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January 29, 2021

Abstract

In some cases, the nilpotent approximation of an almost-Riemannian structure can degenerate into a constant rank sub-Riemannian one. In those cases, the nilpotent approximation can be replaced by a solvable one that turns out to be a linear ARS on a nilpotent Lie group or a homogeneous space.

The distance defined by the solvable approximation is analyzed in the 3D-generic cases. It is shown that it is a better approximation of the original distance than the nilpotent one.

Key words: Almost-Riemannian geometry, Nilpotent approximation.
AMS subject classifications: 53C15, 53C17, 22E25, 53B99.

1 Introduction

The aim of this paper is to locally approximate almost-Riemannian structures (ARS in short) at singular points, by ARSs on Lie groups and to show that this approximation is generally better than the nilpotent one.

An ARS on an n-dimensional differential manifold is a rank-varying sub-Riemannian structure that can be defined, at least locally, by a set of n vector fields satisfying the Lie algebra rank condition (Larc in short). We denote by Δ_p the linear span of the vector fields at the point p. The set of points where $\dim(\Delta_p) < n$ is called the singular locus or the singular set and denoted by $\mathcal{Z}$. Many papers dedicated to the study of ARSs can be found in the literature, for instance [2], [8], [9], [11] [12].

In the generic 3-dimensional case, in which we are particularly interested, the singular set is a codimension one embedded submanifold and the points where $\Delta_p = T_p\mathcal{Z}$ are isolated (see [3] and [10]).

We are likewise interested in the so-called **simple ARSs** on Lie groups (or homogeneous space) because they will be used as approximating structures for general

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ARSs: a simple ARS on an n-dimensional Lie group is an almost-Riemannian structure defined by $n-1$ left-invariant vector fields and one vector field whose flow is a one-parameter group of automorphisms, called linear in the sequel. Under some conditions, the singular set of such structures is a subgroup or an analytic, embedded, codimension one submanifold (see [5] and [17]).

In some cases the nilpotent approximation of an ARS degenerates, because it is no longer an ARS but a constant rank sub-Riemannian structure. In other words, it may happen that some of the vector fields of the nilpotent approximation vanish, changing the almost-Riemannian structure into a constant rank sub-Riemannian one. For instance, if

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \quad \text{and} \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ x^2 \end{pmatrix},$$

then its nilpotent approximation is

$$\widehat{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \quad \text{and} \quad \widehat{X}_3 = 0.$$

It is what happens in some generic 3-dimensional cases (see for instance [10]). In this paper we are interested in the case where only one of the vector fields vanishes and the other ones are independent: then they define a left-invariant sub-Riemannian structure on a Lie group (or a homogeneous space).

Our aim consists in recovering the almost-Riemannian structure lost in the nilpotent approximation, thanks to a vector field, denoted $\widetilde{X}_n$ which is the homogeneous component of degree 0 of the Taylor expansion in privileged coordinates of the vector field that vanishes. The new family of vector fields composed by the nilpotent approximation and $\widetilde{X}_n$ is called the solvable approximation.

The Lie algebra generated by this new family of vector fields is finite dimensional and solvable. However, we are interested in some nilpotent Lie group on which $\widetilde{X}_n$ acts as a linear vector field (these vector fields were generalized in [6]). Thanks to the equivalence theorem of [16] we know that the space $\mathbb{R}^n$ is diffeomorphic to a homogeneous space. Through this diffeomorphism, the solvable approximation is equivalent to a simple ARS on a homogeneous space or a Lie group. It is important to notice that in the 3D-generic case the non-degenerated nilpotent approximation, and the solvable one in the degenerated case, are simple ARSs on Lie groups or homogeneous space (see Section 3.3).

On the other hand, the solvable approximation gives rise to a distance denoted by $\widetilde{d}$. This distance has the advantage to be really almost-Riemannian unlike the distance $\widehat{d}$ associated to the nilpotent approximation in the degenerated cases. The distance $\widetilde{d}$ is not homogeneous but always satisfies $\widetilde{d} \leq \widehat{d}$.

Denoting by d the distance associated to the original structure we show that in some 3D-generic cases the order of $|d - \widetilde{d}|$ is strictly better than the one of $|d - \widehat{d}|$. More accurately, the order of $|d - \widehat{d}|$ is $d^{\frac{3}{2}}$ and the one of $|d - \widetilde{d}|$ is d^2 in the cases we consider.

Moreover, the nilpotent distance $\widehat{d}$ is left-invariant while d and $\widetilde{d}$ are not. Using this fact we prove that for some pairs (q, q') of points translated from the singular locus the difference $|d(q, q') - \widetilde{d}(q, q')|$ is strictly smaller than $|d(q, q') - \widehat{d}(q, q')|$.

The paper is organized as follows. Section 2 contains generalities about ARSs, nonholonomic order, privileged coordinates, the nilpotent approximation, linear vector fields and simple ARS on Lie groups or homogeneous spaces.

In Section 3 we introduce the definition of a solvable approximation, we analyze its algebraic structure and an example is detailed.

Section 4 is divided in two parts. In the first one, we state two propositions about the almost-Riemannian distance $\widetilde{d}$ defined by the solvable approximation. The second part is devoted to analyze $\widetilde{d}$ in the 3-dimensional generic case.

Finally, in Section 5 we provide the Hamiltonian associated to the flow defined by the solvable approximation in the 3D generic case and we compute the geodesics with initial condition $x(0) = y(0) = z(0) = 0$ and $p(0) = \cos(\theta)$, $q(0) = \sin(\theta)$, $r(0) = r$ in a particular case.

2 Preliminaries

In this section some definitions and results are reviewed and come from [4], [7] and [15].

2.1 Almost-Riemannian structures

An almost-Riemannian structure can always be locally defined by a set of n vector fields, where n is the dimension of the state space. Since we are interested in local questions, the following definition will be enough in this paper, and the reader is referred to [4], [5] and [15] for the global definition on manifolds.

We denote by $Lie(X_1, \ldots, X_n)$ the Lie algebra generated by the vector fields $X_1, \ldots, X_n$ on $\mathbb{R}^n$.

Definition 1. *We say that the vector fields $X_1, \ldots, X_n$ satisfy the Lie algebra rank condition (Larc in short) on an open set Ω of $\mathbb{R}^n$ if*

$$Lie(X_1, \ldots, X_n)(p) = T_p\mathbb{R}^n,$$

for all $p \in \Omega$.

Definition 2. *The set $\{X_1, \ldots, X_n\}$ defines an almost-Riemannian structure (ARS in short) on the open and connected subset Ω of $\mathbb{R}^n$ if:*

(i) It satisfies Larc.

(ii) The singular locus, that is $\mathcal{Z} = \{p \in \Omega / rank(X_1(p), X_2(p), \ldots X_n(p)) < n\}$ is non-empty, but with empty interior.

The metric is defined by declaring the frame to be orthonormal.

Remark 1.

1. *The structure is Riemannian out of $\mathcal{Z}$.*

2. *Let $v \in T_p\Omega$. If p is a Riemannian point then $v = \sum_{i=1}^{n} u_i X_i(p)$ in a unique way and its length is $||v|| = \left(\sum_{i=1}^{n} {u_i}^2\right)^{\frac{1}{2}}$. If $p \in \mathcal{Z}$ then the decomposition of v, if it exists, is not unique and $||v|| = \inf\left\{\left(\sum_{i=1}^{n} {u_i}^2\right)^{\frac{1}{2}} : v = \sum_{i=1}^{n} u_i X_i(p)\right\}$.*

An absolutely continuous curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ is admissible if there exists a measurable essentially bounded function $t \mapsto u(t)$ from $[0,T]$ into $\mathbb{R}^n$ called control function such that $\dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)) + \ldots + u_n(t)X_n(\gamma(t))$ for almost every $t \in [0,T]$. Given an admissible curve $\gamma : [0,T] \longrightarrow \mathbb{R}^n$, the length of γ is

$$l(\gamma) = \int_0^T ||\dot{\gamma}(t)||dt.$$

The almost-Riemannian distance (or Carnot-Caratheodory distance) on Ω associated with the n-ARS is defined by

$$d(p_0, p_1) = \inf\left\{l(\gamma) : \gamma(0) = p_0,\ \gamma(T) = p_1,\ \gamma \text{ admissible}\right\}.$$

It induces the usual topology on Ω.

2.2 Nonholonomic orders

In what follows $\{X_1, X_2, \ldots, X_n\}$ defines an ARS on $\Omega \subset \mathbb{R}^n$.

Definition 3. *Let $f : M \to \mathbb{R}$ be a continuous function. The nonholonomic order of f at p, denoted $ord_p(f)$, is the real number defined by*

$$ord_p(f) = \sup\left\{s \in \mathbb{R} : f(q) = O\left(d(p,q)^s\right)\right\}.$$

This order is always nonnegative.

Let C^∞(p) denote the set of germs of smooth functions at p. For $f \in C^\infty(p)$, we call nonholonomic derivative of order 1 of f the Lie derivatives $X_1 f, \cdots, X_n f$. We call further $X_i X_j f, X_i X_j X_k f, \ldots$, the nonholonomic derivatives of f of order $2, 3, \ldots$ of f. The nonholonomic derivative of order 0 of f at p is $f(p)$.

As a consequence, the nonholonomic order of a smooth (germ of) function is given by the formula

$$\mathrm{ord}_p(f) = \min\left\{s \in \mathbb{N} : \exists\, i_1, \ldots, i_s \in \{1, \ldots, n\} \text{ s.t. } \left(X_{i_1} \ldots X_{i_s} f\right)(p) \neq 0\right\},$$

where as usual we adopt the convention that $\min \emptyset = +\infty$.

Let $VF(p)$ denote the set of germs of smooth vector fields at p.

Definition 4. *Let $X \in VF(p)$. The nonholonomic order of X at p, denoted by $ord_p(X)$, is the real number defined by:*

$$ord_p(X) = \sup\left\{\sigma \in \mathbb{R} : ord_p(Xf) \geq \sigma + ord_p(f),\ \forall\, f \in C^\infty(p)\right\}.$$

2.3 Privileged coordinates

We adopt the notation of [15] to define privileged coordinates. Let $VF(\Omega)$ denote the set of smooth vector fields on Ω. We define $\Delta^1 = \text{span}\{X_1, \ldots, X_n\}$. For $s \geq 1$, define $\Delta^{s+1} = \Delta^s + [\Delta^1, \Delta^s]$, where

$$[\Delta^1, \Delta^s] = \text{span}\left\{[X, Y] : X \in \Delta^1,\ Y \in \Delta^s\right\}.$$

For $p \in \Omega$, we set for $s \geq 1$, $\Delta^s(p) = \{X(p) : X \in \Delta^s\}$. By definition these sets are linear subspaces of $T_p\Omega$.

The evaluation of these sets at p forms a flag of subspaces of $T_p\Omega$, and since $X_1, \ldots, X_n$ satisfy Larc, we get,

$$\Delta^1(p) \subset \Delta^2(p) \subset \cdots \subset \Delta^{r-1}(p) \subsetneq \Delta^r(p) = T_p\Omega, \tag{1}$$

where $r = r(p)$ is called the degree of nonholonomy at p. Let $n_i(p) = \dim \Delta^i(p)$. The r-tuple of integers $(n_1(p), \ldots, n_r(p))$ is called the growth vector at p. The first integer in the growth vector is the rank $n_1(p) \leq n$ of the family $X_1(p), \ldots, X_n(p)$, and the last one $n_r(p) = n$ is the dimension of $\mathbb{R}^n$.

Definition 5. *The point p is regular if the growth vector is constant in some neighborhood of p. Otherwise we say that p is a singular point.*

The structure of the flag (1) may also be described by another sequence of integers. We define the weights at p, $w_i = w_i(p)$, $i = 1, \ldots, n$, by setting $w_j = s$ if $n_{s-1}(p) < j \leq n_s(p)$, where $n_0 = 0$. In other words, we have

$$w_1 = \cdots = w_{n_1} = 1, w_{n_1+1} = \cdots = w_{n_2} = 2, \ldots, w_{n_{r-1}+1} = \cdots = w_{n_r} = r.$$

Definition 6. *A system of privileged coordinates at p is a system of local coordinates $(x_1, \ldots, x_n)$ such that $ord_p(x_j) = w_j$, for $j = 1, \ldots, n$.*

On the other hand, given a sequence of integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ we define the weight of the monomial $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ to be $w(\alpha) = w_1\alpha_1 + \cdots + w_n\alpha_n$ and the weighted degree of the monomial vector field $x^\alpha \dfrac{\partial}{\partial x_j}$ to be $w(\alpha) - w_j$. The weighted degrees allow to compute the orders of functions and vector fields in a purely algebraic way.

Constructions of privileged coordinates can be found in [7] and [15].

Proposition 1 ([15], Proposition 2.2)**.** *For a smooth function f with a Taylor expansion in privileged coordinates*

$$f(x) \sim \sum_\alpha c_\alpha x^\alpha,$$

the order of f is the least weighted degree of monomials having a nonzero coefficient in the Taylor series.
For a vector field X with a Taylor expansion in privileged coordinates

$$X(x) \sim \sum_{\alpha,j} a_{\alpha,j} x^\alpha \frac{\partial}{\partial x_j},$$

the order of X is the least weighted degree of a monomial vector field having a nonzero coefficient in the Taylor series.

Remark 2. *A vector field of degree $< -r$ vanishes.*

The one-parameter family of dilations $\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by $\delta_\lambda(x) = (\lambda^{w_1}x_1, \lambda^{w_2}x_2, \ldots, \lambda^{w_n}x_n)$, $\lambda \geq 0$. A dilation δ_λ acts also on functions and vector fields by pull-back: $\delta_\lambda^* f = f \circ \delta_\lambda$ and $\delta_\lambda^* X$ is the vector field such that $(\delta_\lambda^* X)(\delta_\lambda^* f) = \delta_\lambda^*(Xf)$. So we have the following definition.

Definition 7. *A function f is homogeneous of degree s if $\delta_\lambda^* f = \lambda^s f$. A vector field X is homogeneous of degree s if $\delta_\lambda^* X = \lambda^s X$.*

Proposition 2 ([7], Proposition 5.16)**.** *Let X and Y be vector fields on M. If X and Y are homogeneous of degree k and l respectively (in the chosen system of privileged coordinates) then $[X, Y]$ is homogeneous of degree $k + l$ or vanishes.*

Definition 8. *The function defined by $x \mapsto ||x||_p = \sum_{i=1}^{n} |x_i|^{\frac{1}{w_i}}$ is the so-called pseudo-norm at p.*

Remark 3. *Let $x = (x_1, \ldots, x_n)$ be a system of privileged coordinates defined on an open neighborhood U of the point p. When composed with the coordinate functions, the pseudo-norm at p is (non smooth) homogeneous of order 1, that is, $||x(q)||_p = O(d(p, q))$, where $x(q)$ are the coordinates of $q \in U$.*

2.4 Nilpotent approximation

Fix a system of privileged coordinates $(x_1, \ldots, x_n)$ at p. Every vector field X_i is of order ≥ -1, hence it has, in x coordinates, a Taylor expansion

$$X_i(x) \sim \sum_{\alpha,j} a_{\alpha,j} x^\alpha \frac{\partial}{\partial x_j},$$

where $w(\alpha) \geq w_j - 1$ if $a_{\alpha,j} \neq 0$. Grouping together the monomial vector fields of same weighted degree we express X_i as a series of homogeneous vector fields of the form

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + X_i^{(2)} + \cdots, \tag{2}$$

where $X_i^{(s)}$ has degree s. We set

$$\widehat{X}_i = X_i^{(-1)}, \quad i = 1, \ldots, n.$$

Definition 9. *The family of vector fields $\left(\widehat{X}_1, \ldots, \widehat{X}_n\right)$ is called the nilpotent approximation of the system $(X_1, \ldots, X_n)$ at p.*

Proposition 3 ([7], Proposition 5.17)**.** *The vector fields $\widehat{X}_i$, $i = 1, \ldots, n$, generate a nilpotent Lie algebra Lie $\left(\widehat{X}_1, \ldots, \widehat{X}_n\right)$ of step $r = w_n$. They satisfy Larc at every point $y \in \mathbb{R}^n$, and the distance $\widehat{d}$ is finite for every $x, y \in \mathbb{R}^n$.*

The following results are found in [7] and [15] and will be useful in this work.

Proposition 4 ([7] Proposition 7.25 and [15], Lemma 2.1)**.** *There exist positive constants C, C' such that for all $q \in \mathbb{R}^n$ we have*

$$C||q||_p \leq \widehat{d}_p(p,q) \leq C'||q||_p.$$

Lemma 1 ([15], Lemma 2.2)**.** *There exists constant C and $\varepsilon > 0$, such that, for any $z^0 \in \mathbb{R}^n$ and any $t \in \mathbb{R}^+$ satisfying $\tau = \max(||z^0||_p, t) < \varepsilon$, we have*

$$||z(t) - \widehat{z}(t)||_p \leq C\tau t^{\frac{1}{r}},$$

where $z(\cdot)$ and $\widehat{z}(\cdot)$ are trajectories of the nonholonomic systems defined respectively by $X_1, \ldots, X_n$ and $\widehat{X}_1, \ldots, \widehat{X}_n$ starting at the same point z^0, associated with the same control function $u(\cdot)$, and satisfying $||u(t)|| = 1$ a.e.

To finish, we recall the very important Theorem 7.32 of [7] stated here with a slight modification.

Theorem 1 (Theorem 7.32 in [7])**.** *There exist constants $\varepsilon > 0$ and $C > 0$ such that for any $q, q' \in B(p, \varepsilon)$, we have*

$$-C\tau d(q,q')^{\frac{1}{r}} \leq d(q,q') - \widehat{d}(q,q') \leq C\widehat{\tau}\widehat{d}(q,q')^{\frac{1}{r}},$$

where τ is as in Lemma 1 and $\widehat{\tau}$ is similarly defined, this is $\tau = \max(||q||_p, d(q,q'))$ and $\widehat{\tau} = \max\left(||q||_p, \widehat{d}(q,q')\right)$.

2.5 Linear vector fields

The definition of linear vector fields comes from [6] and [16].

Let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra (the set of left-invariant vector fields, identified with the tangent space at the identity). The set of analytic vector fields on G is denoted by $V^\omega(G)$, and the normalizer of $\mathfrak{g}$ in $V^\omega(G)$ is by definition

$$\mathcal{N} = \mathrm{norm}_{V^\omega(G)}\mathfrak{g} = \{X \in V^\omega(G) : \forall\, Y \in \mathfrak{g} \quad [X,Y] \in \mathfrak{g}\}.$$

Definition 10. *A vector field $\mathcal{X}$ on G is said to be linear or to be infinitesimal automorphism (see [13]), if $\mathcal{X}$ belongs to $\mathcal{N}$ and $\mathcal{X}(e) = 0$, where e is the identity of G.*

We can see in [16] that a vector field $\mathcal{X}$ on G if and only its flow $(\phi_t)_{t\in\mathbb{R}}$ is a one-parameter group of automorphisms of G and a linear vector field is consequently analytic and complete.

2.5.1 Simple ARS's on Lie groups

Linear and invariant vector fields make it possible to define almost-Riemannian structures on Lie groups. The following definition is given in [5].

Definition 11. *A simple ARS is an ARS defined on a connected Lie group G by a set of n vector fields $\{\mathcal{X}, Y_1, \ldots, Y_{n-1}\}$ where $\mathcal{X}$ is linear, $Y_1, \ldots, Y_{n-1}$ are left-invariant,* $\dim G = n$ *and the rank of $\mathcal{X}, Y_1, \ldots, Y_{n-1}$ is full on a non empty subset of G and the set $\{\mathcal{X}, Y_1, \ldots, Y_{n-1}\}$ satisfies Larc.*

For instance, the famous Grushin plane on the Abelian Lie group $\mathbb{R}^2$ is a simple ARS. This structure was introduced in [5] and its isometries have been study in [17].

In Section 3.3 a 3-dimensional example will be provided.

2.5.2 Simple ARS's on homogeneous spaces

Consider a homogeneous space G/H of a connected and simply connected Lie group G by a closed subgroup H (the elements of G/H are right cosets of H because we deal with left-invariant vector fields). Since we are interested in simply connected quotients we assume H to be connected. Let $\mathfrak{g}$ be the Lie algebra of G, identified with the space of left-invariant vector fields. The projection of such a vector field Y on G/H is well-defined, is referred to as a left-invariant vector field, and we can assume that it vanishes identically only if Y is the zero field (see details in [16]). On the other hand the projection of a linear field $\mathcal{X}$ of G does exist on G/H if and only if H is invariant under its flow, or equivalently, because H is connected, if the Lie algebra of H is $\mathrm{ad}(\mathcal{X})$-invariant. This allows to define linear vector fields and simple ARS on G/H:

Let $Y_1, \ldots, Y_{n-1}, \mathcal{X}$ be a set of $n = \dim(G/H)$ vectors fields on G/H, where $Y_1, \ldots, Y_{n-1}$ are invariant and $\mathcal{X}$ is linear. It defines a simple ARS if

1. *They satisfy Larc.*
2. *The singular set $\mathcal{Z}$ where their rank is less than n is proper with empty interior.*

In the sequel, we will need (a simplified version of) the equivalence Theorem (see [16] and [5]).

Theorem 2 (Equivalence Theorem)**.** *Let $f_1, \ldots, f_n$ be a set of n complete vector fields on $\mathbb{R}^n$ and let us assume:*

1. *$f_1, \ldots, f_n$ define an Almost-Riemannian Structure on $\mathbb{R}^n$;*
2. *The Lie algebra $\mathcal{L}$ generated by $f_1, \ldots, f_n$ is finite dimensional;*
3. *The ideal $\mathfrak{g}$ generated in $\mathcal{L}$ by $f_1, \ldots, f_{n-1}$ is nilpotent and of codimension 1 in $\mathcal{L}$.*

Then $\mathbb{R}^n$ is diffeomorphic to a homogeneous space G/H of the nilpotent simply connected group G generated by $\mathfrak{g}$ and $f_1, \ldots, f_n$ defines a simple ARS on G/H. More accurately the vector fields $f_1, \ldots, f_{n-1}$ are left-invariant and f_n is linear on this homogeneous space.

3 Solvable Approximation

In this section we introduce the solvable approximation of an ARS and we analyze its algebraic structure.

3.1 Definition

Let $\{X_1, \ldots, X_n\}$ be a set of vector fields defining an almost-Riemannian structure on an open neighborhood of $0 \in \mathbb{R}^n$. The point $p = 0$ is assumed to belong to the singular locus, the natural coordinates of $\mathbb{R}^n$ to be privileged and we consider the nilpotent approximation $\left\{\widehat{X}_1, \ldots, \widehat{X}_n\right\}$ of $\{X_1, \ldots, X_n\}$ at $p = 0$.

It may happen that some of the vector fields $\widehat{X}_i$ vanish, possibly changing the almost-Riemannian structure defined by $X_1, \ldots, X_n$ into a constant rank sub-Riemannian one. It is what happens in some cases of generic 3-dimensional ARSs that are described in detail in Section 3.3. In what follows we are interested in the case where only one of the $\widehat{X}_i$'s vanishes, say $\widehat{X}_n = 0$, and the other ones are independent and define a left-invariant sub-Riemannian structure on a Lie group, or a homogeneous space, the underlying manifold of which is $\mathbb{R}^n$. Recall that each X_i can be expanded in a series of homogeneous vector fields in the system of privileged coordinates at $p = 0$, this is

$$X_i = X_i^{(-1)} + X_i^{(0)} + X_i^{(1)} + \ldots, \qquad \forall\, i \in \{1, \ldots, n\},$$

where $X_i^{(k)}$ is the homogeneous component of degree k. Denoting $\widetilde{X}_n = X_n^{(0)}$, we introduce the following definition:

Definition 12 (Solvable approximation)**.** *The family* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *is the solvable approximation of* $\{X_1, \ldots, X_n\}$.

Proposition 5. $\mathcal{L} = Lie\left(\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right)$ *is a finite dimensional solvable Lie algebra. Its step of solvability is less than or equal to* $\log_2(r) + 1$*, where* r *is the degree of nonholonomy at* $p = 0$.

Proof. Let $\mathcal{D}^k\mathcal{L}$ stand for the k^{th} derived algebra of $\mathcal{L}$, with $\mathcal{L} = \mathcal{D}^0\mathcal{L}$. According to Proposition 2 and Remark 2 the algebra $\mathcal{L}$ is generated by homogeneous vector fields of degree $0, -1, \ldots, -r$ because the $\widehat{X}_i$'s are homogeneous of degree -1, for $i = 1, \ldots, n-1$, and $\widetilde{X}_n$ is homogeneous of degree 0. According to Proposition 2 again $\mathcal{D}^1\mathcal{L}$ is generated by homogeneous vector fields of degree $-1, \ldots, -r$. More generality $\mathcal{D}^s\mathcal{L}$ is generated by homogeneous vector fields of degree $-2^{s-1}, -2^s, \ldots, -r$, so that $\mathcal{D}^s\mathcal{L} = 0$ if $2^s > r$. Therefore $\mathcal{L}$ is solvable and the step of solvability σ of $\mathcal{L}$ satisfies $\sigma \leq \log_2(r) + 1$. On the other hand, the Lie algebra $\mathcal{L}$ splits into homogeneous components

$$\mathcal{L} = \mathcal{L}^0 \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \ldots \oplus \mathcal{L}^{-r},$$

where $\mathcal{L}^{-s}$ is the set of homogeneous vector fields of degree $-s$ under the action of δ_λ. A homogeneous vector field X of degree $w \in \{0, -1, -2, \ldots, -r\}$ writes in coordinates $X(x) = \sum_{i=1}^{n} f_i(x)\dfrac{\partial}{\partial x_i}$, where $f_i(x)$ is a homogeneous polynomial function of degree $w + \mathrm{ord}(x_i)$. Since the space of polynomials of degree $w + \mathrm{ord}(x_i)$ is finite dimensional, $\mathcal{L}^{-w}$ is finite dimensional. Therefore $\mathcal{L}$ is finite dimensional. □

Remark 4. *It is clear that the families of vector fields* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *and* $\{X_1, \ldots, X_n\}$ *have the same nilpotent approximation. Consequently the family of vector fields* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *satisfies Larc on* $\mathbb{R}^n$ *and the growth vector at* 0 *of* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *is equal to the one of* $\{X_1, \ldots, X_n\}$.

3.2 Structure of the approximating system

Fundamental remark. *Despite the previous result we are not interested in the solvable Lie group associated to the Lie algebra Lie* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *but in some nilpotent Lie group on which* $\widetilde{X}_n$ *acts as a linear vector field.*

For this reason we denote by $\mathfrak{h}$ the Lie algebra generated by $\widehat{X}_1, \ldots, \widehat{X}_{n-1}$ and by $\mathfrak{g}$ the ideal generated by $\mathfrak{h}$ in $\mathcal{L} = \text{Lie}\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$.

Proposition 6. *The ideal* $\mathfrak{g}$ *is the space of vector fields of* $\mathcal{L}$ *whose nonholonomic order is negative. It is a nilpotent Lie algebra and*

$$\mathcal{L} = \mathfrak{g} \oplus \mathbb{R}\widetilde{X}_n.$$

Moreover $D = -ad(\widetilde{X}_n)$ *is a derivation of* $\mathfrak{g}$.

Proof. Since $\mathfrak{g}$ is the ideal generated by $\mathfrak{h}$ in $\mathcal{L}$, we have $\mathcal{L} = \mathbb{R}\widetilde{X}_n \oplus \mathfrak{g}$ and $\text{ad}\left(\widetilde{X}_n\right)$ is a derivation of $\mathfrak{g}$. □

Let G be the simply connected Lie group whose Lie algebra is isomorphic to $\mathfrak{g}$. According to [16] there exists a linear vector field on G associated to the derivation $D = -\text{ad}\left(\widetilde{X}_n\right)$. With a clear abuse of notation we will denote it by $\widetilde{X}_n$. Thanks to the equivalence theorem we have the following:

Theorem 3. *The space* $\mathbb{R}^n$ *is diffeomorphic to a homogeneous space* G/G_0 *of* G. *Through this diffeomorphism* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ *is equivalent to a simple ARS on* G/G_0, *and the Lie algebra* $\mathfrak{g}_0$ *of* G_0 *is isomorphic to the set of vector fields of* $\mathfrak{g}$ *that vanish at* 0.

Proof. First of all, notice that the vector fields $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ are complete from their triangular form, namely, in the equation $\dot{x} = \sum_{i=1}^{n} u_i \widehat{X}_i(x) + u_n \widetilde{X}_n(x)$, $\dot{x}_j$ is linear with respect to the coordinates of weight w_j and polynomial with respect to coordinates of weight $< w_j$ (see [7] or [15] for details). They define an ARS, hence in particular satisfy Larc and generate a finite dimensional Lie algebra (Proposition 5). According to Theorem 2 $\mathbb{R}^n$ is diffeomorphic to a homogeneous space G/G_0 of G, where G_0 is the connected subgroup of G whose Lie algebra is, after identification of $L(G)$ with $\mathfrak{g}$, the set of elements of $\mathfrak{g}$ that vanish at 0. Thanks to the diffeomorphism between $\mathbb{R}^n$ and G/G_0 the system $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ can be identified to a simple ARS on G/G_0. □

We are also interested in conditions for which $G = \mathbb{R}^n$.

Theorem 4. *With the previous notations the following assertions are equivalent:*

(i) $ad\left(\widetilde{X}_n\right).\widehat{X}_i$ *belongs to Span* $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}\right\}$ *for* $i = 1, \ldots, n-1$*;*

(ii) $\mathfrak{h}$ *is* $ad\left(\widetilde{X}_n\right)$*-invariant;*

(iii) $\mathfrak{h} = \mathfrak{g}$.

Under these conditions $\widetilde{X}_n$ *is a linear vector field on* $\exp(\mathfrak{h})$.

Proof.

(i) $\Rightarrow$(ii) It is an immdiate consequence of the Jacobi identity.

(ii)$\Rightarrow$(iii) Condition (ii) implies $\mathcal{L} = \mathfrak{h} \oplus \mathbb{R}\widetilde{X}_n$, hence $\mathfrak{h} = \mathfrak{g}$.

(iii)$\Rightarrow$(i) Condition (iii) implies that $\mathfrak{h}$ is ad $\left(\widetilde{X}_n\right)$-invariant. According to Proposition 2 the set of elements of $\mathfrak{h}$ of order -1 is Span $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}\right\}$. For $i = 1, \ldots, n-1$ the bracket $\left[\widehat{X}_i, \widetilde{X}_n\right]$ belongs to $\mathfrak{h}$, since $\mathfrak{h} = \mathfrak{g}$, and is of order -1 or is equal to 0. Therefore it belongs to Span $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}\right\}$.

□

3.3 Example. The 3D-generic case

The local representation of a generic ARS in dimension 3 is detailed in Section 4.2. Its nilpotent approximation at a point of the singular locus is the following:

$$\widehat{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} 0 \\ 1 \\ x\cos\sigma \end{pmatrix}, \quad \widehat{X}_3 = \begin{pmatrix} 0 \\ 0 \\ x\sin\sigma \end{pmatrix}, \quad \text{where } \sigma \in \left[0, \frac{\pi}{2}\right].$$

There are two particular cases, according to the value of the parameter σ. The first one for $\sigma = \frac{\pi}{2}$, because the bracket of $\widehat{X}_1$ and $\widehat{X}_2$ vanishes, and the second one for $\sigma = 0$, because $\widehat{X}_3$ vanishes, it is the tangent case.

More accurately the Lie brackets are:

$$\left[\widehat{X}_1, \widehat{X}_2\right] = \begin{pmatrix} 0 \\ 0 \\ \cos\sigma \end{pmatrix}, \quad \left[\widehat{X}_1, \widehat{X}_3\right] = \begin{pmatrix} 0 \\ 0 \\ \sin\sigma \end{pmatrix}, \quad \left[\widehat{X}_2, \widehat{X}_3\right] = 0.$$

The analysis of the different cases is as follows:

1. **General case** $\sigma \in \left]0, \frac{\pi}{2}\right[$.

   We can set $Z = \begin{pmatrix} 0 \\ 0 \\ \cos\sigma \end{pmatrix}$ so that $\left[\widehat{X}_1, \widehat{X}_2\right] = Z$ and $(\widehat{X}_1, \widehat{X}_2, Z)$ is the Heisenberg Lie algebra. Since $\widehat{X}_3$ vanishes at $(0,0,0)$ and since its Lie brackets with $\widehat{X}_1$, $\widehat{X}_2$ and Z are

   $$\left[\widehat{X}_1, \widehat{X}_3\right] = \tan(\sigma) Z \quad \text{and} \quad \left[\widehat{X}_2, \widehat{X}_3\right] = \left[Z, \widehat{X}_3\right] = 0,$$

   it is a linear vector field on the Heisenberg group, the associated derivation of which is

   $$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tan\sigma & 0 & 0 \end{pmatrix}.$$

   The conclusion is that $\left(\widehat{X}_1, \widehat{X}_2, \widehat{X}_3\right)$ defines a simple ARS on the 3D Heisenberg group.

2. **Particular case** $\sigma = \frac{\pi}{2}$. Then

   $$\widehat{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \widehat{X}_3 = \begin{pmatrix} 0 \\ 0 \\ x \end{pmatrix}.$$

   It is a simple ARS on the Abelian Lie group $\mathbb{R}^3$. Indeed $\widehat{X}_1$ and $\widehat{X}_2$ are (left and right) invariant and $\widehat{X}_3$ is linear.

3. **Tangent case** $\sigma = 0$. Here the nilpotent approximation degenerates into the following sub-Riemannian structure on the Heisenberg group

   $$\widehat{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \quad \widehat{X}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

   However we will see after the next remark that in case where the component $\widetilde{X}_3$ of order 0 of the vector field X_3 does not vanish then $\left(\widehat{X}_1, \widehat{X}_2, \widetilde{X}_3\right)$ defines a simple ARS on a homogeneous space.

Remark 5. *In the cases $\sigma = \frac{\pi}{2}$ and $\sigma \in \left]0, \frac{\pi}{2}\right[$ the Lie algebra generated by $\widehat{X}_1, \widehat{X}_2, \widehat{X}_3$ is 4-dimensional and two points of view are possible. The usual one consists in considering $\mathbb{R}^3$ as a homogeneous space of a nilpotent 4-dimensional Lie group. Our point of view is to consider $\widehat{X}_3$ as a linear vector field on the 3-dimensional Lie group $\mathbb{R}^3$ endowed with the Abelian structure if $\sigma = \frac{\pi}{2}$ and the Heisenberg one if $\sigma \in \left]0, \frac{\pi}{2}\right[$.*

Following this way we will consider the solvable approximation whenever $\sigma = 0$, and finally all approximations of generic 3D-ARS will appear as being simple ARS.

The solvable approximation of the tangent case $\sigma = 0$.

The homogeneous component of nonholonomic order 0 of X_3 is

$$\widetilde{X}_3 = \begin{pmatrix} 0 \\ 0 \\ az + bx^2 + cy^2 \end{pmatrix} = \left(az + bx^2 + cy^2\right) \frac{\partial}{\partial z} \quad \text{(See Section 4.2 again).}$$

As well as in the general case the Lie algebra generated by $\widehat{X}_1$ and $\widehat{X}_2$ is:

$$\mathfrak{h} = \text{Span}\left\{\widehat{X}_1, \widehat{X}_2, Z = \left[\widehat{X}_1, \widehat{X}_2\right]\right\} \quad \text{where} \quad Z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{\partial}{\partial z},$$

that is the Heisenberg algebra. On the other hand the algebra generated by $\widehat{X}_1$, $\widehat{X}_2$ and $\widetilde{X}_3$ is $\text{Span}\left\{\widehat{X}_1, \widehat{X}_2, Z, \left[\widehat{X}_1, \widetilde{X}_3\right], \left[\widehat{X}_2, \widetilde{X}_3\right], \widetilde{X}_3\right\}$, where:

$$\left[\widehat{X}_1, \widetilde{X}_3\right] = \begin{pmatrix} 0 \\ 0 \\ 2bx \end{pmatrix} = 2bx\frac{\partial}{\partial z} \quad \text{and} \quad \left[\widehat{X}_2, \widetilde{X}_3\right] = \begin{pmatrix} 0 \\ 0 \\ 2cy + ax \end{pmatrix} = (2cy + ax)\frac{\partial}{\partial z},$$

and the ideal generated by $\widehat{X}_1$ and $\widehat{X}_2$ is:

$$\mathfrak{g} = \text{Span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, 2bx\frac{\partial}{\partial z}, 2cy\frac{\partial}{\partial z} + ax\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\}.$$

A straightforward computation shows that $\widetilde{X}_3$ acts as a derivation on $\mathfrak{g}$. If we assume $b \neq 0$ and $c \neq 0$ then we have also:

$$\mathfrak{g} = \text{Span}\left\{\frac{\partial}{\partial x}, x\frac{\partial}{\partial z}, \frac{\partial}{\partial y}, y\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right\}.$$

This is the 5-dimensional Heisenberg Lie algebra $\mathfrak{h}^2$ and in this basis the derivation $D = -\text{ad}(\widetilde{X}_3)$ is given by the following matrix:

$$D = \begin{pmatrix} 0 & 0 & & & \\ 2b & a & & & \\ & & 0 & 0 & \\ & & 2c & a & \\ & & & & a \end{pmatrix}.$$

Finally the solvable approximation $\left(\widehat{X}_1, \widehat{X}_2, \widetilde{X}_3\right)$ is a simple ARS on $\mathbb{R}^3$ diffeomorphic to a quotient of the 5-dimensional group Heisenberg $\mathbb{H}^2$.

4 Distances

We can distinguish three different families of vector fields from the above section: $\{X_1, X_2, \ldots, X_n\}$, $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}\right\}$ and $\left\{\widehat{X}_1, \ldots, \widehat{X}_{n-1}, \widetilde{X}_n\right\}$ which satisfy Larc. Assuming orthonormality, they define three different distances: d, $\widehat{d}$ and $\widetilde{d}$ respectively,

where $\widetilde{d}$ and $\widehat{d}$ are defined on $\mathbb{R}^n$. This section is divided in two parts. In the first one, we give two propositions about the almost-Riemannian distance $\widetilde{d}$ defined by the solvable approximation. The second part is devoted to analyze $\widetilde{d}$ in the 3-dimensional generic case.

4.1 The almost-Riemannian distance $\widetilde{d}$

The following proposition establishes a relation between $\widetilde{d}$ and $\widehat{d}$. It is important because it allows us to find an upper bound for $\widetilde{d}$ (see Section 4.2.2) and to compare the distances of the solvable and nilpotent approximation.

Proposition 7. *For all $x, y \in \mathbb{R}^n$, $\widetilde{d}(x,y) \leq \widehat{d}(x,y)$.*

Proof. Let $x, y \in \mathbb{R}^n$ and let γ be a minimizing geodesic for $\dot{x} = \sum_{i=1}^{n-1} u_i \widehat{X}_i$, such that $\gamma(0) = x$, $\gamma(T) = y$. Setting $u_n = 0$ the curve γ is admissible for $\dot{x} = \sum_{i=1}^{n-1} u_i \widehat{X}_i + u_n \widetilde{X}_n$. Since $u_n = 0$ the length of γ is the same for both metrics, hence $\widetilde{d}(x,y) \leq l(\gamma) = \widehat{d}(x,y)$. □

Let δ_λ be the dilation related to the privileged coordinates and the weights at $p = 0$. We know that the distance $\widehat{d}$ is homogeneous of degree 1 with respect to δ_λ (see [15]). However $\widetilde{d}$ does not possess this property. This is due to the fact that $\widetilde{X}_n$ and the $\widehat{X}_i$'s do not have the same degree of homogeneity.

Proposition 8. *The almost-Riemannian distance $\widetilde{d}$ is not homogeneous.*

Proof. Let γ be an admissible curve for $\widetilde{d}$, that is

$$\dot{\gamma}(t) = \sum_{i=1}^{n-1} u_i \widehat{X}_i(\gamma(t)) + u_n \widetilde{X}_n(\gamma(t)). \tag{3}$$

Since $\widehat{X}_i$ and $\widetilde{X}_n$ are homogeneous of degree -1 and 0 respectively and the pullback by δ_λ of a vector field X (see [1]) is defined by

$$d\delta_\lambda(q)\,(\delta_\lambda^* X(q)) = X\big(\delta_\lambda(q)\big), \tag{4}$$

we get

$$\begin{aligned} d\delta_\lambda(q).\widehat{X}_i(q) &= d\delta_\lambda(q).\lambda \cdot \delta_\lambda^* \widehat{X}_i(q) = \lambda \widehat{X}_i\big(\delta_\lambda(q)\big) \quad \text{and} \\ d\delta_\lambda(q).\widetilde{X}_n(q) &= d\delta_\lambda(q).\delta_\lambda^* \widetilde{X}_n(q) = \widetilde{X}_n\big(\delta_\lambda(q)\big). \end{aligned} \tag{5}$$

Therefore

$$\begin{aligned} \frac{d}{dt}\left(\delta_\lambda \circ \gamma\right)(t) &= d\delta_\lambda\left(\gamma(t)\right).\dot{\gamma}(t) \\ &= \sum_{i=1}^{n-1} u_i d\delta_\lambda\big(\gamma(t)\big).\widehat{X}_i\big(\gamma(t)\big) + u_n d\delta_\lambda\big(\gamma(t)\big).\widetilde{X}_n\big(\gamma(t)\big) \\ &= \sum_{i=1}^{n-1} \lambda u_i \widehat{X}_i\Big(\delta_\lambda\big(\gamma(t)\big)\Big) + u_n \widetilde{X}_n\Big(\delta_\lambda\big(\gamma(t)\big)\Big). \end{aligned}$$

This implies that $l(\delta_\lambda\gamma) \neq \lambda l(\gamma)$, except if $u_n(t)$ vanishes a.e. This proves the non homogeneity of $\widetilde{d}$. □

4.2 The 3D-tangential case

In Section 3, we have established a model to locally approximate an n-ARS whose nilpotent approximation is a constant rank sub-Riemannian structure, by a solvable approximation. In this context we want to determine conditions for $|d - \widetilde{d}|$ to be smaller than $|d - \widehat{d}|$.

Recall that $\Delta(p) = \text{span}\{X_1(p), \ldots, X_n(p)\}$ and the singular locus $\mathcal{Z}$ is the set of points of $\mathbb{R}^n$ where the rank of the linear span of the vector fields is less than n. From [10] we take the following.

Proposition 9. *Consider a 3-ARS. The following conditions are generic for 3-ARSs*

(G1) $\dim(\Delta(p)) \geq 2$ *and* $\Delta(p) + [\Delta(p), \Delta(p)] = T_pM$ *for every* $p \in M$*;*

(G2) $\mathcal{Z}$ *is an embedded (possibly empty) two-dimensional submanifold of* M*;*

(G3) the points where $\Delta(p) = T_p\mathcal{Z}$ *are isolated.*

Proposition 10. *Under the previous conditions there are three types of points:*

1. *Riemannian points where* $\Delta(p) = T_pM$*.*
2. *type-1 points where* $\Delta(p)$ *has dimension 2 and is transversal to* $\mathcal{Z}$*.*
3. *type-2 points where* $\Delta(p)$ *has dimension 2 and is tangent to* $\mathcal{Z}$*.*

Moreover type-2 points are isolated, type-1 points form a 2 dimensional manifold and all other points are Riemannian.

The local representation of the 3-dimensional ARS at type-2 points is given by the vector fields

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 \\ 1 + \delta(x,y,z) \\ x\,(1 + \theta(x,y,z)) \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ az + bx^2 + cy^2 + o\,(x^2 + y^2 + |z|) \end{pmatrix},$$

where δ and θ are smooth functions of order greater than or equal to 1 and a, b, c are not all zero. Furthermore, from Subsection 3.3, the nilpotent approximation in privileged coordinates is

$$\widehat{X}_1 = X_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \widehat{X}_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix}, \quad \widehat{X}_3 = 0.$$

and

$$\widetilde{X}_3 = \begin{pmatrix} 0 \\ 0 \\ az + bx^2 + cy^2 \end{pmatrix}.$$

$\left(\widehat{X}_1, \widehat{X}_2, \widetilde{X}_3\right)$ is the solvable approximation at $p = 0$ in case when 0 is a tangential (type-2) point.

4.2.1 Divergence of curves

Let $p = 0$ be a type-2 point such that the coordinates centered at p are privileged and q, q' belonging to the ball centered at p and radius ϵ, denoted by $B(p, \epsilon)$.

In this subsection we analyze the divergence of curves respectively admissible for d and $\widetilde{d}$, defined by the same control functions and starting at the same point q. More accurately: let γ be the geodesic for d such that $\gamma(0) = q$, $\gamma(T) = q'$ with $u_1^2 + u_2^2 + u_3^2 = 1$ and let $\widetilde{\gamma}$ be the admissible curve for $\widetilde{d}$ defined by the same control functions as γ with $\widetilde{\gamma}(0) = q$. We have the following:

$$\begin{aligned}
\dot{\gamma}(t) - \dot{\widetilde{\gamma}}(t) &= u_1 \left(X_1(\gamma) - \widehat{X}_1(\widetilde{\gamma}) \right) + u_2 \left(X_2(\gamma) - \widehat{X}_2(\widetilde{\gamma}) \right) + u_3 \left(X_3(\gamma) - \widetilde{X}_3(\widetilde{\gamma}) \right) \\
\begin{pmatrix} \dot{x}(t) - \dot{\widetilde{x}}(t) \\ \dot{y}(t) - \dot{\widetilde{y}}(t) \\ \dot{z}(t) - \dot{\widetilde{z}}(t) \end{pmatrix} &= \begin{pmatrix} 0 \\ u_2 \delta(x,y,z) \\ u_2 x \theta(x,y,z) + u_3 \left(a(z - \widetilde{z}) + b\left(x^2 - \widetilde{x}^2\right) + c\left(y^2 - \widetilde{y}^2\right) + o\left(x^2 + y^2 + |z|\right) \right) \end{pmatrix}
\end{aligned}$$

We have successively:

- $\dot{x}(t) = \dot{\widetilde{x}}(t)$, hence $x(t) = \widetilde{x}(t)$.
- $\dot{y}(t) - \dot{\widetilde{y}}(t) = u_2 \delta(x,y,z)$, hence $y(t) - \widetilde{y}(t) = \displaystyle\int_0^t u_2(s)\delta(x,y,z)ds$.

  We denote by $\rho \geq 1$ the order of δ. Then $|\delta(x,y,z)| \leq \text{Cst} \cdot ||\gamma(s)||_p^\rho \leq \text{Cst} \cdot \tau^\rho$ because $||\gamma(s)||_p \leq \text{Cst} \cdot \tau$, where $\tau = \max\left(||q||_p, t\right)$ (the proof of the above inequality is given in the proof of Lemma 1 of [15]), hence

  $$|y(t) - \widetilde{y}(t)| \leq \int_0^t \text{Cst} \cdot ||\gamma(s)||_p^\rho ds \leq \int_0^t \text{Cst} \cdot \tau^\rho \, ds = \text{Cst} \cdot \tau^\rho \cdot t. \qquad (6)$$

- $\dot{z}(t) - \dot{\widetilde{z}}(t) = u_2 x \theta(x,y,z) + u_3 \left(a(z - \widetilde{z}) + c\left(y^2 - \widetilde{y}^2\right) + o\left(x^2 + y^2 + |z|\right) \right)$, hence $z(t) - \widetilde{z}(t) = \displaystyle\int_0^t u_2(s) x\theta(x,y,z)ds + \int_0^t u_3(s) a(z - \widetilde{z})ds + \int_0^t u_3(s) c\left(y^2 - \widetilde{y}^2\right) ds +$ $\displaystyle\int_0^t u_3(s) o\left(x^2 + y^2 + |z|\right) ds$.

  Since (x, y, z) are privileged coordinates at 0, then $x^2 + y^2 + |z| \leq C \cdot d(0, (x,y,z))^2$. Moreover, if $f((x,y,z)) = o(x^2 + y^2 + |z|)$, then $f((x,y,z)) = o(d(0,(x,y,z))^2)$. This implies that $\text{ord}_p(f) > 2$, hence $f(\gamma(t)) = O\left(d(0, \gamma(t))^3\right)$. Therefore $|f(\gamma(t))| \leq \text{Cst} \cdot \tau^3$. On the other hand, let us denote by $m \geq 1$ the order of θ. Then $|x \cdot \theta(x,y,z)| \leq \text{Cst} \cdot \tau \cdot ||\gamma(s)||_p^m \leq \text{Cst} \cdot \tau^{m+1}$ because $\dot{x} = u_1$, hence $|x| \leq \text{Cst} \cdot t + ||q||_p \leq \text{Cst} \cdot \tau$. Also notice

  $$\left|y^2 - \widetilde{y}^2\right| = |y + \widetilde{y}| \, |y - \widetilde{y}| \leq \text{Cst} \cdot \tau^\rho \cdot t \cdot \tau = \text{Cst} \cdot t \cdot \tau^{\rho+1},$$

  because $\dot{y} = u_2(1 + \delta(x,y,z))$, hence $|y| \leq \text{Cst} \cdot t + ||q||_p \leq \text{Cst} \cdot \tau$. Similarly for $\widetilde{y}$.

  Then

  $$\begin{aligned}
  |z(t) - \widetilde{z}(t)| &\leq \text{Cst} \cdot t \cdot \tau^{m+1} + \text{Cst} \cdot t^2 \cdot \tau^{\rho+1} + \text{Cst} \cdot t \cdot \tau^3 + \int_0^t |a||z - \widetilde{z}|ds \\
  |z(t) - \widetilde{z}(t)| &\leq \text{Cst} \cdot \left(t \cdot \tau^{m+1} + t^2 \cdot \tau^{\rho+1} + t \cdot \tau^3 \right) e^{|a|t} \\
  |z(t) - \widetilde{z}(t)| &\leq \text{Cst} \cdot t \cdot \tau^{\min(m+1, \rho+1, 3)}. \qquad (7)
  \end{aligned}$$

Finally,

$$\begin{aligned}||\gamma(t)-\widetilde{\gamma}(t)||_p &= |x(t)-\widetilde{x}(t)|+|y(t)-\widetilde{y}(t)|+|z(t)-\widetilde{z}(t)|^{\frac{1}{2}}\\ &\leq \text{Cst}\left(t\cdot\tau^{\rho}+t^{\frac{1}{2}}\cdot\tau^{\frac{\min(m+1,\rho+1,3)}{2}}\right)\\ &\leq \text{Cst}\cdot t^{\frac{1}{2}}\cdot\tau^{\frac{\min(m+1,\rho+1,3)}{2}}. \end{aligned} \tag{8}$$

Remark 6. *The order of $\delta(x,y,z)$ does not change the inequality*

$$||\gamma(t)-\widehat{\gamma}(t)||_p \leq Cst\cdot\tau\cdot t^{\frac{1}{2}},$$

that comes from Lemma 1. Indeed,

$$||\gamma(t)-\widehat{\gamma}(t)||_p \leq Cst\left(t\cdot\tau^{\rho}+t^{\frac{1}{2}}\cdot\tau\right)\leq Cst\cdot t^{\frac{1}{2}}\cdot\tau.$$

4.2.2 Upper bounds

In order to state our main result in the next section, we need upper bounds for the distances d and $\widetilde{d}$.

We know that the distance $\widehat{d}$ is left-invariant, this is to say, $\widehat{d}(q,q')=\widehat{d}(a\cdot q,a\cdot q')$, for all $a\in\mathbb{R}^3$. Here $\cdot$ stands for the Heisenberg product defined by

$$(x,y,z)\cdot(x',y',z')=(x+x',y+y',z+z'+xy'). \tag{9}$$

Recall also that $\widetilde{d}(q,q')\leq\widehat{d}(q,q')$. Considering the above, we have

$$\widetilde{d}(q,q')\leq\widehat{d}(q,q')=\widehat{d}(0,q^{-1}q')\leq C||q^{-1}q'||_p. \tag{10}$$

Considering $q=(x,y,z)$ and $q'=(x',y',z')$ and from (10) and (9) we have

$$\begin{aligned}\widetilde{d}(q,q')\leq C||q^{-1}\cdot q'||_p &= C\left(|x'-x|+|y'-y|+|z'-z+x(y-y')|^{\frac{1}{2}}\right)\\ &\leq C\left(|x-x'|+|y-y'|+|z-z'|^{\frac{1}{2}}+|x(y-y')|^{\frac{1}{2}}\right)\\ &\leq C\left(||q-q'||_p+||q||_p^{\frac{1}{2}}|y-y'|^{\frac{1}{2}}\right). \end{aligned} \tag{11}$$

4.2.3 Upper bound for d

From Theorems 7.31 and 7.26 of [7] we get

$$d(q,q')\leq\text{Cst}\sum_{k,j|w_k\leq w_j}||q||_p^{1-\frac{w_k}{w_j}}|q_k-q'_k|^{\frac{1}{w_j}}, \tag{12}$$

Since $w_1=w_2=1$ and $w_3=2$ we obtain that

$$d(q,q')\leq\text{Cst}\left(||q-q'||_p+||q||_p^{\frac{1}{2}}\left(|y-y_2|^{\frac{1}{2}}+|x-x_2|^{\frac{1}{2}}\right)\right). \tag{13}$$

4.2.4 Main result

We have seen that the order of δ does not change the estimation of $||\gamma(t) - \widehat{\gamma}(t)||_p$ according to Lemma 1, moreover, it does not change the estimates of $\widehat{d}$ as well, this is to say, $\widehat{d}(\gamma(t), \widehat{\gamma}(t)) \leq \text{Cst} \cdot t^{\frac{1}{2}} \cdot \tau$. Indeed, from inequality (10) and Remark 6, we get

$$\widehat{d}(\gamma(t), \widehat{\gamma}(t)) \leq \text{Cst} \left(t^{\frac{1}{2}} \cdot \tau + t \cdot \tau^{\frac{1}{2}+\rho} \right),$$

then $\widehat{d}(\gamma(t), \widehat{\gamma}(t)) \leq \text{Cst} \cdot t^{\frac{1}{2}} \cdot \tau$. However, the above is not true when we want to estimate $\widetilde{d}$, because the estimates depend of ρ and m. Indeed, from inequalities (8) and (11) we get

$$\widetilde{d}(\gamma(t), \widetilde{\gamma}(t)) \leq \text{Cst} \cdot \left(t^{\frac{1}{2}} \cdot \tau^{\frac{\min(m+1,\rho+1,3)}{2}} + t^{\frac{1}{2}} \cdot t^{\frac{1}{2}} \cdot \tau^{\frac{\rho}{2}} \right) \leq \text{Cst} \cdot t^{\frac{1}{2}} \cdot \tau^{\frac{\min(m+1,\rho+1,3)}{2}}.$$

Therefore, we can conclude that if $\rho \geq 2$ and $m \geq 2$ then

$$\widetilde{d}(\gamma(t), \widetilde{\gamma}(t)) \leq \text{Cst} \cdot t^{\frac{1}{2}} \cdot \tau^{\frac{3}{2}}.$$

Finally, we obtain:

Proposition 11. *If $ord_p(\delta) \geq 2$ and $ord_p(\theta) \geq 2$, then*

1. $\widetilde{d}(\gamma(t), \widetilde{\gamma}(t)) \leq Cst \cdot t^{\frac{1}{2}} \cdot \tau^{\frac{3}{2}}$.
2. $d(\gamma(t), \widetilde{\gamma}(t)) \leq Cst \cdot t^{\frac{1}{2}} \cdot \tau^{\frac{3}{2}}$.

Proof. The proof of item 2 follows from inequalities (8) and (13). □

Theorem 5. *If $m \geq 2$ and $\rho \geq 2$, then there exists constants C and $\epsilon > 0$, such that, for all $q, q' \in B(p, \epsilon)$, we have*

$$-C\tau^{\frac{3}{2}} d(q, q')^{\frac{1}{2}} \leq d(q, q') - \widetilde{d}(q, q') \leq C \cdot \widetilde{\tau}^{\frac{3}{2}} \widetilde{d}(q, q')^{\frac{1}{2}}, \tag{14}$$

where

$$\tau = \max\left(||q||_p, d(q, q')\right)$$
$$\widetilde{\tau} = \max\left(||q||_p, \widetilde{d}(q, q')\right).$$

Proof. Let q belonging to $B(p, \epsilon)$. Let us consider the geodesics $\gamma : [0, T] \to M$ for the distance d such that $\gamma(0) = q$, $\gamma(T) = q'$ and associated with the control function $u(\cdot)$ satisfying $||u(t)|| = 1$ and $\widetilde{\gamma}$ the admissible curve for $\widetilde{d}$ defined by the same control functions that γ with $\widetilde{\gamma}(0) = q$. By Proposition 11 item 1

$$\widetilde{d}\left(\gamma(T), \widetilde{\gamma}(T)\right) \leq \text{Cst} \cdot T^{\frac{1}{2}} \cdot \tau^{\frac{3}{2}}. \tag{15}$$

On the other hand, note that

$$d(q, q') = l(\gamma) = l(\widetilde{\gamma}) \geq \widetilde{d}\left(q, \widetilde{\gamma}(T)\right).$$

Moreover, by triangle inequality, we have

$$\widetilde{d}\,(q,\widetilde{\gamma}(T)) \geq \widetilde{d}\,(q,q') - \widetilde{d}\,(q',\widetilde{\gamma}(T))\,,$$

Then, from (15), transitivity and since $\gamma(T) = q'$, we get

$$\begin{aligned} d(q,q') &\geq \widetilde{d}\,(q,q') - \mathrm{Cst}\cdot T^{\frac{1}{2}}\cdot\tau^{\frac{3}{2}} \\ d(q,q') - \widetilde{d}\,(q,q') &\geq -\mathrm{Cst}\cdot T^{\frac{1}{2}}\cdot\tau^{\frac{3}{2}} \end{aligned} \tag{16}$$

Now, we change the roles of d and $\widetilde{d}$ and by Proposition 11 item 2, we obtain

$$d(q,q') - \widetilde{d}\,(q,q') \leq \mathrm{Cst}\cdot\widetilde{T}^{\frac{1}{2}}\cdot\widetilde{\tau}^{\frac{3}{2}}, \tag{17}$$

where $\widetilde{T} = \widetilde{d}(q,q')$.

Therefore from (16) and (17)

$$-C\tau^{\frac{3}{2}}d(q,q')^{\frac{1}{2}} \leq d(q,q') - \widetilde{d}(q,q') \leq C\cdot\widetilde{\tau}^{\frac{3}{2}}\widetilde{d}(q,q')^{\frac{1}{2}}.$$

The proof is complete. □

4.2.5 Translations

We mentioned in Section 4.2.2 that the distance $\widehat{d}$ is left-invariant. It is not the case of $\widetilde{d}$. Let q be a point in a neighborhood of 0 and $g \in \mathbb{R}^3$. We are interested in conditions under which $\widetilde{d}(g, g\cdot q) \leq \widetilde{d}(0,q)$ (the product is the Heisenberg one).

Let $\gamma(t) = \big(x(t), y(t), z(t)\big)$ be a geodesic of $\widetilde{d}$ such that $\gamma(0) = 0$ with control functions u_1, u_2 and u_3. We consider $g = (g_1, g_2, g_3) \in \mathbb{R}^3$. Let $\gamma_g(t) = L_g\,(\gamma(t)) = (x_g(t), y_g(t), z_g(t))$ and $\overline{u}_1, \overline{u}_2, \overline{u}_3$ its control functions. Note that γ_g is admissible for $\widetilde{d}$ as long as it does not meet $\mathcal{Z}$. Indeed, all absolutely continuous curves are admissible out of the singular locus since the metric is Riemannian. The goal is to find conditions for g such that γ_g has a length less than γ. Since $\mathrm{Lie}\Big\{\widehat{X}_1, \widehat{X}_2\Big\}$ is the Heisenberg algebra, then

$$L_g\,(\gamma(t)) = (x(t) + g_1,\ y(t) + g_2,\ z(t) + g_1 y(t) + g_3)\,.$$

We set $h(x,y,z) = az + bx^2 + cy^2$. Then

$$\begin{aligned} h(\gamma_g) &= a(z + g_1 y + g_3) + b(x + g_1)^2 + c(y + g_2)^2 \\ &= h(\gamma) + h(g) + (2bx + ay)g_1 + 2cyg_2 = h(\gamma) + h(g) + f(g,\gamma), \end{aligned}$$

where $f(g,\gamma) = (2bx + ay)g_1 + 2cyg_2$.

We assume that $h(\gamma_g)$ does not vanish, this is to say γ_g is not on $\mathcal{Z}$. In particular for $t = 0$, $h(\gamma_g) = h(g)$ then $h(g) \neq 0$ this is equivalent to $g \notin \mathcal{Z}$.

We have the following result.

Theorem 6. *Let $\gamma : [0,T] \longrightarrow \mathbb{R}^n$ be a length minimizer of $\widetilde{d}$ with control functions $u_1(t)$, $u_2(t)$, $u_3(t)$ with $u_3(t) \neq 0$ a.e, and $h\,(\gamma_g) \neq 0$. If $|h(\gamma)| \leq |h(\gamma_g)|$ then $\widetilde{d}\,(\gamma_g(0), \gamma_g(T)) \leq \widetilde{d}\,(\gamma(0), \gamma(T))$.*

Proof. Since $x_g(t) = x(t) + g_1$ then $\dot{x}_g(t) = \dot{x}(t) = u_1(t)$. This implies that $u_1(t) = \overline{u}_1(t)$. In the same way, we obtain $u_2(t) = \overline{u}_2(t)$ because $y_g(t) = y(t) + g_2$. Furthermore, $z_g(t) = z(t) + g_1 y(t) + g_3$ then

$$\dot{z}_g(t) = \dot{z}(t) + g_1\dot{y}(t) = u_2(t)x_g(t) + u_3(t)h\left(\gamma(t)\right). \tag{18}$$

Besides the above equation, $z_g(t)$ satisfies the equation

$$\dot{z}_g(t) = \overline{u}_2(t)x_g(t) + \overline{u}_3(t)h\left(\gamma_g(t)\right), \tag{19}$$

because γ_g is an admissible curve for $\widetilde{d}$. Finally, from the equations (19) and (18) and as $u_2(t) = \overline{u}_2(t)$ we get

$$\overline{u}_3(t) = \frac{u_3(t)h\left(\gamma(t)\right)}{h\left(\gamma_g(t)\right)}.$$

The condition $|h(\gamma)| \leq |h(\gamma_g)|$ implies that $|\overline{u}_3(t)| \leq |u_3(t)|$, hence $\overline{u}_3(t)^2 \leq u_3(t)^2$. Therefore the length of γ_g decreases and consequently $\widetilde{d}\left(\gamma_g(0), \gamma_g(T)\right) \leq \widetilde{d}\left(\gamma(0), \gamma(T)\right)$. □

In the same sense of the above theorem, the following result gives us a sufficient condition to determine when the distance of the points translated by g is less than the distance from the origin to $\gamma(T)$.

Theorem 7. *With the same conditions of the above. If* $\dfrac{\dfrac{\partial}{\partial g_i}\left(h(g) + f(g,\gamma)\right)}{h\left(\gamma_g\right)} > 0$ *then* $\widetilde{d}\left(\gamma_g(0), \gamma_g(T)\right) < \widetilde{d}\left(\gamma(0), \gamma(T)\right)$.

Proof. From Theorem 6, the control functions of γ_g are $u_1(t)$, $u_2(t)$ and $\overline{u}_3(t)$, hence

$$\begin{aligned}
l\left(\gamma_g\right) &= \int_0^T \left(u_1(t)^2 + u_2(t)^2 + \overline{u}_3(t)^2\right)^{\frac{1}{2}} dt \\
&= \int_0^T \left(u_1(t)^2 + u_2(t)^2 + \frac{u_3(t)^2 h(\gamma(t))^2}{h(\gamma_g(t))^2}\right)^{\frac{1}{2}} dt \\
\frac{\partial}{\partial g_i} l\left(\gamma_g\right) &= \int_0^T \frac{1}{2}\left(\frac{\left(u_1(t)^2 + u_2(t)^2\right)h(\gamma_g(t))^2 + u_3(t)^2 h(\gamma(t))^2}{h(\gamma_g(t))^2}\right)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial g_i}\left(\frac{u_3(t)^2 h(\gamma(t))^2}{h(\gamma_g(t))^2}\right) dt \\
&= \int_0^T \frac{-|h(\gamma_g(t))|u_3(t)^2 h(\gamma(t))^2}{\left(\left(u_1(t)^2 + u_2(t)^2\right)h(\gamma_g(t))^2 + u_3(t)^2 h(\gamma(t))^2\right)^{\frac{1}{2}} h(\gamma_g(t))^3} \cdot \frac{\partial}{\partial g_i}\left(h(g) + f(g,\gamma)\right) dt \\
&= -\int_0^T \frac{u_3(t)^2 h(\gamma(t))^2 \dfrac{\partial}{\partial g_i}\left(h(g) + f(g,\gamma)\right)}{\left(\left(u_1(t)^2 + u_2(t)^2\right)h(\gamma_g(t))^2 + u_3(t)^2 h(\gamma(t))^2\right)^{\frac{1}{2}} |h(\gamma_g(t))| h(\gamma_g(t))} dt.
\end{aligned}$$

Note that the function S defined by

$$S(t) = \frac{u_3(t)^2 h(\gamma(t))^2 \dfrac{\partial}{\partial g_i}\left(h(g) + f(g,\gamma)\right)}{\left(u_1(t)^2 + u_2(t)^2 h(\gamma_g(t))^2 + u_3(t)^2 h(\gamma(t))^2\right)^{\frac{1}{2}} |h(\gamma_g(t))| h(\gamma_g(t))}$$

is positive if and only if

$$\frac{\frac{\partial}{\partial g_i}\left(h(g)+f(g,\gamma)\right)}{h(\gamma_g(t))} > 0.$$

In this case $\dfrac{\partial l(\gamma_g)}{\partial g_i} < 0$ and $\widetilde{d}\,(\gamma_g(0),\gamma_g(T)) < \widetilde{d}\,(\gamma(0),\gamma(T))$.

In particular at $g=(0,0,0)$,

$$\frac{\partial}{\partial g_i} l\,(\gamma_g)\Big|_{(g_1,g_2,g_3)=(0,0,0)} = -\int_0^T \frac{u_3(t)^2 \frac{\partial}{\partial g_i}\left(h(g)+f(g,\gamma)\right)}{h(\gamma(t))} dt.$$

□

4.2.6 Conclusion

In Section 3, we have shown that in case where the nilpotent approximation of an ARS degenerates, that is when it is no longer an ARS but a sub-Riemannian structure, we can replace it by a simple ARS on a Lie group or a homogeneous space. Thanks to formula (14) of Theorem 5 we know that, at least in some 3D generic cases, the order of the approximation of d by $\widetilde{d}$ is better than the one of the approximation of d by $\widehat{d}$. Indeed, this order is d^2 in the first case and $d^{\frac{3}{2}}$ in the second one. However, this does not prove that the solvable approximation is really better than the nilpotent one, and anyway it is certainly not true for any pair of points.

Since under left translations the nilpotent distance $\widehat{d}$ is invariant while the solvable distance $\widetilde{d}$ may be decreasing, we can expect to prove that the approximation by $\widetilde{d}$ is strictly better than the one by $\widehat{d}$ for pairs of points translated in a suitable direction.

For this purpose, we consider here the 3D-generic case of Section 4.2 with the particular values $a=1$, $b=c=0$, that is $\widetilde{X}_3 = z\frac{\partial}{\partial z}$. The singular locus of the solvable approximation is then the plane $\{z=0\}$.

In what follows we consider a (normal) geodesic γ for $\widetilde{d}$, originated at $(0,0,0)$ and parametrized by arc length on $[0,T]$. Denoting $\gamma(t)=(x(t),y(t),z(t))$ it is moreover assumed that $z(t)>0$ on $]0,T]$.

This geodesic is translated by $\mathbf{g}=(0,0,g)$, with $g\geq 0$, into $\gamma_g = L_{\mathbf{g}}\gamma$. Since $\mathbf{g}$ belongs to the center of the Heisenberg group the curve γ_g is simply $\gamma_g(t) = (x(t),y(t),z(t)+g)$.

The different distances between g and $\gamma_g(T)$ are analyzed in several steps.

1. Since the controls associated to γ_g are u_1, u_2, and $\dfrac{z(t)}{z(t)+g}u_3$ the length of γ_g related to $\widetilde{d}$ is

$$\widetilde{l}(\gamma_g) = \int_0^T \left(u_1^2+u_2^2+\left(\frac{z(t)}{z(t)+g}\right)^2 u_3^2\right)^{\frac{1}{2}} dt.$$

Since $z \longmapsto \dfrac{z}{z+g}$ is increasing we have $\widetilde{l}(\gamma_g) \leq I_g$, where I_g stands for

$$I_g = \int_0^T \left(u_1^2 + u_2^2 + \left(\frac{z_m}{z_m + g} \right)^2 u_3^2 \right)^{\frac{1}{2}} dt,$$

with $z_m = \max\{z(t); \quad t \in [0,T]\}$.

2. We apply now formula (14), which writes here:

$$d(\mathbf{g}, \gamma_g(T)) \leq \widetilde{d}(\mathbf{g}, \gamma_g(T)) + C \cdot \widetilde{\tau}^{\frac{3}{2}} \widetilde{d}(\mathbf{g}, \gamma_g(T))^{\frac{1}{2}},$$

where $\widetilde{\tau} = \max\left\{ \|\mathbf{g}\|_p, \ \widetilde{d}(\mathbf{g}, \gamma_g(T)) \right\}$. It will always be assumed that $\|\mathbf{g}\|_p \leq \widetilde{d}(\mathbf{g}, \gamma_g(T))$, that is $g^{\frac{1}{2}} \leq \widetilde{d}(\mathbf{g}, \gamma_g(T))$. Taking into account $\widetilde{d}(\mathbf{g}, \gamma_g(T)) \leq \widehat{d}(\mathbf{g}, \gamma_g(T)) = \widehat{d}(0, \gamma(T))$, we get

$$\begin{aligned} d(\mathbf{g}, \gamma_g(T)) &\leq \widetilde{d}(\mathbf{g}, \gamma_g(T)) + C \cdot \widetilde{d}(\mathbf{g}, \gamma_g(T))^2 \\ &\leq \frac{I_g}{T} \widetilde{d}(0, \gamma(T)) + C \cdot \left(\frac{I_g}{T} \right)^2 \widetilde{d}(0, \gamma(T))^2 \\ &\leq \frac{I_g}{T} \widehat{d}(0, \gamma(T)) \left(1 + \frac{I_g}{T} C \cdot \widetilde{d}(0, \gamma(T)) \right). \end{aligned}$$

3. In order to approximate $z(t)$ and $u_3(t)$, we consider the Hamiltonian equations (see details in the next section), for the values $a = 1$, $b = c = 0$. They are:

$$\left\{ \begin{array}{ll} \dot{x} & = p \\ \dot{y} & = q + rx \\ \dot{z} & = (q + rx)x + rz^2 \end{array} \right. \qquad \left\{ \begin{array}{ll} \dot{p} & = -(q + rx)r \\ \dot{q} & = 0 \\ \dot{r} & = -r^2 z \end{array} \right.$$

It is important to notice that r_0 can be chosen arbitrarily large because $H(t = 0) = \frac{1}{2}(p_0^2 + q_0^2)$. We make the choice $p_0 = q_0$ and the following approximations hold:

$$x(t) \approx p_0 t, \quad \dot{z} \approx q_0 p_0 t = \frac{1}{2} t, \quad z(t) \approx \frac{1}{4} t^2, \quad u_3(t) = r(t) z(t) \approx \frac{1}{4} r_0 t^2.$$

In order to compute $\widetilde{l}(\gamma_g)$, we need to apply the condition $g^{\frac{1}{2}} \leq \widetilde{d}(\mathbf{g}, \gamma_g(T))$ of point 2. We do not know $\widetilde{d}(\mathbf{g}, \gamma_g(T))$ but we can set $\widetilde{d}(\mathbf{g}, \gamma_g(T)) = \beta T$ with $0 < \beta < 1$ (this estimation "a priori" will be justified later), and set $g = \beta^2 T^2$. Then we get

$$\frac{z}{z+g} \leq \frac{z_m}{z_m + g} \approx \frac{\frac{1}{4} T^2}{\frac{1}{4} T^2 + \beta^2 T^2} = \frac{1}{1 + 4\beta^2}.$$

Therefore

$$\begin{aligned}\widetilde{l}(\gamma_g) \leq I_g &= \int_0^T \left(u_1^2 + u_2^2 + \left(\frac{1}{1+4\beta^2}\right)^2 u_3^2\right)^{\frac{1}{2}} dt \\ &= \int_0^T \left(1 - u_3^2 + \left(\frac{1}{1+4\beta^2}\right)^2 u_3^2\right)^{\frac{1}{2}} dt = \int_0^T \left(1 - \frac{16\beta^4+8\beta^2}{(1+4\beta^2)^2} u_3^2\right)^{\frac{1}{2}} dt \\ &\approx \int_0^T \left(1 - \frac{16\beta^4+8\beta^2}{(1+4\beta^2)^2}\frac{1}{16} r_0^2 t^4\right)^{\frac{1}{2}} dt = \int_0^T \left(1 - \frac{2\beta^4+\beta^2}{2(1+4\beta^2)^2} r_0^2 t^4\right)^{\frac{1}{2}} dt.\end{aligned}$$

We write $\delta = \frac{2\beta^4+\beta^2}{2(1+4\beta^2)^2}$ and we set $\delta r_0^2 T^4 = \frac{1}{2}$. Notice that this is possible, even if T is small, by increasing r_0. Thanks to $(1-c)^{\frac{1}{2}} \leq 1 - 0.5c$ whenever $0 \leq c \leq 1$ we get:

$$\widetilde{l}(\gamma_g) \leq I_g \leq \int_0^T \left(1 - 0.5\delta r_0^2 t^4\right) dt = T - 0.5\delta r_0^2 \frac{T^5}{5} = T(1 - 0.1\delta r_0^2 T^4) \approx 0.95T.$$

4. Assuming $C \cdot \widetilde{d}(0, \gamma(T)) = C \cdot T \leq 0.01$ we get on one hand:

$$\begin{aligned}d(\mathbf{g}, \gamma_g(T)) &\leq \widetilde{d}(\mathbf{g}, \gamma_g(T)) + C \cdot \widetilde{d}(\mathbf{g}, \gamma_g(T))^2 \\ &= \widetilde{d}(\mathbf{g}, \gamma_g(T))(1 + C \cdot \widetilde{d}(\mathbf{g}, \gamma_g(T))) \\ &\leq 1.01\widetilde{d}(\mathbf{g}, \gamma_g(T)).\end{aligned}$$

On the other hand:

$$\begin{aligned}\widehat{d}(\mathbf{g}, \gamma_g(T)) - d(\mathbf{g}, \gamma_g(T)) &\geq \widehat{d}(\mathbf{g}, \gamma_g(T)\left(1 - 1.01\frac{I_g}{T}\right) \\ &= \widetilde{d}(\mathbf{g}, \gamma_g(T))\frac{\widehat{d}(\mathbf{g}, \gamma_g(T))}{\widetilde{d}(\mathbf{g}, \gamma_g(T))}\left(1 - 1.01\frac{I_g}{T}\right) \\ &\geq \widetilde{d}(\mathbf{g}, \gamma_g(T))\frac{T}{I_g}\left(1 - 1.01\frac{I_g}{T}\right) \\ &= \widetilde{d}(\mathbf{g}, \gamma_g(T))\left(\frac{T}{I_g} - 1.01\right).\end{aligned}$$

Therefore $\widehat{d}(\mathbf{g}, \gamma_g(T)) - d(\mathbf{g}, \gamma_g(T)) > d(\mathbf{g}, \gamma_g(T)) - \widetilde{d}(\mathbf{g}, \gamma_g(T))$ as soon as $\frac{T}{I_g} - 1.01 > 0.01$ hence as soon as $\frac{I_g}{T} < 0.98$.

According to Point 3, we can obtain $I_g \leq 0.95T$ and in that case the solvable distance between $\mathbf{g}$ and $\gamma_g(T)$ is strictly closer to the original distance between these points than the nilpotent one.

5 Geodesics

In this section the Hamiltonian for the normal flow defined by the solvable approximation in the 3D generic case is given. We compute the geodesic with initial condition $x(0) = y(0) = z(0) = 0$ and covector $\lambda = (p, q, r) \in T^*\mathbb{R}^3$ with $p(0) = \cos(\theta)$,

$q(0) = \sin(\theta)$, $r(0) = r$.

From the above sections, the solvable approximation is defined by

$$\widehat{X}_1 = X_1, \quad \widehat{X}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \widetilde{X}_3 = \left(az + bx^2 + cy^2\right)\frac{\partial}{\partial z}. \tag{20}$$

From (20), the Hamiltonian for the normal flow is given by

$$H(\lambda) = \frac{1}{2}\left(\left\langle \lambda, \widehat{X}_1(x,y,z)\right\rangle^2 + \left\langle \lambda, \widehat{X}_2(x,y,z)\right\rangle^2 + \left\langle \lambda, \widetilde{X}_3(x,y,z)\right\rangle^2\right)$$
$$H(\lambda) = \frac{1}{2}\left(p^2 + (q+rx)^2 + r^2\left(az + bx^2 + cy^2\right)^2\right),$$

where $\lambda = (p,q,r) \in T^*\mathbb{R}^3$. Hence

$$\begin{aligned} \dot{x}(t) &= p & \dot{p}(t) &= -(q+rx)r - 2bxr^2(az + bx^2 + cy^2) \\ \dot{y}(t) &= q + rx & \dot{q}(t) &= -2cyr^2(az + bx^2 + cy^2) \\ \dot{z}(t) &= (q+rx)x + r(az + bx^2 + cy^2)^2 & \dot{r}(t) &= -ar^2(az + bx^2 + cy^2) \end{aligned}$$

are the associated Hamiltonian equations to the solvable approximation.

The geodesic with initial condition $x(0) = y(0) = z(0) = 0$ and $p(0) = \cos(\theta)$, $q(0) = \sin(\theta)$ and $r(0) = r = 0$ is given by

$$\begin{aligned} x(t) &= t\cos(\theta) \\ y(t) &= t\sin(\theta) \\ z(t) &= \frac{1}{4}t^2\sin(2\theta), \end{aligned} \tag{21}$$

because $p(t) = \cos(\theta)$ and $q(t) = \sin(\theta)$, this is to say p and q are constants.

Notice that the above geodesic for $\widetilde{d}$ is the same as the geodesic for $\widehat{d}$. The above implies that this geodesic is optimal for any time and has no conjugate time (see Theorem 5.1 and 5.2 in [10]). We can see some geodesics in Figure 1 when $r = 0$.

Due to the complexity of the Hamiltonian system of equations, we compute the geodesics considering $a = c = 0$ and $b = 1$. Thus the Hamiltonian is

$$H(\lambda) = \frac{1}{2}\left(p^2 + (q+rx)^2 + r^2x^4\right),$$

hence

$$\begin{aligned} \dot{x}(t) &= p & \dot{p}(t) &= -(q+rx)r - 2r^2x^3 \\ \dot{y}(t) &= q + rx & \dot{q}(t) &= 0 \\ \dot{z}(t) &= xq + rx^2 + rx^4 & \dot{r}(t) &= 0 \end{aligned} \tag{22}$$

Considering the initial condition $x(0) = 0$ then $p(0) = \cos(\theta)$, $q(0) = \sin(\theta)$ and $r(0) = r$. If $r = 0$ then the solution to the differential systems (22) is given by (21). If $r(0) = r \neq 0$, since $\dot{x}(t) = p$, we get

$$\ddot{x} = -r(q+rx) - 2r^2x^3$$
$$\ddot{x} + r^2x + 2r^2x^3 = -rq.$$

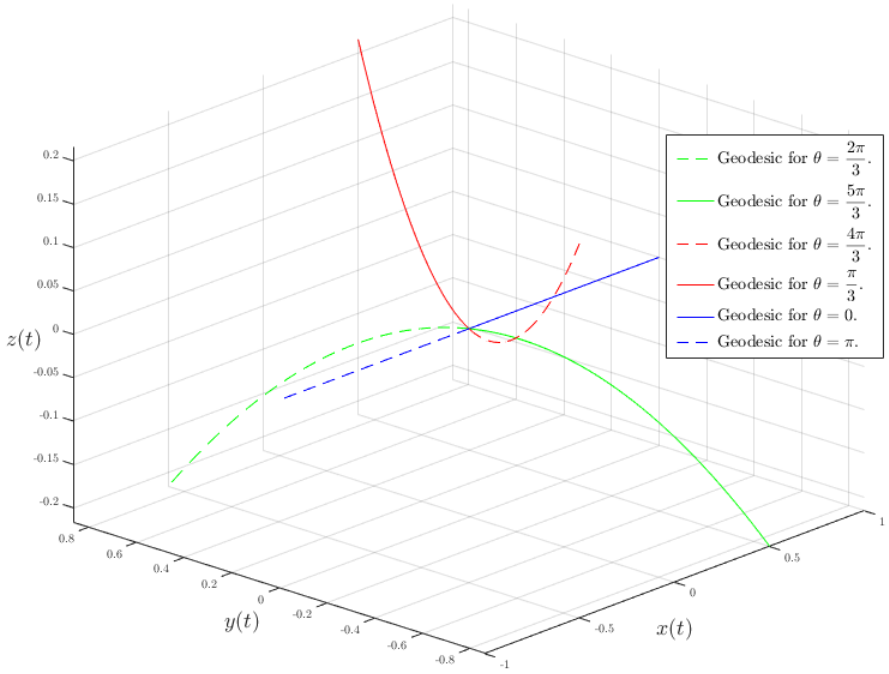

Figure 1: Geodesics for $\theta \in \left\{0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{6}, \frac{5\pi}{6}\right\}$ when $r = 0$.

Since $q(0) = \sin(\theta)$ and $\dot{q} = 0$, then $q = \sin(\theta)$. Hence

$$\ddot{x} + r^2 x + 2r^2 x^3 = -r\sin(\theta). \tag{23}$$

The equation (23) is equivalent to

$$\ddot{x} + r^2 x + 2r^2 x^3 = -r\sin(\theta)\mathrm{cn}(0, k^2), \tag{24}$$

where $\mathrm{cn}(0 \cdot t, k^2)$ is the Jacobian elliptic function that has a period in $0 \cdot t$ equal to $4K(k^2)$ and $K(k^2)$ is the complete elliptic integral of the first kind for the modulus k (see more in [14]). This equivalence is due to the fact that $\mathrm{cn}(0, k^2) = 1$.

In [18] a general solution to

$$\ddot{x} + c_n\dot{x} + w_n x + \epsilon x^3 = F\mathrm{cn}(wt, k^2),$$

is given by

$$x(t) = a_1(t)\mathrm{cn}\left(w_1 t + \phi, k_1^2\right) + A_1(t)\mathrm{cn}\left(wt, k^2\right) + B_1(t) \cdot \mathrm{sn}\left(wt, k^2\right).$$

Therefore, the solution for the equation (24) is given by

$$x(t) = a_1(t)\mathrm{cn}\left(w_1 t + \phi, k_1^2\right) + A_1(t),$$

where $a_1(t), A_1(t), w_1, \phi$ and k_1 need to be determined. Notice that $B_1(t) \cdot \mathrm{sn}\left(wt, k^2\right)$ vanishes because $\mathrm{sn}(0, k^2) = 0$.

From [18] is possible to obtain that $a_1(t)$ and $A_1(t)$ are constants. Then

$$x(t) = a_1 \mathrm{cn}\left(w_1 t + \phi, k_1^2\right) + A_1. \tag{25}$$

Moreover, since $x(0) = 0$

$$-A_1 = a_1 \cdot \mathrm{cn}(\phi, k_1^2). \tag{26}$$

Furthermore, differentiating in (25) and since $\dot{x}(0) = p(0) = \cos(\theta)$, we have

$$a_1 = \frac{\cos(\theta)\mathrm{ns}(\phi, k_1^2)\mathrm{nd}(\phi, k_1^2)}{w_1}.$$

Finally, since $y(0) = z(0) = 0$,

$$\begin{aligned}
x(t) &= a_1 \left(\mathrm{cn}\left(w_1 t + \phi, k_1^2\right) - \mathrm{cn}\left(\phi, k_1^2\right)\right) \\
y(t) &= \left(\sin(\theta) - r a_1 \mathrm{cn}\left(\phi, k_1^2\right)\right) t + \frac{r a_1}{k_1^2 w_1}\Big(\arccos\left(\mathrm{dn}(w_1 t + \phi, k_1^2)\right) - \arccos\left(\mathrm{dn}(\phi, k_1^2)\right) \Big) \\
z(t) &= -\Big(r a_1^4 \mathrm{cn}\left(\phi, k_1^2\right)^4 + r a_1^2 \mathrm{cn}\left(\phi, k_1^2\right)^2 + \sin(\theta)\Big) a_1 \mathrm{cn}\left(\phi, k_1^2\right) t + \frac{r a_1^4}{3 k_1^8 w_1} z_1(t) \\
&\quad + \frac{4 r a_1^4 \mathrm{cn}\left(\phi, k_1^2\right)}{2 k_1^6 w_1} z_2(t) + \frac{6 r a_1^4 \mathrm{cn}\left(\phi, k_1^2\right)^2 + r a_1^2}{k_1^4 w_1} z_3(t) \\
&\quad + \frac{4 r a_1^4 \mathrm{cn}\left(\phi, k_1^2\right)^3 + 2 r a_1^2 \mathrm{cn}\left(\phi, k_1^2\right) + \sin(\theta)}{k_1^2 w_1} z_4(t),
\end{aligned}$$

where $k_1'^2 + k_1^2 = 1$, $E(\cdot)$ is the incomplete elliptic integral of the second kind and

$$\begin{aligned}
z_1(t) &= (2 - 3k_1^4) k_1'^4 w_1 t + 2(2k_1^4 - 1)\big(E(w_1 t + \phi) - E(\phi)\big) \\
&\quad + k_1^4\big(\mathrm{sn}(w_1 t + \phi, k_1^2)\mathrm{cn}(w_1 t + \phi, k_1^2)\mathrm{dn}(w_1 t + \phi, k_1^2) - \mathrm{sn}(\phi, k_1^2)\mathrm{cn}(\phi, k_1^2)\mathrm{dn}(\phi, k_1^2)\big) \\
z_2(t) &= (2k_1^4 - 1)\Big(\arcsin\left(k_1^2 \mathrm{sn}\left(w_1 t + \phi, k_1^2\right)\right) - \arcsin\left(k_1^2 \mathrm{sn}(\phi, k_1^2)\right) \Big) \\
&\quad + k_1^2\Big(\mathrm{sn}(w_1 t + \phi, k_1^2)\mathrm{dn}\left(w_1 t + \phi, k_1^2\right) - \mathrm{sn}(\phi, k_1^2)\mathrm{dn}\left(w_1 t + \phi, k_1^2\right) \Big) \\
z_3(t) &= E(w_1 t + \phi) - E(\phi) - k_1'^4 w_1 t \\
z_4(t) &= \arccos\left(\mathrm{dn}(w_1 t + \phi, k_1^2)\right) - \arccos\left(\mathrm{dn}(\phi, k_1^2)\right)\Big).
\end{aligned}$$

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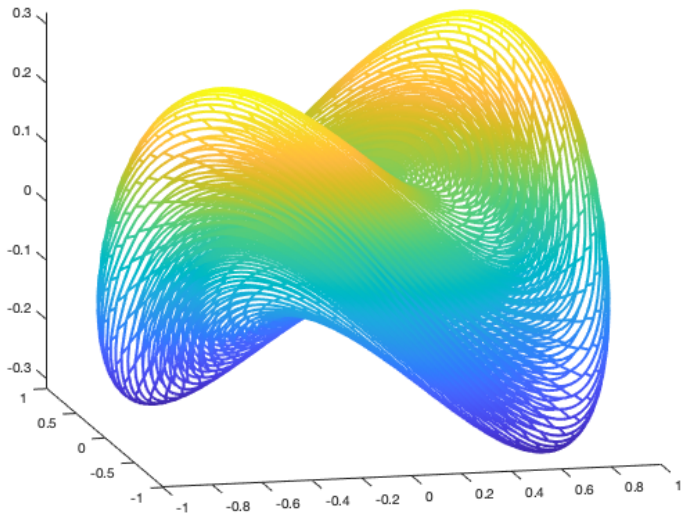

Figure 2: Ball in 3-D generic case.

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