}^{even} = i^n \sqrt{\pi} \sum_{q=0}^{\leq n/2} 2^{\frac{1}{2}+n-2q} \frac{\Gamma(\frac{1}{2} + n - q)}{q!} [b_{n-2q}]_\mu^{even} \tag{103}$$

in which $[b_{n-2q}]_\mu^{e^{even}}$ is given by Eqs.(54), (55) and (60) in [42]. Conversely, for $(n - m)$ odd, we have, from Eqs.(101)-(102):

$$[g^3_{n,TM}]^{odd}_{\mu} = \frac{1}{32}\frac{(-1)^{\frac{n}{2}+1}(\frac{n}{2}-2)!}{\Gamma(\frac{n+5}{2})}S^{odd}_{TM} \tag{104}$$

$$[g^1_{n,TM}]^{odd}_{\mu} = \frac{1}{8}\frac{(-1)^{\frac{n}{2}}(\frac{n}{2}-1)!}{\Gamma(\frac{n+3}{2})}S^{odd}_{TM} \tag{105}$$

in which n is even, $n \succeq 3$ in Eq.104 and $n \succeq 1$ in Eq.105, and in which:

$$S^{even}_{TM} = i^n\sqrt{\pi}\sum_{q=0}^{\leq n/2} 2^{\frac{1}{2}+n-2q}\frac{\Gamma(\frac{1}{2}+n-q)}{q!}[b_{n-2q}]^{odd}_{\mu} \tag{106}$$

in which $[b_{n-2q}]^{odd}_{\mu}$ is given by Eqs.(88), (91) and (93) in [42].

The TE-BSCs are given by Eqs.(104)-(105), whatever the parities of n and m according to:

$$[g^3_{n,TM}]_{\mu} = i[g^3_{n,TE}]_{\mu} \tag{107}$$

$$[g^1_{n,TM}]_{\mu} = -i[g^3_{n,TE}]_{\mu} \tag{108}$$

These equations then readily imply that, whatever the parities:

$$T_1 = g^3_{n,TM}g^{1*}_{p,TM} + g^{1*}_{p,TE}g^3_{n,TE} = 0 \tag{109}$$

$$T_2 = g^3_{n,TM}g^{1*}_{p,TE} - g^{1*}_{p,TM}g^3_{n,TE} = 0 \tag{110}$$

implying that we are facing on-axis axisymmetric beams of the second kind. The facts that BSCs are those of axisymmetric beams agree with the fact, already known, that S_z does not depend on φ. Note that the same kind of beams has been studied using a localized approximation, e.g. [39], but the TE-coefficients have not been worked out. Therefore, this case is omitted from the present paper.

5 Counter-examples.

5.1 Bessel-Gauss beams with ILA.

For this counter-example, we return to Bessel-Gauss beams of subsection 4.4. which studied the case of an axisymmetric non-Maxwellian beam which has been turned to an axisymmetric Maxwellian beam by using a remodelling with the FS-technique. The choice of the remodelling technique (and the details of the implementation of this technique) is exemplified in this section by using the ILA-technique in a way that will turn the original axisymmetric non-Maxwellian beam to a remodelled non-axisymmetric Maxwellian beam [41].

Remodelling using ILA, the on-axis BSCs read as, from Eq.(4) in [41]:

$$g_n^m\left(\begin{array}{c} TM \\ TE \end{array}\right) = \frac{Z_n^m}{2\overline{\mu}}\left[\begin{array}{c} 1 \\ i(1+\frac{\cos\alpha_0-1}{\overline{\mu}}) \end{array}\right]\exp\left(\frac{-ik\sin^2\alpha_0 z_0}{2\overline{\mu}}\right)\exp\left(\frac{-s^2{w_n}^2}{\sin^2\alpha_0\overline{\mu}}\right)e^{-i(m-l+1)\phi_0}$$
$$\times[e^{2i\phi_0}J_{m-1}(w_n)J_{m-1-l}(0)\pm J_{m+1}(w_n)J_{m+1-l}(0)]\exp^{ikz_0} \quad (111)$$

in which Z_n^m is given by Eqs.69-70, w_n is given by Eq.68 and $\overline{\mu}=1+2iks^2z_0$. These BSCs are zero excepted when $m=l\pm 1$. The nonzero BSCs then read as:

$$g_n^{l+1}\left(\begin{array}{c} TM \\ TE \end{array}\right) = \frac{Z_n^{l+1}}{2\overline{\mu}}\left[\begin{array}{c} 1 \\ i(1+\frac{\cos\alpha_0-1}{\overline{\mu}}) \end{array}\right]\exp\left(\frac{-ik\sin^2\alpha_0 z_0}{2\overline{\mu}}\right)\exp\left(\frac{-s^2{w_n}^2}{\sin^2\alpha_0\overline{\mu}}\right)J_v(w_n)\exp^{ikz_0} \quad (112)$$

$$g_n^{l-1}\left(\begin{array}{c} TM \\ TE \end{array}\right) = \pm\frac{Z_n^{l-1}}{2\overline{\mu}}\left[\begin{array}{c} 1 \\ i(1+\frac{\cos\alpha_0-1}{\overline{\mu}}) \end{array}\right]\exp\left(\frac{-ik\sin^2\alpha_0 z_0}{2\overline{\mu}}\right)\exp\left(\frac{-s^2{w_n}^2}{\sin^2\alpha\overline{\mu}}\right)J_v(w_n)\exp^{ikz_0} \quad (113)$$

For $l=0$, the only nonzero BSCs are those of the form $g_n^{\pm 1}$, reading as:

$$g_n^{+1}\left(\begin{array}{c} TM \\ TE \end{array}\right) = \frac{1}{2\overline{\mu}}\left[\begin{array}{c} 1 \\ i(1+\frac{\cos\alpha_0-1}{\overline{\mu}}) \end{array}\right]\exp\left(\frac{-ik\sin^2\alpha_0 z_0}{2\overline{\mu}}\right)\exp\left(\frac{-s^2{w_n}^2}{\sin^2\alpha_0\overline{\mu}}\right)J_0(w_n)\exp^{ikz_0} \quad (114)$$

$$g_n^{-1}\left(\begin{array}{c} TM \\ TE \end{array}\right) = \pm\frac{1}{2\overline{\mu}}\left[\begin{array}{c} 1 \\ i(1+\frac{\cos\alpha_0-1}{\overline{\mu}}) \end{array}\right]\exp\left(\frac{-ik\sin^2\alpha_0 z_0}{2\overline{\mu}}\right)\exp\left(\frac{-s^2{w_n}^2}{\sin^2\alpha_0\overline{\mu}}\right)J_0(w_n)\exp^{ikz_0} \tag{115}$$

in which we have used again $Z_n^{\pm 1} = 1$. For axisymmetric beams of the first kind, we should satisfy Eq.7. From Eqs.114-115, we have:

$$g^1_{n,TM} = g^{-1}_{n,TM} \text{ and } g^1_{n,TE} = -g^{-1}_{n,TE} \tag{116}$$

which indeed implies $K = 1$. But, from Eq.7, we would expect $g^1_{n,TM}/g^{-1}_{n,TE} = i\varepsilon$, which is in conflict with Eqs.114-115 which indeed implies:

$$\frac{g^1_{n,TM}}{g^{-1}_{n,TE}} = \frac{-1}{i(1+\frac{\cos\alpha_0-1}{\overline{\mu}})} \tag{117}$$

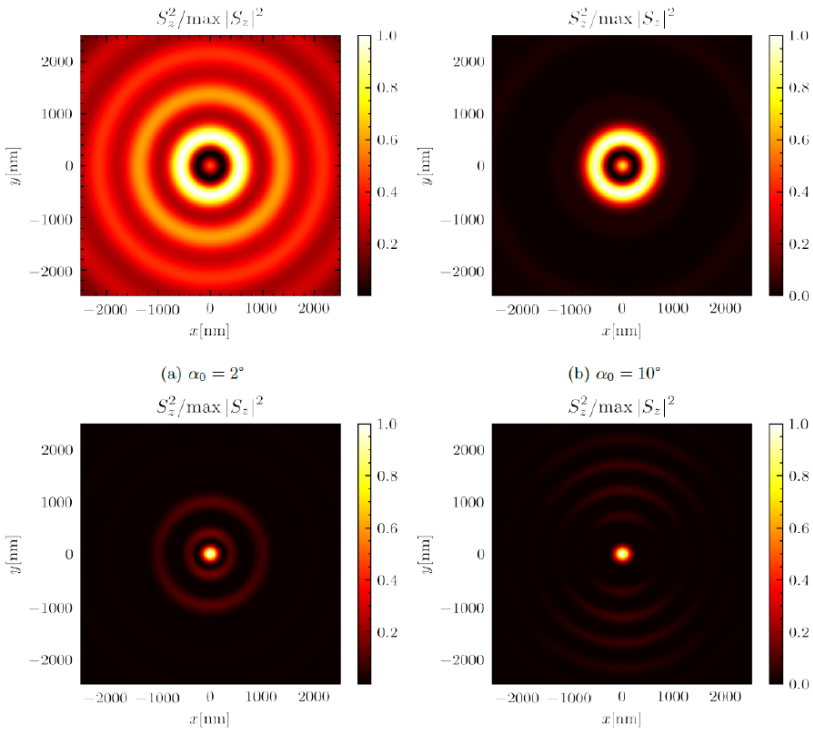
Note however that there is no conflict for $\alpha_0 = 0$, as we should have expected, since Eq.117 then implies that $g^1_{n,TM}/g^{-1}_{n,TE} = i\varepsilon$. Therefore, although the non-Maxwellian original beam is axisymmetric, the remodelling has introduced a dependency with respect to φ. We may check that the same conclusion holds for other values of the order l, Eqs.40-41 being not satisfied but for $\alpha_0 = 0$. ILA would have preserved the axisymmetric of the beam if it had been forced by using Eq.88. This exemplifies the fact that not only the choice of the remodelling technique is significant, but the details of its implementation as well.

The lack of axisymmetry is displayed in the set of Figs.1. exhibiting the normalized S_z^2 for $\alpha_0 = 2°$, $10°$, $30°$ and $60°$. Computations have been made for a wavelength in air equal to 1064 nm, with a beam propagating in water (refractive index equal to 1.33). The order of the beam is $l = 0$, the beam confinement factor is $s = 0.001$ and $z_0 = 0$. This implies $\overline{\mu} = 1$ and $g^1_{n,TM}/g^{-1}_{n,TE} = i/\cos\alpha_0$ and:

$$g^1_{n,TM} = \frac{1}{2}\exp\left(\frac{-s^2w_n^2}{\sin^2\alpha_0}\right)J_0(w_n) = g^{-1}_{n,TM} \tag{118}$$

$$g^1_{n,TE} = \frac{i}{2}\exp\left(\frac{-s^2w_n^2}{\sin^2\alpha_0}\right)J_0(w_n)\cos\alpha_0 = -g^{-1}_{n,TE} \tag{119}$$

We may then state that the loss of axisymmetric is weak insofar as it is not readily perceived for $\alpha_0 = 2°$, $10°$ and $30°$ and can only be detected for $\alpha_0 = 60°$.



(a) $\alpha_0 = 2°$

(b) $\alpha_0 = 10°$

(c) $\alpha_0 = 30°$

(d) $\alpha_0 = 60°$

Figure 1: S_z squared for different values of the axicon angle

5.2 Laguerre-Gauss beams focused by a lens.

Relying on [69], [70], [71], this case has been worked out in [44], [45]. The components E_x, E_y, H_x, H_y of these beams are given in Eqs.(25)-(26), (34) and (38) of [44], from which it is found that S_z still depends on φ. Therefore, we do not expect that the conditions characterizing BSCs of axisymmetric beams would be satisfied in the present case. It is then sufficient to deal with a case as simple as possible. For this, we rely on the BSCs expressed by using the FS-method in [44]. These BSCs are expressed using the N-convention, that is to say assuming that the time-dependence of the fields is $\exp(-i\omega t)$ opposite to the P-convention, which is the usual convention when dealing with GLMT, that is to say assuming that the time-dependence of the fields is of the form $\exp(+i\omega t)$. The relationship between the BSCs expressed in the N-convention and those expressed in the P-convention is given by [72], [44]:

$$\begin{pmatrix} g^{m}_{n,TM}(P) \\ g^{m}_{n,TE}(P) \end{pmatrix} = (-1)^{n+1} \begin{pmatrix} g^{-m*}_{n,TM}(N) \\ g^{-m*}_{n,TE}(N) \end{pmatrix} \tag{120}$$

BSCs with the N-convention are different from 0 only for $m = l + 1$ and $m = l - 1$ [44]. Eq.120 then implies that BSCs with the P-convention are different from 0 only for $m = -l - 1$ and $m = -l + 1$. This is indeed what we observed using quadratures and localized approximations with the P-convention, see Eqs.(47)-(48) and Table 2 in [40]. As a simple illustrative example, we consider the case when the order l of the beam is $+2$, for which $m = -3, -1$ in the P-convention and $m = 1, 3$ in the N-convention. We shall be content to test Eq.40, that is to say, in the P-convention, to ask the question:

$$g^{-3}_{n,TM} g^{-1*}_{p,TM} + g^{-1*}_{p,TE} g^{-3}_{n,TE} = 0 \ ? \tag{121}$$

which, using Eq.120, becomes, in the N-convention to be used here:

$$g^{3}_{n,TM} g^{1*}_{p,TM} + g^{1*}_{p,TE} g^{3}_{n,TE} = 0 \ ? \tag{122}$$

Subsequent equations with FS-technique for Laguerre-Gauss beams focused by a lens will deal with the N-convention, a fact which will be denoted in the notation. Since a simple counter-example to Eq.122 is sufficient to our purposes, we shall use trivial Laguerre-Gauss beams of the form $L^2_0(x) = 1$, with partial waves of order $n = 3$ (i.e. BSCs with a subscript $n = 3$). We recall that the FS-technique distinguishes the BSCs $g^m_{n,TM}$ and $g^m_{n,TE}$ depending on whether $(n - m)$ is even or odd. With $n = 3$ and $m = 1, 3$ we have to deal with $(n - m)$ even. The expressions for the BSCs are then obtained from Eqs (96), (102),

(165) and (176) in [44]. After rearranging and simplifying, they read as, with the subscript 0 referring to the order 0 of the Laguerre-Gauss beam $L_0^2(x)$:

$$[g_{3,TM}^3(N)]_0 = \frac{i\beta - \alpha}{16} A_{20}^3 \tag{123}$$

$$[g_{3,TM}^1(N)]_0 = \frac{\alpha + i\beta}{8}(5A_{20}^3 - 10B_{20}^3 + 4B_{20}^1) \tag{124}$$

$$[g_{3,TE}^3(N)]_0 = \frac{i(i\beta - \alpha)}{32\sqrt{2}} P_{20}^3 \tag{125}$$

$$[g_{3,TE}^1(N)]_0 = \frac{-i(\alpha + i\beta)}{16\sqrt{2}}(5P_{20}^3 + 10Q_{20}^3 - 4Q_{20}^1) \tag{126}$$

in which α and β are complex coefficients which determine the dominant state of polarization [71], [70], [69], and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{P}$, $\mathbf{Q}$ are integrals reading as:

$$A_{l0}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2 - \eta^2}}} \exp[-c(\eta)][(1 + \frac{\sqrt{k^2 - \eta^2}}{k})(\frac{\eta}{k})^{n-1} d\eta \tag{127}$$

$$B_{l0}^n = \int_0^{kNA} \frac{\eta^{|l|+1}}{\sqrt{\sqrt{k^2 - \eta^2}}} \exp[-c(\eta)][(1 - \frac{\sqrt{k^2 - \eta^2}}{k})(\frac{\eta}{k})^{n-1} d\eta \tag{128}$$

$$P_{l0}^n = \int_0^{kNA} (\frac{\eta}{k})^n \frac{\eta^{|l|}}{(k^2 - \eta^2)^{3/4}} \exp[-c(\eta)] A(\eta) d\eta \tag{129}$$

$$Q_{l0}^n = \int_0^{kNA} (\frac{\eta}{k})^n \frac{\eta^{|l|}}{(k^2 - \eta^2)^{3/4}} \exp[-c(\eta)] B(\eta) d\eta \tag{130}$$

in which:

$$A(\eta) = (1 - \frac{\sqrt{k^2 - \eta^2}}{k})\eta^2 - (1 + \frac{\sqrt{k^2 - \eta^2}}{k})(2k^2 - \eta^2) \tag{131}$$

$$B(\eta) = (1 + \frac{\sqrt{k^2 - \eta^2}}{k})\eta^2 - (1 - \frac{\sqrt{k^2 - \eta^2}}{k})(2k^2 - \eta^2) \tag{132}$$

Furthermore, we have:

$$c(\eta) = \eta^2 \frac{k_S^2}{2k^2} R_f^2 \tag{133}$$

in which $k_S = \sqrt{2}/w_0$ (with w_0 being the beam waist radius), R_f is the focal length and NA is the numerical aperture of the imaging system. Inserting Eqs.126-127 into the l.h.s. of Eq.122 with $n = p = 3$, and assuming $\alpha = 1$, $\beta = 0$, we obtain:

$$\begin{aligned} g^3_{3,TM} g^{1*}_{3,TM} + g^{1*}_{3,TE} g^3_{3,TE} &= \frac{-1}{1024}[8A^3_{20}(5A^3_{20} - 10B^3_{20} + 4B^1_{20}) \\ &\quad - \frac{P^3_{20}}{8}(5P^3_{20} + 10Q^3_{20} - 4Q^1_{20})] \end{aligned} \tag{134}$$

which is found to be a real number equal to $\simeq -4.24.10^{15}$, confirming that the beam is not axisymmetric. The lack of axisymmetry is well evidenced in Fig.2 displaying S_z^2 in the $z = 0$ plane.

This may seem to be in conflict with Eqs.109-110 which showed that, for $l = 2$, we were facing a Laguerre-Gauss freely propagating which was axisymmetric, and with this other fact that the focusing lens in the case considered in this subsection is not supposed to alter the axisymmetry of the beam. The explanation is that the axisymmetry of the Laguerre-Gauss beam of subsection 4.5 has been forced by the way used to go from a scalar solution to a vectorial case. In contrast with this freely propagating case which, furthermore, was non-Maxwellian, the case considered in the present section is Maxwellian according to [70], and as confirmed by the excellent agreement between BSCs obtained either by quadratures or by the FS-technique as displayed in Table 3 of [45]. Indeed, in the case of non-Maxwellian beams, quadrature values and FS-values would have been remodelled and, the remodelling processes being different, the values would have expectedly been different as well.

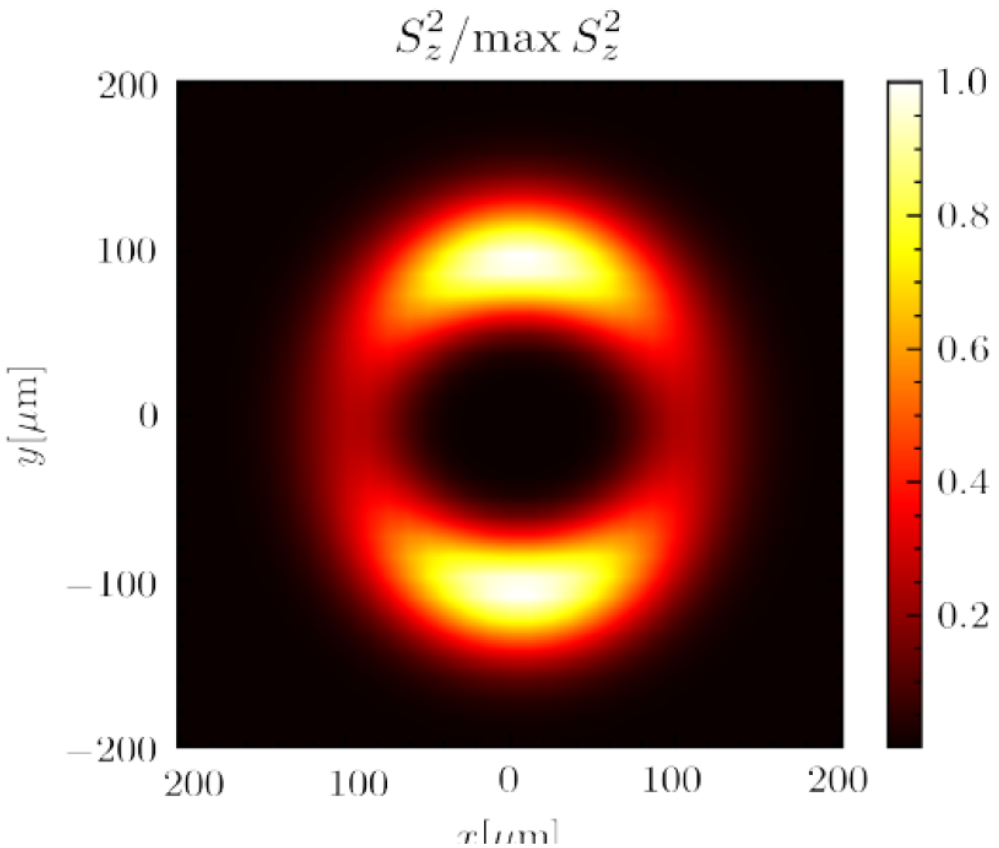


Figure 2: S_z squared

6 Frozen waves laboratories.

Frozen waves are structured light fields constructed from specific superpositions of Bessel beams allowing one to control the longitudinal intensity pattern, a feature which is not possible when dealing with a single nondiffracting Bessel beam [73], [74], [75], [76]. Extensions to the generation of two-dimensional intensity patterns over predetermined surfaces have also been recently explored [77], [78], with applications to transverse intensity control vividly illustrated in Fig. 1 of [77]. Vector frozen wave solutions to the Helmholtz equation are easily constructed from superpositions of vector Bessel beams [79], [80], with several polarization states, and BSCs evaluated either using a localized approximation scheme [81], [82], [37] or using an analytical exact approach derived from quadratures [83], [84], [85].

We are going to exemplify the fact that such frozen waves may provide laboratories to establish new expressions involving BSCs and reflecting symmetry properties of the corresponding beams. To this purpose, let us consider the $(1,0)$-polarization of circularly symmetric beams described by Eqs.(6)-(7) of [83]. For $l=0$, the BSCs then read as:

$$g_{n,TM}^{1(1,0)} = \frac{-(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^1(\cos\alpha_{q0}) + \pi_n^1(\cos\alpha_{q0})] \quad (135)$$

$$g_{n,TM}^{-1(1,0)} = \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^{-1}(\cos\alpha_{q0}) + \pi_n^{-1}(\cos\alpha_{q0})] \quad (136)$$

$$g_{n,TE}^{1(1,0)} = \frac{i(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^1(\cos\alpha_{q0}) + \pi_n^1(\cos\alpha_{q0})] \quad (137)$$

$$g_{n,TE}^{-1(1,0)} = i \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^{-1}(\cos\alpha_{q0}) + \pi_n^{-1}(\cos\alpha_{q0})] \quad (138)$$

in which coefficients B_l are attached to the order of the beam and the coefficients A_{ql} are attached to both the order of the beam and to the q-waves involved in the summation. We then have:

$$g_{n,TM}^{1(1,0)} = i g_{n,TE}^{1(1,0)} \tag{139}$$

$$g_{n,TM}^{-1(1,0)} = -i g_{n,TE}^{-1(1,0)} \tag{140}$$

which satisfy Eq.7 with $\varepsilon = -1$ and:

$$K = \frac{-(n-1)!}{(n+1)!} \tag{141}$$

so that we are facing an on-axis axisymmetric beam of the first kind. For $l = 2$, we have:

$$g_{n,TM}^{1(1,0)} = \frac{(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{i k_{zq}^2 z_0} [\tau_n^1(\cos\alpha_{q2}) - \pi_n^1(\cos\alpha_{q2})] \tag{142}$$

$$g_{n,TM}^{3(1,0)} = \frac{(n-3)!}{(n+3)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{i k_{zq}^2 z_0} [\tau_n^3(\cos\alpha_{q2}) + 3\pi_n^3(\cos\alpha_{q2})] \tag{143}$$

$$g_{n,TE}^{1(1,0)} = \frac{i(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{i k_{zq}^2 z_0} [\tau_n^1(\cos\alpha_{q2}) - \pi_n^1(\cos\alpha_{q2})] \tag{144}$$

$$g_{n,TE}^{3(1,0)} = \frac{-i(n-3)!}{(n+3)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{i k_{zq}^2 z_0} [\tau_n^3(\cos\alpha_{q2}) + 3\pi_n^3(\cos\alpha_{q2})] \tag{145}$$

We therefore have:

$$g_{n,TE}^{1(1,0)} = i g_{n,TM}^{1(1,0)} \tag{146}$$

$$g_{n,TE}^{3(1,0)} = -i g_{n,TM}^{3(1,0)} \tag{147}$$

and we readily check that Eqs.40-41 are satisfied. Indeed, we have:

$$g_{n,TM}^{1(1,0)} g_{p,TM}^{3(1,0)*} + g_{p,TE}^{3(1,0)*} g_{n,TE}^{1(1,0)} = 0 \tag{148}$$

$$g_{n,TM}^{1(1,0)} g_{p,TE}^{3(1,0)*} - g_{p,TM}^{3(1,0)*} g_{n,TE}^{1(1,0)} = 0 \tag{149}$$

so that we are facing an on-axis axisymmetric beam of second kind. Let us now consider the summation of waves of orders 0 and 2. Still from Eqs.(6)-(7) of [83], we now have:

$$g_{n,TM}^{-1(1,0)} = \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^{-1}(\cos\alpha_{q0}) + \pi_n^{-1}(\cos\alpha_{q0})] \tag{150}$$

$$\begin{aligned} g_{n,TM}^{1(1,0)} &= \frac{-(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} \{B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^1(\cos\alpha_{q0}) + \pi_n^1(\cos\alpha_{q0})] \\ &\quad - B_2 A_{q2} g(\alpha_{q2}) e^{ik_{zq}^2 z_0} [\tau_n^1(\cos\alpha_{q2}) - \pi_n^1(\cos\alpha_{q2})]\} \end{aligned} \tag{151}$$

$$g_{n,TM}^{3(1,0)} = \frac{(n-3)!}{(n+3)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{ik_{zq}^2 z_0} [\tau_n^3(\cos\alpha_{q2}) + 3\pi_n^3(\cos\alpha_{q2})] \tag{152}$$

$$g_{n,TE}^{-1(1,0)} = i \sum_{q=-N}^{+N} B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^{-1}(\cos\alpha_{q0}) + \pi_n^{-1}(\cos\alpha_{q0})] \tag{153}$$

$$\begin{aligned} g_{n,TE}^{1(1,0)} &= \frac{i(n-1)!}{(n+1)!} \sum_{q=-N}^{+N} \{B_0 A_{q0} g(\alpha_{q0}) e^{ik_{zq}^0 z_0} [\tau_n^1(\cos\alpha_{q0}) + \pi_n^1(\cos\alpha_{q0})] \\ &\quad + B_2 A_{q2} g(\alpha_{q2}) e^{ik_{zq}^2 z_0} [\tau_n^1(\cos\alpha_{q2}) - \pi_n^1(\cos\alpha_{q2})]\} \end{aligned} \tag{154}$$

$$g_{n,TM}^{3(1,0)} = \frac{-i(n-3)!}{(n+3)!} \sum_{q=-N}^{+N} B_2 A_{q2} g(\alpha_{q2}) e^{ik_{zq}^2 z_0} [\tau_n^3(\cos\alpha_{q2}) + 3\pi_n^3(\cos\alpha_{q2})] \tag{155}$$

from which we have:

$$g_{n,TE}^{-1} = i g_{n,TM}^{-1} \tag{156}$$

$$g_{n,TM}^{1} = h_n^0 + h_n^2 \tag{157}$$

$$g_{n,TE}^{1} = -i(h_n^0 - h_n^2) \tag{158}$$

$$g_{n,TE}^{3} = -i g_{n,TM}^{3} \tag{159}$$

in which we do not need to provide the expressions for h_n^0 and h_n^2. We then observe that the beam is not axisymmetry of the first kind because Eqs.6-7 are not satisfied for two reasons (i) there are BSCs with m different from ± 1 so that Eq.6 is not satisfied and (ii) as can be readily checked, the BSCs with $m = \pm 1$ do not satisfy Eq.7. Indeed, we may determine the coefficient K of this equation by two different ways as follows. We have:

$$\frac{g_{n,TM}^{-1}}{g_{n,TM}^{1}} = K = \frac{g_{n,TM}^{-1}}{h_n^0 + h_n^2} \tag{160}$$

$$\frac{-g_{n,TE}^{-1}}{g_{n,TE}^{1}} = K = \frac{g_{n,TM}^{-1}}{h_n^0 - h_n^2} \tag{161}$$

hence a contradiction. The beam furthermore is not any more of the second kind. Let us for instance consider Eq.40 with $M = 1$, $Q = 3$. We then have:

$$g_{n,TM}^{1} g_{p,TM}^{3*} + g_{p,TE}^{3*} g_{n,TE}^{1} = 2 g_{p,TM}^{3*} h_n^1 \neq 0 \tag{162}$$

Therefore, adding a first kind beam with a second kind beam washes out both symmetries, although the beam, being made from the addition of two

on-axis axisymmetric beams (and even of circularly symmetric beams) is supposed to be an on-axis axisymmetric beam (and a circularly axisymmetric beam as well). This result confirms that frozen waves might constitute formal laboratories relevant to the uncovering of new kinds of BSC-symmetries associated with beam symmetries.

7 Conclusion.

The uncovering of symmetries involving the expressions of BSCs associated with symmetries of a beam is an useful task insofar as it allows simplifications in the framework of scattering theories, such as generalized Lorenz-Mie theories and Extended Boundary Condition Method. An example is the use of symmetries associated with on-axis non-dark axisymmetric beams of the first kind, valid for instance in the cases of a plane wave, an on-axis spherical wavefront, Gaussian beams and zeroth-order Bessel beams as discussed in [14]. This first kind symmetry allows one to reduce the double set of bi-index BSCs $g^m_{n,TM}$ and $g^m_{n,TE}$ to a single set of uni-index BSCs g_n, and, as a matter of fact, allows one to deduce the classical Lorenz-Mie theory from its generalized version, e.g. pp.152-155 in [3].

In the present paper, symmetries of BSCs are completed by two families, the one of non-dark axisymmetric beams of the second kind and the one of dark axisymmetric beams. These new families are illustrated by several examples and counter-examples. It is also shown, relying on the analysis of a class of discrete frozen waves used as formal laboratories that more symmetries are still to be uncovered.

Acknowledgments/Funding.

"National Council for Scientific and Technological Development (CNPq) (426990/2018-8, 307898/2018-0)"

References

[1] G. Gouesbet, B. Maheu, and G. Gréhan. Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation. *Journal of the Optical Society of America A*, 5,9:1427–1443, 1988.

[2] G. Gouesbet, G. Gréhan, and B. Maheu. *Combustion measurements, edited by N. Chigier*, chapter : Generalized Lorenz-Mie theory and applications to optical sizing, pages 339–384. Hemisphere Publishing Corporation, New-York, USA, 1991.

[3] G. Gouesbet and G. Gréhan. *Generalized Lorenz-Mie theories, second edition*. Springer International Publishing AG, 2017.

[4] G. Gouesbet. Generalized Lorenz-Mie theories and mechanical effects of laser light, on the occasion of Arthur Ashkin's receipt ot the 2018 Nobel prize in physics for his pioneering work in optical levitation and manipulation: A review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 225:258–277, 2019.

[5] G. Gouesbet. Van de Hulst Essay: A review on generalized Lorenz-Mie theories with wow stories and epistemological discussion. *Journal of Quantitative Spectroscopy and Radiative Transfer*, Paper 107117, 2020.

[6] P.C. Waterman. Symmetry, unitarity, and geometry in electromagnetic scattering. *Physical Review D*, 3, 4:825–839, 1971.

[7] M.I. Mishchenko, L.D. Travis, and A.A. Lacis. *Scattering, absorption, and emission of light by small particles*. Cambridge University Press, Cambridge, 2002.

[8] G. Gouesbet. T-matrix methods for electromagnetic structured beams: A commented reference database for the period 2014-2018. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 230:247–281, 2019.

[9] K.F. Ren, G. Gréhan, and G. Gouesbet. Symmetry relations in generalized Lorenz-Mie theory. *Journal of the Optical Society of America A*, 11,6:1812–1817, 1994.

[10] G. Gouesbet, C. Letellier, K.F. Ren, and G. Gréhan. Discussion of two quadrature methods of evaluating beam shape coefficients in generalized Lorenz-Mie theory. *Applied Optics*, 35,9:1537–1542, 1996.

[11] G. Gouesbet, L.A. Ambrosio, and J.A. Lock. On an infinite number of quadratures to evaluate beam shape coefficients in generalized Lorenz-Mie theory and extended boundary condition method for structured EM fields. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 242:Article 196779, 4 pages, 2020.

[12] G. Gouesbet. Partial wave expansions and properties of axisymmetric light beams. *Applied Optics*, 35,9:1543–1555, 1996.

[13] G. Gouesbet and G. Gréhan. Sur la généralisation de la théorie de Lorenz-Mie. *Journal of Optics*, 13,2:97–103, 1982.

[14] G. Gouesbet. Poynting theorem in terms of beam shape coefficients and applications to axisymmetric, dark and non-dark, vortex and non-vortex beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:184–196, 2017.

[15] G. Gouesbet and J.A. Lock. A darkness theorem for the beam shape coefficients and its relationship to higher-order non vortex Bessel beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 201:229–235, 2017.

[16] G. Gouesbet. Gradient, scattering and other kinds of longitudinal optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 246:Article 106913, 2020.

[17] G. Gouesbet and L.A. Ambrosio. Axicon optical forces and other kinds of transverse optical forces exerted by off-axis Bessel beams in the Rayleigh regime in the framework of generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopie and Radiative Transfer, virtual special issue of LIP2020*, 258:Article 107356, 2021.

[18] L.A. Ambrosio and G. Gouesbet. On longitudinal radiation pressure cross-sections in the generalized Lorenz-Mie theory and their numerical relationship with the dipole theory of forces. *Journal of the Optical Society of America B*, 38, 3:825–833, 2021.

[19] L.A. Ambrosio and G. Gouesbet. On transverse radiation pressure cross-sections in the generalized Lorenz-Mie theory and their numerical relationships with the dipole theory of forces. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 261:Article 107491, 2021.

[20] L.A. Ambrosio and G. Gouesbet. On the Rayleigh limit of the generalized Lorenz-Mie theory and its formal indentification with the dipole theory of forces. I. The longitudinal case. *Journal of Quantitative Spectroscopie and Radiative Transfer*, 262:Article 107531, 2021.

[21] L.A. Ambrosio and G. Gouesbet. On the Rayleigh limit of the generalized Lorenz-Mie theory and its formal identification with the dipole theory of forces. II. The transverse case. *Journal of Quantitative Spectroscopie and Radiative Transfer*, 266:Article 107591, 2021.

[22] G. Gouesbet. Optical forces exerted by on-axis Bessel beams on Rayleigh particles in the framework of the generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopie and Radiative Transfer*, 260:Article 107471, 2021.

[23] G. Gouesbet and L.A. Ambrosio. Rayleigh limit of the generalized Lorenz-Mie theory for on-axis beams and its relationship with the dipole theory of forces. Part I. non dark axisymmetric beams of the first kind, with the example of Gaussian beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 266:Article 107569, 2021.

[24] L. Robin. *Fonctions sphériques de Legendre et fonctions sphéroidales. Volumes 1, 2, 3.* Gauthier-Villars, Paris, 1957.

[25] G. Gouesbet, J.J. Wang, and Y.P. Han. Transformations of spherical beam shape coefficients in generalized Lorenz-Mie theories through rotations of coordinate system. IV. Plane waves. *Optics Communications*, 283, 17:3244–3254, 2010.

[26] G. Gouesbet, G. Gréhan, and B. Maheu. Expressions to compute the coefficients g_n^m in the generalized Lorenz-Mie theory, using finite series. *Journal of Optics (Paris)*, 19,1:35–48, 1988.

[27] G. Gouesbet, G. Gréhan, and B. Maheu. Computations of the g_n coefficients in the generalized Lorenz-Mie theory using three different methods. *Applied Optics*, 27,23:4874–4883, 1988.

[28] L.A. Ambrosio and G. Gouesbet. Modified finite series technique for the evaluation of beam shape coefficients in the T-matrix methods for structured beams with application to Bessel beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 248:Article 107007 (11 pages), 2020.

[29] K.F. Ren, G. Gouesbet, and G. Gréhan. Integral localized approximation in generalized Lorenz-Mie theory. *Applied Optics*, 37,19:4218–4225, 1998.

[30] G. Gouesbet, J.A. Lock, and G. Gréhan. Generalized Lorenz-Mie theories and description of electromagnetic arbitrary shaped beams: localized approximations and localized beam models, a review. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 112:1–27, 2011.

[31] G. Gouesbet. Second modified localized approximation for use in generalized Lorenz-Mie theories and other theories revisited. *Journal of the Optical Society of America A*, 30, 4:560–564, 2013.

[32] G. Gouesbet and J.A. Lock. Comments on localized and integral localized approximations in spherical coordinates. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 179:132–136, 2016.

[33] G. Gouesbet. On the validity of localized approximations for Bessel beams: all N-Bessel beams are identically equal to zero. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 176:82–86, 2016.

[34] G. Gouesbet and L.A. Ambrosio. On the validity of the use of a localized approximation for helical beams. I. Formal aspects. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 208:12–18, 2018.

[35] G. Gouesbet, J.A. Lock, L.A. Ambrosio, and J.J. Wang. On the validity of localized approximation for an on-axis zeroth-order Bessel beam. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 195:18–25, 2017.

[36] L.A. Ambrosio, J.J. Wang, and G. Gouesbet. On the validity of the integral localized approximation for Bessel beams and associated radiation pressure forces. *Applied Optics*, 56, 19:5377–5387, 2017.

[37] L.A. Ambrosio, L.F.M. Votto, G. Gouesbet, and J.J. Wang. Assessing the validity of the localized approximation for discrete superposition of Bessel beams. *Journal of the Optical Society of America B*, 35, 11:2690–2698, 2018.

[38] A. Chafiq, L.A. Ambrosio, G. Gouesbet, and A. Belafhal. On the validity of the integral localized approximation for on-axis zeroth-order Mathieu beams. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 204:27–34, 2018.

[39] L.A. Ambrosio and G. Gouesbet. On the validity of the use of a localized approximation for helical beams. II. Numerical aspects. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 215:41–50, 2018.

[40] L.A. Ambrosio and G. Gouesbet. On localized approximations for Laguerre-Gauss beams focused by a lens. *Journal of Quantitative Spectroscopy and Radiative Transfer, LIP2018 special issue*, 218:100–114, 2018.

[41] N.L. Valdivia, L.F.M. Votto, G. Gouesbet, J.J. Wang, and L.A. Ambrosio. Bessel-Gauss beams in the generalized Lorenz-Mie theory using three remodeling techniques. *Journal of Quantitative Spectroscopie and Radiative Transfer*, 256:Article 107292, 2020.

[42] G. Gouesbet, L.F.M. Votto, and L.A. Ambrosio. Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam freely propagating. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 227:12–19, 2019.

[43] L.F.M. Votto, L.A. Ambrosio, and G. Gouesbet. Evaluation of beam shape coefficients of paraxial Laguerre-Gauss beam freely propagating by using three remodeling methods. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 239:Article 106618, 2019.

[44] G. Gouesbet, L.A. Ambrosio, and L.F.M. Votto. Finite series expressions to evaluate the beam shape coefficients of a Laguerre-Gauss beam focused by a lens in an on-axis configuration. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 242:Paper 106759, 2019.

[45] L.F.M. Votto, L.A. Ambrosio, G. Gouesbet, and J.J. Wang. Finite series algorithm design for lens-focused Laguerre-Gauss beams in the generalized Lorenz-Mie theory. *Journal of Quantitative Spectroscopie and Radiative Transfer, virtual special of LIP2020*, 261:Paper 107 488, 2021.

[46] J. Durnin, Jr. J.J. Miceli, and J.H. Eberly. Diffraction-free beams. *Physical Review Letters*, 58, 15:1499–1501, 1987.

[47] J. Durnin. Exact solutions for nondiffracting beams. I. The scalar theory. *Journal of the Optical Society of America A*, 4, 4:651–654, 1987.

[48] J.A. Lock. Angular spectrum and localized model of Davis-type beam. *Journal of the Optical Society of America A*, 30, 3:489–500, 2013.

[49] G. Gouesbet and J.A. Lock. Consequences of the angular spectrum decomposition of a focused beam including slower than c beam propagation. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 178:142–151, 2016.

[50] J.J. Wang, T. Wriedt, J.A. Lock, and L. Mädler. General description of circularly symmetric Bessel beams of arbitrary order. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 184:218–232, 2016.

[51] J.J. Wang, T. Wriedt, J.A. Lock, and Y.C. Jiao. General description of transverse mode Bessel beams and construction of basis Bessel fields. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 195:8–17, 2017.

[52] J.J. Wang, T. Wriedt, L.Mädler, Y.P. Han, and P. Hartmann. Multipole expansion of circularly Bessel beams of arbitrary order for scattering calculations. *Optics Communications*, 387:102–109, 2017.

[53] S.R. Mishra. A vector wave analysis of a Bessel beam. *Optics Communications*, 85:159–161, 1991.

[54] T. Cizmar, V. Kollarova, Z. Bouchal, and P. Zemanek. Sub-micron particle organization by self-imaging of non-diffracting beams. *New Journal of Physics*, 8 (3), 43, 2006.

[55] J.M. Taylor and G.D. Love. Multipole expansion of Bessel and Gaussian beams for Mie scattering calculations. *Journal of the Optical Society of America A*, 26, 2:278–282, 2009.

[56] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding. *Optics Letters*, 35, 10:1674–1676, 2010.

[57] J.J. Wang. Characteristics of photonic jets generated by a spherical particle excited by a Bessel beam. *In preparation.*

[58] J. Chen, J. Ng, S. Liu, and Z. Lin. Analytical calculation of axial optical force on a Rayleigh particle illuminated by Gaussian beams beyond the paraxial approximation. *Physical Review E*, 80:Paper 026607, 1–9, 2009.

[59] J. Chen, J. Ng, P. Wang, and Z. Lin. Analytical partial wave expansion of vector Bessel beam and its application to optical binding: erratum. *Optics Letters*, 36, 7:1243, 2011.

[60] Z.X. Wang and D.R. Guo. *Special functions.* World Scientific, Singapore, 1989.

[61] A. Chafiq and A. Belafhal. Scattering of Lommel beams by homogeneous spherical particle in generalized Lorenz-Mie theory. *Optical and Quantum Electronics*, 2/2018, 2018.

[62] F. Gori, G. Guattari, and C. Padovani. Bessel-Gauss beams. *Optics Communication*, 64, 6:491–495, 1987.

[63] L.A. Ambrosio. Analytical descriptions of finite-energy Bessel beams in the generalized Lorenz-Mie theory. *Conference paper, https://www.sbfoton.org.br/, SBFoton conference, Campinas, Brazil, 22-24 October 2018, technical session OS-L-TUE-03 (Lasers 2)*, 2018.

[64] M. Zamboni-Rached and Mo Mojahedi. Shaping finite-energy diffraction- and attenuation-resistant beams through Bessel-Gauss-beam superposition. *Physical Review A*, 92:Article 043839, 2015.

[65] A.E. Siegman. *An introduction to lasers and masers*. McGraw-Hill Book Company, New-York, 1971.

[66] A.E. Siegman. *Lasers*. University Science Books, Sausalito, California, 1986.

[67] G.B. Arfken and H.J. Weber. *Mathematical methods for physicists, sixth edition*. Elsevier Academic Press, Amsterdam, 2005.

[68] M. Abramowitz and I.A. Stegun. *Handbook of mathematical functions*. Dover Publications, Inc, New York, 1972.

[69] P. Török and P.R.T. Munro. The use of Gauss-Laguerre vector beams in STED microscopy. *Optics Express*, 12, 15:3605–3617, 2004.

[70] A.S. van de Nes, S.F. Pereira, and J.J.M. Braat. On the conservation of fundamental optical quantities in non-paraxial imaging systems. *Journal of Modern Optics*, 53, 5-6:677–687, 2006.

[71] A.S. van de Nes and P. Török. Rigorous analysis of spheres in Gauss-Laguerre beams. *Optics Express*, 15,20:13360–13374, 2007.

[72] J.J. Wang and G. Gouesbet. Note on the use of localized beam models for light scattering theories in spherical coordinates. *Applied Optics*, 51, 17:3832–3836, 2012.

[73] M. Zamboni-Rached. Stationary optical wave fields with arbitrary longitudinal shape by superposing equal frequency Bessel beams: Frozen waves. *Optics Express*, 12, 17:4001–4006, 2004.

[74] M. Zamboni-Rached, E. Recami, and H.E. Hernandez-Figueroa. Theory of "frozen waves": modeling the shape of stationary wave fields. *Journal of the Society of America A*, 22, 11:2465–2475, 2005.

[75] M. Zamboni-Rached. Diffraction-attenuation resistant beams in absorbing media. *Optics Express*, 14, 5:1804–1809, 2006.

[76] M. Zamboni-Rached, L.A. Ambrosio, and H.E. Hernandez-Figueroa. Diffraction-attenuation resistant beams: their higher-order versions and finite-aperture generations. *Applied Optics*, 49, 30:5861–5869, 2010.

[77] Leonardo André Ambrosio. Millimeter-structured nondiffracting surface beams. *Journal of the Optical Society of America B*, 36, 3:638–645, 2019.

[78] J.O. de Sarro and L.A. Ambrosio. Surface beams resistant to diffraction and attenuation ans structured at the millimeter scale. *Journal of the Optical Society of America B*, 38, 3:677–684, 2021.

[79] M. Corato-Zanarella and M. Zamboni-Rached. Electromagnetic frozen waves with radial, azimuthal, linear, circular, and elliptical polarizations. *Physical Review A*, 94:Article 053802, 2016.

[80] M. Zamboni-Rached, L.A. Ambrosio, A.H. Dorrah, and M. Mojahedi. Structuring light under different polarization states within micrometer domains: exact analysis from maxwell equations. *Optics Express*, 25, 9:10051–10056, 2017.

[81] L.A. Ambrosio and M. Zamboni-Rached. Optical forces experienced by arbitrary-sized spherical scatterers from superpositions of equal-frequency Bessel beams. *Journal of the Optical Society of America B*, 32, 5:B37–B46, 2015.

[82] L.A. Ambrosio and M. Zamboni-Rached. Analytical approach of ordinary frozen waves for optical trapping and micromanipulation. *Applied Optics*, 54, 10:2584–2593, 2015.

[83] L.A. Ambrosio, M. Zamboni-Rached, and G. Gouesbet. Discrete vector frozen waves in generalized Lorenz-Mie theory: linear, azimuthal and radial polarization. *Applied Optics*, 57, 12:3293–3300, 2018.

[84] L.A. Ambrosio. Circularly symmetric frozen waves: Vector approach for light scattering calculations. *Journal of Quantitative Spectroscopy and Radiative Transfer*, 204:112–119, 2018.

[85] L.A. Ambrosio, M. Zamboni Rached, and G. Gouesbet. Zeroth-order continuous vector frozen waves for light scattering: exact multipole expansion in the generalized Lorenz-Mie theory. *Journal of the Optical Society of America B*, 36, 1:81–89, 2019.