Crystallographic groups and flat manifolds from surface braid groups

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CRYSTALLOGRAPHIC GROUPS AND FLAT MANIFOLDS FROM SURFACE BRAID GROUPS

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Abstract. Let M be a compact surface without boundary, and $n \geq 2$. We analyse the quotient group $B_n(M)/\Gamma_2(P_n(M))$ of the surface braid group $B_n(M)$ by the commutator subgroup $\Gamma_2(P_n(M))$ of the pure braid group $P_n(M)$. If M is different from the 2-sphere $\mathbb{S}^2$, we prove that $B_n(M)/\Gamma_2(P_n(M)) \cong P_n(M)/\Gamma_2(P_n(M)) \rtimes_\varphi S_n$, and that $B_n(M)/\Gamma_2(P_n(M))$ is a crystallographic group if and only if M is orientable.

If M is orientable, we prove a number of results regarding the structure of $B_n(M)/\Gamma_2(P_n(M))$. We characterise the finite-order elements of this group, and we determine the conjugacy classes of these elements. We also show that there is a single conjugacy class of finite subgroups of $B_n(M)/\Gamma_2(P_n(M))$ isomorphic either to S_n or to certain Frobenius groups. We prove that crystallographic groups whose image by the projection $B_n(M)/\Gamma_2(P_n(M)) \longrightarrow S_n$ is a Frobenius group are not Bieberbach groups. Finally, we construct a family of Bieberbach subgroups $\widetilde{G}_{n,g}$ of $B_n(M)/\Gamma_2(P_n(M))$ of dimension $2ng$ and whose holonomy group is the finite cyclic group of order n, and if $\mathcal{X}_{n,g}$ is a flat manifold whose fundamental group is $\widetilde{G}_{n,g}$, we prove that it is an orientable Kähler manifold that admits Anosov diffeomorphisms.

1. Introduction

The braid groups of the 2-disc, or Artin braid groups, were introduced by Artin in 1925 and further studied in 1947 [1, 2]. Surface braid groups were initially studied by Zariski [24], and were later generalised by Fox and Neuwirth to braid groups of arbitrary topological spaces using configuration spaces as follows [7]. Let M be a compact, connected surface, and let $n \in \mathbb{N}$. The *nth ordered configuration space of* M, denoted by $F_n(M)$, is defined by:

$$F_n(M) = \{(x_1, \ldots, x_n) \in M^n \,|\, x_i \neq x_j \text{ if } i \neq j,\ i, j = 1, \ldots, n\}\,.$$

The *n-string pure braid group $P_n(M)$ of M* is defined by $P_n(M) = \pi_1(F_n(M))$. The symmetric group S_n on n letters acts freely on $F_n(M)$ by permuting coordinates, and the *n-string braid group $B_n(M)$ of M* is defined by $B_n(M) = \pi_1(F_n(M)/S_n)$. This gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(M) \longrightarrow B_n(M) \xrightarrow{\sigma} S_n \longrightarrow 1. \tag{1.1}$$

The map $\sigma\colon B_n(M) \longrightarrow S_n$ is the standard homomorphism that associates a permutation to each element of S_n. In [10, 11, 12], three of the authors of this paper studied the quotient $B_n/\Gamma_2(P_n)$, where B_n is the n-string Artin braid group, P_n is the subgroup of B_n of pure braids, and $\Gamma_2(P_n)$ is the commutator subgroup of P_n. In [10], it was proved that this quotient is a crystallographic group. Crystallographic groups play an important rôle in the study of the groups of isometries of Euclidean spaces (see Section 2 for precise definitions, as well as [4, 5, 23] for more details). Using different techniques, Marin extended the results of [10] to generalised braid groups associated to arbitrary complex reflection groups [16]. Beck and Marin showed that other finite non-Abelian groups, not covered by [11, 16], embed in $B_n/\Gamma_2(P_n)$ [3].

In this paper, we study the quotient $B_n(M)/\Gamma_2(P_n(M))$ of $B_n(M)$, where $\Gamma_2(P_n(M))$ is the commutator subgroup of $P_n(M)$, one of our aims being to decide whether it is crystallographic or not.

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The group extension (1.1) gives rise to the following short exact sequence:

$$1 \longrightarrow P_n(M)/\Gamma_2(P_n(M)) \longrightarrow B_n(M)/\Gamma_2(P_n(M)) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1. \tag{1.2}$$

Note that if M is an orientable, compact surface of genus $g \geq 1$ without boundary and $n = 1$ then $B_1(M)/[P_1(M), P_1(M)]$ is the Abelianisation of $\pi_1(M)$, and is isomorphic to $\mathbb{Z}^{2g}$, so it is clearly a crystallographic group.

In Section 2, we recall some definitions and facts about crystallographic groups, and if M is an orientable, compact, connected surface of genus $g \geq 1$ without boundary, we prove that $B_n(M)/\Gamma_2(P_n(M))$ is crystallographic.

Proposition 1. *Let M be an orientable, compact, connected surface of genus $g \geq 1$ without boundary, and let $n \geq 2$. Then there exists a split extension of the form:*

$$1 \longrightarrow \mathbb{Z}^{2ng} \longrightarrow B_n(M)/\Gamma_2(P_n(M)) \xrightarrow{\overline{\sigma}} S_n \longrightarrow 1, \tag{1.3}$$

where the holonomy representation $\varphi\colon S_n \longrightarrow Aut(\mathbb{Z}^{2ng})$ is faithful and where the action is defined by (2.5). In particular, the quotient $B_n(M)/\Gamma_2(P_n(M))$ is a crystallographic group of dimension $2ng$ and whose holonomy group is S_n.

As for $B_n/[P_n, P_n]$, some natural questions arise for $B_n(M)/\Gamma_2(P_n(M))$, such as the existence of torsion, the realisation of elements of finite order and that of finite subgroups, their conjugacy classes, as well as properties of some Bieberbach subgroups of $B_n(M)/\Gamma_2(P_n(M))$. In Theorem 2, we characterise the finite-order elements of $B_n(M)/\Gamma_2(P_n(M))$ and their conjugacy classes, from which we see that the conjugacy classes of finite-order elements of $B_n(M)/\Gamma_2(P_n(M))$ are in one-to-one correspondence with the conjugacy classes of elements of the symmetric group S_n.

Theorem 2. *Let $n \geq 2$, and let M be an orientable surface of genus $g \geq 1$ without boundary.*

(a) Let e_1 and e_2 be finite-order elements of $B_n(M)/\Gamma_2(P_n(M))$. Then e_1 and e_2 are conjugate if and only if their permutations $\overline{\sigma}(e_1)$ and $\overline{\sigma}(e_2)$ have the same cycle type. Thus two finite cyclic subgroups H_1 and H_2 of $B_n(M)/\Gamma_2(P_n(M))$ are conjugate if and only if the generators of $\overline{\sigma}(H_1)$ and $\overline{\sigma}(H_2)$ have the same cycle type.

(b) If H_1 and H_2 are subgroups of $B_n(M)/\Gamma_2(P_n(M))$ that are isomorphic to S_n then they are conjugate.

The results of Theorem 2 lead to the following question: if H_1 and H_2 are finite subgroups of $B_n(M)/\Gamma_2(P_n(M))$ such that $\overline{\sigma}(H_1)$ and $\overline{\sigma}(H_2)$ are conjugate in S_n, then are H_1 and H_2 conjugate? For each odd prime p, we shall consider the corresponding Frobenius group, which is the semi-direct product $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$, the action being given by an automorphism of $\mathbb{Z}_p$ of order $(p-1)/2$. In Proposition 12 we show that the conclusion of Theorem 2 holds for subgroups of $B_5(M)/\Gamma_2(P_5(M))$ that are isomorphic to the Frobenius group $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$.

In Section 3, we study some Bieberbach subgroups of $B_n(M)/\Gamma_2(P_n(M))$ whose construction is suggested by that of the Bieberbach subgroups of $B_n/\Gamma_2(P_n)$ given in [17].

Theorem 3. *Let $n \geq 2$, and let M be an orientable surface of genus $g \geq 1$ without boundary. Let G_n be the cyclic subgroup $\langle (n, n-1, \ldots, 2, 1) \rangle$ of S_n. Then there exists a subgroup $\widetilde{G}_{n,g}$ of $\sigma^{-1}(G_n)/\Gamma_2(P_n(M)) \subset B_n(M)/\Gamma_2(P_n(M))$ that is a Bieberbach group of dimension $2ng$ whose holonomy group is G_n. Further, the centre $Z(\widetilde{G}_{n,g})$ of $\widetilde{G}_{n,g}$ is a free Abelian group of rank $2g$.*

The conclusion of the first part of the statement of Theorem 3 probably does not remain valid if we replace the finite cyclic group G_n by other finite groups. In this direction, if p is an odd prime, in Proposition 13, we prove that there is no Bieberbach subgroup H of $B_p(M)/[P_p(M), P_p(M)]$ for which $\overline{\sigma}(H)$ is the Frobenius group $\mathbb{Z}_p \rtimes \mathbb{Z}_{(p-1)/2}$.

It follows from the definition that crystallographic groups act properly discontinuously and cocompactly on Euclidean space, and that the action is free if the groups are Bieberbach. Thus there exists a flat manifold $\mathcal{X}_{n,g}$ whose fundamental group is the subgroup $\widetilde{G}_{n,g}$ of Theorem 3. Motivated by results about the holonomy representation of Bieberbach subgroups of the Artin braid group quotient

$B_n/[P_n, P_n]$ whose holonomy group is a 2-group obtained in [18], in Section 3, we make use of the holonomy representation of $\widetilde{G}_{n,g}$ given in (3.4) to prove some dynamical and geometric properties of $\mathcal{X}_{n,g}$. To describe these results, we recall some definitions.

If $f\colon M \longrightarrow M$ is a self-map of a Riemannian manifold, M is said to have a hyperbolic structure with respect to f if there exists a splitting of the tangent bundle $T(M)$ of the form $T(M) = E^s \oplus E^u$ such that $\mathrm{D}f\colon E^s \longrightarrow E^s$ (resp. $\mathrm{D}f\colon E^u \longrightarrow E^u$) is contracting (resp. expanding). Further, the map f is called *Anosov* if it is a diffeomorphism and M has hyperbolic structure with respect to f. The classification of compact manifolds that admit Anosov diffeomorphisms is a problem first proposed by Smale [21]. Anosov diffeomorphisms play an important rôle in the theory of dynamical systems since their behaviour is generic in some sense. Porteous gave a criterion for the existence of Anosov diffeomorphisms of flat manifolds in terms of the holonomy representation [20, Theorems 6.1 and 7.1] that we shall use in the proof of Theorem 4.

We recall that a Kähler manifold is a $2n$-real manifold endowed with a Riemannian metric, a complex structure, and a symplectic structure that is compatible at every point. For more about such manifolds, see [22, Chapter 7]. A finitely-presented group is said to be a Kähler group if it is the fundamental group of a closed Kähler manifold. We may now state Theorem 4.

THEOREM 4. *Let $n \geq 2$, and let $\mathcal{X}_{n,g}$ be a $2ng$-dimensional flat manifold whose fundamental group is the Bieberbach group $\widetilde{G}_{n,g}$ of Theorem 3. Then $\mathcal{X}_{n,g}$ is an orientable Kähler manifold with first Betti number $2g$ that admits Anosov diffeomorphisms.*

The proof of Theorem 4 depends mainly on the holonomy representation of the Bieberbach group $\widetilde{G}_{n,g}$, and makes use of the eigenvalues of the matrix representation and the decomposition of the holonomy representation in irreducible representations using character theory.

Finally, in Section 4, we prove in Proposition 17 that the conclusion of Proposition 1 no longer holds if M is the sphere $\mathbb{S}^2$ or a compact, non-orientable surface without boundary. More precisely, if $n \geq 1$ then $B_n(M)/\Gamma_2(P_n(M))$ is not a crystallographic group.

2. Crystallographic groups and quotients of surface braid groups

In this section, we start by recalling some definitions and facts about crystallographic groups. If M is a compact, orientable surface without boundary of genus $g \geq 1$, in Proposition 1, we prove that the quotient $B_n(M)/\Gamma_2(P_n(M))$ is a crystallographic group that is isomorphic to $\mathbb{Z}^{2ng} \rtimes_\varphi S_n$. We also determine the conjugacy classes of the finite-order elements of $B_n(M)/\Gamma_2(P_n(M))$ in Theorem 2.

2.1. Crystallographic groups.

In this section, we recall briefly the definitions of crystallographic and Bieberbach groups, and the characterisation of crystallographic groups in terms of a representation that arises in certain group extensions whose kernel is a free Abelian group of finite rank and whose quotient is finite. We also recall some results concerning Bieberbach groups and the fundamental groups of flat Riemannian manifolds. For more details, see [4, Section I.1.1], [5, Section 2.1] or [23, Chapter 3].

Let G be a Hausdorff topological group. A subgroup H of G is said to be *discrete* if it is a discrete subset. If H is a closed subgroup of G then the quotient space G/H admits the quotient topology for the canonical projection $\pi\colon G \longrightarrow G/H$, and we say that H is *uniform* if G/H is compact. From now on, we identify $\mathrm{Aut}(\mathbb{Z}^m)$ with $\mathrm{GL}(m, \mathbb{Z})$. A discrete, uniform subgroup Π of $\mathbb{R}^m \rtimes \mathrm{O}(m, \mathbb{R}) \subseteq \mathrm{Aff}(\mathbb{R}^m)$ is said to be a *crystallographic group* of dimension m. If in addition Π is torsion free then Π is called a *Bieberbach group* of dimension m.

If Φ is a group, an *integral representation of rank m of Φ* is defined to be a homomorphism $\Theta\colon \Phi \longrightarrow \mathrm{Aut}(\mathbb{Z}^m)$. Two such representations are said to be *equivalent* if their images are conjugate in $\mathrm{Aut}(\mathbb{Z}^m)$. We say that Θ is a *faithful representation* if it is injective. We recall the following characterisation of crystallographic groups.

LEMMA 5 ([10, Lemma 8]). *Let Π be a group. Then Π is a crystallographic group if and only if there exists an integer $m \in \mathbb{N}$, a finite group Φ and a short exact sequence of the form:*

$$0 \longrightarrow \mathbb{Z}^m \longrightarrow \Pi \xrightarrow{\zeta} \Phi \longrightarrow 1, \tag{2.1}$$

such that the integral representation $\Theta\colon \Phi \longrightarrow Aut(\mathbb{Z}^m)$ induced by conjugation on $\mathbb{Z}^m$ and defined by $\Theta(\varphi)(x) = \pi x \pi^{-1}$ for all $x \in \mathbb{Z}^m$ and $\varphi \in \Phi$, where $\pi \in \Pi$ is such that $\zeta(\pi) = \varphi$, is faithful.

If Π is a crystallographic group, the integer m, the finite group Φ and the integral representation $\Theta\colon \Phi \longrightarrow \mathrm{Aut}(\mathbb{Z}^m)$ appearing in the statement of Lemma 5 are called the *dimension*, the *holonomy group* and the *holonomy representation* of Π respectively.

We now recall the connection between Bieberbach groups and manifolds. A Riemannian manifold M is called *flat* if it has zero curvature at every point. By the first Bieberbach Theorem, there is a correspondence between Bieberbach groups and fundamental groups of flat Riemannian manifolds without boundary (see [5, Theorem 2.1.1] and the paragraph that follows it). By [23, Corollary 3.4.6], the holonomy group of a flat manifold M is isomorphic to the group Φ. In 1957, Auslander and Kuranishi proved that every finite group is the holonomy group of some flat manifold (see [23, Theorem 3.4.8] and [4, Theorem III.5.2]). It is well known that a flat manifold determined by a Bieberbach group Π is orientable if and only if the integral representation $\Theta\colon \Phi \longrightarrow \mathrm{GL}(m, \mathbb{Z})$ satisfies $\mathrm{Im}(\Theta) \subseteq \mathrm{SL}(m, \mathbb{Z})$ [5, Theorem 6.4.6 and Remark 6.4.7]. This being the case, Π is said to be an *orientable Bieberbach group*.

2.2. **The group $B_n(M)/\Gamma_2(P_n(M))$.** Let M be a compact, orientable surface without boundary of genus $g \geq 1$. Besides showing that the group $B_n(M)/\Gamma_2(P_n(M))$ is crystallographic, we shall also be interested in the conjugacy classes of its elements by elements of $P_n(M)/\Gamma_2(P_n(M))$, as well as the conjugacy classes of its finite subgroups. In order to study these questions, it is useful to have an algebraic description of this quotient at our disposal. We will make use of the presentations of the (pure) braid groups of M given in [13, Theorems 2.1 and 4.2], where for all $1 \leq i < j \leq n$, $1 \leq r \leq 2g$ and $1 \leq k \leq n$, the elements $a_{k,r}$ and $T_{i,j}$ in $B_n(M)$ are described in [13, Figure 9], and for all $1 \leq i \leq n-1$, the elements σ_i are the classical generators of the Artin braid group that satisfy the *Artin relations*:

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for all } 1 \leq i, j \leq n-1,\ |i-j| \geq 2 \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } 1 \leq i \leq n-2. \end{cases} \tag{2.2}$$

We recall that the full twist braid of $B_n(M)$, denoted by Δ_n^2, is defined by:

$$\Delta_n^2 = (\sigma_1 \cdots \sigma_{n-1})^n, \tag{2.3}$$

and is equal to:

$$\Delta_n^2 = A_{1,2}(A_{1,3}A_{2,3}) \cdots (A_{1,n}A_{2,n} \cdots A_{n-1,n}), \tag{2.4}$$

where for $1 \leq i < j \leq n$, the elements $A_{i,j}$ are the usual Artin generators of P_n defined by $A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}$. By abuse of notation, in what follows, if $\alpha \in B_n(M)$, we also denote its $B_n(M)/\Gamma_2(P_n(M))$-coset by α. The following proposition gives some relations in $B_n(M)$ that will be relevant to our study of $B_n(M)/\Gamma_2(P_n(M))$.

PROPOSITION 6. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, let $1 \leq i \leq n-1$, $1 \leq j \leq n$ and $1 \leq r \leq 2g$, and let $\widetilde{A}_{j,r} = a_{j,1} \cdots a_{j,r-1}a_{j,r+1}^{-1} \cdots a_{j,2g}^{-1}$. The following relations hold in $B_n(M)$:*

$$(a)\ \sigma_i a_{j,r} \sigma_i^{-1} = \begin{cases} a_{i+1,r}\sigma_i^{-2} & \text{if } j = i \text{ and } r \text{ is even} \\ \sigma_i^2 a_{i+1,r} & \text{if } j = i \text{ and } r \text{ is odd} \\ \sigma_i^2 a_i & \text{if } j = i+1 \text{ and } r \text{ is even} \\ a_{i,r}\sigma_i^{-2} & \text{if } j = i+1 \text{ and } r \text{ is odd} \\ a_{j,r} & \text{if } j \neq i, i+1. \end{cases}$$

(b) $T_{i,j} = \sigma_i\sigma_{i+1}\cdots\sigma_{j-2}\sigma_{j-1}^2\sigma_{j-2}\cdots\sigma_i$ *where* $1 \leq i, j \leq n$ *and* $i+1 < j$, *and* $T_{i,i+1} = \sigma_i^2$ *for all* $1 \leq i \leq n-1$.
(c) $T_{i,j} = [a_{i,1}\cdots a_{i,r}, \widetilde{A}_{j,r}]\, T_{i,j-1}$, *for all* $1 \leq i < j \leq n$ *and* $1 \leq r \leq 2g$.
(d) For all $1 \leq i \leq j \leq n$, $T_{i,j}$ *and* Δ_n^2 *belong to* $\Gamma_2(P_n(M))$.

Proof. Part (a) is a consequence of relations (R7) and (R8) of [13, Theorem 4.2, step 3], with the exception of the case $j \neq i, i+1$, which is clear. Part (b) is relation (R9) of [13, Theorem 4.2, step 3], and part (c) is relation (PR3) of [13, Theorem 4.2, presentation 1]. By [13, page 439], $T_{j-1,j-1} = 1$ for all $1 < j \leq n+1$, and it follows from part (c) that $T_{j-1,j} \in \Gamma_2(P_n(M))$ for all $2 \leq j \leq n$, and then by induction on $j-i$ that $T_{i,j} \in \Gamma_2(P_n(M))$ for all $1 \leq i \leq j \leq n$. Using the Artin relations (2.2) and part (b), we have $A_{i,j} = T_{i,j-1}^{-1}T_{i,j}$ for all $1 \leq i < j \leq n$, so $A_{i,j}$ also belongs to $\Gamma_2(P_n(M))$ by part (c), and thus $\Delta_n^2 \in \Gamma_2(P_n(M))$ by (2.4). □

This allows us to compute the Abelianisation of $P_n(M)$.

Corollary 7. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, and let $n \geq 1$. Then the Abelianisation of $P_n(M)$ is a free Abelian group of rank $2ng$, for which $\{a_{i,r} \mid i = 1, \ldots, n$ and $r = 1, \ldots, 2g\}$ is a basis.*

Proof. The result follows from the presentation of $P_n(M)$ given in [13, Theorem 4.2] and the fact that for all $1 \leq i < j \leq n$, $T_{i,j} \in \Gamma_2(P_n(M))$ by Proposition 6(d). □

For all $1 \leq i \leq n-1$, we have $\overline{\sigma}(\sigma_i) = \tau_i$, where τ_i denotes the transposition $(i, i+1)$ in S_n. Using Proposition 6(a), and identifying $\mathbb{Z}^{2ng}$ with $P_n(M)/\Gamma_2(P_n(M))$ via Corollary 7, we obtain the induced action $\varphi\colon S_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2ng})$, that for all $1 \leq i \leq n-1$, $1 \leq j \leq n$ and $1 \leq r \leq 2g$, is defined by:

$$\varphi(\tau_i)(a_{j,r}) = \sigma_i a_{j,r}\sigma_i^{-1} = a_{\tau_i(j),r}. \tag{2.5}$$

The following result is the analogue of [10, Proposition 12] for braid groups of orientable surfaces.

Proposition 8. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, and let $n \geq 1$. Let $\alpha \in B_n(M)/\Gamma_2(P_n(M))$, and let $\pi = \overline{\sigma}(\alpha^{-1})$. Then $\alpha a_{i,r}\alpha^{-1} = a_{\pi(i),r}$ in $B_n(M)/\Gamma_2(P_n(M))$ for all $1 \leq i \leq n$ and $1 \leq r \leq 2g$.*

Proof. The proof is similar to that of [10, Proposition 12], and makes use of (2.5). The details are left to the reader. □

We now give a presentation of $B_n(M)/\Gamma_2(P_n(M))$.

Proposition 9. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, and let $n \geq 1$. The quotient group $B_n(M)/\Gamma_2(P_n(M))$ has the following presentation:*
Generators: $\sigma_1, \ldots, \sigma_{n-1}, a_{i,r}$, $1 \leq i \leq n$, $1 \leq r \leq 2g$.
Relations:
(a) the Artin relations (2.2).
(b) $\sigma_i^2 = 1$, *for all* $i = 1, \ldots, n-1$.
(c) $[a_{i,r}, a_{j,s}] = 1$, *for all* $i, j = 1, \ldots, n$ *and* $r, s = 1, \ldots, 2g$.
(d) $\sigma_i a_{j,r}\sigma_i^{-1} = a_{\tau_i(j),r}$ *for all* $1 \leq i \leq n-1$, $1 \leq j \leq n$ *and* $1 \leq r \leq 2g$.

Proof. The given presentation of $B_n(M)/\Gamma_2(P_n(M))$ may be obtained by applying the standard method for obtaining a presentation of a group extension [14, Proposition 1, p. 139] to the short exact sequence (1.2) for M satisfying the hypothesis, and using Corollary 7, the equality $\overline{\sigma}(\sigma_i) = \tau_i$ for all $i = 1, \ldots, n-1$, and the fact that the relations (a) and (b) constitute a presentation of S_n for the generating set $\{\tau_1, \ldots, \tau_{n-1}\}$. □

We may now prove Proposition 1.

Proof of Proposition 1. Assume that $n \geq 2$. The short exact sequence (1.3) is obtained from (1.2) using Corollary 7. To prove that the short exact sequence (1.3) splits, let $\psi\colon S_n \longrightarrow B_n(M)/\Gamma_2(P_n(M))$ be the map defined on the generating set $\{\tau_1, \ldots, \tau_{n-1}\}$ of S_n by $\psi(\tau_i) = \sigma_i$ for all $i = 1, \ldots, n-1$.

Relations (a) and (b) of Proposition 9 imply that ψ is a homomorphism. Consider the action $\varphi\colon S_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2ng})$ defined by (2.5). By Proposition 8, $\varphi(\theta)$ is the identity automorphism if and only if θ is the trivial permutation, from which it follows that φ is injective. The rest of the statement of Proposition 1 is a consequence of Lemma 5. □

Corollary 10. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, let $n \geq 2$, and let H be a subgroup of S_n. Then the group $\sigma^{-1}(H)/\Gamma_2(P_n(M))$ is a crystallographic group of dimension $2ng$ whose holonomy group is H.*

Proof. The result is an immediate consequence of Proposition 1 and [10, Corollary 10]. □

We now turn to the proof of Theorem 2. For this, we will require the following lemma.

Lemma 11. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$, and let $n \geq 1$. Let $z \in B_n(M)/\Gamma_2(P_n(M))$. Let $z = \omega \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{s_{i,r}} \in B_n(M)/\Gamma_2(P_n(M))$, where $\omega = \psi(\overline{\sigma}(z))$, and $s_{i,r} \in \mathbb{Z}$ for all $1 \leq i \leq n$ and $1 \leq r \leq 2g$. Suppose that $\overline{\sigma}(z)$ is the m-cycle $(l_1, \ldots, l_m)$, where $m \geq 2$, and let $k \in \mathbb{N}$ be such that m divides k. Then $z^k = \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{t_{i,r}}$ where for all $1 \leq i \leq n$ and $1 \leq r \leq 2g$, $t_{i,r} \in \mathbb{Z}$ is given by:*

$$t_{i,r} = \begin{cases} k s_{i,r} & \text{if } i \notin \{l_1, \ldots, l_m\} \\ \frac{k}{m} \Sigma_{j=1}^{m} s_{l_j,r} & \text{if } i = l_j, \text{ where } 1 \leq j \leq m. \end{cases} \tag{2.6}$$

Proof. Let $z \in B_n(M)/\Gamma_2(P_n(M))$ be as in the statement, let $\overline{\sigma}(z) = (l_1, \ldots, l_m)$, where $m \geq 2$, and let $\omega = \psi(\overline{\sigma}(z))$. By Proposition 1, ω is of order m, and the decomposition $z = \omega \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{s_{i,r}}$ arises from (1.3). By Corollary 7 and Proposition 8 and using the fact that $\omega^k = 1$, we obtain:

$$\begin{aligned} z^k &= \left(\omega \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{s_{i,r}}\right)^k = \omega^k \left[\prod_{j=1}^{k} \omega^{j-k} \left(\prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{s_{i,r}}\right) \omega^{k-j}\right] = \prod_{j=1}^{k} \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{\overline{\sigma}(\omega^{k-j})(i),r}^{s_{i,r}} \\ &= \prod_{i=1}^{n} \prod_{r=1}^{2g} \prod_{j=1}^{k} a_{i,r}^{s_{\overline{\sigma}(\omega^{j-k})(i),r}} = \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{t_{i,r}}, \end{aligned} \tag{2.7}$$

where $t_{i,r} = \sum_{j=1}^{k} s_{\overline{\sigma}(\omega^{j-k})(i),r}$ for all $1 \leq i \leq n$ and $1 \leq r \leq 2g$. Equation (2.6) then follows, using the fact that $\overline{\sigma}(\omega^{-u})(l_i) = l_{i-u}$ for all $u \in \mathbb{Z}$, where the index $i - u$ is taken modulo m. □

We now prove Theorem 2.

Proof of Theorem 2.

(a) Let $\psi\colon S_n \longrightarrow B_n(M)/\Gamma_2(P_n(M))$ be the section for the short exact sequence (1.3) given in the proof of Proposition 1, and let e_1 and e_2 be finite-order elements of $B_n(M)/\Gamma_2(P_n(M))$. If e_1 and e_2 are conjugate in $B_n(M)/\Gamma_2(P_n(M))$ then $\overline{\sigma}(e_1)$ and $\overline{\sigma}(e_2)$ are conjugate in S_n, and so have the same cycle type. Conversely, suppose that the permutations $\overline{\sigma}(e_1)$ and $\overline{\sigma}(e_2)$ have the same cycle type. Then they are conjugate in S_n, so there exists $\xi \in S_n$ such that $\overline{\sigma}(e_1) = \xi \overline{\sigma}(e_2) \xi^{-1}$, and up to substituting e_2 by $\psi(\xi^{-1}) e_2 \psi(\xi)$ if necessary, we may assume that $\overline{\sigma}(e_1) = \overline{\sigma}(e_2)$. We claim that if θ is any finite-order element of $B_n(M)/\Gamma_2(P_n(M))$ then θ and $\psi(\overline{\sigma}(\theta))$ are conjugate in $B_n(M)/\Gamma_2(P_n(M))$. This being the case, for $i = 1, 2$, e_i is conjugate to $\psi(\overline{\sigma}(e_i))$, but since $\psi(\overline{\sigma}(e_1)) = \psi(\overline{\sigma}(e_2))$, it follows that e_1 and e_2 are conjugate as required, which proves the first part of the statement. To prove the claim, set $\theta = \omega \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{s_{i,r}} \in B_n(M)/\Gamma_2(P_n(M))$ as in the proof of Lemma 11, where $\omega = \psi(\overline{\sigma}(\theta))$. It thus suffices to prove that θ and ω are conjugate in $B_n(M)/\Gamma_2(P_n(M))$. Let $\overline{\sigma}(\theta) = \tau_1 \cdots \tau_d$ be the cycle decomposition of $\overline{\sigma}(\theta)$, where for $i = 1, \ldots, d$, $\tau_i = (l_{i,1}, \ldots, l_{i,m_i})$ is an m_i-cycle, and $m_i \geq 2$, and let $k = \mathrm{lcm}(m_1, \ldots, m_d)$ be the order of $\overline{\sigma}(\theta)$, which is also the order of θ and of ω. For $t = 1, \ldots, n$ and $r = 1, \ldots, 2g$, we define $p_{t,r} \in \mathbb{Z}$ as follows. If t does not belong to the support $\mathrm{Supp}(\overline{\sigma}(\theta))$ of $\overline{\sigma}(\theta)$, let $p_{t,r} = 0$. If $t \in \mathrm{Supp}(\overline{\sigma}(\theta))$ then $t = l_{i,q}$ for some $1 \leq i \leq d$ and $1 \leq q \leq m_i$, and we define $p_{t,r} = -\sum_{j=1}^{q} s_{l_{i,j},r}$.

It follows from Lemma 11 and the fact that θ is of order k that $p_{l_{i,m_i},r} = 0$, for all $i = 1, \dots, d$. Then for all $1 \leq i \leq d$, $2 \leq q \leq m_i$ and $1 \leq r \leq 2g$, we have:

$$p_{l_{i,q-1},r} - p_{l_{i,q},r} = s_{l_{i,q},r} \text{ and } p_{l_{i,m_i},r} - p_{l_{i,1},r} = -p_{l_{i,1},r} = s_{l_{i,1},r}. \tag{2.8}$$

Let $\alpha = \prod_{i=1}^{n} \prod_{r=1}^{2g} a_{i,r}^{p_{i,r}} \in P_n(M)/\Gamma_2(P_n(M))$. By Corollary 7 and Proposition 8, we have:

$$\begin{aligned}\alpha\omega\alpha^{-1} &= \omega.\omega^{-1}\left(\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{p_{i,r}}\right)\omega.\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{-p_{i,r}} = \omega\left(\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{p_{\overline{\sigma}(\omega^{-1})(i),r}}\right).\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{-p_{i,r}} \\ &= \omega\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{p_{\overline{\sigma}(\omega^{-1})(i),r}-p_{i,r}} = \omega\prod_{i=1}^{n}\prod_{r=1}^{2g} a_{i,r}^{s_{i,r}} = \theta,\end{aligned}$$

where we have also made use of (2.8). Thus θ is conjugate to ω, which proves the claim, and thus the first part of the statement.

(b) We start by showing that if $H \subset B_n(M)/\Gamma_2(P_n(M))$ is isomorphic to S_n then H and $\psi(\overline{\sigma}(H))$ are conjugate. This being the case, it follows that each of the subgroups H_1, H_2 is conjugate to $\psi(\overline{\sigma}(H))$, and the result follows. Suppose that H is a group isomorphic to S_n that embeds in $(\mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n) \rtimes S_n$. Using the $\mathbb{Z}[S_n]$-module structure of $\mathbb{Z}^{2ng}$ given above, it follows that H embeds in $\mathbb{Z}^n \rtimes S_n$, for any one of $2g$ summands of $\mathbb{Z}^n$. Let us first prove that the result holds for such an embedding. Let $s\colon H \longrightarrow \mathbb{Z}^n \rtimes S_n$ be an embedding, let $\pi\colon \mathbb{Z}^n \rtimes S_n \longrightarrow S_n$ be projection onto the second factor, and let $\psi\colon S_n \longrightarrow \mathbb{Z}^n \rtimes S_n$ be inclusion into the second factor. Since $\mathrm{Ker}\,(\pi) = \mathbb{Z}^n$ is torsion free, the restriction of π to $s(H)$ is injective, and so $\pi \circ s\colon H \longrightarrow S_n$ is an isomorphism. Let us prove that the subgroups $s(H)$ and S_n of $\mathbb{Z}^n \rtimes S_n$ are conjugate. Let $\{\tau_1, \dots, \tau_{n-1}\}$ be the generating set of S_n defined previously, and for $i = 1, \dots, n-1$, let α_i be the unique element of H for which $\pi \circ s(\alpha_i) = \tau_i$. Then H is generated by $\{\alpha_1, \dots, \alpha_{n-1}\}$, subject to the Artin relations and $\alpha_i^2 = 1$ for all $i = 1, \dots, n-1$. There exist $a_{i,j} \in \mathbb{Z}$, where $1 \leq i \leq n-1$ and $1 \leq j \leq n$, such that $s(\alpha_i) = (a_{i,1}, \dots, a_{i,n})\tau_i$ in $\mathbb{Z}^n \rtimes S_n$. Using the action of S_n on $\mathbb{Z}^n$ and the fact that $\alpha_i^2 = 1$, it follows that $a_{i,j} = 0$ for all $j \neq i, i+1$, and $a_{i,i+1} = -a_{i,i}$. Then $s(\alpha_i) = (0, \dots, 0, a_i, -a_i, 0, \dots, 0)\tau_i$ for all $1 \leq i \leq n-1$, where $a_i = a_{i,i}$, and the element a_i is in the ith position. One may check easily that these elements also satisfy the Artin relations in $\mathbb{Z}^n \rtimes S_n$. Let $x_1 \in \mathbb{Z}$, and for $i = 2, \dots, n$, let $x_i = x_1 - \sum_{j=1}^{i-1} a_j$. Thus $x_i - x_{i+1} = a_i$ for all $i = 1, \dots, n-1$, and so $(x_1, x_2, \dots, x_i, x_{i+1}, \dots, x_n)\tau_i(-x_1, -x_2, \dots, -x_i, -x_{i+1}, \dots - x_n) = (0, \dots, 0, a_i, -a_i, 0, \dots, 0)\tau_i = s(\alpha_i)$. We conclude that the subgroups $s(H)$ and S_n of $\mathbb{Z}^n \rtimes S_n$ are conjugate.

Returning to the general case where H embeds in $(\mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n) \rtimes S_n$, the previous paragraph shows that the embedding of H into each $\mathbb{Z}^n \rtimes S_n$ is conjugate by an element of the same factor $\mathbb{Z}^n$ to the factor S_n. The result follows by conjugating by the element of $\mathbb{Z}^n \oplus \cdots \oplus \mathbb{Z}^n$ whose ith factor is the conjugating element associated to the ith copy of $\mathbb{Z}^n \rtimes S_n$ for all $i = 1, \dots, 2g$. □

With the statement of Theorem 2 in mind, one may ask whether the result of the claim extends to other finite subgroups. More precisely, if G is a finite subgroup of $B_n(M)/\Gamma_2(P_n(M))$, are G and $\psi(\overline{\sigma}(G))$ conjugate? We have a positive answer in the following case.

PROPOSITION 12. *Let M be a compact, orientable surface without boundary of genus $g \geq 1$. If H_1 and H_2 are subgroups of $B_5(M)/\Gamma_2(P_5(M))$ that are isomorphic to the Frobenius group $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$ then they are conjugate.*

Proof. Using Proposition 1, we identify $B_5(M)/\Gamma_2(P_5(M))$ with $\mathbb{Z}^{10g} \rtimes S_5$. As in the proof of Theorem 2(b), we decompose the first factor as a direct sum $\mathbb{Z}^{10g} = \mathbb{Z}^5 \oplus \cdots \oplus \mathbb{Z}^5$ of $2g$ copies of $\mathbb{Z}^5$, which we interpret as a $\mathbb{Z}[S_5]$-module, the module structure being given by Proposition 8.

Let H be a group isomorphic to a subgroup of S_5 that embeds in $(\mathbb{Z}^5 \oplus \cdots \oplus \mathbb{Z}^5) \rtimes S_5 \cong B_5(M)/\Gamma_2(P_5(M))$. Using the $\mathbb{Z}[S_5]$-module structure of $\mathbb{Z}^{10g}$ given above, it follows that H embeds in $\mathbb{Z}^5 \rtimes S_5$, for any one of the $2g$ summands of $\mathbb{Z}^5$. We will first prove the statement for the embedding of the Frobenius group $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$ in $\mathbb{Z}^5 \rtimes S_5$, and then deduce the result in the general

case. Let H be this Frobenius group, let $s\colon H \longrightarrow \mathbb{Z}^5 \rtimes S_5$ be an embedding, let $\pi\colon \mathbb{Z}^5 \rtimes S_5 \longrightarrow S_5$ be projection onto the first factor, and let $\psi\colon S_5 \longrightarrow \mathbb{Z}^5 \rtimes S_5$ be inclusion into the second factor. Since $\mathrm{Ker}\,(\pi) = \mathbb{Z}^5$ is torsion free, the restriction of π to $s(H)$ is injective, and so $\pi \circ s\colon H \longrightarrow S_5$ is an embedding of H into S_5. Let us prove that the subgroups $s(H)$ and $\psi \circ \pi \circ s(H)$ of $\mathbb{Z}^5 \rtimes S_5$ are conjugate. First observe that the Frobenius group $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$ embeds in S_5 by sending a generator ι_5 of $\mathbb{Z}_5$ to the permutation $w_1 = (1,2,3,4,5)$ and the generator ι_2 of $\mathbb{Z}_2$ to $w_2 = (1,4)(2,3)$. The group S_5 contains six cyclic subgroups of order 5 that are conjugate to $\langle w_1\rangle$, from which we deduce the existence of six pairwise conjugate subgroups of S_5 isomorphic to $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$, each of which contains one of the cyclic subgroups of order 5. We claim that these are exactly the subgroups of S_5 isomorphic to $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$. To see this, let K be such a subgroup. Then the action of an element k of K of order 2 on a 5-cycle $(a_1,\dots,a_5)$ of K inverts the order of the elements $a_1,\dots,a_5$, and this can only happen if k is a product of two disjoint transpositions. Further, there are exactly five products of two disjoint transpositions whose action by conjugation on $(a_1,\dots,a_5)$ inverts the order of the elements $a_1,\dots,a_5$, and these are precisely the elements of K of order 2. In particular, each cyclic subgroup of S_5 of order 5 is contained in exactly one subgroup of S_5 isomorphic to $\mathbb{Z}_5 \rtimes \mathbb{Z}_2$. This proves the claim. So up to composing π by an inner automorphism of S_5 if necessary, we may assume that $\pi \circ s(H) = \langle w_1, w_2\rangle$. Applying methods similar to those of the proof of Lemma 11, the relations $\iota_5^5 = 1$ and $\iota_2^2 = 1$ imply that there exist $a_1,\dots,a_5,x,y \in \mathbb{Z}$ such that:

$$s(w_1) = (a_1,a_2,a_3,a_4,-a_1-a_2-a_3-a_4)w_1 \text{ and } s(w_2) = (x,y,-y,-x,0)w_2. \tag{2.9}$$

Taking the image of the relation $w_2w_1w_2^{-1} = w_1^{-1}$ by s, using (2.9) and simplifying the resulting expression, we obtain:

$$x = -a_2-a_3-a_4 \text{ and } y = -a_3. \tag{2.10}$$

Any map $s\colon H \longrightarrow \mathbb{Z}^5 \rtimes S_5$ of the form (2.9) for which the relations (2.10) hold gives rise to an embedding of H in $\mathbb{Z}^5 \rtimes S_5$. We claim that the image of the embedding is conjugate to the group $\langle w_1, w_2\rangle$ (viewed as a subgroup of the second factor of $\mathbb{Z}^5 \rtimes S_5$). To do so, let $s\colon H \longrightarrow \mathbb{Z}^5 \rtimes S_5$ of the form (2.9) for which the relations (2.10) hold. We will show that there exists $a \in \mathbb{Z}^5$ such that $aw_ia^{-1} = s(w_i)$ for $i = 1,2$. Let $\lambda_5 \in \mathbb{Z}$, and for $i = 1,\dots,4$, let $\lambda_i = \lambda_5 + \sum_{j=1}^{i} a_i$, and let $a = (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5) \in \mathbb{Z}^5$. Then in $\mathbb{Z}^5 \rtimes S_5$, using (2.9) and (2.10), we have:

$$\begin{aligned} aw_1a^{-1} &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)w_1(-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5)\\ &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5).(-\lambda_5,-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4)w_1\\ &= (\lambda_1-\lambda_5,\lambda_2-\lambda_1,\lambda_3-\lambda_2,\lambda_4-\lambda_3,\lambda_5-\lambda_4)w_1\\ &= (a_1,a_2,a_3,a_4,-a_1-a_2-a_3-a_4)w_1 = s(w_1), \end{aligned}$$

and

$$\begin{aligned} aw_2a^{-1} &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)w_2(-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5)\\ &= (\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5).(-\lambda_4,-\lambda_3,-\lambda_2,-\lambda_1,-\lambda_5)w_2\\ &= (\lambda_1-\lambda_4,\lambda_2-\lambda_3,\lambda_3-\lambda_2,\lambda_4-\lambda_1,0)w_2\\ &= (-a_2-a_3-a_4,-a_3,a_3,a_2+a_3+a_4,0)w_2 = (x,y,-y,-x,0)w_2 = s(w_2). \end{aligned}$$

It follows that the subgroups $s(H)$ and $\langle w_1, w_2\rangle$ are conjugate in $\mathbb{Z}^5 \rtimes S_5$.

As in the proof of Theorem 2(b), returning to the general case where H embeds in $(\mathbb{Z}^5 \oplus \cdots \oplus \mathbb{Z}^5) \rtimes S_5$, the previous paragraph shows that the embedding of H into each $\mathbb{Z}^5 \rtimes S_5$ is conjugate by an element of the same factor $\mathbb{Z}^5$ to the factor S_5. The result follows by conjugating by the element of $\mathbb{Z}^5 \oplus \cdots \oplus \mathbb{Z}^5$ whose ith factor is the conjugating element associated to the ith copy of $\mathbb{Z}^5 \rtimes S_5$ for all $i = 1,\dots,2g$. □

3. Some Bieberbach subgroups of $B_n(M)/\Gamma_2(P_n(M))$ and Kähler flat manifolds

By Corollary 10, for any subgroup H of S_n, the quotient group $\sigma^{-1}(H)/\Gamma_2(P_n(M))$ is a crystallographic group that is not Bieberbach since it has torsion elements. We start by proving Theorem 3, which states that $B_n(M)/\Gamma_2(P_n(M))$ admits Bieberbach subgroups. More precisely, for all $n \geq 2$, we will consider the cyclic subgroup $G_n = \langle (n, n-1, \ldots, 2, 1) \rangle$ of S_n, and we show that $\sigma^{-1}(G_n)/\Gamma_2(P_n(M))$ admits a Bieberbach subgroup $\widetilde{G}_{n,g}$ of dimension $2ng$ whose holonomy group is G_n. The group $\widetilde{G}_{n,g}$ is thus the fundamental group of a flat manifold. In Theorem 4, we will prove that this flat manifold is orientable and admits a Kähler structure as well as Anosov diffeomorphisms.

Proof of Theorem 3. Let $\alpha_{n-1} = \sigma_1 \cdots \sigma_{n-1} \in B_n(M)/\Gamma_2(P_n(M))$. By equation (1.3), $\overline{\sigma}(\alpha_{n-1})$ generates the subgroup G_n of S_n. Further, the full twist of $B_n(M)$ is a coset representative of α_{n-1}^n by (2.3), hence $\alpha_{n-1}^n = 1$ in $B_n(M)/\Gamma_2(P_n(M))$ using Proposition 6(d). By Proposition 8, the action by conjugation of α_{n-1} on the elements of the basis $\{a_{i,r} \mid i = 1, \ldots, n \text{ and } r = 1, \ldots, 2g\}$ of $P_n(M)/\Gamma_2(P_n(M))$ given by Corollary 7 is as follows:

$$\alpha_{n-1} : a_{1,r} \longmapsto a_{2,r} \longmapsto \cdots \longmapsto a_{n-1,r} \longmapsto a_{n,r} \longmapsto a_{1,r} \text{ for all } r = 1, \ldots, 2g. \tag{3.1}$$

Using (3.1) and the fact that $\alpha_{n-1}^n = 1$ in $B_n(M)/\Gamma_2(P_n(M))$, we have:

$$(a_{1,1}\alpha_{n-1})^n = a_{1,1}a_{2,1}\cdots a_{n,1}(\alpha_{n-1})^n = a_{1,1}a_{2,1}\cdots a_{n,1}\Delta_n^2 = \prod_{i=1}^{n} a_{i,1}. \tag{3.2}$$

Let $X = \{a_{1,1}\alpha_{n-1},\, a_{i,r}^n \,|\, 1 \leq i \leq n \text{ and } 1 \leq r \leq 2g\}$, let $Y = \{\prod_{i=1}^n a_{i,1},\, a_{i,r}^n \,|\, 1 \leq i \leq n \text{ and } 1 \leq r \leq 2g\}$, and let $\widetilde{G}_{n,g}$ (resp. L) be the subgroup of $\sigma^{-1}(G_n)/\Gamma_2(P_n(M))$ generated by X (resp. Y). Then the restriction $\overline{\sigma}\big|_{\widetilde{G}_{n,g}} : \widetilde{G}_{n,g} \longrightarrow G_n$ is surjective, and using (3.2), we see that L is a subgroup of $\widetilde{G}_{n,g}$. We claim that $L = \mathrm{Ker}\big(\overline{\sigma}\big|_{\widetilde{G}_{n,g}}\big)$. Clearly, $L \subset \mathrm{Ker}\big(\overline{\sigma}\big|_{\widetilde{G}_{n,g}}\big)$. Conversely, let $w \in \mathrm{Ker}\big(\overline{\sigma}\big|_{\widetilde{G}_{n,g}}\big)$. Writing w in terms of the generating set X of $\widetilde{G}_{n,g}$ and using (3.1) and Corollary 7, it follows that $w = (a_{1,1}\alpha_{n-1})^m \prod_{\substack{1\leq i\leq n\\ 1\leq r\leq 2g}} a_{i,r}^{n\delta_{i,r}}$, where $m \in \mathbb{Z}$ and $\delta_{i,r} \in \mathbb{Z}$ for all $1 \leq i \leq n$ and $1 \leq r \leq 2g$. The fact that $w \in \mathrm{Ker}\big(\overline{\sigma}\big|_{\widetilde{G}_{n,g}}\big)$ implies that n divides m, and so $w = ((a_{1,1}\alpha_{n-1})^n)^{m/n} \prod_{\substack{1\leq i\leq n\\ 1\leq r\leq 2g}} a_{i,r}^{n\delta_{i,r}} \in L$, which proves the claim. Thus the following extension:

$$1 \longrightarrow L \longrightarrow \widetilde{G}_{n,g} \xrightarrow{\overline{\sigma}\big|_{\widetilde{G}_{n,g}}} G_n \longrightarrow 1, \tag{3.3}$$

is short exact. Now L is also a subgroup of $P_n(M)/\Gamma_2(P_n(M))$, which is free Abelian of rank $2ng$ by Corollary 7. Since $\{a_{i,r} \mid i = 1, \ldots, n \text{ and } r = 1, \ldots, 2g\}$ is a basis of $P_n(M)/\Gamma_2(P_n(M))$, it follows from analysing Y that $Y' = \{\prod_{i=1}^n a_{i,1},\, a_{i,1}^n, a_{j,r}^n \,|\, 2 \leq i \leq n,\, 1 \leq j \leq n \text{ and } 2 \leq r \leq 2g\}$ is a basis of L. In particular L is free Abelian of rank $2ng$. In terms of the basis Y', the holonomy representation $\rho\colon G_n \longrightarrow \mathrm{Aut}(L)$ associated with the short exact sequence (3.3) is given by the block diagonal matrix:

$$\rho((1, n, n-1, \ldots, 2)) = \begin{pmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_{2g} \end{pmatrix}, \tag{3.4}$$

where $M_1, \ldots, M_{2g}$ are the n-by-n matrices satisfying:

$$M_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & n \\ 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -1 \\ 0 & 0 & 1 & \cdots & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & -1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -1 \end{pmatrix} \text{ and } M_2 = \cdots = M_{2g} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix},$$

where we have used the relation $a_{1,1}^n = (\prod_{i=1}^n a_{i,1})^n \cdot \prod_{i=2}^n (a_{i,1}^n)^{-1}$. It follows from this description that ρ is injective, and from Lemma 5 and (3.3) that $\widetilde{G}_{n,g}$ is a crystallographic group of dimension $2ng$ and whose holonomy group is $\mathbb{Z}_n$.

Now we prove that $\widetilde{G}_{n,g}$ is torsion free. Let $\omega \in \widetilde{G}_{n,g}$ be an element of finite order. Since L is torsion free, the order of ω is equal to that of $\overline{\sigma}(\omega)$ in the cyclic group G_n, in particular $\omega^n = 1$. Using (3.1) and (3.2), as well as the fact that L is torsion free, there exist $\theta \in L$ and $j \in \{0, 1, 2, \dots, n-1\}$ such that $\omega = \theta(a_{1,1}\alpha_{n-1})^j$. Making use of the basis Y' of L,

$$\theta = \left(\prod_{i=1}^{n} a_{i,1}\right)^{\lambda_{1,1}} \cdot \prod_{i=2}^{n} a_{i,1}^{n\lambda_{i,1}} \cdot \prod_{r=2}^{2g} \prod_{i=1}^{n} a_{i,r}^{n\lambda_{i,r}}, \tag{3.5}$$

where $\lambda_{i,r} \in \mathbb{Z}$ for all $i = 1, \dots, n$ and $r = 1, \dots, 2g$. On the other hand:

$$1 = \omega^n = \left(\prod_{k=0}^{n-1} (a_{1,1}\alpha_{n-1})^{jk} \theta (a_{1,1}\alpha_{n-1})^{-jk}\right) (a_{1,1}\alpha_{n-1})^{nj}. \tag{3.6}$$

By (3.2), $(a_{1,1}\alpha_{n-1})^{nj} = \prod_{i=1}^n a_{i,1}^j$, and thus the right-hand side of (3.6) belongs to $P_n(M)/\Gamma_2(P_n(M))$. We now compute the coefficient of $a_{1,1}$ in (3.6) considered as one of the elements of the basis of $P_n(M)/\Gamma_2(P_n(M))$ given by Corollary 7. From (3.1), the terms appearing in the product $\prod_{r=2}^{2g} \prod_{i=1}^{n} a_{i,r}^{n\lambda_{i,r}}$ of (3.5) do not contribute to this coefficient. Since conjugation by $a_{1,1}\alpha_{n-1}$ permutes cyclically the elements $a_{1,1}, a_{2,1}, \dots, a_{n,1}$ by (3.1), it follows that conjugation by $a_{1,1}\alpha_{n-1}$ leaves the first term $\prod_{i=1}^n a_{i,1}^{\lambda_{1,1}}$ of (3.5) invariant, and so with respect to (3.6), it contributes $n\lambda_{1,1}$ to the coefficient of $a_{1,1}$. In a similar manner, with respect to (3.6), the second term of (3.5) contributes $n(\lambda_{2,1} + \cdots + \lambda_{n,1})$ to the coefficient of $a_{1,1}$. Putting together all of this information, the computation of the coefficient of $a_{1,1}$ in (3.6) yields the relation $n(\lambda_{1,1} + \lambda_{2,1} + \cdots + \lambda_{n,1}) + j = 0$. We conclude that $j = 0$, so $\omega = \theta = 1$ because L is torsion free, which using the first part of the statement, shows that $\widetilde{G}_{n,g}$ is a Bieberbach group.

To prove the last part of the statement, using [22, Lemma 5.2(3)] and the fact that G_n is cyclic, the centre $Z(\widetilde{G}_{n,g})$ of $\widetilde{G}_{n,g}$ is given by:

$$Z(\widetilde{G}_{n,g}) = \{\theta \in L \,|\, \rho(g)(\theta) = \theta \text{ for all } g \in G_n\} = \{\theta \in L \,|\, \rho((1, n, \dots, 2))(\theta) = \theta\}. \tag{3.7}$$

To compute $Z(\widetilde{G}_{n,g})$, let $\theta \in L$. Writing θ with respect to the basis Y' of L as a vector $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{2g} \end{pmatrix}$, where for all $i = 1, \dots, 2g$, β_i is a column vector with n elements, and using the description of the action ρ given by (3.4), it follows that $\theta \in Z(\widetilde{G}_{n,g})$ if and only if β_i belongs to the eigenspace of M_i with respect to the eigenvalue 1 for all $i = 1, \dots, 2g$. It is straightforward to see that these eigenspaces are of dimension 1, and are generated by $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ if $i = 1$ and by $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ if $i = 2, \dots, 2g$. We conclude using (3.7) that $Z(\widetilde{G}_{n,g})$ is the free Abelian group generated by $\{\prod_{i=1}^n a_{i,1}, \prod_{i=1}^n a_{i,r}^n \,|\, 2 \leq r \leq 2g\}$. This generating set may be seen to be a basis of $Z(\widetilde{G}_{n,g})$, in particular, $Z(\widetilde{G}_{n,g})$ is free Abelian of rank $2g$. □

We do not know whether $B_n(M)/\Gamma_2(P_n(M))$ admits a Bieberbach subgroup of maximal rank whose holonomy group is non Abelian. The following proposition shows that a certain Frobenius group cannot be the holonomy of any Bieberbach subgroup of $B_n(M)/\Gamma_2(P_n(M))$.

PROPOSITION 13. *Let p be an odd prime, and let M be a compact, orientable surface without boundary of genus $g \geq 1$. In $B_p(M)/\Gamma_2(P_p(M))$ there is no Bieberbach subgroup H such that $\overline{\sigma}(H)$ is isomorphic to the Frobenius group $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_{(p-1)/2}$, where the automorphism $\theta(\iota_{(p-1)/2})$ is of order $(p-1)/2$, $\iota_{(p-1)/2}$ being a generator of $\mathbb{Z}_{(p-1)/2}$.*

Proof. Let H be a subgroup of $B_p(M)/\Gamma_2(P_p(M))$ such that $\overline{\sigma}(H)$ is isomorphic to the Frobenius group $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_{(p-1)/2}$. Let us show that H has non-trivial elements of finite order. Using Proposition 1, we also identify $B_p(M)/\Gamma_2(P_p(M))$ with $\mathbb{Z}^{2gp} \rtimes S_p$. Each element $B_p(M)/\Gamma_2(P_p(M))$ may thus be written as $x.\psi(w)$ where $x \in P_p(M)/\Gamma_2(P_p(M))$ and $w \in S_p$, which we refer to as its normal form. As in the proofs of Theorem 2(b) and Proposition 12, $P_p(M)/\Gamma_2(P_p(M))$ splits as a direct sum $\oplus_1^{2g}\mathbb{Z}^p$ that we interpret as a $\mathbb{Z}[S_p]$-module, the module structure being given by Proposition 8. If $z \in P_p(M)/\Gamma_2(P_p(M))$ then for $j = 1, \ldots, 2g$, let z_j denote its projection onto the jth summand of $\oplus_1^{2g}\mathbb{Z}^p$, and for $(z,\tau) \in \mathbb{Z}^{2gp} \rtimes S_p$, let $(z,\tau)_j = (z_j, \tau) \in \mathbb{Z}^p \rtimes S_p$. Let $\varepsilon\colon \mathbb{Z}^p \longrightarrow \mathbb{Z}$ denote the augmentation homomorphism. We extend ε to a map from $\mathbb{Z}^p \rtimes S_p$ to $\mathbb{Z}$, also denoted by ε, by setting $\varepsilon(z,\tau) = \varepsilon(z)$ for all $(z,\tau) \in \mathbb{Z}^p \rtimes S_p$. Using the $\mathbb{Z}[S_p]$-module structure, observe that:

$$\varepsilon((\lambda z \lambda^{-1})_j) = \varepsilon(z_j) \text{ for all } \lambda \in B_p(M)/\Gamma_2(P_p(M)) \text{ and } z \in P_p(M)/\Gamma_2(P_p(M)). \tag{3.8}$$

Hence for all $(z,\tau), (z',\tau') \in \mathbb{Z}^p \rtimes S_p$:

$$\varepsilon(z\tau.\, z'\tau') = \varepsilon(z\tau z'\tau^{-1}.\, \tau\tau') = \varepsilon(z.\, \tau z'\tau^{-1}) = \varepsilon(z).\varepsilon(\tau z'\tau^{-1}) = \varepsilon(z) + \varepsilon(z') = \varepsilon(z,\tau) + \varepsilon(z',\tau'),$$

and thus $\varepsilon\colon \mathbb{Z}^p \rtimes S_p \longrightarrow \mathbb{Z}$ is a homomorphism. Identifying $\overline{\sigma}(H)$ with the Frobenius group $\mathbb{Z}_p \rtimes_\theta \mathbb{Z}_{(p-1)/2}$, let $w_1, w_2 \in S_p$ be generators of $\mathbb{Z}_p$ and $\mathbb{Z}_{(p-1)/2}$ respectively. For $i = 1, 2$, let $v_i \in H$ be such that $\overline{\sigma}(v_i) = w_i$, and let $a_i \in P_p(M)/\Gamma_2(P_p(M))$ be such that $v_i = a_i\psi(w_i)$, where $\psi\colon S_p \longrightarrow B_p(M)/\Gamma_2(P_p(M))$ is the section for $\overline{\sigma}$ given in the proof of Proposition 1. Using the relation $w_2 w_1 w_2^{-1} = w_1^l$ in the Frobenius group, where l is an element of the multiplicative group $\mathbb{Z}_p^*$ of order $(p-1)/2$, we have $v_2 v_1 v_2^{-1} v_1^{-l} \in H \cap P_p(M)/\Gamma_2(P_p(M))$. Further:

$$\begin{aligned} v_2 v_1 v_2^{-1} v_1^{-l} =& a_2\psi(w_2) a_1\psi(w_1)\psi(w_2)^{-1} a_2^{-1} (a_1\psi(w_1))^{-l} \\ =& a_2\psi(w_2) a_1\psi(w_2)^{-1} a_2^{-1}.\, a_2.\, \psi(w_2 w_1 w_2^{-1}) a_2^{-1} \psi(w_2 w_1 w_2^{-1})^{-1}.\, \psi(w_2 w_1 w_2^{-1} w_1^{-l}). \\ & \prod_{k=1}^{l} (\psi(w_1)^{l-k} a_1^{-1} \psi(w_1)^{k-l}), \end{aligned}$$

and applying (3.8) and using the relation $w_2 w_1 w_2^{-1} = w_1^l$, it follows that:

$$\varepsilon((v_2 v_1 v_2^{-1} v_1^{-l})_i) = (1-l)\varepsilon((a_1)_i) \tag{3.9}$$

for all $1 \le i \le 2g$. Let $v = v_2 v_1 v_2^{-1} v_1^{-l} v_1^{l-1}$. The element v_1^{l-1} also belongs to H, so $v \in H$, and since $v_1^{l-1} = (a_1\psi(w_1))^{l-1} = \left(\prod_{k=0}^{l-2} (\psi(w_1)^k a_1 \psi(w_1)^{-k})\right) \psi(w_1)^{l-1}$, for all $1 \le i \le 2g$, it follows that $(v)_i = \beta_i \psi(w_1)^{l-1}$, where $\beta_i \in \mathbb{Z}^p$ is given by:

$$\beta_i = (v_2 v_1 v_2^{-1} v_1^{-l})_i .\, \left(\prod_{k=0}^{l-2} (\psi(w_1)^k a_1 \psi(w_1)^{-k}) \right)_i . \tag{3.10}$$

Using (3.8), (3.9) and (3.10), we see that:

$$\varepsilon(\beta_i) = \varepsilon((v_2 v_1 v_2^{-1} v_1^{-l})_i) + \varepsilon\left(\left(\prod_{k=0}^{l-2} (\psi(w_1)^k a_1 \psi(w_1)^{-k}) \right)_i \right) = (1-l)\varepsilon((a_1)_i) + (l-1)\varepsilon((a_1)_i) = 0 \tag{3.11}$$

for all $1 \le i \le 2g$. Now in normal form, v may be written $v = (\beta_1, \ldots, \beta_{2g})\psi(w_1)^{l-1}$. Since w_1^{l-1} is non trivial, it follows that v is non trivial. Taking $z = v$ and $k = p$ in Lemma 11 and using (3.11) it follows that v is of order p, and hence H has non-trivial torsion elements. In particular, H is not a Bieberbach group. □

It seems to be an interesting question to classify the subgroups of S_n which can be the holonomy of a Bieberbach subgroup of $B_n(M)/\Gamma_2(P_n(M))$ of maximal rank. In the case where the subgroup is a semi-direct product, the argument given in the proof of Proposition 13 may be helpful in the study of the problem.

Using the holonomy representation of the Bieberbach group $\widetilde{G}_{n,g}$ of Theorem 3, given in equation (3.4), we now prove some dynamical and geometric properties of the flat manifold $\mathcal{X}_{n,g}$ whose fundamental group is $\widetilde{G}_{n,g}$.

Proof of Theorem 4. Let $n \geq 2$, let $g \geq 1$, let $\mathcal{X}_{n,g}$ be a Riemannian compact flat manifold $\mathcal{X}_{n,g}$ whose fundamental group is $\widetilde{G}_{n,g}$, the Bieberbach group given in the statement of Theorem 3, and let G_n be the cyclic group of that theorem. Let 1 denote the generator $(1, n, n-1, \ldots, 2)$ of G_n, and consider the holonomy representation $\rho\colon \mathbb{Z}_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2ng})$ of $\widetilde{G}_{n,g}$ given in the proof of Theorem 3. By (3.4), if the characteristic polynomial of $\rho(1)$ is equal to $(x^n-1)^{2g}$, where $\rho\colon \mathbb{Z}_n \longrightarrow \mathrm{Aut}(\mathbb{Z}^{2ng})$ is the holonomy representation of $\widetilde{G}_{n,g}$. To see this, if $2 \leq i \leq 2g$ then M_i is the companion matrix of the polynomial $x^n - 1$, and if we remove the first row and column of M_1, we obtain the companion matrix of the polynomial $1 + x + x^2 + \cdots + x^{n-1}$, so the characteristic polynomial of M_1 is also equal to $(x-1)(1 + x + x^2 + \cdots + x^{n-1}) = x^n - 1$. In particular, $\det(\rho(1)) = 1$, from which it follows from the end of Section 2.1 that $\mathcal{X}_{n,g}$ is orientable. Further, the eigenvalues of $\rho(1)$ are the nth roots of unity each with multiplicity $2g$, and we conclude from [20, Theorem 7.1] that $\mathcal{X}_{n,g}$ admits Anosov diffeomorphisms. By [5, Theorems 6.4.12 and 6.4.13], the first Betti number of $\mathcal{X}_{n,g}$ is given by:

$$\beta_1(\mathcal{X}_{n,g}) = 2ng - \mathrm{rank}(\rho(1) - I_{2ng}) = 2ng - 2g(n-1) = 2g.$$

It remains to show that $\mathcal{X}_{n,g}$ admits a Kähler structure. In order to do this, we make use of the following result from [15, Theorem 3.1 and Proposition 3.2] (see also [6, Theorem 1.1 and Proposition 1.2]) that a Bieberbach group Γ of dimension m is the fundamental group of a Kähler flat manifold with holonomy group H if and only if m is even, and each $\mathbb{R}$-irreducible summand of the holonomy representation $\psi\colon H \longrightarrow \mathrm{GL}(m, \mathbb{R})$ of Γ, which is also $\mathbb{C}$-irreducible, occurs with even multiplicity. Since $\dim(\mathcal{X}_{n,g}) = 2ng$ and the character vector of the representation ρ is equal to $\begin{pmatrix} 2ng \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, it follows that each real irreducible representation of ρ appears $2g$ times in its decomposition, and hence that $\mathcal{X}_{n,g}$ admits a Kähler structure. □

The Betti numbers of the Kähler manifold $\mathcal{X}_{n,g}$ may be computed using the formula $\beta_i(\mathcal{X}_{n,g}) = \dim(\Lambda^i(\mathbb{R}^{2n}))^{G_n}$ of [6, Page 370]. For real dimensions 4 and 6, we may identify the fundamental group of $\mathcal{X}_{n,g}$ in the CARAT classification [19] and [6, Tables, pp. 368 and 370]. More precisely, the CARAT symbol of the 4-dimensional Bieberbach group $\widetilde{G}_{2,1}$ (resp. the 6-dimensional Bieberbach group $\widetilde{G}_{3,1}$) is 18.1 (resp. 291.1).

4. The cases of the sphere and non-orientable surfaces without boundary

Let M be a compact, connected surface without boundary. In this section, we describe the quotient group $B_n(M)/\Gamma_2(P_n(M))$ for the cases not covered by Proposition 1, namely M is either the 2-sphere $\mathbb{S}^2$ or a compact, non-orientable surface of genus $g \geq 1$ without boundary, which we denote by N_g. If $g = 1$ then N_1 is the projective plane $\mathbb{R}P^2$.

Lemma 14. *Let M be either the 2-sphere $\mathbb{S}^2$ or a compact, non-orientable surface of genus $g \geq 1$ without boundary, and let $n \in \mathbb{N}$. The group $P_n(M)/\Gamma_2(P_n(M))$ is isomorphic to:*

(a) $\mathbb{Z}_2 \oplus \mathbb{Z}^{n(n-3)/2}$ if $M = \mathbb{S}^2$ and $n \geq 3$. Further, the $\Gamma_2(P_n(M))$-coset of the full twist Δ_n^2 is the generator of the summand $\mathbb{Z}_2$.

(b) $\mathbb{Z}_2^n \oplus \mathbb{Z}^{(g-1)n}$ if $M = N_g$.

Proof. Part (a) follows from [8, page 674]. For part (b), if $g = 1$ then $M = \mathbb{R}P^2$, and the result is a consequence of [9, Proposition 8]. So suppose that $g \geq 2$. We make use of [13, Theorem 5.1, Presentation 3]. Since $T_{i,i} = 1$ for all $1 \leq i \leq n$ [13, page 439] then by relation (Pr3) of that presentation, $T_{i,j} = T_{i,j-1}$ in $P_n(M)/\Gamma_2(P_n(M))$ for all $1 \leq i < j \leq n$, and it follows by induction on $j - i$ that $T_{i,j} = 1$. We conclude from the presentation that $P_n(M)/\Gamma_2(P_n(M))$ is generated by

$\{a_{j,r} \mid 1 \leq j \leq n \text{ and } 1 \leq r \leq g\}$, subject to the relations $a_{j,1}^2 a_{j,2}^2 \cdots a_{j,g}^2 = 1$ for all $1 \leq j \leq n$, from which we obtain the given isomorphism. □

PROPOSITION 15. *Let $M = N_g$, where $g \geq 1$. In terms of the presentation of $B_n(M)$ given by* [13, Theorem 2.2]*, the map $\psi\colon\ S_n \longrightarrow B_n(M)/\Gamma_2(P_n(M))$ defined on the generating set $\{\tau_i \,|\, i = 1, \ldots, n-1\}$ of S_n by $\psi(\tau_i) = \sigma_i$ for all $1 \leq i \leq n-1$ is an injective homomorphism. Consequently, the short exact sequence (1.2) splits, and $B_n(M)/\Gamma_2(P_n(M)) \cong (\mathbb{Z}_2^n \oplus \mathbb{Z}^{n(g-1)}) \rtimes S_n$*

Proof. Since $\tau_1, \ldots, \tau_{n-1}$ and $\sigma_1, \ldots, \sigma_{n-1}$ satisfy the Artin relations, and using (1.2), it suffices to show that $\sigma_i^2 = 1$ in $B_n(M)/\Gamma_2(P_n(M))$ for all $i = 1, \ldots, n-1$. We now prove that this is the case. If $g = 1$ (resp. $g \geq 2$), with the notation of [9, Theorem 7] (resp. [13, Theorem 2.2]), $\sigma_i^2 = A_{i,i+1}$ (resp. $\sigma_i^2 = T_{i,i+1}$) in $P_n(M)$ for all $1 \leq i \leq n-1$, so σ_i^2 belongs to $\Gamma_2(P_n(M))$ by [9, Proposition 8] (resp. by the proof of Lemma 14), and thus its $\Gamma_2(P_n(M))$-coset is trivial in $B_n(M)/\Gamma_2(P_n(M))$ as required. □

By Lemma 14, if $M = \mathbb{S}^2$ or N_g, where $g \geq 1$, $P_n(M)/\Gamma_2(P_n(M))$ has torsion, and we cannot use the methods of the proof of Proposition 1 . But in fact, in these cases, the group $B_n(M)/\Gamma_2(P_n(M))$ is not crystallographic. To see this, we first prove the following lemma.

LEMMA 16. *Let Π be a group, and suppose that there exists a group extension of the form:*

$$1 \longrightarrow T \times H \longrightarrow \Pi \longrightarrow \Phi \longrightarrow 1,$$

where H is torsion free, and T is finite and non-trivial. Then Π is not a crystallographic group.

Proof. To prove that Π is not crystallographic, by [5, page 34], it suffices to show that it possesses a non-trivial, normal finite subgroup. Let us show that T is such a subgroup. By the hypotheses, it suffices to prove that T is normal. To see this, we view the kernel $T \times H$ as a subgroup of Π. Any inner automorphism of Π thus restricts to an automorphism of $T \times H$. Since the image by such an automorphism of any element of T is also a torsion element, it follows from the fact that H is torsion free that T is indeed normal in Π. □

PROPOSITION 17. *Let $M = \mathbb{S}^2$ or N_g, where $g \geq 1$. Then for all $n \geq 1$, the quotient $B_n(M)/\Gamma_2(P_n(M))$ is not a crystallographic group.*

Proof. The result follows from (1.2) and Lemmas 14 and 16. □

REMARK 18. If $M = \mathbb{S}^2$ (resp. $M = N_g$, where $g \geq 2$), the subgroup T is that generated by the class of the full twist braid (resp. by $\{a_{j,1}a_{j,2}\cdots a_{j,g} \mid j = 1, \ldots, n\}$ using the notation of [13, Theorem 5.1]). If $M = \mathbb{R}P^2$ then $T = P_n(\mathbb{R}P^2)/\Gamma_2(P_n(\mathbb{R}P^2))$.

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