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GLOBAL-IN-TIME $L^p - L^q$ ESTIMATES FOR SOLUTIONS OF THE KRAMERS-FOKKER-PLANCK EQUATION

XUE PING WANG AND LU ZHU

ABSTRACT. In this work, we prove an optimal global-in-time $L^p - L^q$ estimate for solutions to the Kramers-Fokker-Planck equation with short range potential in dimension three. Our result shows that the decay rate as $t \rightarrow +\infty$ is the same as the heat equation in x -variables and the divergence rate as $t \rightarrow 0_+$ is related to the sub-ellipticity with loss of $1/3$ derivatives of the Kramers-Fokker-Planck operator.

1. INTRODUCTION

The Kramers-Fokker-Planck equation is the evolution equation for the distribution functions describing the Brownian motion of particles in an external field:

$$\frac{\partial W}{\partial t} = \left(-v \cdot \nabla_x + \nabla_v \cdot \left(\gamma v - \frac{F(x)}{m} \right) + \frac{\gamma k T}{m} \Delta_v \right) W, \quad (1.1)$$

where $F(x) = -m \nabla V(x)$ is the external force and $W = W(t; x, v)$ is the distribution function of particles for $x, v \in \mathbb{R}^n$ and $t > 0$. In this equation, x and v represent the position and velocity variables of particles, m the mass, k the Boltzmann constant, γ the friction coefficient and T the temperature of the media. This equation, called the Kramers equation in the book of H. Risken [14], was initially derived and used by H. A. Kramers [8] to describe kinetics of chemical reaction. Later on it turned out that it had more general applicability to different fields such as supersonic conductors, Josephson tunneling junction and relaxation of dipoles. Equation 1.1, also often called the Fokker-Planck equation, is in fact a special case of the more general Fokker-Planck equation ([14]) or the Kolmogorov forward equation for continuous-time diffusion processes ([7]).

After appropriate normalisation of physical constants and change of unknowns, the KFP equation can be written into the form

$$\partial_t u(t; x, v) + Pu(t; x, v) = 0, \quad (x, v) \in \mathbb{R}^n \times \mathbb{R}^n, t > 0, \quad (1.2)$$

with initial data

$$u(0; x, v) = u_0(x, v), \quad (1.3)$$

where P is the KFP operator defined by

$$P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x - \nabla V(x) \cdot \nabla_v. \quad (1.4)$$

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In this work, $V(x)$ is supposed to be a real-valued C^1 function verifying

$$|V(x)| + \langle x \rangle |\nabla V(x)| \leq C \langle x \rangle^{-\rho}, \quad x \in \mathbb{R}^n, \quad (1.5)$$

for some $\rho \geq -1$. Here $\langle x \rangle = (1 + |x|^2)^{1/2}$. Remark that $V(x)$ is determined up to an additive constant. (1.5) implies that when with $\rho > 0$, this constant is chosen such that

$$\lim_{|x| \rightarrow \infty} V(x) = 0$$

which can be interpreted as a normalization condition for $V(x)$. Let \mathbf{m} be the function defined by

$$\mathbf{m}(x, v) = \frac{1}{(2\pi)^{\frac{n}{4}}} e^{-\frac{1}{2}(\frac{v^2}{2} + V(x))}. \quad (1.6)$$

Then $\mathfrak{M} = \mathbf{m}^2$ is the Maxwellian ([14]) and \mathbf{m} verifies the stationary KFP equation

$$P\mathbf{m} = 0 \quad \text{in } \mathbb{R}_{x,v}^{2n}. \quad (1.7)$$

The large-time asymptotics of the solution to the KFP equation is mostly motivated by mathematical analysis of trend to equilibrium in statistical physics and is studied by many authors for confining potentials. See, for example, [3, 4, 5, 6]. In these works, the potential $V(x)$ is supposed to be confining so that the spectre of P is discrete. The typical result is return to the equilibrium with exponential rate: $\exists \sigma > 0$ such that

$$u(t) = \langle \mathbf{m}, u_0 \rangle \mathbf{m} + O(e^{-\sigma t}), \quad t \rightarrow +\infty, \quad (1.8)$$

where $V(x)$ is assumed to be normalized by

$$\int_{\mathbb{R}^n} e^{-V(x)} dx = 1.$$

In [9], sub-exponential convergence rate is obtained for weakly confining potential. For quickly decreasing potentials (or more precisely, for quickly decreasing $|\nabla V(x)|$), it is shown in [12] for $n = 1$ and [16] for $n = 3$ that

$$u(t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} (\langle \mathbf{m}, u_0 \rangle \mathbf{m} + O(t^{-\epsilon})), \quad t \rightarrow +\infty, \quad (1.9)$$

in weighted L^2 -spaces with weight in x -variables.

In this work, we consider potentials $V(x)$ satisfying (1.5) with $\rho \geq -1$ and study $L^p - L^q$ estimates of $u(t)$ for $t > 0$. Here

$$L^p = L^p(\mathbb{R}_{x,v}^{2n}; dx dv)$$

is equipped with the natural norm. For $f \in L^p$ and T bounded linear operator from L^p to L^q , we denote :

$$\|f\|_p = \|f\|_{L^p}, \quad \|T\|_{p \rightarrow q} = \|T\|_{\mathcal{L}(L^p, L^q)}. \quad (1.10)$$

By an abuse of notation, for a closed linear operator T in L^2 with $C_0^\infty(\mathbb{R}^{2n})$ as a core and for $p \in [1, \infty[$, we still denote by the same letter T its minimal closed extension in L^p (i.e., the closure in L^p of the restriction of T to $C_0^\infty(\mathbb{R}^{2n})$). Similarly, the notation $e^{-tP} : L^p \rightarrow L^q$ means that the restriction of e^{-tP} on C_0^∞ extends to a map from L^p to

L^q . Under fairly general condition, e^{-tP} is a strongly continuous positivity preserving contraction semigroup in L^p . Since for $1 \leq p < \infty$,

$$\overline{(e^{-tP}|_{C_0^\infty})|_{L^p}} = e^{-t\overline{(P|_{C_0^\infty})|_{L^p}}},$$

our notation is consistent in some sense. The main result of this work is the following

Theorem 1.1. *Let $n = 3$ and condition (1.5) be satisfied with $\rho > 1$. For $1 \leq p < q \leq \infty$, there exists some constant $C > 0$ such that*

$$\|e^{-tP}\|_{p \rightarrow q} \leq \frac{C}{(\gamma(t))^{\frac{3}{2p}(1-\frac{p}{q})}}, \quad t \in]0, \infty[, \quad (1.11)$$

where $\gamma(t) = \sigma(t)\theta(t)$ with

$$\sigma(t) = t - 2 \coth(t) + 2 \operatorname{cosech}(t), \quad \theta(t) = 4\pi e^{-t} \sinh(t). \quad (1.12)$$

The function $\gamma(t)$ appears in the explicit formula of fundamental solution for the free KFP equation (see Section 2) and behaves like: $\gamma(t) \sim t$ as $t \rightarrow \infty$ and $\gamma(t) \sim ct^4$ as $t \rightarrow 0$, $c > 0$. The two factors of $\gamma(t)$ have different meanings. $\theta(t)$ arises from the semigroup generated by the harmonic oscillator

$$H = \Re P = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}$$

in $L^p(\mathbb{R}_v^3)$. For $\sigma(t)$, remark that

$$\sigma(t) \sim t, \text{ as } t \rightarrow +\infty; \quad \sigma(t) \sim \frac{t^3}{6}, \text{ as } t \rightarrow 0_+. \quad (1.13)$$

For $p = 1, q = \infty$,

$$(\sigma(t))^{-\frac{3}{2}} \sim \frac{C_1}{t^{\frac{3}{2}}}, \quad t \rightarrow \infty; \quad (\sigma(t))^{-\frac{3}{2}} \sim \frac{C}{t^{\frac{9}{2}}}, \quad t \rightarrow 0_+.$$

One sees that the term $(\sigma(t))^{-\frac{3}{2}}$ is of the same order as that of the heat semigroup $e^{t\Delta_x}$ as map from $L^1(\mathbb{R}^3)$ to $L^\infty(\mathbb{R}^3)$ as $t \rightarrow \infty$ and of the same order as that of $e^{-t|D_x|^{\frac{2}{3}}}$ as $t \rightarrow 0_+$. This may be explained by the fact that at low energies, the KFP operator P behaves like a Witten Laplacian ([4, 10]), while globally it is sub-elliptic in x with the loss of $\frac{1}{3}$ derivatives.

To prove Theorem 1.1, we first study the semigroup e^{-tP_0} in $L^p - L^q$ setting, where

$$P_0 = -\Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2} + v \cdot \nabla_x.$$

Then we consider P as perturbation of P_0 and use Duhamel's formula to prove (1.11). The short-time estimate for e^{-tP} can be easily obtained (see Theorem 4.1) and is valid for $n \geq 1$ and $\rho > -1$. The proof of (1.11) for t large is based on a result of time-decay of e^{-tP} in weighted L^2 spaces obtained in [16].

In this work, we often use an argument of duality which is based on the relation

$$P^* = JPJ \quad (1.14)$$

in L^2 , where J is the reflection in v variable: $Jf(x, v) = f(x, -v)$. If one has some estimates for P or e^{-tP} in L^p , one can often use the duality between L^p and L^q , $p^{-1} + q^{-1} = 1$,

to affirm that the same statements are true for P^* or $(e^{-tP})^*$ in L^q . Since J preserves any L^p norm, the same estimates hold true for P or e^{-tP} in L^q .

The remaining part of this work is organized as follows. In Section 2, we establish an explicit useful formula for the fundamental solution of the free KFP operator P_0 . Global-in-time $L^p - L^q$ estimates are obtained for e^{-tP_0} in Section 3. Theorem 1.1 is proved in Section 4.

2. FONDAMENTAL SOLUTION OF THE FREE KFP EQUATION

In this Section, we use the method of complex deformation to calculate the fundamental solution of the free KFP equation. Let P_0 be the free KFP operator:

$$P_0 = v \cdot \nabla_x - \Delta_v + \frac{1}{4}|v|^2 - \frac{n}{2}, (x, v) \in \mathbb{R}^{2n}. \quad (2.1)$$

In L^2 , using the partial Fourier transform in x -variables, we have for $u \in D(P_0)$

$$P_0 u(x, v) = \mathcal{F}_{x \rightarrow \xi}^{-1} \hat{P}_0(\xi) \hat{u}(\xi, v), \quad \text{where} \quad (2.2)$$

$$\hat{P}_0(\xi) = -\Delta_v + \frac{1}{4} \sum_{j=1}^n (v_j + 2i\xi_j)^2 - \frac{n}{2} + |\xi|^2 \quad (2.3)$$

$$\hat{u}(\xi, v) = (\mathcal{F}_{x \rightarrow \xi} u)(\xi, v) \triangleq \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x, v) dx. \quad (2.4)$$

Denote

$$D(\hat{P}_0) = \{f \in L^2(\mathbb{R}_{\xi, v}^{2n}); \hat{P}_0(\xi) f \in L^2(\mathbb{R}_{\xi, v}^{2n})\}. \quad (2.5)$$

Then $\hat{P}_0 \triangleq \mathcal{F}_{x \rightarrow \xi} P_0 \mathcal{F}_{x \rightarrow \xi}^{-1}$ is a direct integral of the family of complex harmonic operators $\{\hat{P}_0(\xi); \xi \in \mathbb{R}^n\}$. $\{\hat{P}_0(\xi), \xi \in \mathbb{R}^n\}$ is a holomorphic family of type (A) in sense of Kato with constant domain $D = D(-\Delta_v + \frac{v^2}{4})$ in $L^2(\mathbb{R}_v^n)$. Let $F_j(s) = (-1)^j e^{\frac{s^2}{2}} \frac{d^j}{ds^j} e^{-\frac{s^2}{2}}$, $j \in \mathbb{N}$, be the Hermite polynomials and

$$\varphi_j(s) = (j! \sqrt{2\pi})^{-\frac{1}{2}} e^{-\frac{s^2}{4}} F_j(s)$$

the normalized Hermite functions. For $\xi \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, define

$$\psi_\alpha(v) = \prod_{j=1}^n \varphi_{\alpha_j}(v_j) \quad \text{and} \quad \psi_\alpha^\xi(v) = \psi_\alpha(v + 2i\xi). \quad (2.6)$$

Then

$$\hat{P}_0(\xi) \psi_\alpha^\xi = (|\alpha| + |\xi|^2) \psi_\alpha^\xi. \quad (2.7)$$

For $\alpha, \beta \in \mathbb{N}^n$, $\xi \rightarrow \langle \psi_\alpha^\xi, \psi_\beta^{-\xi} \rangle$ extends to an entire function for $\xi \in \mathbb{C}$ and is constant on $i\mathbb{R}$. Therefore $\langle \psi_\alpha^\xi, \psi_\beta^{-\xi} \rangle$ is constant for $\xi \in \mathbb{C}$ and one has

$$\langle \psi_\alpha^\xi, \psi_\beta^{-\xi} \rangle = \delta_{\alpha\beta} = \begin{cases} 1, & \alpha = \beta, \\ 0, & \alpha \neq \beta. \end{cases}, \quad \forall \alpha, \beta \in \mathbb{N}^n, \xi \in \mathbb{R}^n. \quad (2.8)$$

e^{-tP_0} is a contraction semigroup in $L^2(\mathbb{R}_{x, v}^{2n})$. Its distributional kernel can be explicitly computed, using Mehler's formula for harmonic oscillator ([11])(see also [1]), where this fundamental solution is calculated with different method and expressed in slightly

different way. Recall ([2]) that for the heat kernel of n - dimensional harmonic oscillator $-\Delta + x^2$ is given by

$$E(x, y; t) = \frac{1}{(2\pi \sinh(2t))^{\frac{n}{2}}} \exp\left(-\frac{\coth(2t)}{2}(x^2 + y^2) + \operatorname{cosech}(2t)x \cdot y\right), \quad t > 0. \quad (2.9)$$

Lemma 2.1. *Let $n \geq 1$. The distributional kernel of e^{-tP_0} is given by*

$$F(x, v, x', v'; t) = \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \exp\left(-\frac{1}{4\sigma(t)}|x - x' - \omega(t)(v + v')|^2\right) K(v, v'; t). \quad (2.10)$$

where

$$\begin{aligned} K(v, v'; t) &= \frac{1}{(4\pi \sinh(t))^{\frac{n}{2}}} \exp\left(\frac{nt}{2} - \frac{\coth(t)}{4}(|v|^2 + |v'|^2) + \frac{\operatorname{cosech}(t)}{2}v \cdot v'\right) \\ \omega(t) &= \coth(t) - \operatorname{cosech}(t) \\ \sigma(t) &= t - 2\coth(t) + 2\operatorname{cosech}(t). \end{aligned} \quad (2.11)$$

Proof. Since the n -dimensional free KFP operator P_0 is a direct sum of n one dimensional operators, it suffices to prove the lemma for $n = 1$. Applying (2.9) and making use of change of scale, we deduce that the Mehler's formula for the heat kernel of the one-dimensional harmonic oscillator $H = -\frac{d^2}{dv^2} + \frac{1}{4}v^2 - \frac{1}{2}$ is given by:

$$e^{-tH}u = \int_{\mathbb{R}} K(v, v'; t)u(v')dv', \quad t > 0, u \in C_0^\infty, \quad (2.12)$$

where

$$K(v, v'; t) = \frac{1}{\sqrt{4\pi \sinh(t)}} \exp\left(\frac{t}{2} - \frac{\coth(t)}{4}(v^2 + v'^2) + \frac{\operatorname{cosech}(t)}{2}vv'\right), \quad (2.13)$$

which is an entire function in v and v' in \mathbb{C} . Set

$$\tilde{K}(v, v', \xi; t) = e^{-|\xi|^2 t} K(v + 2i\xi, v' + 2i\xi; t). \quad (2.14)$$

Since ψ_l^ξ is an eigenfunction of $\hat{P}_0(\xi)$ associated with the eigenvalue $l + |\xi|^2$, one has

$$e^{-t\hat{P}_0(\xi)}\psi_l^\xi = e^{-t(l+|\xi|^2)}\psi_l^\xi$$

On the other hand, one has

$$\int_{\mathbb{R}} K(v + 2i\xi, v'; t)\psi_l(v')dv' = e^{-tl}\psi_l^\xi,$$

since the both sides are entire functions in $v \in \mathbb{C}$. Using deformation of contour and the decay properties of $K(v, v'; t)$, one obtains

$$\int_{\mathbb{R}} K(v + 2i\xi, v' + 2i\xi; t)\psi_l^\xi(v')dv' = \int_{\mathbb{R}} K(v + 2i\xi, v'; t)\psi_l(v')dv' \quad (2.15)$$

for $\xi \in \mathbb{R}$. It follows that

$$e^{-t\hat{P}_0(\xi)}\psi_l^\xi = \int_{\mathbb{R}} \tilde{K}(v, v', \xi; t)\psi_l^\xi(v')dv' = e^{-t(l+|\xi|^2)}\psi_l^\xi.$$

Since the span of $\{\psi_l^\xi, l \in \mathbb{N}\}$ is dense in $L^2(\mathbb{R}_v)$, one concludes that the heat kernel of $\hat{P}_0(\xi)$ is equal to $\tilde{K}(v, v', \xi; t)$ for $t > 0$. $\tilde{K}(v, v', \xi; t)$ can be written as

$$\tilde{K}(v, v', \xi; t) = K(v, v'; t)\hat{g}(v, v', \xi; t) \quad (2.16)$$

where

$$\hat{g}(v, v', \xi; t) = \exp(-i\omega(t)(v + v')\xi - |\xi|^2\sigma(t)) \quad (2.17)$$

with

$$\omega(t) = \coth(t) - \operatorname{cosech}(t), \quad \sigma(t) = t - 2\coth(t) + 2\operatorname{cosech}(t). \quad (2.18)$$

Since $\sigma(t) > 0$ for $t > 0$, the inverse Fourier transform of \hat{g} in ξ can be explicitly calculated:

$$\begin{aligned} g(v, v', x; t) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{g}(v, v', \xi; t) d\xi \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(x - \omega(t)(v + v'))\xi} e^{-\sigma(t)|\xi|^2} d\xi \\ &= \frac{1}{\sqrt{4\pi\sigma(t)}} \exp\left(-\frac{1}{4\sigma(t)}(x - \omega(t)(v + v'))^2\right). \end{aligned}$$

Therefore, the integral kernel of e^{-tP_0} is given by

$$F(x, v, x', v'; t) = \frac{1}{\sqrt{4\pi\sigma(t)}} \exp\left(-\frac{1}{4\sigma(t)}(x - x' - \omega(t)(v + v'))^2\right) K(v, v'; t). \quad (2.19)$$

□

The fundamental solution $F(x, v, x', v'; t)$ for the free KFP equation has several nice properties. For example, one has for $f \in C_0^\infty(\mathbb{R}^{2n})$,

$$\int (e^{-tP_0} f)(x, v) dx = (e^{-tH} g)(v), \quad v \in \mathbb{R}^n, \quad (2.20)$$

where $g(v) = \int f(x, v) dx$ and H is the harmonic oscillator: $H = -\Delta_v + \frac{1}{4}v^2 - \frac{n}{2}$.

3. GLOBAL-IN-TIME ESTIMATES FOR THE FREE KFP OPERATOR

In this section, we give some global-in-time $L^p - L^q$ estimates for e^{-tP_0} needed in the proof of Theorem 1.1.

Proposition 3.1. *Let $n \geq 1$. For $t > 0$, e^{-tP_0} defined on $C_0^\infty(\mathbb{R}^{2n})$ extends to an operator bounded from L^1 to L^∞ and the following estimate is true for the free KFP operator:*

$$\|e^{-tP_0}\|_{1 \rightarrow \infty} \leq \frac{1}{(4\pi\gamma(t))^{\frac{n}{2}}} \quad (3.1)$$

for $t > 0$. Here where

$$\gamma(t) = \sigma(t)\theta(t), \quad \theta(t) = 4\pi e^{-t} \sinh(t).$$

Proof. Let $f \in C_0^\infty(\mathbb{R}^{2n})$. By Lemma 2.1, one has

$$|e^{-tP_0} f(x, v)| \leq \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \int_{\mathbb{R}^{2n}} K(v, v'; t) |f(x', v')| dx' dv' \quad (3.2)$$

which gives

$$\begin{aligned} |e^{-tP_0} f(\cdot, v)|_{L_x^\infty} &\leq \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} K(v, v'; t) \|f(\cdot, v')\|_{L_x^1} dv' \\ &= \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} (e^{-tH} g)(v) \end{aligned} \quad (3.3)$$

where $g(v') = \|f(\cdot, v')\|_{L_x^1}$, since $K(v, v', t)$ is the distributional kernel of e^{-tH} . From (2.11), it follows that

$$\|e^{-tP_0} f\|_\infty \leq \frac{1}{(4\pi\gamma(t))^{\frac{n}{2}}} \|f\|_1, \quad f \in C_0^\infty(\mathbb{R}^{2n}). \quad (3.4)$$

(refe3.1) is derived by an argument of density. \square

Part of following results may be known. We include a proof for reason of completeness.

Corollary 3.2. (a). *One has*

$$\|e^{-tP_0}\|_{p \rightarrow p} \leq 1 \quad (3.5)$$

for $1 \leq p \leq \infty$ and

$$\|e^{-tP_0}\|_{L^p \rightarrow L^q} \leq \frac{1}{(4\pi\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}}, \quad t > 0, \quad (3.6)$$

for $1 \leq p \leq q \leq \infty$.

(b). e^{-tP_0} , $t \geq 0$, is a strongly continuous positivity preserving contraction semigroup in L^p for $1 \leq p < \infty$.

Proof. (a). P_0 is closed and accretive in L^2 . Therefore e^{-tP_0} , $t \geq 0$, is a strongly continuous contraction semigroup in L^2 . In particular, (3.5) is true for $p = 2$. We denote by the same symbol e^{-tP_0} the operator induced in $L^p(\mathbb{R}^{2n})$. By (2.20), one has

$$\|e^{-tP_0} f\|_1 \leq \|e^{-tH} f\|_1 \leq \|f\|_1 \quad (3.7)$$

$$\|(e^{-tP_0} - 1)f\|_1 \leq \|(e^{-tH} - 1)f\|_1 \quad (3.8)$$

for $f \in L^1$. The first estimate implies (3.5) for $p = 1$. By arguments of duality and interpolation, we obtain (3.5) for $p \in [1, \infty]$. (3.6) follows from (3.1) and (3.5) by interpolation.

(b). Since $C_0^\infty(\mathbb{R}^{2n})$ is a common core of P in L^p , $1 \leq p < \infty$, the semigroup property of e^{-tP} in L^p follows from that of e^{-tP} in L^2 . By Theorem X.55 in [13], e^{-tH} is a strongly continuous contraction semigroup in $L^p(\mathbb{R}_v^n)$, $1 \leq p < \infty$. The strongly continuity of e^{-tP_0} in L^1 follows from (3.8). The general case $1 < p < \infty$ can be deduced from the cases $p = 1$ and $p = 2$. e^{-tP_0} is positivity preserving, because its distributional kernel $F(x, v, x', v'; t)$ is positive. \square

To study the full KFP operator P , we want to treat the $W = -\nabla V(x) \cdot \nabla_v$ as perturbation and need some more estimates for e^{-tP_0} .

Proposition 3.3. *Let $k \in \mathbb{N}$. The following estimates are true for the free KFP equation:*

$$\|\langle v \rangle^k e^{-tP_0}\|_{1 \rightarrow \infty} + \|\langle D_v \rangle^k e^{-tP_0}\|_{1 \rightarrow \infty} \leq \frac{C}{(\gamma(t))^{\frac{n}{2}}} \left(1 + t^{-\frac{k}{2}}\right) \quad (3.9)$$

and for any $p \in [1, \infty]$,

$$\|\langle v \rangle^k e^{-tP_0}\|_{p \rightarrow p} + \|\langle D_v \rangle^k e^{-tP_0}\|_{p \rightarrow p} \leq C \left(1 + t^{-\frac{k}{2}}\right) \quad (3.10)$$

for $t > 0$.

Proof. Remark that the distributional kernel $K(v, v', t)$ of e^{-tH} satisfies the estimate

$$0 \leq K(v, v', t) \leq \frac{1}{(4\pi\theta(t))^{\frac{n}{2}}} e^{-\frac{\cosh^2(t)-1}{2\sinh(2t)}|v|^2}$$

uniformly in v' and that $\frac{\cosh^2(t)-1}{2\sinh(2t)} \sim ct$ as $t \rightarrow 0$, $c > 0$. As in the proof of Proposition 3.1, one has for $f \in C_0^\infty(\mathbb{R}^{2n})$

$$\begin{aligned} \|\langle v \rangle^k e^{-tP_0} f(\cdot, v)\|_{L_x^\infty} &\leq \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \int_{\mathbb{R}^n} \langle v \rangle^k K(v, v', t) \|f(\cdot, v')\|_{L_x^1} dv' \\ &\leq \frac{1}{(4\pi\sigma(t))^{\frac{n}{2}}} \sup_{v, v'} \langle v \rangle^k K(v, v', t) \|f\|_1 \\ &\leq \frac{C}{(\gamma(t))^{\frac{n}{2}}} (1 + t^{-\frac{k}{2}}) \|f\|_1, \quad t > 0. \end{aligned}$$

This shows

$$\|\langle v \rangle^k e^{-tP_0}\|_{1 \rightarrow \infty} \leq \frac{C}{(\gamma(t))^{\frac{n}{2}}} (1 + t^{-\frac{k}{2}}), \quad t > 0.$$

Similarly, one can estimate $\|\partial_v^\alpha e^{-tP_0}\|_{1 \rightarrow \infty}$ by evaluating $\sup_{v, v'} |\partial_v^\alpha K(v, v', t)|$ for $t > 0$ and $\alpha \in \mathbb{N}^n$. (3.9) is proved.

In the same way, one has

$$\begin{aligned} \|\langle v \rangle^k e^{-tP_0} f\|_1 &\leq C \int_{\mathbb{R}^n} \langle v \rangle^k K(v, v', t) dv' \|f(\cdot, v')\|_{L_x^1} dv' \\ &\leq C \int \langle v \rangle^k \sup_{v'} K(v, v'; t) dv \|f\|_1 \\ &\leq C_1 (1 + t^{-\frac{k}{2}}) \|f\|_1, \quad t > 0. \end{aligned}$$

The same result holds true in L^2 , because

$$\|H^k e^{-tH}\|_{L_v^2 \rightarrow L_v^2} \leq t^{-k}$$

by the Spectral Theorem for positive selfadjoint operators and $(\langle v \rangle^{2k} + \langle D_v \rangle^{2k})(H+1)^{-k}$ is bounded in L^2 . By arguments of duality and interpolation, we obtain for $p \in [1, \infty]$

$$\|\langle v \rangle^k e^{-tP_0}\|_{p \rightarrow p} \leq C \left(1 + t^{-\frac{k}{2}}\right), \quad t > 0.$$

Again using the formula of $K(v, v', t)$, one can show

$$\|\partial_v^\alpha e^{-tP_0}\|_{p \rightarrow p} \leq C_\alpha \left(1 + t^{-\frac{|\alpha|}{2}}\right), \quad t > 0.$$

for any $\alpha \in \mathbb{N}^n$. This proves (3.10). \square

As consequence of Proposition 3.3, one obtains the following

Corollary 3.4. *For $1 \leq p \leq q \leq \infty$ and for any $k \in \mathbb{N}$, one has*

$$\|\langle v \rangle^k e^{-tP_0}\|_{p \rightarrow q} + \|\langle D_v \rangle^k e^{-tP_0}\|_{p \rightarrow q} \leq \frac{C}{(\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}} \left(1 + t^{-\frac{k}{2}}\right), \quad (3.11)$$

and

$$\|e^{-tP_0} \langle v \rangle^k\|_{p \rightarrow q} + \|e^{-tP_0} \langle D_v \rangle^k\|_{p \rightarrow q} \leq \frac{C}{(\gamma(t))^{\frac{n}{2p}(1-\frac{p}{q})}} \left(1 + t^{-\frac{k}{2}}\right), \quad (3.12)$$

for $t > 0$.

4. GLOBAL-IN-TIME ESTIMATES FOR e^{-tP}

Set $P = P_0 + W$ with $W = -\nabla V(x) \cdot \nabla_v$. Under the condition $\rho \geq -1$, W is relatively bounded perturbation of P_0 with relative bound 0 and P is closed with $D(P) = D(P_0)$. Since

$$e^{-tW} f(x, v) = f(x, v + t\nabla V(x)),$$

e^{-tW} preserves L^p norm. e^{-tP_0} and e^{-tW} are strongly continuous semigroups of contractions in L^p , $1 \leq p < \infty$. By theorem on perturbation of semigroups of contractions ([13]), e^{-tP} is a strongly continuous semigroup of contractions in L^p , $p \in [1, \infty[$. It follows from Trotter's formula that e^{-tP} is positivity preserving. We are interested in e^{-tP} when it is regarded as map from L^p to L^q , $q > p$.

4.1. Short-time estimates for e^{-tP} .

Theorem 4.1. *Let $n \geq 1$ and (1.5) be satisfied with $\rho \geq -1$. Then one has for $1 \leq p < q \leq \infty$*

$$\|e^{-tP}\|_{p \rightarrow q} \leq \frac{C}{\gamma(t)^{\frac{n}{2p}(1-\frac{p}{q})}}, \quad t \in]0, 1]. \quad (4.1)$$

Proof. The proof is based on Duhamel's formula

$$e^{-tP} = e^{-tP_0} + \int_0^t e^{-(t-s)P_0} W e^{-sP} ds. \quad (4.2)$$

Set

$$\alpha(p, q) = \frac{n}{2} \left(\frac{1}{p} - \frac{1}{q} \right).$$

Remark that $\gamma(t) \sim ct^4$ as $t \rightarrow 0_+$. For $1 \leq p \leq p' \leq 2$ such that $\frac{1}{p} - \frac{1}{p'} < \frac{1}{4n}$, one has: $4\alpha(p, p') < \frac{1}{2}$. Since e^{-tP} is a contraction semigroup in L^p , by (3.11), one has

$$\begin{aligned} \|e^{-tP}\|_{p \rightarrow p'} &\leq \|e^{-tP_0}\|_{p \rightarrow p'} + C \int_0^t \|\nabla_v e^{-(t-s)P_0}\|_{p \rightarrow p'} \|e^{-sP}\|_{p \rightarrow p} ds \\ &\leq C \left(\gamma(t)^{-\alpha(p, p')} + \int_0^t |t-s|^{-\frac{1}{2}-4\alpha(p, p')} ds \right) \\ &\leq C_1 \gamma(t)^{-\alpha(p, p')}, \quad \text{for } t \in]0, 1]. \end{aligned}$$

For each $n \geq 1$, take $k = k(n)$ numbers p_1, \dots, p_k such that

$$1 = p_1 < p_2 < \dots < p_{k-1} < p_k = 2 \quad \text{and} \quad \frac{1}{p_j} - \frac{1}{p_{j+1}} < \frac{1}{4n}.$$

Writing e^{-tP} as $(e^{-\frac{t}{k}P})^k$, one obtains

$$\begin{aligned} \|e^{-tP}\|_{1 \rightarrow 2} &\leq \|e^{-\frac{t}{k}P}\|_{p_1 \rightarrow p_2} \cdots \|e^{-\frac{t}{k}P}\|_{p_{k-1} \rightarrow 2} \\ &\leq C\gamma(t)^{-\alpha(1,p_2) - \dots - \alpha(p_{k-1},2)} \\ &= C\gamma(t)^{-\alpha(1,2)} \end{aligned}$$

for $t \in]0, 1]$. This proves (4.1) for $p = 1$ and $q = 2$. The general case follows by duality and interpolation. \square

4.2. Large-time estimate for e^{-tP} .

Theorem 4.2. *Assume $n = 3$ and that (1.5) is satisfied with $\rho > 1$. One has for $1 \leq p < q \leq \infty$*

$$\|e^{-tP}\|_{p \rightarrow q} \leq Ct^{-\frac{3}{2p}(1-\frac{p}{q})} \quad (4.3)$$

for $t \in [1, \infty[$.

Under the conditions of Theorem 4.2, it is proved in [16] that for $s > \frac{3}{2}$,

$$\|\langle x \rangle^{-s} e^{-tP} \langle x \rangle^{-s}\|_{L^2 \rightarrow L^2} \leq C\langle t \rangle^{-\frac{3}{2}}, t \geq 0. \quad (4.4)$$

It follows that for $0 < r \leq \frac{3}{2}$ and $s > r$, one has

$$\|e^{-tP}\|_{\mathcal{L}^{2,s} \rightarrow \mathcal{L}^{2,-s}} \leq C\langle t \rangle^{-r}, t \geq 0. \quad (4.5)$$

Here $\mathcal{L}^{2,s} = L^2(\mathbb{R}_{x,v}^{2n}, \langle x \rangle^{2s} dx dv)$.

Proof of Theorem 4.2 To obtain large time $L^p - L^q$ estimate for e^{-tP} , we use the following decomposition which is deduced from Duhamel's formula :

$$e^{-tP} = e^{-tP_0} + I(t) + J(t) \quad (4.6)$$

where

$$I(t) = \int_0^t e^{-(t-s)P_0} W e^{-sP_0} ds, \quad (4.7)$$

$$J(t) = \int_0^t \int_0^s e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} d\tau ds. \quad (4.8)$$

Decompose $I(t) = I_1(t) + I_2(t)$ and $J(t) = J_1(t) + J_2(t)$ where

$$\begin{aligned} I_1(t) &= \int_0^{\frac{t}{2}} e^{-(t-s)P_0} W e^{-sP_0} ds, \\ I_2(t) &= \int_{\frac{t}{2}}^t e^{-(t-s)P_0} W e^{-sP_0} ds, \\ J_1(t) &= \int_0^{\frac{t}{2}} \int_0^s e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} d\tau ds, \\ J_2(t) &= \int_{\frac{t}{2}}^t \int_0^s e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} d\tau ds \end{aligned}$$

We estimate each term on the right hand side in $L^1 - L^\infty$ norm. Remark first that since $\nabla V(x)$ is bounded, the portion of the integral in $I_1(t)$ related to $s \in [0, \frac{1}{4}]$ can be bounded by

$$\begin{aligned} & \left\| \int_0^{\frac{1}{4}} e^{-(t-s)P_0} W e^{-sP_0} ds \right\|_{1 \rightarrow \infty} \\ & \leq \int_0^{\frac{1}{4}} \|\nabla V\|_\infty \|e^{-(t-s)P_0}\|_{1 \rightarrow \infty} \|\nabla_v e^{-sP_0}\|_{1 \rightarrow 1} ds \\ & \leq Ct^{-\frac{3}{2}} \int_0^{\frac{1}{4}} s^{-\frac{1}{2}} ds \leq Ct^{-\frac{3}{2}} \end{aligned}$$

for $t \geq 1$. Under the assumption (1.5), $\nabla V(x) \in L^r(\mathbb{R}^3)$ for any $r > \frac{3}{1+\rho}$ and $r \geq 1$. By Hölder's inequality and (3.11), $\nabla V e^{-\delta P_0}$, $\delta > 0$, maps continuously L^p to L^1 where

$$3 < p = \frac{r}{r-1} < 1 + \frac{1+\rho}{2-\rho}.$$

This is possible, because $\frac{1+\rho}{2-\rho} > 2$ for $\rho > 1$. It follows that

$$\begin{aligned} \|\nabla V e^{-sP_0}\|_{1 \rightarrow 1} & \leq \|\nabla V e^{-\frac{1}{8}P_0}\|_{p \rightarrow 1} \|e^{-(s-\frac{1}{8})P_0}\|_{1 \rightarrow p} \\ & \leq Cs^{-\frac{3}{2}(1-\frac{1}{p})} (1 + s^{-\frac{1}{2}}). \end{aligned} \tag{4.9}$$

This shows that $s \rightarrow \|\nabla V e^{-sP_0}\|_{1 \rightarrow 1}$ is integrable in $s \in [\frac{1}{4}, \infty[$. Consequently, $I_1(t)$ can be bounded as follows

$$\begin{aligned} \|I_1(t)\|_{1 \rightarrow \infty} & \leq Ct^{-\frac{3}{2}} + \int_{\frac{1}{4}}^{\frac{t}{2}} \|\nabla_v e^{-(t-s)P_0}\|_{1 \rightarrow \infty} \|\nabla V e^{-sP_0}\|_{1 \rightarrow 1} ds \\ & \leq C_1 t^{-\frac{3}{2}} \left(1 + \int_{\frac{1}{4}}^{\frac{t}{2}} \|\nabla V e^{-sP_0}\|_{1 \rightarrow 1} ds \right) \\ & \leq C_2 t^{-\frac{3}{2}}, \quad \text{for } t \geq 1. \end{aligned}$$

Since $\nabla V \in L^r(\mathbb{R}_x^3)$, by (3.12), $e^{-\delta P_0} \nabla V$ is bounded from L^∞ to L^r . Corollary 3.2 shows

$$\begin{aligned} \|e^{-(t-s)P_0} \nabla V\|_{\infty \rightarrow \infty} & \leq \|(e^{-(t-s-\delta)\tau P_0}\|_{r \rightarrow \infty} \|e^{-\delta P_0} \nabla V\|_{\infty \rightarrow r} \\ & \leq C \langle t-s \rangle^{-\frac{3}{2r}} = C \langle t-s \rangle^{-\frac{1+\rho}{2} + \epsilon}, \end{aligned}$$

for $s \in [\frac{t}{2}, t - \frac{1}{4}]$ and $t \geq 1$. Therefore, $I_2(t)$ can be estimated by

$$\begin{aligned} \|I_2(t)\|_{1 \rightarrow \infty} &\leq Ct^{-\frac{3}{2}} + \int_{\frac{t}{2}}^{t-\frac{1}{4}} \|e^{-(t-s)P_0} W e^{-sP_0}\|_{1 \rightarrow \infty} ds \\ &\leq C_1 t^{-\frac{3}{2}} \left(1 + \int_{\frac{t}{2}}^{t-\frac{1}{4}} \langle t-s \rangle^{-\frac{1+\rho}{2}+\epsilon} ds \right) \\ &\leq C_2 t^{-\frac{3}{2}}, \quad \text{for } t \geq 1. \end{aligned}$$

It follows that

$$\|I(t)\|_{1 \rightarrow \infty} \leq Ct^{-\frac{3}{2}}, \quad \text{for } t \geq 1. \quad (4.10)$$

For the term $J_1(t)$, we split the domain of integration $\Omega = \{(\tau, s); 0 \leq s \leq \frac{t}{2}, 0 \leq \tau \leq s\}$ into two parts: $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{(\tau, s) \in \Omega; \text{ either } \tau \leq \frac{1}{4} \text{ or } s - \tau \leq \frac{1}{4}\}, \quad \Omega_2 = \Omega \setminus \Omega_1.$$

By Corollary 3.2 and the fact that e^{-tP} is contraction in L^p , one can show as above that the $L^1 - L^\infty$ norm of the piece of $J_1(t)$ related to the integration with respect to $(\tau, s) \in \Omega_1$ can be bounded by $Ct^{-\frac{3}{2}}$. To treat the remaining part, let $p > 3$ be close enough to 3. Then

$$\begin{aligned} &\| \int_{\Omega_2} e^{-(t-s)P_0} W e^{-\tau P} W e^{-(s-\tau)P_0} d\tau ds \|_{1 \rightarrow \infty} \\ &\leq Ct^{-\frac{3}{2}} \int_{\Omega_2} \|e^{-\frac{1}{8}P_0} \nabla V e^{-\tau P} \nabla V e^{-\frac{1}{8}P_0}\|_{p \rightarrow 1} \langle s - \tau \rangle^{-\frac{3(p-1)}{2p}} d\tau ds \\ &\leq C_1 t^{-\frac{3}{2}} \int_{\frac{1}{4}}^{\frac{t}{2}} \|e^{-\frac{1}{8}P_0} \nabla V e^{-\tau P} \nabla V e^{-\frac{1}{8}P_0}\|_{p \rightarrow 1} \end{aligned}$$

$\langle x \rangle^{-(\frac{1}{2}+\epsilon)} : L^p(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ is bounded. Condition (1.5) and (3.11) show that $\nabla V e^{-\frac{1}{8}P_0}$ is bounded from $L^p(\mathbb{R}^6)$ to $\mathcal{L}^{2, \rho + \frac{1}{2} - \epsilon}$. By (3.12), $e^{-\frac{1}{8}P_0} \nabla V$ is bounded from $\mathcal{L}^{2, -\rho + \frac{1}{2} + \epsilon}$ to L^1 . Using (4.5), we obtain

$$\|e^{-\frac{1}{8}P_0} \nabla V e^{-\tau P} \nabla V e^{-\frac{1}{8}P_0}\|_{p \rightarrow 1} \leq C_\epsilon \langle \tau \rangle^{-\rho + \frac{1}{2} + \epsilon}$$

It follows that

$$\|J_1(t)\|_{L^1 \rightarrow L^\infty} \leq \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } \rho > \frac{3}{2} \\ C_\epsilon t^{-\rho + \epsilon}, & \text{if } 1 < \rho \leq \frac{3}{2}. \end{cases} \quad (4.11)$$

The same estimates hold true for $J_2(t)$. Putting them together, we obtain

$$\|J(t)\|_{L^1 \rightarrow L^\infty} \leq \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } \rho > \frac{3}{2} \\ C_\epsilon t^{-\rho + \epsilon}, & \text{if } 1 < \rho \leq \frac{3}{2}. \end{cases} \quad (4.12)$$

From Corollary 3.2), (4.6), (4.10) and (4.12), we obtain

$$\|e^{-tP}\|_{L^1 \rightarrow L^\infty} \leq \begin{cases} Ct^{-\frac{3}{2}}, & \text{if } \rho > \frac{3}{2} \\ C_\epsilon t^{-\rho + \epsilon}, & \text{if } 1 < \rho \leq \frac{3}{2}. \end{cases}$$

If $\rho > \frac{3}{2}$, then Theorem 4.2 is proved by interpolation. If $1 < \rho \leq \frac{3}{2}$, one obtains

$$\|e^{-tP}\|_{L^p \rightarrow L^q} \leq C_\epsilon t^{-\rho(\frac{1}{p} - \frac{1}{q}) + \epsilon}, \quad (4.13)$$

for $t > 1$ and $1 \leq p < q \leq \infty$.

We now use (4.13) instead of (4.5) to improve large time decay of e^{-tP} for $\rho \in]1, \frac{3}{2}]$. Let $\beta_0(p, q) = \rho(\frac{1}{p} - \frac{1}{q})$. Using (4.6) and the results for e^{-tP_0} , we can show as before

$$\|e^{-tP}\|_{1 \rightarrow \infty} \leq Ct^{-\frac{3}{2}} + \|J_1(t)\|_{1 \rightarrow \infty} + \|J_2(t)\|_{1 \rightarrow \infty} \quad (4.14)$$

for $t \geq 1$. To estimate $\|J_1(t)\|_{1 \rightarrow \infty}$. Again we split $\Omega = \Omega_1 \cup \Omega_2$ as before. The $L^1 - L^\infty$ norm of the piece of $J_1(t)$ given by integration over Ω_1 is bounded by $Ct^{-\frac{3}{2}}$. For $(\tau, s) \in \Omega_2$, we have

$$\begin{aligned} & \|e^{-(t-s)P_0} W e^{-(s-\tau)P} W e^{-\tau P_0}\|_{1 \rightarrow \infty} \\ & \leq \|e^{-(t-s-\delta)P_0} \nabla_v\|_{1 \rightarrow \infty} \|e^{-\delta P_0} \nabla V e^{-(s-\tau)P} \nabla V e^{-\delta P_0}\|_{p \rightarrow 1} \|\nabla_v e^{-(\tau-\delta)P_0}\|_{1 \rightarrow p} \end{aligned}$$

where $p = 3 + \epsilon'$, $\epsilon' > 0$, $\delta > 0$. By Hölder's inequality, (3.11) and (3.12), $\nabla V e^{-\delta P_0}$ is bounded from $L^p \rightarrow L^{p_1}$ and $e^{-\delta P_0} \nabla V$ is bounded from $L^{q_1} \rightarrow L^1$, where

$$\frac{1}{p_1} = \frac{1}{p} + \frac{1}{r} \quad \text{and} \quad \frac{1}{q_1} + \frac{1}{r} = 1.$$

By choosing p close to 3 and r close to $\frac{3}{1+\rho}$, p_1 can be any number smaller than $\frac{3}{2+\rho}$ and q_1 can be any number bigger than $\frac{3}{2-\rho}$. Set

$$r_1 = \beta_0\left(\frac{3}{2+\rho}, \frac{3}{2-\rho}\right) = \frac{2\rho^2}{3}$$

Making use of (4.13) instead of (4.5), one obtains

$$\|J_1(t)\|_{1 \rightarrow \infty} \leq \frac{C}{t^{\frac{3}{2}}} \left(1 + \int_{\frac{1}{4}}^{\frac{t}{2}} s^{-r_1+\epsilon} ds\right)$$

In a similar way, one can show that $\|J_2(t)\|_{1 \rightarrow \infty}$ satisfies the same estimate. If $\rho > \sqrt{\frac{3}{2}}$, then $r_1 > 1$ and Theorem 4.2 is proved. If $1 < \rho \leq \sqrt{\frac{3}{2}}$, one obtains for any $\epsilon > 0$

$$\|e^{-tP}\|_{L^p \rightarrow L^q} \leq C_\epsilon t^{-(\frac{1}{2}+r_1)(\frac{1}{p}-\frac{1}{q})+\epsilon}, \quad (4.15)$$

for $t > 1$ and $1 \leq p < q \leq \infty$. Set $\beta_1(p, q) = (\frac{1}{2} + r_1)(\frac{1}{p} - \frac{1}{q})$ and

$$r_2 = \beta_1\left(\frac{3}{2+\rho}, \frac{3}{2-\rho}\right) = \frac{\rho(1+2r_1)}{3}.$$

Repeating the arguments from (4.13) to (4.15) with (4.13) replaced by (4.15), one concludes that if $r_2 > 1$, Theorem 4.2 is proved. Otherwise, one has

$$\|e^{-tP}\|_{L^p \rightarrow L^q} \leq C_\epsilon t^{-(\frac{1}{2}+r_2)(\frac{1}{p}-\frac{1}{q})+\epsilon}, \quad (4.16)$$

for $t > 1$ and $1 \leq p \leq q \leq \infty$. For $k \geq 3$, set $\beta_{k-1}(p, q) = (\frac{1}{2} + r_{k-1})(\frac{1}{p} - \frac{1}{q})$ and

$$r_k = \beta_{k-1}\left(\frac{3}{2+\rho}, \frac{3}{2-\rho}\right) = \frac{\rho(1+2r_{k-1})}{3}, \quad (4.17)$$

Let $\rho > 1$ be fixed. By an induction on k , one can prove that for each k , either $r_k > 1$, then (4.3) is proved by the above argument; or $0 < r_k \leq 1$, then one has

$$\|e^{-tP}\|_{L^p \rightarrow L^q} \leq C_\epsilon t^{-(\frac{1}{2}+r_k)(\frac{1}{p}-\frac{1}{q})+\epsilon}, \quad (4.18)$$

for $t > 1$ and $1 \leq p < q \leq \infty$. We affirm that for each $\rho > 1$, there exists $k \in \mathbb{N}$ such that $r_k > 1$. In fact, if $r_k \leq 1$ for all $k \in \mathbb{N}$, then $\{r_k\}$ would be an increasing sequence bounded by 1. Let $\ell = \lim_{k \rightarrow \infty} r_k$. Then $\ell \in]0, 1]$. However, taking the limit $k \rightarrow \infty$ in (4.17), one has

$$\ell = \frac{\rho(1+2\ell)}{3}$$

which gives $\ell = \frac{\rho}{3-2\rho} > 1$, because $\rho > 1$. This contradiction in ℓ proves that for each $\rho > 1$, there exists some k such that $r_k > 1$. Therefore (4.3) follows by repeating at most k times the arguments from (4.13) to (4.15) with (4.13) replaced by newly improved estimate. This achieves the proof of Theorem 4.2 for any $\rho > 1$. \square

Theorem 1.1 follows from Theorems 4.1 and 4.2.

Remark 4.3. *Let $n = 1$ and condition (1.5) be satisfied with $\rho > 4$. It is known ([12]) that for $s > \frac{5}{2}$*

$$\|e^{-tP}\|_{\mathcal{L}^s \rightarrow \mathcal{L}^{-s}} \leq C \langle t \rangle^{-\frac{1}{2}}, t \geq 1. \quad (4.19)$$

The method used in the proof of Theorem 4.2 does not allow to deduce from (4.19) any decay of e^{-tP} in $L^1 - L^\infty$ for t large. For example, for the term $I_1(t)$ given in (4.6), the method used in the proof of Theorem 4.2 only gives

$$\|I_1(t)\|_{1 \rightarrow \infty} \leq C \left(t^{-\frac{1}{2}} + \int_{\frac{1}{4}}^{\frac{t}{2}} \langle t-s \rangle^{-\frac{1}{2}} \langle s \rangle^{-\frac{1}{2}} ds \right), \quad t \geq 1.$$

The last integral does not decay as $t \rightarrow \infty$.

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