



HAL
open science

Recovering time-dependent inclusion in heat conductive bodies

Olivier Poisson

► **To cite this version:**

| Olivier Poisson. Recovering time-dependent inclusion in heat conductive bodies. 2021. hal-03278958

HAL Id: hal-03278958

<https://hal.science/hal-03278958>

Preprint submitted on 6 Jul 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Recovering time-dependent inclusion in heat conductive bodies

O. Poisson*

July 2, 2021

Abstract

We consider an inverse boundary value problem for the heat equation with a nonsmooth coefficient of conductivity which models the displacement of a moving body inside an isotrop but nonhomogeneous background. We prove the uniqueness of the moving inclusion from the knowledge of the Dirichlet-to-Neumann operator by using a dynamical probe method.

Keywords: Inverse problem, Heat equation, Dynamical probe method.

AMS : 35R30, 35K05.

1 Introduction

1.1 Inverse heat conductivity problem

Let $T > 0$ and let Ω be a bounded domain in \mathbb{R}^3 , with a lipschitzian boundary $\Gamma = \partial\Omega$. Let us consider the anisotropic heat equation

$$\partial_t v - \operatorname{div}(\mathbf{a}\nabla v) = 0 \quad \text{in } \Omega_{0,T} \equiv \Omega \times (0, T), \quad (1)$$

where the operators div , the divergence, and ∇ , the gradient, are relative to the spatial variable x . In our model, the conductivity $\mathbf{a} = (a_{ij})_{1 \leq i, j \leq 3}$ is a 3×3 real symmetric matrix with positive bounded measurable coefficients of x . It satisfies the uniform elliptic condition:

there exists $\gamma_\infty > 0$ such that

$$\gamma_\infty^{-1}|\xi|^2 \leq \mathbf{a}\xi \cdot \xi \leq \gamma_\infty|\xi|^2, \quad \xi \in \mathbb{R}^3. \quad (2)$$

It is well-known (see [19]) that, for all $f \in L^2(0, T; H^{1/2}(\Gamma))$ and $v_0 \in L^2(\Omega)$, there exists only one solution $v = v(\mathbf{a}, v_0; f) \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of (1) with the following initial boundary value problem:

$$\begin{cases} v = f & \text{on } \Gamma_{0,T} \equiv \Gamma \times (0, T), \\ v|_{t=0} = v_0 & \text{on } \Omega. \end{cases} \quad (3)$$

Then, we can define the Dirichlet-to-Neumann map (D-N map) as

$$\Lambda_{\mathbf{a}; v_0} : L^2((0, T); H^{1/2}(\Gamma)) \ni f \mapsto \mathbf{a}\nabla v(\mathbf{a}, v_0; f) \cdot \nu \in L^2((0, T); H^{-1/2}(\Gamma)),$$

*Aix Marseille Université, I2M, UMR CNRS 6632, France (olivier.poisson@univ-amu.fr).

where ν denotes the outer unit normal to Γ . In physical terms, $f = f(t, x)$ is the temperature distribution on the boundary and $\Lambda_{\mathbf{a}, v_0}(f)$ is the resulting heat flux through the boundary.

In this article we are concerned with the Calderón inverse problem for (1) which is to determine \mathbf{a} from the knowledge of the D-N map $\Lambda_{v_0, \mathbf{a}}$. The conductivity \mathbf{a} consists in a non necessarily smooth background and an unknown inclusion $t \mapsto D_t \subset \Omega$ which moves continuously inside the body Ω . Thus, in our inverse problem, the function $\mathbf{a}|_{\Omega \setminus D_t}$ coincides with a measurable real matrix-function $\mathbf{b} \in L^\infty(\Omega)$ which satisfies (2) and represents the conductivity of a background medium, and so, is known. The inverse problem we address is to determine the moving inclusion $D = \cup_{0 \leq t \leq T} (D_t \times \{t\}) \subset \Omega_{0,T}$ from the knowledge of $\Lambda_{\mathbf{a}, v_0}$.

Remark 1. *In our problem the value of the conductivity inside the inclusion, $\mathbf{a}|_{D_t}$, and the initial value of v , v_0 , are unknown but the article does not deal with their determination.*

1.2 Main assumptions

The two following assumptions were already considered in [3],[13],[17].

(H0): There exists a positive constant δ_1 such that

$$(H0a) : \quad ((\mathbf{b} - \mathbf{a})\mathbf{a}^{-1}\mathbf{b})_S \geq \delta_1 > 0, \quad \mathbf{b} - \mathbf{a} \geq \delta_1 > 0 \quad \text{in } D,$$

or

$$(H0b) : \quad ((\mathbf{b} - \mathbf{a})\mathbf{a}^{-1}\mathbf{b})_S \leq -\delta_1 < 0, \quad \mathbf{b} - \mathbf{a} \leq -\delta_1 < 0 \quad \text{in } D.$$

(Here, if \mathbf{m} is a 3×3 real matrix then $\mathbf{m}_S = \frac{1}{2}(\mathbf{m} + \mathbf{m}^T)$ denotes its symmetric part.)

(H1): for all $t \in [0, T]$, the set $\mathbb{R}^3 \setminus \overline{D_t}$ is connected.

Because of technical limitations of our method when \mathbf{b} is not sufficiently smooth, we need some additional geometrical assumptions on D . For a point $x \in \mathbb{R}^3$ and a non-empty set $E \subset \mathbb{R}^3$ we denote by $d(x, E)$ the quantity $\inf_{z \in E} |x - z|$ and by $|E|$ the Lebesgue-measure of E .

(H2): $t \mapsto D_t$ is lipschitzian in the following sense:

there exists $K_D > 0$ such that for all $x \in \overline{\Omega}$ the mapping $t \mapsto d(x, \Omega \setminus D_t)$ is lipschitzian in $[0, T]$ with lipschitzian constant K_D and the mapping $t \mapsto d(x, D_t)$ is lipschitzian at all $s \in [0, T]$ such that $D_s \neq \emptyset$ with lipschitzian constant K_D .

(H3): D_t is lipschitzian, uniformly as $t \in [0, T]$, i.e, there exists $L_D \in (0, 1)$ such that $|D_t \cap B(x, r)| \geq L_D \min(|D_t|, |B(x, r)|)$, $\forall r > 0$, $x \in \partial D_t$, $t \in [0, T]$.

Runge approximation in the dynamical probe method (see [13]) is based on the uniqueness property (UC) which holds if the conductivity is constant but may fail if it is not sufficiently smooth. Therefore we add the following assumption on \mathbf{b} :

(UC) in Ω - Let a sufficiently smooth domain $\omega \subset \Omega$, $a < b$ and let $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ such that $\partial_t u - \operatorname{div}(\mathbf{b}\nabla u) = 0$ in $\omega \times (a, b)$ and $u = \mathbf{a}\nabla u \cdot \nu = 0$ on $S \times (a, b)$, where S is a non-empty open subset of $\partial\omega$. Then, necessarily, $u = 0$ in $\omega \times (a, b)$.

Remark 2. *The above definition of (UC) is independent of the choice of the time-interval $[0, T]$ since in our work we assume that \mathbf{b} does not depend on the variable t .*

Remark 3. *Condition (UC) holds if \mathbf{b} is lipschitzian or piecewise smooth: see Vessella [18, chap 5].*

1.3 Main Result

Here we state our uniqueness result for the above inverse problem. Let $v_0, v'_0 \in L^2(\Omega)$, two conductivities \mathbf{a}, \mathbf{a}' satisfying (H0)-(H3) and (UC). Let D' the inclusion related to \mathbf{a}' .

Theorem 1. *Assume that $\Lambda_{v_0, \mathbf{a}} = \Lambda_{v'_0, \mathbf{a}'}$. Then, $D = D'$.*

Remark 4. *Our proof of Theorem 1 is not completely constructive, although it is based on a dynamical method as in [17], where a (theoretical) reconstruction of D from the knowledge of $\Lambda_{v_0, \mathbf{a}}$ was developed.*

Remark 5. *We shall proof Theorem 1 with the following assumption:*

$$\overline{D(t)} \subset \Omega, \quad t \in [0, T].$$

Therefore we replace (H1) by:

(H1'): *one has $\overline{D(t)} \subset \Omega$, and the set $\Omega \setminus \overline{D_t}$ is connected, for all $t \in [0, T]$.*

The general proof of Theorem 1 where $D(t)$ may touch $\partial\Omega$ is easily get from the following modification on the case (H1'):

- *We consider a large smooth bounded domain Ω' containing $\overline{\Omega}$ and we put $\mathbf{b} = I_3$ (the 3×3 identity matrix) in $\Omega' \setminus \Omega$.*
- *(If necessary)¹ (UC) is assumed with Ω replaced by Ω' .*

Remark 6. *The proof of Theorem 1 will show that (H0) can be extended to the following situation:*

(H0') *There exist positive constants ε_0, δ_1 , such that for $(x, t) \in \overline{D}$,*

$$((\mathbf{b} - \mathbf{a})\mathbf{a}^{-1}\mathbf{b})_S(x) \geq \delta_1 > 0, \quad \mathbf{b}(x) - \mathbf{a}(x) \geq \delta_1 > 0 \quad \text{if } d(x, \partial D_t) \leq \varepsilon_0,$$

or

$$((\mathbf{b} - \mathbf{a})\mathbf{a}^{-1}\mathbf{b})_S \leq -\delta_1 < 0, \quad \mathbf{b}(x) - \mathbf{a}|_{D_t}(x) \leq -\delta_1 < 0 \quad \text{if } d(x, \partial D_t) \leq \varepsilon_0,$$

¹the question that (UC) in Ω would imply (UC) in Ω' is out of the scope of this article

1.4 Outline

In Section 2 we recall the basis of the dynamical probe method, Runge approximation, and we construct indicator and pre-indicator functions from special Cauchy boundary data. In Section 3 we give lower and upper estimates on the indicator function which we achieve to prove in Section 4 after several technical results. In Section 5 we achieve the proof of our main Theorem 1.

1.5 Literature review

Assumption was already considered in [3],[13], but not in [17].

2 The dynamical probe method (DPM) with special solutions of the heat equation

2.1 Notations

Let us give some notations for this paper. For $E \subset \mathbb{R}^3$, $a < b$, and for $U \subset \mathbb{R}^3 \times \mathbb{R}$, we put $E_{a,b} = E \times (a, b)$ and $U_t \equiv \{x \in \mathbb{R}^3 \mid (x, t) \in U\}$.

For non-negative integers p, q or $p = 1/2$, $H^p(\Omega)$, $H^p(\partial\Omega)$ and $H^{p,q}(\Omega_{(a,b)})$ denote the usual Sobolev spaces where the superscripts p and q indicate the regularity with respect to x and t , respectively. For an open set $U \subset \mathbb{R}^4$ with Lipschitz boundary ∂U , $H^{p,q}(U)$ is defined likewise. More precisely, $g \in H^{p,q}(U)$ if and only if there exists $G \in H^{p,q}(\mathbb{R}^4)$ with $G = g$ in U . If it is the case, $\|g\|_{H^{p,q}(U)}$ is defined to be

$$\|g\|_{H^{p,q}} := \inf \|G\|_{H^{p,q}(\mathbb{R}^4)},$$

where the infimum is taken over all G such that $G = g$ in U . Let X be a normed space of functions. A function $f(x, t)$ is said to be in $L^2((0, T); X)$ if $f(\cdot, t) \in X$ for almost all $t \in (0, T)$ and

$$\|f\|_{L^2((0,T);X)}^2 := \int_0^T \|f(\cdot, t)\|_{L^2(X)}^2 dt < \infty.$$

(see [15] for more details).

We write $\mathcal{L}_{\mathbf{a}} := \partial_t - \operatorname{div}(\mathbf{a}\nabla\cdot)$, so $\mathcal{L}_I := \partial_t - \Delta$ for the homogeneous case. Similarly, we consider operator for the backward related heat equation, $\mathcal{L}_{\mathbf{a}}^* := -\partial_t - \operatorname{div}(\mathbf{a}\nabla\cdot)$. We denote by $B(r)$ any ball of radius $r > 0$ in \mathbb{R}^3 . The open ball $\{x \in \mathbb{R}^3; |y - x| < r\}$, $r > 0$, is denoted $B(y, r)$.

We denote by $\mathbf{d}(t)$ the distance between $y(t)$ and D_t if $D_t \neq \emptyset$, i.e., $\mathbf{d}(t) = d(y(t), D_t)$. If $D_t = \emptyset$ then we put $\mathbf{d}(t) = +\infty$, $1/\mathbf{d}(t) = 0$.

If \mathbf{m} is a 3×3 real matrix then $\mathbf{m}_S = \frac{1}{2}(\mathbf{m} + \mathbf{m}^T)$ denotes its symmetric part and if $\xi \in \mathbb{R}^3$ then $\mathbf{m}_S(\xi)^2 := \mathbf{m}_S \xi \cdot \xi$.

2.2 Brief history of the determination of an inclusion from the D-N map

The determination (i.e, the problem of uniqueness) of a sufficiently smooth inclusion inside an homogeneous body was stated in [4] with a proof by contradiction. DPM for (1) is an extension of Ikehata's probe method which was developed for the

elliptic equation $\operatorname{div}(\mathbf{a}\nabla v) = 0$. It was firstly presented in [3] where the background is homogeneous and $D_t \in C^2$ for all t . Although a part of DPM in [3] works for all spatial dimension n , the reconstruction of D was proved in the case $n = 1$ only. DPM in [3], corrected in [13], uses Runge approximation of the fundamental solution of $\mathcal{L}_I u = \delta_{(y,s)}$, $(y, s) \in \Omega_{(0,T)}$. Extending [4] in a more quantitative version which requires a little bit more smoothness and differs to the DPM of [3] and [13], [2] proved the log-stability of $\Lambda_{\mathbf{a},0} \mapsto D$.

Returning to DPM, the author of this article used in [17] "special solutions" for the classical heat operator which are more convenient functions than the basic fundamental solutions $\Gamma(x - y, t - s)$, because their behaviour in time and space are sufficiently separated. Since the background in [17] is homogeneous, [17] showed that DPM can reconstruct any spatially irregular inclusion as in the elliptic situation. But in our situation we are limited to inclusions with not too spatially irregularities (see (H3)), and we have no reconstruction process.

2.3 Runge approximation

Runge approximation for the operator unperturbed operator \mathcal{L}_I with the homogeneous conductivity $\mathbf{a} = I_3$ was developed in [3] and [13].

Let a lipschitzian curve $\Sigma : [0, T] \ni t \mapsto y(t) \in \mathbb{R}^3 \setminus \overline{D_t}$ which does not touch D . We extend Σ to $t \in \mathbb{R}$ by putting $y(t) = y(T)$ for $t \geq T$ and $y(t) = y(0)$ for $t \leq 0$. Then, thanks to (H1'), there exists an open set $U \subset \Omega \times \mathbb{R}$ containing \overline{D} and satisfying

$$\begin{cases} \partial U \text{ is lipschitzian,} \\ \operatorname{dist}(U, \Sigma) := \inf\{|x - y|; x \in U, y \in \Sigma\} > 0, \\ \Omega \setminus \overline{U_t} \text{ is connected, } t \in \mathbb{R}. \end{cases}$$

Runge-type approximation as in [3, 13, 17] works thanks to (UC) notably, and yields the following result. For $\tau > 0$ we denote $\Sigma_\tau = \cup_{t \in \mathbb{R}} B(y(t), 1/\tau) \times \{t\}$.

Proposition 1. *Assume (H1') and (UC). Let Σ and U be as above. Let $u \in H^{1,0}(\Omega_{(0,T)}) \cap H^{0,1}(\Omega_{(0,T)})$ be a solution of $\mathcal{L}_{\mathbf{b}} u = 0$ in $\Omega_{(-1,T+1)} \setminus \Sigma_\tau$. Then for $\tau > \inf\{\tau > 0 \mid \operatorname{dist}(U, \Sigma_\tau) > 0\}$ there exists a sequence $u_j \in H^{1,0}(\Omega_{(-1,T+1)}) \cap H^{0,1}(\Omega_{(-1,T+1)})$ such that*

$$\begin{cases} \mathcal{L}_{\mathbf{b}} u_j = 0 & \text{in } \Omega_{(-1,T+1)}, \\ u_j \rightarrow u & \text{in } H^{1,0}(U) \cap H^{0,1}(U), \\ u_j(0) = u(0) & \text{in } L^2(\Omega). \end{cases}$$

2.4 Heat Kernels

In many researchs devoted to inverse problems for parabolic equations, the background is homogeneous, i.e, $\mathbf{b} = I_3$. In such a classical situation, the heat operator is $\partial_t - \Delta$ and its usual kernel $\Gamma(x, t)$ has many properties, as

1. It is explicit:

$$\Gamma(x, t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{x^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}^3.$$

2. It satisfies

$$\Gamma(x, t) \leq \frac{C}{\sqrt{t}} |\nabla \Gamma(x, t)|, \quad t > 0, \quad x \in \mathbb{R}^3,$$

for some $C > 0$. Hence, $\Gamma(x, t)$ is small compared to $|\nabla\Gamma(x, t)|$ as $t \rightarrow 0$.

3. Thanks to the Laplace transform $\int_0^\infty \cdot e^{-\tau^2 t} dt$ of $\partial_t - \Delta$, we consider similarly the elliptic operator $-\Delta + \tau^2$ with the (large) real parameter $\tau > 0$. Its kernel $E(x; \tau)$ is explicit too:

$$E(x; \tau) = \int_0^\infty \Gamma(x, t) e^{-\tau^2 t} dt = \frac{e^{-\tau|x|}}{4\pi|x|}, \quad x \in \mathbb{R}^3.$$

4. It satisfies

$$E(x; \tau) \leq \tau |\nabla E(x; \tau)|, \quad x \in \mathbb{R}^3.$$

Hence, $E(x; \tau)$ is small compared to $|\nabla E(x; \tau)|$ as $\tau \rightarrow \infty$, uniformly in all bounded set of $\mathbb{R}^3 \setminus \{0\}$. This fact was exploited in [17].

Let us come back to the heat equation with a general conductivity \mathbf{b} . We put $\mathbf{b}(x) = I_3$ for $x \in \mathbb{R}^3 \setminus \bar{\Omega}$.

For $y \in \mathbb{R}^3$, we denote by $G_y \in C(\mathbb{R}; L^2(\mathbb{R}^3))$ the fundamental solution of

$$\mathcal{L}_{\mathbf{b}} G_y = \delta_{(y, 0)},$$

which satisfies

$$G_y(x, t) = 0, \quad t < 0.$$

We have the estimate:

$$\frac{\kappa e^{-\frac{|x-y|^2}{4\kappa^2 t}}}{t^{3/2}} \leq G_y(x, t) \leq \frac{e^{-\frac{\kappa^2 |x-y|^2}{4t}}}{\kappa t^{3/2}}, \quad x \in \mathbb{R}^3, t > 0, \quad (4)$$

for some constant $\kappa = \kappa(\mathbf{b}) \in (0, 1)$ (see [1, 16]).

For $\tau > 0$ we put the Laplace Transform of $G_y(x, t)$ as

$$p_\tau(x; y) := e^{-\tau^2 t} \int_{-\infty}^t e^{\tau^2 s} G_y(x, t-s) ds = \int_0^\infty e^{-\tau^2 s} G_y(x, s) ds. \quad (5)$$

Let us observe that $p_\tau(\cdot; y)$ belongs to $H_{loc}^1(\mathbb{R}^3 \setminus \{y\})$ and, thanks to (4), satisfies

$$(-\operatorname{div}(\mathbf{b}\nabla\cdot) + \tau^2)p_\tau(\cdot; y) = \delta_y(\cdot), \quad (6)$$

$$2\sqrt{\pi} \frac{\kappa^2 e^{-\frac{\tau}{\kappa}|x-y|}}{|x-y|} \leq p_\tau(x; y) \leq 2\sqrt{\pi} \frac{e^{-\kappa\tau|x-y|}}{\kappa^2|x-y|}, \quad x \in \mathbb{R}^3 \setminus \{y\}. \quad (7)$$

This is also a consequence of the works of Nash and Aronson.

2.5 Special solutions

Let us consider a lipschitzian curve $\Sigma \subset \mathbb{R}^3 \times \mathbb{R}$ as in Section 2.3, and fix $\theta \in (0, T)$. Let another positive parameter $\mu \geq 1$ that we shall precise later.

In [17] with $\mathbf{b} \equiv I_3$ the author considered special solutions related to the following functions:

$$U_{OP}(x, t) := e^{\tau^2(T+t)} \int_0^\infty e^{\tau\mu(|t-\theta-s|-|t-\theta|)} \Gamma(x-y(t-s), s) e^{-\tau^2 s} ds,$$

$$U_{OP}^*(x, t) := e^{-\tau^2(T+t)} \int_0^\infty e^{\tau\mu(|t-\theta+s|-|t-\theta|)} \Gamma(x-y(t+s), s) e^{-\tau^2 s} ds.$$

In fact, U_{OP} and U_{OP}^* are respectively solutions of the following forward and backward heat equations:

$$\begin{aligned}\mathcal{L}_I U_{OP}(x, t) &= e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \\ \mathcal{L}_I^* U_{OP}^*(x, t) &= e^{-\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}.\end{aligned}$$

Moreover they satisfies (see [17, Lemma 1]):

$$\begin{aligned}U_{OP}(x, t) &= \varphi(x, t) e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)), \\ U_{OP}^*(x, t) &= \varphi^*(x, t) e^{-\tau^2(t+T)} e^{-\tau\mu|t-\theta|} p_\tau(x; y(t)),\end{aligned}$$

such that, for some $C = C(R, \mu) > 0$ and all $\tau \geq C$,

$$\frac{1}{C} \leq |\varphi(x, t)| + |\varphi^*(x, t)| \leq C \quad \text{in } B(0, R) \times \mathbb{R}, \quad (8)$$

$$|\nabla\varphi(x, t)| + |\nabla\varphi^*(x, t)| \leq C \quad \text{in } B(0, R) \times \mathbb{R}. \quad (9)$$

With the general conductivity \mathbf{b} , we construct here special solutions u_τ and u_τ^* as follows. Let us put

$$m_\tau(x, t) = M_0(\tau|x - y(t)|), \quad t \in \mathbb{R}, \quad (10)$$

where M_0 is defined by $M_0(r) = |1 - r| \mathbf{1}_{|r| \leq 1}$. Hence m_τ is a lipschitzian function with support closed to Σ as $\tau \gg 1$. We then put, for $(x, t) \in \mathbb{R}^3 \times \mathbb{R}$,

$$\begin{aligned}u_\tau(x, t) &= \int_{s \in \mathbb{R}} \int_{y \in \mathbb{R}^3} e^{\tau^2(s+T)} e^{-\tau\mu|s-\theta|} m(y, s) G_y(x, t - s) dy ds \quad (11) \\ &= \int_{s=0}^{\infty} \int_{y \in \mathbb{R}^3} e^{\tau^2(T+t-s)} e^{-\tau\mu|t-\theta-s|} m(y, t - s) G_y(x, s) dy ds,\end{aligned}$$

$$\begin{aligned}u_\tau^*(x, t) &= \int_{s \in \mathbb{R}} \int_{y \in \mathbb{R}^3} e^{-\tau^2(T+s)} e^{-\tau\mu|s-\theta|} m_\tau(y, s) G_y(x, s - t) dy ds \quad (12) \\ &= \int_{s=0}^{\infty} \int_{y \in \mathbb{R}^3} e^{-\tau^2(T+t+s)} e^{-\tau\mu|t-\theta+s|} m_\tau(y, t + s) G_y(x, s) dy ds.\end{aligned}$$

The functions u_τ and $u_\tau^*(x, t)$ are positive and satisfy

$$\begin{aligned}\mathcal{L}_{\mathbf{b}} u_\tau(x, t) &= e^{\tau^2(t+T)} e^{-\tau\mu|t-\theta|} m(x, t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \quad (13) \\ \mathcal{L}_{\mathbf{b}}^* u_\tau^*(x, t) &= e^{-\tau^2(T+t)} e^{-\tau\mu|t-\theta|} m_\tau(x, t) \quad \text{in } \mathbb{R}^3 \times \mathbb{R}.\end{aligned}$$

Remark 7. If $m_\tau(x, t)$ was replaced by $\delta(x - y(t))$ then it would be difficult to make the estimation of $\dot{y}(s) \nabla_y G_{y(t-s)}(x, t)$ that would appear in the expression of $\partial_t u_\tau$.

We then expect that

$$u_\tau(x, t) \stackrel{\tau \rightarrow \infty}{\simeq} e^{\tau^2(T+t)} e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)), \quad (14)$$

$$u_\tau^*(x, t) \stackrel{\tau \rightarrow \infty}{\simeq} e^{-\tau^2(T+t)} e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)), \quad (15)$$

where the meaning of " \simeq " will be clarified shortly. Since the comparison requires the time-derivatives of $u_\tau(x, t)$ or $u_\tau^*(x, t)$ and remembering Remark 7, we introduce the following smooth approximation of $p_\tau(x; y(t))$:

$$P_\tau(x, t) := \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2 s} m_\tau(y, t) G_y(x, s) dy ds. \quad (16)$$

We then put

$$q_\tau(x, t) := e^{-\tau^2(T+t)}u_\tau(x, t) - e^{-\tau\mu|t-\theta|}P_\tau(x, t), \quad (17)$$

$$q_\tau^*(x, t) := e^{\tau^2(T+t)}u_\tau^*(x, t) - e^{-\tau\mu|t-\theta|}P_\tau(x, t). \quad (18)$$

The main difficulty in the proof of Theorem 1 is to prove that the quantity

$$R_0 := \int_D (|\nabla q_\tau(x, t)|^2 + |\nabla q_\tau^*(x, t)|^2) dx dt, \quad (19)$$

is negligible compared to $\int_D \tau^{-6} e^{-2\tau\mu|t-\theta|} |\nabla p_\tau(x, t)|^2 dx dt$ or, in an equivalent way (see Lemmas 4.3 and 4.4), to $\int_D \tau^{-4} e^{-2\tau\mu|t-\theta|} |p_\tau(x, t)|^2 dx dt$. We shall prove in Appendix the following Lemma.

Lemma 2.1. (*Estimate of ∇q_τ in D_t*). *Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Under (H3) there exist two positive constants $C = C(\mathbf{b}, L_D, d_\Omega, \mu, M)$, $C_0 = C_0(\kappa, d_\Omega, M)$ such that if $t \in [0, T]$ and $\tau > \max(2\mu, C_0/d(t))$ then*

$$\int_{D_t} (|\nabla q_\tau(x, t)|^2 + |\nabla q_\tau^*(x, t)|^2) dx \leq C \tau^{-4} e^{-2\tau\mu|t-\theta|} \mathbf{d}(t)^2 \int_{D_t} |p_\tau(x, y(t))|^2 dx. \quad (20)$$

(Remember that $\mathbf{d}(t) = d(y(t), D_t)$.) So R_0 is effectively "negligible" when the curve Σ is sufficiently close to D at least at time θ . This constraint is new compared to consequences of (8) and (9) (for which Assumption (H3) is in addition superfluous) and makes a theoretical reconstruction of D problematic, as opposite to the possible reconstruction proposed in [17].

2.6 Pre-indicator sequence and indicator function

As in section 2.3, we can consider sequences $(u_j)_j$ and $(u_j^*)_j$ such that $u_j \rightarrow u_\tau$ and $u_j^* \rightarrow u_\tau^*$ in the sense of Proposition 1. Considering $v_j = v(\mathbf{a}, v_0; u_j|_{\Gamma(0,T)})$ and the solution $v_\tau \in H^1((0, T); L^2(\Omega)) \cap L^2((0, T); H^1(\Omega))$ of

$$\begin{cases} \mathcal{L}_{\mathbf{a}} v_\tau &= \mathcal{L}_{\mathbf{b}} u_\tau, \\ v_\tau &= u_\tau \quad \text{on } \Gamma_{0,T}, \\ v_\tau|_{t=0} &= v_0 \quad \text{on } \Omega, \end{cases} \quad (21)$$

we put

$$w_\tau = v_\tau - u_\tau \quad (22)$$

and

$$\begin{aligned} I_j(\tau) &:= \int_{\Gamma \times [0, T]} (\Lambda_{\mathbf{a}; v_0}(u_j|_{\Gamma(0, T)}) - \mathbf{b} \nabla u_j \cdot \nu) u_j^*|_{\Gamma \times [0, T]} d\sigma(x) dt, \\ I_\infty(\tau) &:= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_\tau \nabla u_\tau^* dx dt + \int_{\Omega} [w_\tau u_\tau^*]_0^T dx, \end{aligned} \quad (23)$$

where $d\sigma(x)$ is the usual measure on the boundary Γ . The knowledge of $\Lambda_{\mathbf{a}; v_0}$ involves that of $I_j(\tau)$'s. Furthermore, as for the proofs in [3, 17], Proposition 1 implies that

$$I_j(\tau) \rightarrow I_\infty(\tau) \in \mathbb{R} \quad \text{as } j \rightarrow \infty. \quad (24)$$

Hence, if (UC) holds, then the knowledge of $\Lambda_{\mathbf{a}; v_0}$ involves that of $I_\infty(\tau)$'s.

3 Estimates on the indicator function

Lemma 3.1. *Under assumption (H0b) we have*

$$\begin{aligned} I_\infty(\tau) &\leq C \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \\ &\quad + C \int_D (|\nabla q_\tau|^2 + |\nabla q_\tau^*|^2) dx dt + 10(\|v_0\|_{L^2(\Omega)}^2 + d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (25)$$

and

$$\begin{aligned} I_\infty(\tau) &\geq \frac{1}{C} \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \\ &\quad - C \int_D (|\nabla q_\tau|^2 + |\nabla q_\tau^*|^2) dx dt - 10(\|v_0\|_{L^2(\Omega)}^2 + d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (26)$$

for some $C = C(\mathbf{a}) \geq 1$, for all $\tau > \mu + 1$, $\mu > 0$.

Proof in Appendix.

We put also

$$d_\Omega := \sup\{|x - y|; x, y \in \Omega\}, \quad (27)$$

$$\varepsilon_\Sigma := \inf_{t \in [0, T]} \mathbf{d}(t) > 0. \quad (28)$$

Lemma 3.2. *Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Then, under assumption (H0b), there exist positive constants $c = c(\mathbf{a}, \mu, M)$, $C_1 = C_1(v_0, d_\Omega)$, $C_2 = C_2(\kappa, d_\Omega, M)$, $C_3 = C_3(d_\Omega, \kappa, M, \mu)$, such that, if $\tau > \max(2\mu, C_2/\varepsilon_\Sigma)$, we then have*

$$\begin{aligned} I_\infty(\tau) &\leq c\tau^{-4} \int_D e^{-2\tau\mu|t-\theta|} |p_\tau(x, y(t))|^2 dx dt \\ &\quad + C_1 e^{-\tau\mu \min(T-\theta, \theta)}, \end{aligned} \quad (29)$$

and

$$\begin{aligned} I_\infty(\tau) &\geq c\tau^{-4} \int_D (1 - C_3 \mathbf{d}(t)^2) e^{-2\tau\mu|t-\theta|} |p_\tau(x, y(t))|^2 dx dt \\ &\quad - C_1 e^{-\tau\mu \min(T-\theta, \theta)}. \end{aligned} \quad (30)$$

The proof of Lemma 3.2 requires the developments of Section 4

Let us extend $\|n$ to $(-\infty, 0]$ by putting $\|n(I) = -\infty$ if $I \leq 0$.

Lemma 3.3. *Let $\theta \in (0, T)$. Let us fix $\mu \geq \mu_1 = 4\kappa^{-1}d_\Omega \max((T-\theta)^{-1}, \theta^{-1})$.*

1) *Let a lipschitzian curve Σ such that $|\dot{y}| \leq M$ and $\varepsilon_\Sigma > 0$. We then have*

$$\limsup_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \leq -2\kappa\varepsilon_\Sigma. \quad (31)$$

2) *Assume that $D_\theta \neq \emptyset$. Let $\alpha \in (0, (8C_3)^{-1/2})$ where C_3 is the constant in (30). Let $M, \alpha > 0$ and a family $\Sigma = \Sigma(\varepsilon)$ for $0 < \varepsilon \leq \alpha^3$ such that we have $|\dot{y}(\cdot)|_\infty \leq M$ and*

$$\begin{cases} |\dot{y}(\cdot)|_\infty &\leq M, \\ \mathbf{d}(t) = d(y(t), D_t) &\leq 2\alpha \quad \text{for } |t - \theta| \leq \alpha^4, \\ \mathbf{d}(t) = d(y(t), D_t) &\in [\varepsilon/2, 2\alpha^{-3}|t - \theta|] \quad \text{for } |t - \theta| \geq \varepsilon\alpha, \\ \mathbf{d}(t) = d(y(t), D_t) &\in (\varepsilon/2, 2\varepsilon\alpha^{-2}) \quad \text{for } |t - \theta| \leq \varepsilon\alpha. \end{cases} \quad (32)$$

Then there exists $\varepsilon_1 = \varepsilon_1(\kappa, \theta, D, \alpha, M, \mu) \in (0, \alpha]$ such that for $0 < \varepsilon \leq \varepsilon_1$ we have $I_\infty(\tau) > 0$ if $\tau > \tau_1(\varepsilon)$, and

$$\liminf_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \geq -8\kappa^{-1}\alpha^{-2}\varepsilon \quad (33)$$

Proof in Appendix.

4 Technical Results

4.1 First technical aspects

Lemma 4.1. *Let $t \in [0, T]$ such that $D_t \neq \emptyset$. Then there exists a non empty finite family I , and points $x_i \in D_t$, $i \in I$, such that*

$$\cup_{i \in I} B_i(1/\tau) \subset \overline{D_t} \subset \cup_{i \in I} B_i(3/\tau),$$

and $B_i(1/\tau) \cap B_j(1/\tau) = \emptyset$ if $i, j \in I$, $i \neq j$, where $B_i(R)$ denotes the open euclidian ball of radius $R > 0$ and centered at x_i .

Proof. The lemma is a straightforwardly consequence of the compactness of $\overline{D_t}$ and Vitali's lemma.

We have the following proposition (see [5] for example).

Proposition 2. *(Parabolic Harnack inequality). There exists $c = c(\kappa) > 0$ depending on the constant κ of (4) only such that if $r > 0$, $t \in \mathbb{R}$, if $y \in \mathbb{R}^3 \setminus B(2r)$ or $0 \notin (t - r^2, t + r^2)$, then we have*

$$\max_{x \in \overline{B(r)}, s \in [t - \frac{3}{4}r^2, t - \frac{1}{4}r^2]} G_y(x, s) \leq c \min_{x \in \overline{B(r)}, s \in [t + \frac{1}{4}r^2, t + r^2]} G_y(x, s). \quad (34)$$

Let us recall that p_τ is defined by (5). From Proposition 2, we prove the following Lemma.

Lemma 4.2. *(Elliptic Harnack inequality). Let $\beta > 0$. There exists $c_\beta = c(\mathbf{b}, \beta)$ such that for all $\tau > 0$, all ball $B(\beta/\tau) \subset \mathbb{R}^N$, if $y \notin B(2\beta/\tau)$ we then have*

$$\max_{x \in B(\beta/\tau)} p_\tau(x; y) \leq c_\beta \min_{x \in B(\beta/\tau)} p_\tau(x; y). \quad (35)$$

Proof. Applying (34) with $s = t$, $r = \beta/\tau$, we have, for all $x, z \in \overline{B(\beta/\tau)}$,

$$\begin{aligned} p_\tau(z; y) &= \int_0^\infty e^{-\tau^2 s} G_y(z, s) ds \\ &= \int_{\frac{1}{2}\beta^2/\tau^2}^\infty e^{-\tau^2(s - \frac{1}{2}\beta^2/\tau^2)} G_y(z, s - \frac{1}{2}\beta^2/\tau^2) ds \\ &\leq \int_{\frac{1}{2}\beta^2/\tau^2}^\infty e^{-\tau^2(s - \frac{1}{2}\beta^2/\tau^2)} c G_y(x, s) ds \\ &\leq c e^{\frac{1}{2}\beta^2} \int_0^\infty e^{-\tau^2 s} G_y(x, s) ds \\ &= c e^{\frac{1}{2}\beta^2} p_\tau(x; y). \end{aligned}$$

We then obtain (35). □

Let us recall that $y(\cdot)$ and Σ were defined in Section 2.5 and P_τ by (16).

Lemma 4.3. (*Caccioppoli's Inequality for P_τ*). Let P_τ be defined by (16). Let $\beta > 0$. Then there exists $c = c(\beta, \mathbf{b}) > 0$ such that for all $\tau > 0$, if $B(\beta/\tau) \cap B(y(t); \frac{1}{\tau}) = \emptyset$ we then have

$$\frac{1}{c} \int_{B(\frac{\beta}{4\tau})} \tau^2 P_\tau^2(x, t) dx \leq \int_{B(\frac{\beta}{2\tau})} |\nabla P_\tau|^2(x, t) dx \leq c \int_{B(\frac{\beta}{\tau})} \tau^2 P_\tau^2(x, t) dx. \quad (36)$$

Proof in Appendix.

4.2 Comparison between u_τ , P_τ and p_τ

Lemma 4.4. (*Comparison between P_τ and p_τ*). There exists $c = c(\mathbf{b}) > 0$ such that for all $\tau > 0$, $t \in \mathbb{R}$, if $x \notin B(y(t); \frac{2}{\kappa^5 \tau})$ we then have

$$\frac{1}{c} \tau^3 P_\tau(x, t) \leq p_\tau(x, y(t)) \leq c \tau^3 P_\tau(x, t), \quad (37)$$

where κ is the constant of (4) or (7).

Proof in Appendix.

Lemma 4.5. (*Comparison between u_τ and p_τ*). Let $M > 0$ and assume that $|y|_\infty \leq M$. Then there exist positive constants $C = C(\mathbf{b}, d_\Omega, \mu, M)$, $C_1 = C_1(\kappa, d_\Omega, M)$ such that for $\tau \geq \max(2\mu, 2\kappa^{-1}M)$, $t \in [0, T]$, $x \in \Omega \setminus B(y(t), C_1/\tau)$, we have:

$$e^{-\tau^2(T+t)} u_\tau(x, t) \leq C e^{-\tau\mu|t-\theta|} \tau^{-3} p_\tau(x, y(t)). \quad (38)$$

Proof in Appendix.

Lemma 4.6. Let $t \in [0, T]$. Let $M > 0$ and assume that $|y|_\infty \leq M$. Then there exist positive constants $C = C(\mathbf{b}, d_\Omega, \mu, M)$, $C_1 = C_1(\kappa, d_\Omega, M)$ such that for $\tau \geq \max(2\mu, 2\kappa^{-1}M)$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$ we have:

$$|\partial_t(e^{-\tau^2(t+T)} u_\tau(x, t))| \leq C e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (39)$$

Proof in Appendix.

Remember that q_τ is defined by (17).

Lemma 4.7. (*Estimate of q_τ*). Let $t \in [0, T]$. Let $M > 0$ and assume that $|y|_\infty \leq M$. Then there exist $C = C(\mathbf{b}, d_\Omega, \mu, M) > 0$, $C_1 = C_1(\kappa, d_\Omega, M)$, such that for $\tau \geq 2\mu$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$, we have

$$|q_\tau(x, t)| \leq C \tau^{-3} e^{-\tau\mu|t-\theta|} |x - y(t)| p_\tau(x, y(t)). \quad (40)$$

Proof in Appendix.

4.3 Estimates in D_t

Lemma 4.8. (*Estimates of P_τ in D_t*). Let $t \in [0, T]$. Let us put $\tau_0 = \frac{12}{\kappa^5 \mathbf{d}(t)}$. Then, there exists $c = c(\mathbf{b}) \geq 1$ such that for all $\tau > \tau_0$, we have

$$\frac{1}{c} \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx \leq \int_{D_t} |\nabla P_\tau(x, t)|^2 dx \leq c \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx. \quad (41)$$

Proof in Appendix.

Lemma 4.9. (Estimate of ∇q_τ in D_t). Let $t \in [0, T]$. Let $M > 0$ and assume that $|\dot{y}|_\infty \leq M$. Then there exist two positive constants $C = C(\mathbf{b}, d_\Omega, \mu, M)$, $C_2 = C_2(\kappa, d_\Omega, M)$ such that if $\tau > \tau_1 := \max(2\mu, C_2/d(t)) \geq \tau_0$, where τ_0 is defined in Lemma 4.8, then

$$\int_{D_t} |\nabla q_\tau|^2(x, t) dx \leq C\tau^{-4} e^{-2\tau\mu|t-\theta|} \int_{D_t} |x - y(t)|^2 |p_\tau(x, y(t))|^2 dx. \quad (42)$$

Proof in Appendix.

Lemma 4.10. Under (H3) there exists a positive constant $C = C(L_D, \kappa)$ such that we have, for $\tau > \kappa^{-3}d(t)^{-1}$, $t \in [0, T]$,

$$\int_{D_t} |x - y(t)|^2 |p_\tau(x, y(t))|^2 dx \leq Cd(t)^2 \int_{D_t} |p_\tau(x, y(t))|^2 dx. \quad (43)$$

Proof in Appendix.

Now we are ready to prove Lemma 3.2.

4.4 Proof of Lemma 3.2

We obtain (30) and (29) from (26), (25), (20) of Lemma 2.1, (41) of Lemma 4.8. \square

5 Proof of Theorem 1

We may assume that (H0b) holds, since the case where (H0a) holds is similar. Thanks to Remark 5 we have $\overline{D_t} \cup \overline{D'_t} \subset \Omega$, $t \in [0, T]$. Let us assume that $D \neq D'$. Then there exists $(z, \theta) \in \Omega \times [0, T]$ with $D_\theta \neq \emptyset$, $z \in \partial D_\theta$ and $z \notin \overline{D'_\theta}$ or with $D'_\theta \neq \emptyset$, $z \in \partial D'_\theta$ and $z \notin \overline{D_\theta}$. Thus, we consider for simplicity that $z \in \partial D_\theta$ and $z \notin \overline{D'_\theta}$. Thanks to (H2), $t \mapsto (D_t, D'_t)$ is continuous so we consider also that $0 < \theta < T$. In fact let us explain why can consider also that $z \in \partial D_\theta \setminus \overline{D'_t}$ if $|t - \theta| < \beta$ for some $\beta > 0$. If D'_t is void for $|t - \theta|$ sufficiently small then it is immediate, but if D'_t is not void for $|t - \theta|$ sufficiently small then we can't be sure that $d(z, D'_t) > 0$ when $t \simeq \theta$. However, in such a case, thanks to (H2), there exists a sequence $\theta_n \rightarrow \theta$ satisfying $D'_{\theta_n} \neq \emptyset$ and $\overline{D_{\theta_n}} \setminus \overline{D'_{\theta_n}} \neq \emptyset$. We then replace (z, θ) by another couple (z_n, θ_n) with $z_n \in \partial D_{\theta_n} \setminus \overline{D'_{\theta_n}}$. Then, since $D'_{\theta_n} \neq \emptyset$ and thanks to (H2), we have $z_n \notin \partial D'_t$ if $t \simeq \theta_n$.

So we can consider that

$$z \in \partial D_\theta \setminus \overline{D'_t} \text{ if } |t - \theta| \leq \beta \text{ for some } \beta > 0. \quad (44)$$

We then construct a family of curves $\Sigma = \Sigma_\varepsilon$ for $0 < \varepsilon \leq \alpha^3 =: \varepsilon_1$, for some positive α such that (32) and

$$d(y(t), D'_t) \geq \alpha/2 \quad \forall t, \quad (45)$$

hold. Proof. Since D_θ and D'_θ are lipschitzian and simply connected (see (H1')) then there exists a lipschitzian curve $\tilde{y} : [0, 1] \ni s \mapsto \tilde{y}(s) \in \mathbb{R}^3$ with a constant

Lipschitz \tilde{M} such that $\tilde{y}(0) = z$, $\tilde{y}(1) \notin \Omega$, $\tilde{y}(s) \notin \overline{D_\theta}$ for $s \neq 0$ and $\tilde{y}(s) \notin \overline{D'_t}$ for $s \in [0, 1]$ and $|t - \theta| \leq \beta$. Thanks to (H2), (H3) and to (44) we have

$$d(\tilde{y}(s), D'_t) \geq \alpha - K_{D'}|t - \theta|, \quad \text{for } s \in [0, 1], \quad (46)$$

$$d(\tilde{y}(s), D_t) \in [\alpha s - K_D|t - \theta|, \frac{1}{\alpha}s + K_D|t - \theta|], \quad s \in [0, 1], \quad (47)$$

for some sufficiently small positive $\alpha \leq \max(1, \theta, T - \theta, d(D_t, \partial\Omega), (2K_D)^{-1}, (2K_{D'})^{-1}, \sqrt{\beta})$. We put $y_0(t) = \tilde{y}(|t - \theta|/\alpha^2)$ for $|t - \theta| \leq \alpha^2$ and $y_0(t) = \tilde{y}(1)$ for $|t - \theta| \geq \alpha^2$. From (46) and since $y_0(t') \notin \Omega$ for $|t' - \theta| \geq \alpha^2$ we then have

$$d(y_0(t'), D'_t) \geq \alpha/2, \quad \text{for } |t - \theta| \leq \alpha^2 \text{ or } |t' - \theta| \geq \alpha^2. \quad (48)$$

From (47) and since $y_0(t') \notin \Omega$ for $|t' - \theta| \geq \alpha^2$ we then have for $|t - \theta| \leq |t' - \theta|$

$$d(y_0(t'), D_t) \in [\frac{1}{2}\alpha^{-1}|t - \theta|, 2\alpha^{-3}|t - \theta|]. \quad (49)$$

Then for all $\varepsilon \in (0, \alpha^3]$ we put

$$y(t) = y_0(\varepsilon) \text{ for } |t - \theta| \leq \varepsilon, \quad y(t) = y_0(t) \text{ for } |t - \theta| \geq \varepsilon.$$

Thanks to (48), (49) we obtain (32) and (45).

Let us denote by $I'_\infty(\tau)$ the indicator function for the conductivity \mathbf{a}' . Thanks to Lemma 3.3 we have for $\varepsilon \in (0, \varepsilon_1]$

$$\limsup_{\tau \rightarrow \infty} \tau^{-1} \ln(I'_\infty(\tau)) \leq -\kappa\alpha,$$

and

$$\liminf_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \geq -8\kappa^{-1}\alpha^{-2}\varepsilon.$$

Then, $I'_\infty(\tau) \neq I_\infty(\tau)$ for all $\varepsilon < \min(\varepsilon_1, \kappa^2\alpha^4/4)$ and τ sufficiently large. The result at §2.6 implies that $\Lambda_{v_0, \mathbf{a}} \neq \Lambda_{v'_0, \mathbf{a}'}$. \square

Appendix

Proof of Lemma 3.1. We put

$$X_1 := \int_{\Omega \times [0, T]} ((\mathbf{a} - \mathbf{b})\mathbf{a}^{-1}\mathbf{b})_S (\nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt,$$

$$X_2 := \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) (\nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt,$$

$$w_\tau := v_\tau - u_\tau, \quad (50)$$

$$\begin{aligned} \Psi_\tau &:= (\mathbf{a} - \mathbf{b})\nabla v_\tau + \mathbf{b}\nabla w_\tau = \mathbf{a}\nabla v_\tau - \mathbf{b}\nabla u_\tau \\ &= (\mathbf{a} - \mathbf{b})\nabla u_\tau + \mathbf{a}\nabla w_\tau, \end{aligned} \quad (51)$$

$$\begin{aligned}
B_1 &:= \int_{\Omega \times [0, T]} \mathbf{a}^{-1} (\Psi_\tau)^2 \, dx \, e^{-2\tau^2(T+t)} \, dt \\
B_2 &:= \int_{\Omega \times [0, T]} \mathbf{a} (\nabla w_\tau)^2 \, dx \, e^{-2\tau^2(T+t)} \, dt, \\
B_3 &:= \int_{\Omega \times [0, T]} \tau^2 w_\tau^2 \, dx \, e^{-2\tau^2(T+t)} \, dt,
\end{aligned}$$

and

$$\begin{aligned}
R_1 &:= \int_{\Omega} [w_\tau u_\tau^*]_0^T \, dx, \\
R_2 &:= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_\tau \nabla (e^{\tau^2(T+t)} u_\tau^* - e^{-\tau^2(T+t)} u_\tau) \, dx \, e^{-\tau^2(t+T)} \, dt \\
&= \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_\tau \cdot (\nabla q_\tau^*(x, t) - \nabla q_\tau(x, t)) \, dx \, e^{-\tau^2(t+T)} \, dt, \\
R_3 &:= \frac{1}{2} \int_{\Omega} [w_\tau^2 e^{-2\tau^2(T+t)}]_0^T \, dx.
\end{aligned}$$

Step 1. Let us prove that

$$I_\infty(\tau) = X_1 + B_1 + B_3 + R_1 + R_2 + R_3, \quad (52)$$

$$I_\infty(\tau) = X_2 - B_2 - B_3 + R_1 + R_2 - R_3. \quad (53)$$

From (23) we have

$$I_\infty(\tau) = \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b}) \nabla v_\tau \nabla u_\tau \, dx \, e^{-2\tau^2(T+t)} \, dt + R_1 + R_2. \quad (54)$$

1. Let us put

$$\begin{aligned}
A_1 &:= \int_{\Omega \times [0, T]} \mathbf{a}^{-1} \Psi_\tau \cdot (\mathbf{a} - \mathbf{b}) \nabla u_\tau \, dx \, e^{-2\tau^2(T+t)} \, dt, \\
A_2 &:= \int_{\Omega \times [0, T]} \nabla w_\tau \Psi_\tau \, dx \, e^{-2\tau^2(T+t)} \, dt.
\end{aligned}$$

Then, since $(\mathbf{a} - \mathbf{b}) \nabla u_\tau = \Psi_\tau - \mathbf{a} \nabla w_\tau$, we then have $A_1 = B_1 - A_2$.

By integration by parts we have

$$\begin{aligned}
A_2 &= - \int_{\Omega \times [0, T]} w_\tau \operatorname{div} \Psi_\tau \, dx \, e^{-2\tau^2(T+t)} \, dt = - \int_{\Omega \times [0, T]} w_\tau \partial_t w_\tau \, dx \, e^{-2\tau^2(T+t)} \, dt \\
&= -B_3 - R_3.
\end{aligned} \quad (55)$$

We thus have

$$A_1 = B_1 + B_3 + R_3. \quad (56)$$

For any 3×3 real matrix \mathbf{m} we have $\mathbf{m} \nabla u_\tau \cdot \nabla u_\tau = \mathbf{m}_S \nabla u_\tau \cdot \nabla u_\tau$. Then, thanks to

$$\nabla v_\tau = \mathbf{a}^{-1} \Psi_\tau + \mathbf{a}^{-1} \mathbf{b} \nabla u_\tau, \quad (57)$$

we obtain (52) from (54) and (56).

2. Let us consider (54) again. Thanks to (50) then to (51) we have

$$\begin{aligned} (\mathbf{a} - \mathbf{b})\nabla v_\tau \nabla u_\tau &= (\mathbf{a} - \mathbf{b})\nabla u_\tau \nabla u_\tau + \mathbf{a}\nabla w_\tau \nabla u_\tau - \mathbf{b}\nabla w_\tau \nabla u_\tau \\ &= (\mathbf{a} - \mathbf{b})\nabla u_\tau \nabla u_\tau + \mathbf{a}\nabla w_\tau (\nabla v_\tau - \nabla w_\tau) - \mathbf{b}\nabla w_\tau \nabla u_\tau \\ &= (\mathbf{a} - \mathbf{b})\nabla u_\tau \nabla u_\tau - \mathbf{a}\nabla w_\tau \nabla w_\tau + \nabla w_\tau \Psi_\tau. \end{aligned}$$

Hence

$$I_\infty(\tau) = \int_{\Omega \times [0, T]} (\mathbf{a} - \mathbf{b})(\nabla u_\tau)^2 dx e^{-2\tau^2(T+t)} dt - B_2 + A_2 + R_1 + R_2,$$

which yields (53) with the help of (55). \square

Step 2. We put

$$X_0 := \int_D e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dx dt \quad (58)$$

and

$$\begin{aligned} R_4 &:= \frac{1}{2} \int_\Omega e^{4\tau^2 T} |u_\tau^*(T)|^2 dx + \frac{1}{2} \int_\Omega e^{2\tau^2 T} |u_\tau^*(0)|^2 dx \\ &\quad + 2 \int_\Omega e^{-2\tau^2 T} |u_\tau(0)|^2 dx + 2 \int_\Omega e^{-2\tau^2 T} |v_0|^2 dx, \\ R_5 &:= \int_D |\nabla q_\tau|^2 dx dt, \quad R_5^* := \int_D |\nabla q_\tau^*|^2 dx dt. \end{aligned}$$

Under assumption (H0b) we have the following estimates:

$$I_\infty(\tau) \geq CX_0 + \frac{1}{2}B_1 + B_3 - 2R_4 - \frac{1}{C}(R_5 + R_5^*), \quad (59)$$

$$I_\infty(\tau) \leq \frac{1}{C}X_0 - \frac{1}{2}B_2 - B_3 + 2R_4 + \frac{1}{C}(R_5 + R_5^*), \quad (60)$$

for some $C = C(\mathbf{a}) \in (0, 1)$.

Proof. Thanks to Cauchy-Minkovski inequality and to the definition (22) we have

$$\begin{aligned} R_1 + R_3 &= \int_\Omega (w_\tau u_\tau^* + \frac{1}{2}w_\tau^2 e^{-4\tau^2 T})|_{t=T} dx \\ &\quad - \int_\Omega (w_\tau u_\tau^* + \frac{1}{2}w_\tau^2 e^{-2\tau^2 T})|_{t=0} dx \\ &\geq -R_4. \end{aligned} \quad (61)$$

Similarly we have

$$R_1 - R_3 \leq R_4. \quad (62)$$

We observe that, thanks to (52) and (53),

$$X_1 = I_\infty(\tau) - B_1 - B_3 - R_1 - R_2 - R_3, \quad (63)$$

$$X_2 = I_\infty(\tau) + B_2 + B_3 - R_1 - R_2 + R_3. \quad (64)$$

Thanks to (57) again we have

$$\begin{aligned} |R_2| &\leq \int_{\Omega \times [0, T]} e^{-\tau^2(T+t)} |\mathbf{a} - \mathbf{b}| |\mathbf{a}^{-1}| |\Psi_\tau| |\nabla q_\tau^* - \nabla q_\tau| dxdt \\ &\quad + \int_{\Omega \times [0, T]} e^{-\tau^2(T+t)} |\mathbf{a} - \mathbf{b}| |\mathbf{a}^{-1}| |\mathbf{b}| |\nabla u_\tau| |\nabla q_\tau^* - \nabla q_\tau| dxdt \end{aligned} \quad (65)$$

$$\leq \frac{1}{2} B_1 + \frac{1}{2} X_1 + C(R_5 + R_5^*). \quad (66)$$

From (63) and (66) we get

$$|R_2| \leq \frac{1}{2} I_\infty(\tau) - \frac{1}{2} (B_3 + R_3 + R_1 + R_2) + C(R_5 + R_5^*). \quad (67)$$

Estimates (52), (61) and (67) imply

$$I_\infty(\tau) \geq \frac{1}{2} X_1 + \frac{1}{2} B_1 + B_3 - R_4 - C(R_5 + R_5^*). \quad (68)$$

By using (17), (18), (H0b), and the basic estimate $a^2 \geq \frac{1}{2}(a+b)^2 - b^2$, we have

$$\begin{aligned} X_1 &\geq \int_D \delta_1 e^{-2\tau\mu|t-\theta|} |\nabla P_\tau(x, t)|^2 dxdt \\ &\quad - \int_{\Omega \times [0, T]} |\mathbf{b}| |\mathbf{a}^{-1}| |\mathbf{a} - \mathbf{b}| |\nabla q_\tau|^2 dxdt \geq CX_0 - \frac{1}{C} R_5, \end{aligned}$$

for some $C = C(\mathbf{a}) \in (0, 1)$. Then with (68) we obtain (59).

Similarly, by using (53), (58), (64) we obtain (60).

Step 3. We prove that for $\tau > \mu + 1$ we have

$$|R_4| \leq (2\|v_0\|_{L^2(\Omega)}^2 + 5d_\Omega) e^{-\tau\mu \min(T-\theta, \theta)}. \quad (69)$$

Proof. Firstly, we have

$$\begin{aligned} 0 \leq u_\tau(x, 0) &= \int_0^\infty \int_{\mathbb{R}^3} e^{\tau^2(T-s)} e^{-\tau\mu|\theta+s|} m(y, -s) G_y(x, s) dyds \\ &\leq e^{\tau^2 T} e^{-\tau\mu\theta} \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2 s} G_y(x, s) dyds = \frac{1}{\tau^2} e^{\tau^2 T} e^{-\tau\mu\theta}. \end{aligned}$$

Here we used the notorious relation

$$\int_{\mathbb{R}^3} G_y(x, s) dy = 1. \quad (70)$$

Hence

$$0 \leq e^{-\tau^2 T} u_\tau(x, 0) \leq e^{-\tau\mu\theta}, \quad \tau \geq 1. \quad (71)$$

Similarly we have

$$0 \leq e^{2\tau^2 T} u_\tau^*(x, T) \leq e^{-\tau\mu(T-\theta)}, \quad \tau \geq 1. \quad (72)$$

Secondly, since $\tau > \mu + 1 > 1$ we have

$$\begin{aligned}
0 \leq u_\tau^*(x, 0) &= \int_0^\infty \int_{\mathbb{R}^3} e^{-\tau^2(T+s)} e^{-\tau\mu|\theta-s|} m(y, s) G_y(x, s) dy ds \\
&\leq e^{-\tau^2 T} e^{-\tau\mu\theta} \int_0^\infty e^{-(\tau^2-\tau\mu)s} \int_{\mathbb{R}^3} G_y(x, s) dy ds \\
&= \frac{1}{\tau^2 - \tau\mu} e^{-\tau^2 T} e^{-\tau\mu\theta} \leq e^{-\tau^2 T} e^{-\tau\mu\theta}.
\end{aligned} \tag{73}$$

From (73), (72), (71), we obtain for $\tau > \mu + 1 > 1$:

$$R_4 \leq 2\|v_0\|_{L^2(\Omega)}^2 e^{-2\tau^2 T} + 2d_\Omega e^{-\tau\mu(T-\theta)} + 3d_\Omega e^{-\tau\mu\theta},$$

which implies (69).

Estimates (26) and (25) come immediately from (59), (60), (69) and the fact that $B_j \geq 0$ for $j = 1, 2, 3$. \square

Proof of Lemma 4.3. We observe that for all t , the function $P_\tau(\cdot; t)$ the unique solution in $H^1(\mathbb{R}^3)$ of

$$(-\operatorname{div}(\mathbf{b}\nabla\cdot) + \tau^2)P_\tau(\cdot; t) = m_\tau(\cdot, t). \tag{74}$$

Let $\phi \in C^1(\mathbb{R}; [0, 1])$ with $\phi(r) = 1$ for $|r| \leq 1/2$ and $\phi(r) = 0$ for $|r| \geq 1$. Put $\psi(x) = \phi(\tau(x - x_0)/\beta)$ where x_0 is the center of the ball $B(\beta/\tau)$. We multiply (74) by $P_\tau(\cdot, t)\psi^2$ and integrate it over Ω . Since $\operatorname{supp}(\psi) \cap \operatorname{supp}(m_\tau(\cdot, t))$ has Lebesgue measure zero, we then have

$$\int_\Omega [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b}\nabla P_\tau(\cdot, t)\psi P_\tau(\cdot, t)\nabla\psi + \tau^2 P_\tau^2(\cdot, t)\psi^2] = 0. \tag{75}$$

Then, from Cauchy-Minkovski's inequality,

$$\begin{aligned}
\int_\Omega [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + \tau^2 P_\tau^2(\cdot, t)\psi^2] &\leq \int_\Omega |2\mathbf{b}\nabla P_\tau(\cdot, t)\psi P_\tau(\cdot, t)\nabla\psi| \\
&\leq \int_\Omega \left[\frac{1}{2}\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b}P_\tau^2(\cdot, t)(\nabla\psi)^2 \right].
\end{aligned}$$

Thus, for some $C' = C'(\mathbf{b}) > 0$,

$$\int_\Omega [|\nabla P_\tau(\cdot, t)|^2 + \tau^2 P_\tau^2(\cdot, t)] \psi^2(x) dx \leq C' \int_\Omega P_\tau^2(\cdot, t) |\nabla\psi|^2(x) dx,$$

with $C'' = C''(\mathbf{b}) > 0$. Since $\operatorname{supp} \psi \subset B(\beta/\tau)$ with $|\nabla\psi(x)| \leq \frac{\tau}{\beta} \max|\phi'|$, $\psi \geq 0$, and $\psi = 1$ in $B(\frac{\beta}{2\tau})$, we then have

$$\int_{B(\frac{\beta}{2\tau})} |\nabla P_\tau(\cdot, t)|^2(x) dx \leq C'' \frac{\tau^2}{\beta^2} \int_{B(\frac{\beta}{\tau})} P_\tau^2(\cdot, t) dx,$$

which proves the second inequality in (36).

From (75) and thanks to Cauchy-Minkovski's inequality we have also

$$\begin{aligned}
\int_\Omega [\mathbf{b}(\nabla P_\tau(\cdot, t))^2 \psi^2 + \tau^2 P_\tau^2(\cdot, t)\psi^2] &\leq \int_\Omega |2\mathbf{b}\nabla P_\tau(\cdot, t)\nabla\psi P_\tau(\cdot, t)\psi| \\
&\leq \int_\Omega \left[\frac{2}{\tau^2} \gamma_\infty^2 |\nabla P_\tau(\cdot, t)|^2 |\nabla\psi|^2 + \frac{1}{2} \tau^2 P_\tau^2(\cdot, t)\psi^2 \right].
\end{aligned}$$

Thus, for some $C' = C'(\mathbf{b}) > 0$,

$$\int_{\Omega} \tau^2 P_{\tau}^2(\cdot, t) \psi^2(x) dx \leq C' \frac{1}{\tau^2} \int_{\Omega} |\nabla P_{\tau}(\cdot, t)|^2 |\nabla \psi|^2(x) dx.$$

We then obtain

$$\int_{B(\frac{\beta}{2\tau})} \tau^2 P_{\tau}^2(\cdot, t) dx \leq C'' \frac{1}{\beta^2} \int_{B(\frac{\beta}{\tau})} |\nabla P_{\tau}(\cdot, t)|^2 dx,$$

$C'' = C''(\mathbf{b})$, which proves the first inequality in (36) with β replaced by 2β . \square

Proof of Lemma 4.4. Since $G_x(y, s) = G_y(x, s)$ and thanks to (34) with $r = 1/\tau$, we have for all $x \notin B(y(t), 2/\tau)$:

$$\begin{aligned} \int_{B(y(t), 1/\tau)} G_y(x, s) dy &= \int_{B(y(t), 1/\tau)} G_x(y, s) dy \leq |B(1/\tau)| \max_{B(y(t), 1/\tau)} G_x(\cdot, s) \\ &\leq c\tau^{-3} G_x(y(t), s + \frac{1}{2\tau^2}) = c\tau^{-3} G_{y(t)}(x, s + \frac{1}{2\tau^2}). \end{aligned}$$

Then, since $\tau|x - y(t)| \geq 2/\kappa^5 \geq 2$, since $m_{\tau} \leq 1$ and $\text{supp } m_{\tau} = B(y(t), \frac{1}{\tau})$ we have

$$\begin{aligned} P_{\tau}(x, t) &\leq \int_0^{\infty} e^{-\tau^2 s} \int_{B(y(t), \frac{1}{\tau})} G_y(x, s) dy ds \quad (76) \\ &\leq c\tau^{-3} \int_0^{\infty} e^{-\tau^2 s} G_{y(t)}(x, s + \frac{1}{2\tau^2}) ds \\ &= c\tau^{-3} \int_{\frac{1}{2\tau^2}}^{\infty} e^{-\tau^2(s - \frac{1}{2\tau^2})} G_{y(t)}(x, s) ds \\ &\leq c'\tau^{-3} \int_0^{\infty} e^{-\tau^2 s} G_{y(t)}(x, s) ds = c'\tau^{-3} p_{\tau}(x; y(t)). \end{aligned}$$

We obtain the first inequality of (37). Let us prove the second one. Since $m_{\tau} \geq 1/2$ in $B(y(t), \frac{1}{2\tau})$ we then have

$$\begin{aligned} P_{\tau}(x, t) &\geq \frac{1}{2} \int_0^{\infty} e^{-\tau^2 s} \int_{B(y(t), \frac{1}{2\tau})} G_y(x, s) dy ds \\ &\geq c\tau^{-3} \int_0^{\infty} e^{-\tau^2 s} \inf_{y \in B(y(t), \frac{1}{2\tau})} G_y(x, s) ds. \end{aligned}$$

By applying (34) with $r = 1/\tau$ and observing that $G_y(x, s) = G_x(y, s)$ we then have for all $x \notin B(y(t), 2/\tau)$:

$$\begin{aligned} P_{\tau}(x, t) &\geq c\tau^{-3} \int_0^{\infty} e^{-\tau^2 s} \inf_{y \in B(y(t), \frac{1}{2\tau})} G_y(x, s) ds \\ &\geq c\tau^{-3} \int_0^{\infty} e^{-\tau^2 s} G_{y(t)}(x, s - \frac{1}{2\tau^2}) ds \\ &= c\tau^{-3} \int_{\frac{1}{2\tau^2}}^{\infty} e^{-\tau^2(s + \frac{1}{2\tau^2})} G_{y(t)}(x, s) ds \\ &= c'\tau^{-3} \left(p_{\tau}(x, y(t)) - \int_0^{\frac{1}{2\tau^2}} e^{-\tau^2 s} G_{y(t)}(x, s) ds \right), \quad (77) \end{aligned}$$

with some $c' > 0$. We put $R := \int_0^{\frac{1}{2\tau^2}} e^{-\tau^2 s} G_{y(t)}(x, s) ds$. Thanks to (4) and (7) we have

$$\begin{aligned} R &\leq \int_0^{\frac{1}{2\tau^2}} \frac{e^{-\frac{\kappa^2|x-y(t)|^2}{4s}}}{\kappa s^{3/2}} ds \leq \sqrt{2}\kappa^{-1}\tau \int_1^\infty e^{-\kappa^2|x-y(t)|^2\tau^2 r/2} dr \\ &= 2\sqrt{2}\kappa^{-3}\tau^{-1}|x-y(t)|^{-2}e^{-\kappa^2|x-y(t)|^2\tau^2/2} \\ &\leq \frac{1}{2}p_\tau(x, y(t)) \frac{2}{\kappa^5\tau|x-y(t)|} \exp(-\kappa^{-1}\tau|x-y(t)|(\kappa^2\tau|x-y(t)|/2-1)). \end{aligned}$$

Since $\tau|x-y(t)| \geq 2/\kappa^5 > 2/\kappa^2$, we then have $R \leq \frac{1}{2}p_\tau(x, y(t))$. Hence

$$P_\tau(x, t) \geq \frac{1}{2}c'\tau^{-3}p_\tau(x, y(t)).$$

The conclusion follows. \square

Proof of Lemma 4.5. Let us observe that

$$e^{-\tau\mu|t-\theta-s|} \leq e^{-\tau\mu|t-\theta|}e^{\tau\mu s} \quad s > 0, t \in \mathbb{R}. \quad (78)$$

Hence

$$\begin{aligned} e^{-\tau^2(T+t)}u_\tau(x, t) &= \int_0^\infty e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s) G_y(x, s) dy ds \\ &\leq e^{-\tau\mu|t-\theta|} H, \end{aligned} \quad (79)$$

where we put

$$H := \int_0^\infty e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s) dy ds \equiv H_1 + H_2 \quad (80)$$

with

$$\begin{aligned} H_1 &:= \int_{s>\lambda/\tau} e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s) dy ds, \\ H_2 &:= \int_0^{\lambda/\tau} e^{-(\tau^2-\tau\mu)s} \int_{B(y(t-s), 1/\tau)} G_y(x, s) dy ds, \end{aligned}$$

and where $\lambda := 2|x-y(t)|/\kappa$. We put also $M' := M\lambda + 1$, $C_1 = \max(1, 8\kappa^{-7}(1 + M^2 d_\Omega^2))$. Since $|y(t-s) - y(t)| \leq Ms$, we then have $B(y(t-s), 1/\tau) \subset B(y(t), Ms + 1/\tau)$ and so

$$\begin{aligned} H_2 &\leq e^{\mu\lambda} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), Ms+1/\tau)} G_y(x, s) dy ds \\ &\leq e^{\mu\lambda} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), M'/\tau)} G_y(x, s) dy ds. \end{aligned}$$

Since $\tau \geq 2\kappa^{-1}M$, $|x-y(t)| \geq 1/\tau$, we then have $|x-y(t)| \geq M'/\tau$ and so we can

apply (34) where x and y are exchanged and with $r = M'/(2\tau)$. Hence

$$\begin{aligned}
H_2 &\leq ce^{\mu\lambda} \int_0^\infty e^{-\tau^2 s} |B(y(t), M'/\tau)| G_{y(t)}(x, s + M'^2/(2\tau^2)) ds \\
&\leq ce^{\mu\lambda} \int_0^\infty e^{-\tau^2 s} (2M')^3 |B(y(t), 1/(2\tau))| G_{y(t)}(x, s + M'^2/(2\tau^2)) ds \\
&= ce^{\mu\lambda} (2M')^3 \int_{M'^2/(2\tau^2)}^\infty e^{-\tau^2(s - M'^2/(2\tau^2))} |B(y(t), 1/(2\tau))| G_{y(t)}(x, s) ds \\
&= ce^{\mu\lambda + M'^2/2} M'^3 \tau^{-3} (p_\tau(x, y(t)) - R),
\end{aligned}$$

with $R := \int_0^{M'^2/(2\tau^2)} e^{-\tau^2 s} G_{y(t)}(x, s) ds$ and $c = c(\mathbf{b})$ is the constant (34). Thanks to (4) we have

$$\begin{aligned}
R &\leq \int_0^{\frac{M'^2}{2\tau^2}} \frac{e^{-\frac{\kappa^2|x-y(t)|^2}{4s}}}{\kappa s^{3/2}} ds \leq \sqrt{2}\kappa^{-1}\tau M'^{-1} \int_1^\infty e^{-\kappa^2|x-y(t)|^2\tau^2 r/(2M'^2)} dr \\
&= 2\sqrt{2}M'\kappa^{-3}\tau^{-1}|x-y(t)|^{-2} e^{-\kappa^2|x-y(t)|^2\tau^2/(2M'^2)} \\
&\leq \frac{1}{2}p_\tau(x, y(t)) \frac{M'}{\kappa^5\tau|x-y(t)|} \exp(-\kappa^{-1}\tau|x-y(t)|(M'^{-2}\kappa^2\tau|x-y(t)|/2 - 1)).
\end{aligned}$$

Since $\tau|x-y(t)| \geq C_1$ then $M'^{-2}\kappa^2\tau|x-y(t)| \geq 2$ and $M'^{-1}\kappa^5\tau|x-y(t)| \geq 1$. Hence $R \leq \frac{1}{2}p_\tau(x, y(t))$ and

$$\begin{aligned}
H_2 &\leq CM'^3 e^{2\kappa^{-1}\mu|x-y(t)|} \tau^{-3} p_\tau(x, y(t)) \\
&\leq CM'^3 e^{2\kappa^{-1}\mu d_\Omega} \tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T} \\
&= C(\mathbf{b}, d_\Omega, \mu, M) \tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T}.
\end{aligned} \tag{81}$$

Let us estimate H_1 . Since $G_y(x, s) \leq \kappa^{-1}s^{-3/2}$ and $\tau \geq 2\mu$ we then have

$$\begin{aligned}
H_1 &\leq \kappa^{-1}\tau^{-3} \int_{s>\lambda/\tau} s^{-3/2} e^{-\tau^2 s/2} ds \\
&\leq \kappa^{-1}\tau^{-3} \begin{cases} \int_{s>\lambda/\tau} (\lambda/\tau)^{-3/2} e^{-\tau^2 s/2} ds &= 2(\lambda/\tau)^{-3/2} \tau^{-2} e^{-\tau\lambda/2} \\ \int_{s>\lambda/\tau} s^{-3/2} e^{-\tau\lambda/2} ds &= 2(\lambda/\tau)^{-1/2} e^{-\tau\lambda/2} \end{cases}.
\end{aligned}$$

Hence

$$H_1 \leq 2\kappa^{-1}\lambda^{-1}\tau^{-3} e^{-\tau\lambda/2}.$$

Thanks to (7) we then obtain

$$H_1 \leq \kappa^{-2}\tau^{-3} p_\tau(x, y(t)), \quad (x, t) \in \Omega_{0,T}. \tag{82}$$

Then, thanks to Lemma 4.4 and from (82), (81), (79), the conclusion follows. \square

Proof of Lemma 4.6. We have

$$\partial_t(e^{-\tau^2(t+T)} u_\tau(x, t)) = Y_1 + Y_2,$$

with

$$\begin{aligned}
Y_1 &:= -\tau\mu \int_{s=0}^t e^{-\tau^2 s} \text{sign}(t-\theta-s) e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s) G_y(x, s) dy ds, \\
Y_2 &:= \int_{s=0}^t e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} \partial_t m_\tau(y, t-s) G_y(x, s) dy ds.
\end{aligned}$$

Let us estimate Y_1 . We have

$$\begin{aligned} |Y_1(x, t)| &\leq \tau\mu \int_{s=0}^{\infty} e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{\mathbb{R}^3} m_\tau(y, t-s) G_y(x, s) dy ds \\ &= \tau\mu e^{-\tau^2(t+T)} u_\tau(x, t). \end{aligned}$$

Thanks to Lemma 4.5 we obtain for

$$|Y_1(x, t)| \leq C e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (83)$$

Let us estimate Y_2 . Remember that $\text{supp } m_\tau(\cdot, t) \subset \overline{B(y(t), 1/\tau)}$ and that

$$|\partial_t m_\tau(y, t)| = \tau |\dot{y}(t) \nabla M_0(\tau(y - y(t)))| \leq M\tau.$$

Hence we have, as in the estimates of (76) we obtain

$$\begin{aligned} |Y_2| &\leq C\tau \int_{s=0}^{\infty} e^{-\tau^2 s} e^{-\tau\mu|t-\theta-s|} \int_{B(y(t-s), 1/\tau)} G_y(x, s) dy ds \\ &\leq e^{-\tau\mu|t-\theta|} H, \end{aligned}$$

where H is defined by (80). Hence

$$|Y_2| \leq C' e^{-\tau\mu|t-\theta|} \tau^{-2} p_\tau(x, y(t)). \quad (84)$$

From (83), (84) we obtain (39). \square

Proof of Lemma 4.7. We write $q_\tau(x, t) = \int_0^\infty e^{-\tau^2 s} \int_{\mathbb{R}^3} (A-B) G_y(x, s) dy ds$ with

$$\begin{aligned} A &\equiv e^{-\tau\mu|t-\theta-s|} m_\tau(y, t-s), \\ B &\equiv e^{-\tau\mu|t-\theta|} m_\tau(y, t). \end{aligned}$$

Let us observe that, since $e^{\tau\mu s} - 1 \leq \mu\tau s e^{\tau\mu s}$ and thanks to (78), then

$$|A - B| \leq e^{-\tau\mu|t-\theta|} (\mu\tau s e^{\tau\mu s} 1_{B(y(t-s), 1/\tau)} + M\tau s \max(1_{B(y(t), 1/\tau)}, 1_{B(y(t-s), 1/\tau)})).$$

Hence

$$|q_\tau(x, t)| \leq \tau e^{-\tau\mu|t-\theta|} (\mu R_1 + M R_2) \quad (85)$$

with

$$R_1 := \int_0^\infty e^{-\tilde{\tau}^2 s} \int_{B(y(t-s), 1/\tau)} s G_y(x, s) dy ds, \quad (86)$$

$$R_2 := \int_0^\infty e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds, \quad (87)$$

where $\tilde{\tau} := \sqrt{\tau^2 - \tau\mu}$ and $\tilde{B} := B(y(t-s), 1/\tau) \cup B(y(t), 1/\tau)$.

Let us put again $\lambda = 2\kappa^{-1}|x - y(t)|$. We write $R_2 = R_{21} + R_{22}$ with

$$\begin{aligned} R_{21} &:= \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds, \\ R_{22} &:= \int_{\lambda/\tau}^\infty e^{-\tau^2 s} \int_{\tilde{B}} s G_y(x, s) dy ds. \end{aligned}$$

As for the estimate of H_1 in the proof of Lemma 4.5 we have

$$\begin{aligned}
|R_{22}| &\leq 2\kappa^{-1}\tau^{-3} \int_{\lambda/\tau}^{\infty} s^{-1/2} e^{-\tau^2 s/2} ds \\
&\leq 2\kappa^{-1}\tau^{-3+1/2}\lambda^{-1/2} \int_{\lambda/\tau}^{\infty} e^{-\tau^2 s/2} ds = 4\kappa^{-1}\tau^{-5+1/2}\lambda^{-1/2} e^{-\tau\lambda/2} \\
&= 2\sqrt{2}\kappa^{-1/2}\tau^{-5+1/2}|x-y(t)|^{-1/2} e^{-\tau\kappa^{-1}|x-y(t)|}.
\end{aligned}$$

Thanks to (7) and since $\tau|x-y(t)| \geq 1$ we then have

$$\begin{aligned}
|R_{22}| &\leq \kappa^{-5/2}\tau^{-4}(\tau|x-y(t)|)^{-1/2}|x-y(t)|p_{\tau}(x, y(t)) \\
&\leq C(\kappa)\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)).
\end{aligned} \tag{88}$$

For $s \leq \lambda/\tau$ we have $\tilde{B} \subset B(y(t), M'/\tau)$ with $M' := \lambda M + 1$. Hence, as for the estimate of H_2 in the proof of Lemma 4.5 and since $\tau|x-y(t)| \geq 2M'^2\kappa^{-5}$ we then have

$$\begin{aligned}
|R_{21}| &\leq \lambda\tau^{-1} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), M'/\tau)} G_y(x, s) dy ds \\
&\leq \lambda\tau^{-1} C(\mathbf{b}, d_{\Omega}, M)\tau^{-3} p_{\tau}(x, y(t)) \\
&\leq C(\mathbf{b}, d_{\Omega}, M)\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)).
\end{aligned} \tag{89}$$

From (88) and (89), we obtain that for $\tau|x-y(t)| \geq 2M'^2\kappa^{-5}$ we have

$$|R_2| \leq C(\mathbf{b}, d_{\Omega}, M)\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)). \tag{90}$$

Now, we estimate R_1 as R_2 by splitting the integral in (87) with $s < \lambda/\tau$ or $s > \lambda/\tau$. We observe that $\tilde{\tau} = \tau\sqrt{1 - \mu\tau^{-1}} \geq \frac{1}{\sqrt{2}}\tau$. Hence $R_1 = R_{11} + R_{12}$ with, since $\tilde{\tau}|x-y(t)| \geq \sqrt{2}\kappa^{-5}$ and $\tau \geq 2\mu$,

$$\begin{aligned}
|R_{12}| &\leq \tau^{-3+1/2}\lambda^{-1/2} \int_{\lambda/\tau}^{\infty} e^{-\tilde{\tau}^2 s/2} ds \\
&= 2\kappa^{-1}\tilde{\tau}^{-2}\tau^{-3+1/2}\lambda^{-1/2} e^{-\tau\lambda+\lambda\mu} \\
&\leq C(\kappa, \mu, d_{\Omega})\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)).
\end{aligned} \tag{91}$$

Finally, since $\tau|x-y(t)| \geq 2\kappa^{-5}$, we have, as for the estimate of R_{21} ,

$$\begin{aligned}
|R_{11}| &\leq \lambda\tau^{-1}e^{\lambda\mu} \int_0^{\lambda/\tau} e^{-\tau^2 s} \int_{B(y(t), 1/\tau)} G_y(x, s) dy ds \\
&\leq C(\mathbf{b}, \mu, d_{\Omega}, M)\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)).
\end{aligned} \tag{92}$$

Thanks to (91) and (92), we obtain

$$|R_1| \leq C(\mathbf{b}, \mu, d_{\Omega}, M)\tau^{-4}|x-y(t)|p_{\tau}(x, y(t)). \tag{93}$$

Thanks to (93) and (90), (85), we obtain (40) for $\tau \geq 2\mu$, $t \in [0, T]$ and $x \in \Omega \setminus B(y(t), C_1/\tau)$. \square

Proof of Lemma 4.8. We consider a family of balls $B_i(1/\tau)$, $i \in I$, $\tau > \tau_0$, as in Lemma 4.1. By using (35) (with $\beta = 6$), (36) and (37), and by observing that $B_i(6/\tau_0) \cap B(y(t), 2\kappa^5/\tau_0) = \emptyset$, we can write

$$\begin{aligned}
\int_{D_t} |\nabla P_\tau|^2(x, t) dx &\leq \sum_{i \in I} \int_{B_i(3/\tau)} |\nabla P_\tau|^2(x, t) dx \\
&\leq C_1 \sum_{i \in I} \int_{B_i(6/\tau)} \tau^2 P_\tau^2(x, t) dx \\
&\leq C_2 \sum_{i \in I} \int_{B_i(6/\tau)} \tau^{-4} p_\tau^2(x, y(t)) dx \\
&\leq C_3 \sum_{i \in I} |B_i(6/\tau)| \tau^{-4} p_\tau^2(x_i, y(t)) \\
&\leq C_4 \sum_{i \in I} |B_i(1/\tau)| \tau^{-4} \min_{B_i(1/\tau)} p_\tau^2(\cdot, y(t)) \\
&\leq C_5 \sum_{i \in I} \int_{B_i(1/\tau)} \tau^{-4} p_\tau^2(x, y(t)) dx \\
&\leq C_5 \int_{D_t} \tau^{-4} p_\tau^2(x, y(t)) dx, \tag{94}
\end{aligned}$$

with $C_j = C_j(\mathbf{b})$. Hence, the second inequality of (41) is proved. The proof of the first one is similar. \square

Proof of Lemma 4.9. We put $C'_1 = C_1 + 6$, $C_2 = \max(C'_1, 12\kappa^{-5})$ where $C_1 = C_1(\kappa, d_\Omega, M)$ is the constant in Lemma 4.7. We consider again the balls $B(1/\tau)$, $B(3/\tau)$, defined in Lemma 4.1. Thus

$$J := \int_{D_t} |\nabla q_\tau(x, t)|^2 dx \leq \sum_i \int_{B_i(3/\tau)} |\nabla q_\tau(x, t)|^2 dx. \tag{95}$$

Let us fix i and denote $B(3/\tau) = B_i(3/\tau)$. We consider again the functions $\phi \in C^1(\mathbb{R}; [0, 1])$ and $\psi(x) = \phi(\tau(x - x_0)/6)$ where x_0 is the center of a ball $B(6/\tau)$, as in the proof of Lemma 4.3 (with $\beta = 6$).

Thanks to Lemma 4.6, there exists a positive constant $C = C(\mathbf{b}, d_\Omega, \mu, M)$ such that for $\tau \geq 2\mu$, $t \in [0, T]$, $x \in \Omega \setminus B(y(t); C_1/\tau)$, we have

$$\begin{aligned}
|(-\operatorname{div} \mathbf{b} \nabla + \tau^2) q_\tau(x, t)| &= \left| \partial_t (e^{-\tau^2(t+T)} u_\tau(x, t)) \right| \\
&\leq C \tau^{-2} e^{-\tau\mu|t-\theta|} p_\tau(x, y(t)). \tag{96}
\end{aligned}$$

We observe that

$$\begin{aligned}
x \in \operatorname{supp}(\psi) = \overline{B(6/\tau)} &\Rightarrow |x - y(t)| \geq |x_0 - y(t)| - 6/\tau \geq \\
&\mathbf{d}(t) - 6/\tau > C'_1/\tau - 6/\tau = C_1/\tau.
\end{aligned}$$

Hence we can multiply (96) by $q_\tau(x, t) \psi^2(x)$ and integrate it over Ω . This implies

$$\begin{aligned}
\int_{\Omega} (\mathbf{b}(\nabla q_\tau(\cdot, t))^2 \psi^2 + 2\mathbf{b} \nabla q_\tau(\cdot, t) \psi q_\tau(\cdot, t) \nabla \psi + \tau^2 q_\tau^2(\cdot, t) \psi^2) \\
\leq C \tau^{-2} e^{-\tau\mu|t-\theta|} \int_{\Omega} |q_\tau(\cdot, t)| p_\tau(\cdot, y(t)) \psi^2.
\end{aligned}$$

Then, from Cauchy-Minkovski's inequality, and as in the proof of Lemma 4.3, we obtain

$$\begin{aligned} \int_{\Omega} (|\nabla q_{\tau}(\cdot, t)|^2 + \tau^2 q_{\tau}^2(\cdot, t)) \psi^2 &\leq C \int_{\Omega} q_{\tau}^2(\cdot, t) (\nabla \psi)^2 + C e^{-\tau \mu |t - \theta|} \tau^{-2} \cdot \\ &\quad \cdot \left(\int_{\Omega} |q_{\tau}(\cdot, t)|^2 \psi^2 \right)^{1/2} \left(\int_{\Omega} |p_{\tau}(\cdot, y(t))|^2 \psi^2 \right)^{1/2}. \end{aligned}$$

Since $\text{supp } \psi = \overline{B(6/\tau)}$ with $|\nabla \psi(x)| \leq \tau \max |\phi'|/6$, $\psi \geq 0$, and $\psi = 1$ in $B(\frac{3}{\tau})$, we then have

$$\begin{aligned} \int_{B(\frac{3}{\tau})} |\nabla q_{\tau}(\cdot, t)|^2 &\leq C \tau^2 \int_{B(\frac{6}{\tau})} q_{\tau}^2(\cdot, t) + C e^{-\tau \mu |t - \theta|} \tau^{-2} \cdot \\ &\quad \cdot \left(\int_{B(\frac{6}{\tau})} |q_{\tau}(\cdot, t)|^2 \right)^{1/2} \left(\int_{B(\frac{6}{\tau})} |p_{\tau}(\cdot, y(t))|^2 \right)^{1/2}. \end{aligned}$$

Thanks to Lemma 4.7 and by using $\tau^{-1} \leq C_1 |x - y(t)|$ for $x \in B(\frac{6}{\tau})$, we then have

$$\begin{aligned} \int_{B(\frac{3}{\tau})} |\nabla q_{\tau}(\cdot, t)|^2 &\leq C \tau^{-4} e^{-2\tau \mu |t - \theta|} \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx + \\ &\quad C \tau^{-5} e^{-2\tau \mu |t - \theta|} \cdot \left(\int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx \right)^{1/2} \cdot \\ &\quad \cdot \left(\int_{B(\frac{6}{\tau})} |p_{\tau}(x, y(t))|^2 dx \right)^{1/2} \\ &\leq C' e^{-2\tau \mu |t - \theta|} \tau^{-4} \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx. \quad (97) \end{aligned}$$

By putting (97) in (95) we obtain

$$J \leq C' e^{-2\tau \mu |t - \theta|} \tau^{-4} \sum_i \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx. \quad (98)$$

Finally, as in (94) we have

$$\sum_i \int_{B(\frac{6}{\tau})} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx \leq C \int_{D_t} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx.$$

This with (98) prove (42). \square

Proof of Lemma 4.10. We can assume that $D_t \neq \emptyset$. We put $\lambda = 2\kappa^{-2} \mathbf{d}(t)$ and

$$\begin{aligned} J &:= \int_{D_t} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx = J_1 + J_2, \\ J_1 &:= \int_{D_t \cap B(y(t), \lambda)} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx, \\ J_2 &:= \int_{D_t \setminus B(y(t), \lambda)} |x - y(t)|^2 |p_{\tau}(x, y(t))|^2 dx, \\ \tilde{J} &:= \int_{D_t} |p_{\tau}(x, y(t))|^2 dx. \end{aligned}$$

We then have

$$J_1 \leq \lambda^2 \tilde{J} = 4\kappa^{-4} \mathbf{d}(t)^2 \tilde{J}. \quad (99)$$

On the one hand thanks to (7) we have

$$J_2 \leq |D_t| 4\pi\kappa^{-4} \exp(-2\kappa\lambda\tau),$$

and, on the other hand,

$$\begin{aligned} J_2 &\leq 4\pi\kappa^{-4} \int_{|x-y(t)|>\lambda} e^{-2\kappa\tau|x-y(t)|} dx \leq 16\pi^2\kappa^{-4} \int_{r>\lambda} e^{-2\kappa\tau r} r^2 dr \\ &\leq 20\pi^2\lambda^2\kappa^{-5}\tau^{-1} \exp(-2\kappa\tau\lambda) \leq 20\pi^2\lambda^3\kappa^{-4} \exp(-2\kappa\tau\lambda) \\ &\leq 20\pi^2\kappa^{-10} (2\mathbf{d}(t))^3 \exp(-2\kappa\tau\lambda). \end{aligned}$$

Here we used $\lambda \geq \kappa^{-1}\tau^{-1}$. Hence

$$J_2 \leq 4\pi\kappa^{-4} \min(40\pi\kappa^{-6}\mathbf{d}(t)^3, |D_t|) \exp(-2\kappa\tau\lambda). \quad (100)$$

Let us fix $x_0 \in \partial D_t$ such that $\mathbf{d}(t) = |x_0 - y(t)|$. Then $B(y(t), 2\mathbf{d}(t)) \supset B(x_0, \mathbf{d}(t))$ so, thanks to (7) and to (H3), we have

$$\begin{aligned} \tilde{J} &\geq \int_{D_t \cap B(y(t), 2\mathbf{d}(t))} |p_\tau(x, y(t))|^2 dx \\ &\geq |D_t \cap B(y(t), 2\mathbf{d}(t))| 4\pi\kappa^4 (2\mathbf{d}(t))^{-2} \exp(-4\kappa^{-1}\tau\mathbf{d}(t)) \\ &\geq \pi\kappa^4 |D_t \cap B(x_0, \mathbf{d}(t))| \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda) \\ &\geq \pi\kappa^4 L_D \min(|D_t|, |B(x_0, \mathbf{d}(t))|) \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda) \\ &\geq \pi\kappa^4 L_D \min(|D_t|, \frac{4}{3}\pi\mathbf{d}(t)^3) \mathbf{d}(t)^{-2} \exp(-2\kappa\tau\lambda). \end{aligned}$$

Then

$$\frac{J_2}{\mathbf{d}(t)^2 \tilde{J}} \leq C L_D^{-1} \kappa^{-14}, \quad (101)$$

for some numerical parameter $C > 0$. From (101) and (99) we obtain

$$J \leq C' L_D^{-1} \mathbf{d}(t)^2 \tilde{J},$$

which is the estimate to prove. \square

Proof of Lemma 2.1. It is the direct consequence of Lemma 4.9 and Lemma 4.10.

Proof of Lemma 3.3

1) Thanks to (7) and to Lemma 3.2 there exists $\tau_1 > 0$ such that we have, for all $\tau > \tau_1$,

$$\begin{aligned} I_\infty(\tau) &\leq c\tau^{-4} \int_0^T \int_{D_t} 16\kappa^{-4} \varepsilon_\Sigma^{-2} e^{-2\kappa\tau\varepsilon_\Sigma} dx dt + C_1 e^{-4\kappa^{-1}d_\Omega\tau} \\ &\leq (c'|D|\tau^{-4}\varepsilon_\Sigma^{-2} + C_1) e^{-2\kappa\tau\varepsilon_\Sigma}. \end{aligned}$$

We then obtain (31).

2) Let us fix $T_1 \in (0, \min(\frac{1}{8}|D_\theta|^{1/3}, \theta, T - \theta, \alpha^4)]$ such that we have, thanks to (H3),

$$|D_t| \geq \frac{1}{2}|D_\theta| \geq T_1^3 \quad \text{for } |t - \theta| \leq T_1. \quad (102)$$

Thanks to (32) we have $\mathbf{d}(t)^2 \leq \frac{1}{2C_3}$ for $|t - \theta| \leq T_1$. Thanks to (30) in Lemma 3.2 we then have, for $\tau > \tau_1$,

$$I_\infty(\tau) \geq I_0 - R_0$$

with

$$I_0 := \frac{c}{2}\tau^{-4} \int_{|t-\theta| < T_1} e^{-2\tau\mu|t-\theta|} \int_{D_t} p_\tau^2(x, y(t)) dx dt,$$

$c = c(\mathbf{a}, \mu, M) > 0$ and, thanks to (7),

$$\begin{aligned} R_0 &:= c\tau^{-4} \int_{|t-\theta| > T_1} \int_{D_t} e^{-2\tau\mu|t-\theta|} p_\tau^2(x, y(t)) dx dt + C_1 e^{-\tau\mu \min(\theta, T-\theta)} \\ &\leq (16c\kappa^{-4}\tau^{-4}|\Omega|\varepsilon_\Sigma^{-2} + C_1) e^{-2\tau\mu T_1} \leq C_4 \varepsilon_\Sigma^{-2} \tau^{-4} e^{-2\tau\mu T_1}, \end{aligned}$$

with $C_4 = C_4(\mathbf{a}, |\Omega|, M, v_0)$. Thanks to (7) and to (32) we have also, for $\varepsilon \in (0, T_1/\alpha]$ (and observe that $(0, T_1/\alpha] \subset (0, \alpha^3]$),

$$\begin{aligned} I_0 &\geq \frac{c}{2}\tau^{-4} \int_{|t-\theta| < \alpha\varepsilon} e^{-2\tau\mu|t-\theta|} \int_{D_t \cap B(y(t), 2\mathbf{d}(t))} \kappa^4 d_\Omega^{-2} e^{-4\tau\kappa^{-1}\mathbf{d}(t)} \mathbf{d}(t) dx dt \\ &\geq \frac{c}{2}\kappa^4 d_\Omega^{-2} \tau^{-4} e^{-4\tau\kappa^{-1}\alpha^{-1}\varepsilon} \int_{|t-\theta| < \alpha\varepsilon} e^{-2\tau\mu|t-\theta|} |D_t \cap B(y(t), 2\mathbf{d}(t))| dt. \end{aligned}$$

We have $D_t \cap B(y(t), 2\mathbf{d}(t)) \supset D_t \cap B(x(t), \mathbf{d}(t))$ for some $x(t) \in \partial D_t$. Thanks to (102) and to (H3) we then have $|D_t \cap B(x(t), \mathbf{d}(t))| \geq L_D |B(0, 1)| \mathbf{d}(t)^3$. Since $\varepsilon/2 \leq \mathbf{d}(t) \leq 2\alpha^{-2}\varepsilon$ for $|t - \theta| < \alpha\varepsilon$ we then have $|D_t \cap B(y(t), 4\alpha^{-2}\varepsilon)| \geq L_D |B(0, 1)| \varepsilon^3/8$ and so

$$\begin{aligned} I_0 &\geq C'(\Omega, \mathbf{a}, M, \alpha) \varepsilon^3 \tau^{-4} e^{-8\tau\kappa^{-1}\alpha^{-2}\varepsilon} \int_{|t-\theta| < \varepsilon} e^{-2\tau\mu|t-\theta|} dt \\ &\geq C''(\Omega, \mathbf{a}, M, \alpha) \varepsilon^3 \tau^{-5} e^{-8\tau\kappa^{-1}\alpha^{-2}\varepsilon}. \end{aligned}$$

Let us put $\varepsilon_1 = T_1 \min(1, \mu\kappa\alpha^2/8)$. Then for $\varepsilon \in (0, \varepsilon_1]$ we have

$$R_0/I_0 \leq C''' \varepsilon^{-3} \varepsilon_\Sigma^{-2} \tau e^{-\tau\mu T_1}.$$

Then we have $R_0/I_0 \leq \frac{1}{2}$ for $\tau \geq \tau_0$ sufficiently large and so

$$I_\infty(\tau) \geq \frac{1}{2} I_0 \geq C(\mathbf{a}, \Omega, M, \alpha) \varepsilon^{-3} \tau^{-5} e^{-8\tau\kappa^{-1}\alpha^{-2}\varepsilon}.$$

Hence

$$\liminf_{\tau \rightarrow \infty} \tau^{-1} \ln(I_\infty(\tau)) \geq -8\kappa^{-1}\alpha^{-2}\varepsilon.$$

The conclusion follows. \square

References

- [1] , D. G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bull. Amer. Math. Soc. **73**, Number 6 (1967), 890-896.
- [2] M. Di Cristo and S. Vessella, *Stable determination of the discontinuous conductivity coefficient of a parabolic equation*, arXiv:0904 (2009).
- [3] Y. Daido, H. Kang and G. Nakamura, *A probe method for the inverse boundary value problem of non-stationary heat equations*, Inverse Problems **23** (2007), 1787-1800.
- [4] A. Elayyan and V. Isakov, *On uniqueness of the recovery of the discontinuous conductivity coefficient of a parabolic equation*, SIAM. J. Math. Anal. **28** (1997), 49-59.
- [5] E.B. Fabes and D.W. Stroock, *A new proof of Moser's parabolic Harnack inequality via the old ideas of Nash*, Arch. Rat. Mech. Anal. , 96 (1986) 327-338.
- [6] P. Gaitan, H. Isozaki, O. Poisson, S. Siltanen and J. Tamminen, *Inverse problems for time-dependent singular heat conductivities - One dimensional case*, SIAM Journal of Mathematical Analysis 45(3), pp. 1675-1690 (2013).
- [7] P. Gaitan, H. Isozaki, O. Poisson, S. Siltanen, J. Tamminen, *Inverse problems for time-dependent singular heat conductivities. Multi dimensional case*, Communications in Partial Differential Equations 40(5), pp. 837-877 (2014).
- [8] P. Gaitan, H. Isozaki, O. Poisson, S. Siltanen and J. Tamminen, *Probing for inclusions in heat conductive bodies*, Inverse Problems and Imaging **6** (2012), pp. 423-446.
- [9] M. Ikehata and M. Kawashita, *The enclosure method for the heat equation*, Inverse Problems **25** (2009), 075005.1 .
- [10] M. Ikehata, *Probe method and a Carleman function*, Inverse Problems **23** (2007), pp.1871-1894, doi:10.1088/0266-5611/23/5/006 .
- [11] M. Ikehata and M. Kawashita, *On the reconstruction of inclusions in a heat conductive body from dynamical boundary data over a finite time interval*, Inverse Problems **26** (2010), 095004.
- [12] V. Isakov, *Inverse Problems for partial differential equations*, Appl.Math.Sci., **127**, Berlin Springer (1988).
- [13] V. Isakov, K. Kim, G. Nakamura, *Reconstruction of an unknown inclusion by thermography*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **Vol. IX**, issue 4 (2010), p. 725-758.
- [14] H. Kawakami and M. Tsuchiya, *Uniqueness in shape identification of a time-varying domain and related parabolic equations on non-cylindrical domains* Inverse Problems **26** (2010), 125007 (34pp), doi:10.1088/0266-5611/26/12/125007
- [15] J.L. Lions and E. Magenes *Non-Homogeneous Boundary Value Problems and Applications II* Berlin: Springer (1972).

- [16] J. Nash, *Continuity of solutions of a parabolic and elliptic equations*, Amer. J. Math., **80** (1958), p. 931-954.
- [17] O. Poisson, *Recovering time-dependent inclusion in heat conductive bodies using a dynamical probe method*, Journal of Mathematical Analysis and Applications, 441 (2): 862-844 (2016).
- [18] S. Vessella, *Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates*, Inverse Problems, **24/2** (2008), 023001.
- [19] J. Wloka, *Partial Differential Equations*, London: Cambridge University Press (1987).