Approximate viscosity solutions of path-dependent PDEs and Dupire's vertical differentiability

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Abstract

We introduce a notion of approximate viscosity solution for a class of nonlinear path-dependent PDEs (PPDEs), including the Hamilton-Jacobi-Bellman type equations. Existence, comparaison and stability results have been established under fairly general conditions. It is also consistent with smooth solutions when the dimension is less or equal to two, or the non-linearity is concave in the second order space derivative. We finally investigate the regularity (in the sense of Dupire) of the solution to the PPDE.

1 Introduction

The main objective of the paper is to study nonlinear path-dependent PDEs (PPDEs) of the form

$$-\partial_t \mathrm{v}(t,\mathrm{x}) - F\big(t,\mathrm{x},\mathrm{v}(t,\mathrm{x}),\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})\big) = 0, \quad (t,\mathrm{x}) \in \Theta, \tag{1.1}$$

where $\Theta = [0,T] \times C([0,T])$ or $\Theta = [0,T] \times D([0,T])$, $C([0,T])$ being the space of all $\mathbb{R}^d$–valued continuous paths on $[0,T]$, and $D([0,T])$ being the Skorokhod space of all $\mathbb{R}^d$–valued càdlàg paths on $[0,T]$. In the above, $\partial_t \mathrm{v}$ (resp. $\nabla_{\mathrm{x}}\mathrm{v}$ and $\nabla^2_{\mathrm{x}}\mathrm{v}$) represents the horizontal derivative (resp. first and second order vertical derivatives) of v in the sense of Dupire [7], see also Section 2.4 below.

Extending classical PDEs, PPDEs have various applications in the study of dynamic stochastic systems in non-Markovian settings (hedging/super-hedging of path-dependent options [7], path-dependent optimal control problems and stochastic differential games [16], etc.). Pioneering works, such as [7, 3, 15], investigated the existence of classical solutions, but it is generally very hard to obtain the required regularity of solutions, even for the

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most simple path-dependent heat equation. For this reason, much effort has been devoted to introduce appropriate weak notions of solution.

In a first stream of literature, see e.g. [8, 9, 10, 18], the authors investigate a relaxed notion of viscosity solutions for PPDEs. As for the classical notion of viscosity solutions for PDEs on $\mathbb{R}^d$, see e.g. [6], they consider families of smooth test functions and use the derivatives of the test functions to define the viscosity supersolution and subsolution properties. However, in contrast to the pointwise tangent property used in the classical setting, their definition appeals to an optimality property with respect to a stopping problem under a nonlinear expectation. For fully nonlinear equations, they require F to be non-degenerate as well as a strong uniform continuity conditions (which for instance are not satisfied in the case of linear equations with path-dependent coefficients). In [18], the non-degeneracy condition has been removed, but more restrictive continuity conditions on F are needed. Only very recently, it was introduced in [5] a notion of Crandall-Lions viscosity solutions, where comparison is only studied for the path-dependent heat equation with constant coefficients.

Another approach consists in approximating the generator F of the PPDE by a sequence $(F^n)_{n\geq 1}$, for which strong existence holds, and then define the limit of the corresponding sequence of solutions $(v^n)_{n\geq 1}$ as a solution of the original PPDE. This is the case of the notion of strong viscosity solution studied in [4], according to which a function v is called a strong viscosity solution of the PPDE with generator F if there exists a sequence $(v^n, F^n)_{n\geq 1}$ that converges to (v, F) pointwise, and each v^n is a smooth solution to the PPDE with generator F^n. Existence and uniqueness of strong viscosity solutions was proved for a class of parabolic semilinear PPDEs, which can be expected because of their link with BSDEs and the associated stability results. Finally, in [11], the authors introduced a notion of pseudo-Markovian viscosity solutions for fully nonlinear PPDEs by considering a sequence of PPDEs with generators F^n associated to paths frozen up to the exit time of certain domains. In their case, existence in the fully nonlinear setting relies on the convergence of a suitable sequence of functions, which is a-priori non trivial.

Let us also mention the notion of Sobolev solutions investigated in [14] for a class of PPDEs, which uses global properties to define a solution, in contrast to the local property required in the definition of viscosity solutions.

In this paper, we introduce a novel weak notion of solutions in the same vein as the strong viscosity solutions of [4] and the pseudo-Markovian viscosity solutions of [11], in the sense that it is defined by an approximation argument. Similar to [11], we consider PPDEs with fully nonlinear, and possibly degenerate, generator F, which can exhibit a non-uniformly continuous linear part. As in [11], our construction is based on a finite dimensional approximation of paths, that are frozen on some time intervals. The difference is that we do not consider a sequence of (possible infinitely many) hitting times to define the frozen-path PPDE, but use a simple sequence $\pi = (\pi^n)_{n\geq 1}$ of discrete time grids on $[0, T]$, with $\pi^n = (0 = t_0^n < \cdots < t_n^n = T)$, and then freeze the path argument on each time interval $[t_i^n, t_{i+1}^n)$ to obtain our sequence of generators F^n, see (2.7) below. Importantly, the path argument is not only frozen, it is also approximated by a piecewise constant or a piecewise linear approximation. For PPDEs having a probabilistic interpretation in

terms of BSDEs or optimal control problems, this amounts to replacing in the coefficients the original diffusion process by the corresponding piecewise constant or piecewise linear approximation. From this point of view, this definition is very natural and simple. The fact of freezing the path allows us to reduce to a finite dimensional setting, on which standard viscosity solution techniques can be applied. The fact of making an additional approximation of the original path allows us to view the corresponding PDEs as PDEs with a finite number of parameters on which a-priori regularity estimates can be obtained.

In this setting, we say that a function v is a π-approximate viscosity solution of the PPDE if it is the limit of the (classical) viscosity solutions $(v^n)_{n\geq 1}$ associated to the sequence $(F^n)_{n\geq 1}$, see Definition 2.6 for a precise statement. Without defining explicitly a notion of solution, this approximation approach was already used successfully in [1] to obtain $\mathbb{C}^{1+\alpha}$-type estimates on the candidate solution of a specific class of fully-non linear PPDEs. We show in this paper that it can actually allow one to consider fairly general situations.

Using this definition, the uniqueness and comparison are direct consequences of the comparison principle of classical PDEs. As for the existence, it is easy to obtain for a class of semilinear PPDE or HJB type equations, where the solution has a probabilistic representation by the path-dependent BSDEs or optimal control problems. In fact, it is generally enough to assume some uniform continuity conditions on the coefficient functions, and then to apply the standard stability results of the BSDEs and SDEs, c.f. Proposition 2.8. However, it becomes more challenging to prove existence in the general nonlinear case, without the help of a probabilistic representation. This is our main result, Theorem 2.11. It is proved in a rather general setting ensuring that comparison and existence holds at the level of the path-frozen finite dimensional PDEs, and only uses viscosity solution techniques. In particular, the generator can be degenerate and does not need to be concave when $d > 2$, an assumption made in [10].

At the same time, we are able to show that a classical solution of the PPDE is an approximate viscosity solution, when $d \leq 2$ or the non-linearity is concave (or convexe) in the second order spatial derivative. In particular, our result could induce natural numerical schemes for the class of nonlinear PPDEs, with an explicit convergence rate, see Remark 2.12.

Finally, another advantage of our approximation technique is that it allows one to obtain $\mathbb{C}^{\beta,1+\alpha}$-type regularity estimates (in the sense of Dupire) for solutions to PPDEs, a subject which does not seem to have been investigated so far. In particular, this enables to use the $\mathbb{C}^{0,1}$-functional Itô's formula of [1], see this reference for a specific application in mathematical finance. In this paper, we show that the solution inherits the $C^{1+\alpha}$-regularity of the coefficients and the terminal condition. For semi-linear equations, the results can actually be proved by purely probabilistic arguments. For fully nonlinear equations, the approximation approach is fully exploited. In this case, we need to assume additional structure conditions, and stay in the context where the generator is concave with respect to the Hessian matrix or the dimension does not exceed 2. The (very difficult) question of whether a uniform ellipticity condition can regularize the solution is left for future research.

The rest of the paper is organized as follows. In Section 2, we introduce our notion of π-approximate viscosity solutions to the PPDE, and provide some comparison, existence and stability results, in a fairly general fully nonlinear setting. Then, in Section 3, we provide some first regularity results on the solutions to the PPDE.

2 Approximate viscosity solution of path-dependent PDEs

In this section, we first define our notion of approximate viscosity solution, after introducing some general notations. Its very definition implies that comparison holds under standard assumptions. We next explain how it is naturally connected to optimal control problems. Finally, we prove that existence holds in a fully nonlinear setting, and provide a first stability result.

2.1 Preliminaries

By the parabolic and local nature of the PPDE (1.1), it is natural to consider the case where its coefficients are defined on the space $C([0,T])$ of $\mathbb{R}^d$–valued continuous paths on $[0,T]$. A solution should then be defined on $[0,T]\times C([0,T])$. On the other hand, when one investigates the vertical regularity of this solution in the sense of Dupire, one needs to consider paths with jumps, and hence it is more convenient to define it on the space $D([0,T])$ of $\mathbb{R}^d$–valued càdlàg paths on $[0,T]$. For this reason, we will consider two cases $\Omega = C([0,T])$ or $\Omega = D([0,T])$ equipped with two different semimetrics.

For both cases of Ω, given $\mathrm{x}\in\Omega$, we define its uniform norm by $\|\mathrm{x}\| := \sup_{s\in[0,T]}|\mathrm{x}_s|$, and

$$\mathrm{x}_{t\wedge} := (\mathrm{x}_{t\wedge s})_{s\le T} \ \text{ for } t\le T\ ,\ [\mathrm{x}]^n_i := (\mathrm{x}_{t^n_j})_{j=0,\cdots,i} \text{ for } i\le n,$$

in which the later is defined with respect to a given increasing sequence of discrete time grids $\pi = (\pi^n)_{n\ge1}$, i.e.

- $\pi^n = (t^n_i)_{0\le i\le n}$ with $0 = t^n_0 < \cdots < t^n_n = T$, for each $n\ge1$,
- $\pi^n \subset \pi^{n+1}$, for each $n\ge1$,
- $|\pi^n| := \max_{i<n}|t^n_{i+1} - t^n_i| \longrightarrow 0$ as $n\longrightarrow\infty$.

We define different semimetrics $\rho : \Omega\times\Omega \longrightarrow \mathbb{R}_+$ as well as different projection operators $\Pi^n : \Omega \longrightarrow \Omega$, depending whether $\Omega = C([0,T])$ or $\Omega = D([0,T])$. Let $\varpi_\circ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a modulus of continuity[1] which is concave and satisfies that, for some constant $c>0$, $\varpi_\circ(x)\ge cx$ for all $x\ge0$. We further define

$$\varpi'_\circ(x) \ := \ \sqrt{\varpi_\circ(x^2)}, \quad \text{for all } x\ge0. \tag{2.1}$$

[1] i.e. non-decreasing and satisfying $\varpi_\circ(0) = 0$.

Case I: $\Omega = C([0,T])$**.** Let us define the family of semimetrics ρ and $(\rho_t)_{t\in[0,T]}$ by

$$\rho := \rho_T, \quad \rho_t(\mathrm{x},\mathrm{x}') \;:=\; \varpi'_\circ\big(\|\mathrm{x}_{t\wedge} - \mathrm{x}'_{t\wedge}\|\big), \quad \text{for all } (t,\mathrm{x},\mathrm{x}') \in [0,T]\times\Omega\times\Omega. \tag{2.2}$$

Further, for each $n \geq 1$, $t \in (t^n_k, t^n_{k+1}]$, $k = 0,\cdots,n-1$, we define $\Pi^n_t : \Omega \longrightarrow \Omega$ by

$$\begin{aligned}\Pi^n_t[\mathrm{x}]_s \;:=\; & \sum_{i=0}^{k-1} \mathbf{1}_{\{s\in[t^n_i,t^n_{i+1})\}}\Big(\frac{t^n_{i+1}-s}{t^n_{i+1}-t^n_i}\mathrm{x}_{t^n_i} + \frac{s-t^n_i}{t^n_{i+1}-t^n_i}\mathrm{x}_{t^n_{i+1}}\Big)\\ & + \mathbf{1}_{\{s\in[t^n_k,T]\}}\Big(\frac{t-s\wedge t}{t-t^n_k}\mathrm{x}_{t^n_k} + \frac{s\wedge t - t^n_k}{t-t^n_k}\mathrm{x}_t\Big),\ \mathrm{x}\in C([0,T]).\end{aligned}$$

Namely, $\Pi^n_t[\mathrm{x}]$ is the linear interpolation of the points $(\mathrm{x}_{t^n_0},\cdots,\mathrm{x}_{t^n_k},\mathrm{x}_t)$. By convention, $\Pi^n[\cdot] := \Pi^n_T[\cdot]$.

Case II: $\Omega = D([0,T])$**.** For each $t\in[0,T]$, we define the Skorokhod metric $d_t : \Omega\times\Omega \longrightarrow \mathbb{R}_+$ by

$$d_t(\mathrm{x},\mathrm{x}') \;=\; d_t(\mathrm{x}_{t\wedge},\mathrm{x}'_{t\wedge}) \;:=\; \inf_{\lambda_t\in\Lambda_t} \max\big\{\|\lambda_t - I_t\|, \|\mathrm{x}_{t\wedge} - (\mathrm{x}'\circ\lambda)_{t\wedge}\|\big\},$$

for $(t,\mathrm{x},\mathrm{x}')\in[0,T]\times D([0,T])\times D([0,T])$, where $I_t : [0,t]\longrightarrow[0,t]$ is the identity function, and Λ_t denotes the collection of all strictly increasing and continuous bijections from $[0,t]$ to $[0,t]$. Let μ be a finite positive measure on $[0,T]$ with atoms at finitely number of points in π^n for some $n\geq 1$. We then define the family of semimetrics ρ and $(\rho_t)_{t\in[0,T]}$ by

$$\rho := \rho_T, \quad \rho_t(\mathrm{x},\mathrm{x}') \;:=\; \varpi'_\circ\Big(d_t\big(\mathrm{x}_{t\wedge},\mathrm{x}'_{t\wedge}\big) + \int_0^t \big|\mathrm{x}_s - \mathrm{x}'_s\big|\mu(ds)\Big), \tag{2.3}$$

for $(t,\mathrm{x},\mathrm{x}')\in[0,T]\times\Omega\times\Omega$. For each $n\geq 1$, $t\in(t^n_k,t^n_{k+1}]$, $k=0,\cdots,n-1$, we define $\Pi^n_t : \mathrm{x}\in\Omega\longrightarrow\Pi^n_t[\mathrm{x}]\in\Omega$ as the piecewise constant approximation of x:

$$\Pi^n_t[\mathrm{x}]_s := \sum_{i=0}^{k-1}\mathbf{1}_{\{s\in[t^n_i,t^n_{i+1})\}}\mathrm{x}_{t^n_i} + \mathbf{1}_{\{s\in[t^n_k,t)\}}\mathrm{x}_{t^n_k} + \mathbf{1}_{\{t\}}\mathrm{x}_t, \quad \text{and} \quad \Pi^n[\cdot] := \Pi^n_T[\cdot].$$

Hereafter, we shall consider the two cases simultaneously and only write (Ω,ρ), unless we need to specialize to $\Omega = D([0,T])$ as in Section 2.4 and Section 3 below.

Remark 2.1. *In both cases of the definitions of $(\Omega,\rho,\Pi[\cdot])$, there exists a constant $C>0$ such that $\rho_t(\mathrm{x},\mathrm{x}')\leq C(1+\|\mathrm{x}_{t\wedge}-\mathrm{x}'_{t\wedge}\|)$ and*

$\rho(\mathrm{x},\mathrm{x}')\leq\rho_t(\mathrm{x},\mathrm{x}')$, *for all $t\in[0,T]$ and $\mathrm{x},\mathrm{x}'\in\Omega$ such that $(\mathrm{x}_s-\mathrm{x}_t)_{s\in[t,T]} = (\mathrm{x}'_s-\mathrm{x}'_t)_{s\in[t,T]}$.*

More importantly, one has

$$\rho_t\big(\Pi^n_t[\mathrm{x}],\mathrm{x}\big)\longrightarrow 0 \text{ as } n\longrightarrow\infty, \text{ for all } \mathrm{x}\in\Omega_t := \{\mathrm{x}\in\Omega\ : \mathrm{x} = \mathrm{x}_{t\wedge}\},$$

and in particular, the space (Ω_t,ρ_t) is separable.

2.2 Definition of approximate viscosity solutions

Let $\mathbb{S}^d$ denote the space of all symmetric $d\times d$-dimensional matrices, and $\Theta := [0,T]\times\Omega$. We say that a functional $u : \Theta \longrightarrow \mathbb{R}$ is non-anticipative if $u(t,\mathrm{x}) = u(t,\mathrm{x}_{t\wedge})$ for all $(t,\mathrm{x}) \in \Theta$. We are given a generator function $F : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d \longrightarrow \mathbb{R}$, which is non-anticipative in the sense that $F(t,\mathrm{x},y,z,\gamma) = F(t,\mathrm{x}_{t\wedge},y,z,\gamma)$ for all $(t,\mathrm{x},y,z,\gamma) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$, and a terminal data $g : \Omega \longrightarrow \mathbb{R}$. Throughout the paper, we study the following PPDE:

$$-\partial_t \mathrm{v}(t,\mathrm{x}) - F\big(t,\mathrm{x},\mathrm{v}(t,\mathrm{x}),\nabla_{\mathrm{x}}\mathrm{v}(t,\mathrm{x}),\nabla^2_{\mathrm{x}}\mathrm{v}(t,\mathrm{x})\big) = 0, \quad (t,\mathrm{x}) \in [0,T)\times\Omega, \tag{2.4}$$

with terminal condition

$$\mathrm{v}(T,\mathrm{x}) = g(\mathrm{x}), \quad \mathrm{x} \in \Omega. \tag{2.5}$$

Our definition of approximate viscosity solutions is based on the approximations of the path x, through the operator Π^n defined above, which allows us to reduce to a finite dimensional setting. Let us first introduce some functional spaces. Given $L > 0$, let

$$\mathcal{C}_L := \Big\{ g : \Omega \longrightarrow \mathbb{R} \ : g \ \text{ is continuous w.r.t. } \|\cdot\| \text{ and satisfies } \sup_{\mathrm{x}\in\Omega} \frac{|g(\mathrm{x})|}{1+\|\mathrm{x}\|} \le L \Big\},$$

and let $\mathcal{LSC}^n_L$ (resp. $\mathcal{USC}^n_L$) denote the collection of non-anticipative functions $f : \Theta\times\mathbb{R}^d \longrightarrow \mathbb{R}$ satisfying, for each $i \le n-1$, and $(t,\mathrm{x},x) \in [t^n_i, t^n_{i+1}) \times \Omega \times \mathbb{R}^d$,

$$|f(t,\mathrm{x},x)| \ \le \ L\big(1 + \|\mathrm{x}\| + |x|\big) \ \text{ and } f(t,\mathrm{x},x) = f\big(t,\Pi^n[\mathrm{x}],x\big),$$

and such that $(t,x) \longmapsto f(t,\mathrm{x},x)$ is lower-semicontinuous (resp. upper-semicontinuous) on each domain $[t^n_i, t^n_{i+1}) \times \mathbb{R}^d$, $i = 0,\cdots,n-1$. Then, it is clear that a functional $f \in \mathcal{LSC}^n_L \cup \mathcal{USC}^n_L$ can be represented in the form

$$f(t,\mathrm{x},x) \ = \ \sum_{i=0}^{n-1} f_i\big(t,[\mathrm{x}]^n_i,x\big)\mathbf{1}_{[t^n_i,t^n_{i+1})}(t) + f_n\big([\mathrm{x}]^n_n,x\big)\mathbf{1}_{\{t^n_n\}}(t). \tag{2.6}$$

for a sequence of functions $f_i : [t^n_i, t^n_{i+1}) \times (\mathbb{R}^d)^{i+1} \times \mathbb{R}^d \longrightarrow \mathbb{R}$, $i = 0,\cdots,n-1$, and a map $f_n : (\mathbb{R}^d)^{n+1} \times \mathbb{R}^d \longrightarrow \mathbb{R}$. Let

$$\mathcal{C}^n_L \ := \ \mathcal{LSC}^n_L \cap \mathcal{USC}^n_L.$$

Finally, let us define for $t \in [t^n_i, t^n_{i+1})$

$$F^n(t,\mathrm{x},y,z,\gamma) \ := \ F\big(t,\Pi^n_{t^n_i}[\mathrm{x}],y,z,\gamma\big), \quad \text{for all } (\mathrm{x},y,z,\gamma) \in \Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d. \tag{2.7}$$

Definition 2.2. *We say that $u^n \in \mathcal{LSC}^n_L$ (resp. $\mathcal{USC}^n_L$) of the form (2.6) is a π^n-viscosity supersolution (resp. subsolution) of (2.4) if, for all $\mathrm{x} \in \Omega$ and $i = 0,\cdots,n-1$, the map $(t,x) \longmapsto u^n(t,\mathrm{x},x)$ is a viscosity supersolution (resp. subsolution) of*

$$-\partial_t\varphi(t,x) - F^n\big(t,\mathrm{x},\varphi(t,x),D\varphi(t,x),D^2\varphi(t,x)\big) = 0, \quad \text{on } [t^n_i,t^n_{i+1})\times\mathbb{R}^d, \tag{2.8}$$

satisfying the boundary condition

$$\liminf_{t\nearrow t^n_{i+1},x'\to x} \delta\Big(\varphi(t,x') - u^n\big(t^n_{i+1},\Pi^n[\mathrm{x}]\boxplus_{t^n_{i+1}}x,x\big)\Big) \ge 0, \quad \text{for all } x \in \mathbb{R}^d,$$

with $\delta = 1$ (resp. $\delta = -1$) and[2]

$$\mathrm{x} \boxplus_t \mathrm{x}' := \mathrm{x}\mathbf{1}_{[0,t)} + \mathrm{x}'\mathbf{1}_{[t,T]}, \quad \textit{for all } t \leq T \textit{ and } \mathrm{x}, \mathrm{x}' \in \Omega.$$

Moreover, we say that u^n is a π^n-viscosity solution of (2.4) if it is both a π^n-viscosity supersolution and subsolution of (2.4).

In order to ensure the existence and uniqueness of a π^n-viscosity solution in $\cup_{L'>0}\mathcal{C}^n_{L'}$, we make the following standard assumption throughout the paper, in which the continuity or Lipschitz continuity property of a functional $\mathrm{x} \in \Omega \longmapsto f(\mathrm{x})$ is defined w.r.t. the uniform norm $\|\cdot\|$.

Assumption 2.3. *Let $L > 0$ be a fixed positive constant.*

(i) *For all $(t, \mathrm{x}) \in \Theta$, the map $(y, z, \gamma) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longmapsto F(t, \mathrm{x}, y, z, \gamma)$ is L-Lipschitz continuous, and $|F(t, \mathrm{x}, 0, 0, 0)| \leq L(1 + \|\mathrm{x}\|)$.*

(ii) *There exists a continuous map $\varpi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that $\varpi(0) = 0$ and*

$$F\big(t, \mathrm{x}, y, \alpha(z - z'), \Gamma_1\big) - F\big(t, \mathrm{x}, y, \alpha(z - z'), \Gamma_2\big) \;\leq\; \varpi\big(\alpha|z - z'|^2 + |z - z'|\big),$$

for all $(t, \mathrm{x}, y, z, z') \in \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$, $\Gamma_1, \Gamma_2 \in \mathbb{S}^d$ and $\alpha > 0$ such that

$$-3\alpha \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix} \leq \begin{pmatrix} \Gamma_1 & 0 \\ 0 & -\Gamma_2 \end{pmatrix} \leq 3\alpha \begin{pmatrix} I_d & -I_d \\ -I_d & I_d \end{pmatrix},$$

in which I_d denotes the $d \times d$-dimensional identity matrix.

(iii) *The map $\gamma \longmapsto F(\cdot, \gamma)$ is non-decreasing (for the natural partial order on $\mathbb{S}^d$).*

(iv) *There exists a compact subset A of $\mathbb{R}^{d'}$, $d' \geq 1$, non-anticipative continuous maps $\underline{\sigma}, \overline{\sigma} : \Theta \times A \longrightarrow \mathbb{M}^d$ and L-Lipschitz non-anticipative continuous maps $\underline{F}, \overline{F} : \Theta \times \mathbb{R} \times \mathbb{R}^d \longmapsto \mathbb{R}$ such that*

(a) *$\mathrm{x} \in \Omega \longmapsto (\underline{\sigma}, \overline{\sigma})(t, \mathrm{x}, a)$ is L-Lipschitz continuous, uniformly in $(t, a) \in [0, T] \times A$.*

(b) *For all $\gamma \in \mathbb{S}^d$,*

$$\underline{F} + \inf_{a \in A} \frac{1}{2}\mathrm{Tr}\big[\underline{\sigma}\,\underline{\sigma}^\top(\cdot, a)\,\gamma\big] \;\leq\; F(\cdot, \gamma) \;\leq\; \overline{F} + \sup_{a \in A} \frac{1}{2}\mathrm{Tr}\big[\overline{\sigma}\,\overline{\sigma}^\top(\cdot, a)\,\gamma\big].$$

Remark 2.4. Item (i) enables us to work on an unbounded domain by exhibiting a suitable penalty function to obtain a comparison principle, see the proof of Proposition 2.5 below. The Lipschitz continuity in (z, γ) could be made local in x by allowing the Lipschitz constant to be of the form $L'(1 + \|\mathrm{x}\|)$. We refrain from pursuing with this relaxation to avoid unnecessary complexities. Items (ii)-(iii) of Assumption 2.3 are standard to ensure the well-posedness of the PDE as well as the comparison principle. Item (iv) of Assumption 2.3 will be used to exhibit a π^n-viscosity supersolution and a π^n-viscosity subsolution of (2.4) with terminal condition $g \in \mathcal{C}_L$, having linear growth, uniformly in $n \geq 1$, see (2.10) below. It is essentially induced by the Lipschitz continuity of F, and could be replaced by other conditions leading to similar a-priori estimates.

[2] Hereafter, we identify a constant $x \in \mathbb{R}^d$ to the constant process $x\mathbf{1}_{[0,T]}$.

Proposition 2.5. *Let Assumption 2.3 hold true.*

(i) *Fix $n \geq 1$ and let $u^n \in \mathcal{LSC}^n_L$ (resp. $v^n \in \mathcal{USC}^n_L$) be a π^n-viscosity supersolution (resp. subsolution) of* (2.4) *such that $u^n(t^n_n, \cdot) \geq v^n(t^n_n, \cdot)$. Then, $u^n \geq v^n$ on $\Theta \times \mathbb{R}^d$.*

(ii) *Fix $g \in \mathcal{C}_L$ and $n \geq 1$. Then,* (2.4) *has a unique π^n-viscosity solution $v^n \in \cup_{L'>0}\mathcal{C}^n_{L'}$ satisfying the terminal condition*

$$v^n\big(t^n_n, \mathrm{x}, \mathrm{x}_{t^n_n}\big) = v^n\big(t^n_n, \Pi^n[\mathrm{x}], \mathrm{x}_{t^n_n}\big) = g\big(\Pi^n[\mathrm{x}]\big), \quad \textit{for all } \mathrm{x} \in \Omega. \tag{2.9}$$

(iii) *Fix $g \in \mathcal{C}_L$ and let $(v^n)_{n\geq 1}$ be defined as in* (ii) *above. Then, there exists $C > 0$, that depends only on L, such that*

$$\sup_{n\geq 1} |v^n(t, \mathrm{x}, x)| \leq C(1 + \|\mathrm{x}\| + |x|), \ \textit{for all } (t, \mathrm{x}, x) \in \Theta \times \mathbb{R}^d. \tag{2.10}$$

Proof. (i) Let us $n \geq 1$, $0 \leq i \leq n-1$, $\mathrm{x} \in \Omega$. If w is a viscosity subsolution of (2.8) on $[t^n_i, t^n_{i+1}) \times \mathbb{R}^d$, then Assumption 2.3.(i) ensures that $(t, x) \in [t^n_i, t^n_{i+1}) \times \mathbb{R}^d \longmapsto w(t, x) - \varepsilon(1 + |x|^2)e^{4Lt}$ is a viscosity subsolution of (2.8) on $[t^n_i, t^n_{i+1}) \times \mathbb{R}^d$ for all $\varepsilon > 0$. Combined with Assumption 2.3.(ii), it follows by standard arguments that (2.8) admits a comparison principle among functions having linear growth (see e.g. [6, Section 8]). Item (i) of Proposition 2.5 is then proved by backward induction.

(ii) – (iii) For each $n \geq 1$, the uniqueness of the π^n-viscosity solution v^n of (2.4) with terminal condition (2.9) follows from Item (i). It remains to construct a solution having the (uniform) linear growth property (2.10).

Recall that A is a compact subset of $\mathbb{R}^{d'}$ for some $d' \geq 1$, in Assumption 2.3. Let us consider a probability space equipped with a standard d-dimensional Brownian motion W and the corresponding Brownian filtration, and denote by $\mathcal{A}$ the collection of all predictable processes taking values in A. Then, given $\theta := (t, \mathrm{x}, x) \in \Theta \times \mathbb{R}^d$ and $\mathfrak{a} \in \mathcal{A}$, let $(X^{\theta,\mathfrak{a}}, Y^{\theta,\mathfrak{a}}, Z^{\theta,\mathfrak{a}})$ be the unique triplet of $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$-adapted processes satisfying

$$\mathbb{E}\Big[\|X^{\theta,\mathfrak{a}}\|^2 + \|Y^{\theta,\mathfrak{a}}\|^2 + \int_t^T |Z^{\theta,\mathfrak{a}}_s|^2 ds\Big] < \infty$$

and that solves the (decoupled) forward-backward SDE

$$\begin{aligned} X^{\theta,\mathfrak{a}}_s &= \mathbf{1}_{\{s\in[0,t)\}}\mathrm{x}_s + \mathbf{1}_{\{s\in[t,T]\}}\Big(x + \int_t^s \underline{\sigma}(r, \Pi^n[X^{\theta,\mathfrak{a}}], \mathfrak{a}_r) dW_r\Big), \\ Y^{\theta,\mathfrak{a}}_s &= g\big(\Pi^n[X^{\theta,\mathfrak{a}}]\big) + \int_s^T \underline{F}(r, \Pi^n[X^{\theta,\mathfrak{a}}], Y^{\theta,\mathfrak{a}}_r, Z^{\theta,\mathfrak{a}}_r) dr - \int_s^T Z^{\theta,\mathfrak{a}}_r dW_r, \ s \in [t, T]. \end{aligned}$$

It follows then from standard arguments combined with Item (iv) of Assumption 2.3 that $(t, \mathrm{x}, x) \in \Theta \times \mathbb{R}^d \longmapsto \underline{u}^n(t, \mathrm{x}, x) := \inf\{Y^{\theta,\mathfrak{a}}_t, \ \mathfrak{a} \in \mathcal{A}\}$ is a π^n-viscosity subsolution (actually solution) of (2.4) with terminal condition (2.9), see e.g. [19]. At the same time, using the Lipschitz continuity condition in Item (iv) of Assumption 2.3, it follows by standard estimates on the BSDEs (see e.g. [12]) that we can find $C > 0$, that depends only on L, such that

$$|Y^{\theta,\mathfrak{a}}_t| \leq C(1 + \|\mathrm{x}\| + |x|), \ \forall\, \mathfrak{a} \in \mathcal{A}.$$

Therefore, $\underline{u}^n$ satisfies the linear growth estimate, uniformly in $n \geq 1$. Similarly, we can exhibit a π^n-viscosity supersolution $\overline{u}^n$ of (2.4) with terminal condition (2.9) that satisfies the same growth estimate. Then, the existence of a π^n-viscosity solution v^n and the estimate (2.10) follow from Perron's method, see e.g. [6]. □

Given the existence result of Proposition 2.5, we can now provide our definition of π-approximate viscosity solutions.

Definition 2.6. *Given $g \in \mathcal{C}_L$, let v^n be the unique π^n-viscosity solution of (2.4) satisfying the terminal condition (2.9), for each $n \geq 1$. We say that v is a π-approximate viscosity solution of (2.4)-(2.5) if $(v^n)_{n\geq 1}$ admits a pointwise limit and*

$$\mathrm{v}(t,\mathrm{x}) = \lim_{n\to\infty} v^n(t,\mathrm{x},\mathrm{x}_t), \quad \textit{for all } (t,\mathrm{x}) \in \Theta.$$

2.3 Comparison principle, existence and stability properties

We provide here some results on the comparison, existence and stability of π-approximate viscosity solutions. For a π^n-viscosity solution solution $v^n : \Theta \times \mathbb{R}^d \longrightarrow \mathbb{R}$, we will consider it as a functional on Θ with

$$v^n(t,\mathrm{x}) \;:=\; v^n(t,\mathrm{x},\mathrm{x}_t), \quad \text{for all } (t,\mathrm{x}) \in \Theta.$$

2.3.1 Comparison of solutions

By the comparison principle results for π^n-viscosity solutions of Proposition 2.5, it follows immediately a comparison principle for the π-approximate viscosity solutions, by Definition 2.6. We state the result below and omit its proof.

Proposition 2.7. *For $k = 1, 2$, let $F^k : \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$ satisfy Assumption 2.3, $g^k \in \mathcal{C}_L$, and v^k be a π-approximate viscosity solution of (2.4)-(2.5) associated to (F^k, g^k) in place of (F, g). Assume that $F^1 \leq F^2$ and that $g^1 \geq g^2$. Then, $\mathrm{v}^1 \geq \mathrm{v}^2$ on Θ.*

2.3.2 A first existence result for Hamilton-Jacobi-Bellman equations

When PPDE (2.4) is in form of a path-dependent Hamilton-Jacobi-Bellman (HJB) equation, then, for each $n \geq 1$, its π^n-viscosity solution v^n is the solution to a classical HJB equation, which corresponds to the value function of a controlled diffusion processes problem in which the path of the controlled forward process X is replaced by its approximation $\Pi^n[X]$ in the coefficients. In this case, the convergence of $(v^n)_{n\geq 1}$ follows from the convergence of $\Pi^n[X]$ to X. Let us provide the following example to illustrate the idea. A more abstract result will be stated below.

Proposition 2.8. *Let $g \in \mathcal{C}_L$, and*

$$F(t,\mathrm{x},y,z,\gamma) \;=\; \sup_{a\in A}\Big(F_\circ(t,\mathrm{x},y,z,a) + \frac{1}{2}\mathrm{Tr}[\sigma\sigma^\top(t,\mathrm{x},a)\gamma]\Big), (t,\mathrm{x},y,z) \in \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d,$$

for some Borel set $A \subset \mathbb{R}^{d'}$, $d' \geq 1$, *and continuous functionals* $F_\circ : \Theta \times \mathbb{R} \times \mathbb{R}^d \times A \longrightarrow \mathbb{R}$ *and* $\sigma : \Theta \times A \longrightarrow \mathbb{R}$ *such that* $\sup_{(t,a)\in[0,T]\times A} \big|F(t, \mathbf{0}, 0, 0, a)\big| < \infty$ *and*

$$\sup_{(t,\mathrm{x},a)\in\Theta\times A} \frac{|\sigma(t,\mathrm{x},a)|}{1+|\mathrm{x}|} < \infty,$$

$\mathrm{x} \in \Omega \longmapsto \sigma(t,\mathrm{x},a)$ *is Lipschitz under* $\|\cdot\|$ *uniformly in* $(t,a) \in [0,T]\times A$.

Moreover, assume that, for all $(t,\mathrm{x},\mathrm{x}',y,y',z,z',a) \in [0,T]\times\Omega^2\times\mathbb{R}^2\times(\mathbb{R}^d)^2\times A$,

$$\big|g(\mathrm{x})-g(\mathrm{x}')\big| \leq \rho(\mathrm{x},\mathrm{x}'), \text{ and } \big|F_\circ(t,\mathrm{x},y,z,a)-F_\circ(t,\mathrm{x}',y',z',a)\big| \leq L\big(\rho_t(\mathrm{x},\mathrm{x}')+|y-y'|+|z-z'|\big).$$

Then, PPDE (2.4)-(2.5) *has a unique* π^n*-viscosity solution* v^n *for each* $n \geq 1$, *and* $v^n \longrightarrow \mathrm{v}$ *pointwise for some non-anticipative map* $\mathrm{v} : \Theta \longrightarrow \mathbb{R}$. *In particular,* v *is a* π*-approximate viscosity solution of* (2.4)-(2.5).

Proof. Let us fix $(t,\mathrm{x}) \in \Theta$. First, by Proposition 2.5, there exists a unique π^n-viscosity solution v^n to (2.4) with linear growth. Moreover, by the representation theorem of Markovian 2BSDE (see e.g. [19]), one has $v^n(t,\mathrm{x}) = \sup_{\mathfrak{a}\in\mathcal{A}} Y_t^{n,\mathfrak{a}}$, where $(X^{n,\mathfrak{a}}, Y^{n,\mathfrak{a}}, Z^{n,\mathfrak{a}})$ is the unique solution to the FBSDE

$$\begin{aligned}
X_s^{n,\mathfrak{a}} &= \mathrm{x}_{s\wedge t} + \int_t^{s\vee t} \sigma\big(r, \Pi^n\big[X^{n,\mathfrak{a}}\big], \mathfrak{a}_r\big) dW_r, \ s \in [0,T], \\
Y_s^{n,\mathfrak{a}} &= g\big(\Pi^n\big[X^{n,\mathfrak{a}}\big]\big) + \int_s^T F_\circ\big(r, \Pi^n\big[X^{n,\mathfrak{a}}\big], Y_r^{n,\mathfrak{a}}, Z_r^{n,\mathfrak{a}}\big) dr - \int_s^T Z_r^{n,\mathfrak{a}} dW_r, \ s \in [t,T],
\end{aligned}$$

and $\mathcal{A}$, W are as in Step (ii)-(iii) of the proof of Proposition 2.5. Let $X^{\mathfrak{a}}$ be the controlled diffusion process defined by

$$X_s^{\mathfrak{a}} \ = \ \mathrm{x}_{s\wedge t} + \int_t^{s\vee t} \sigma\big(r, X^{\mathfrak{a}}, \mathfrak{a}_r\big) dW_r, \ s \in [0,T].$$

It follows from standard stability result for SDEs that

$$\sup_{\mathfrak{a}\in\mathcal{A}} \mathbb{E}\Big[\big\|X^{n,\mathfrak{a}} - X^{\mathfrak{a}}\big\|^2\Big] \ \longrightarrow \ 0, \text{ as } n \longrightarrow \infty.$$

Using the concavity of the continuity modulus function $\varpi_\circ$ in both (2.2) and (2.3), one can deduce that

$$\sup_{\mathfrak{a}\in\mathcal{A}} \mathbb{E}\Big[\int_t^T \Big|F_\circ\big(r, \Pi^n\big[X^{n,\mathfrak{a}}\big], 0, 0\big) - F_\circ\big(r, X^{\mathfrak{a}}, 0, 0\big)\Big|^2 dr + \Big|g\big(\Pi^n\big[X^{n,\mathfrak{a}}\big]\big) - g\big(X^{\mathfrak{a}}\big)\Big|^2\Big] \ \longrightarrow \ 0.$$

Let $(Y^{\mathfrak{a}}, Z^{\mathfrak{a}})$ be the unique solution of the BSDE

$$Y_s^{\mathfrak{a}} \ = \ g\big(X^{\mathfrak{a}}\big) + \int_s^T F_\circ\big(r, X^{\mathfrak{a}}, Y_r^{\mathfrak{a}}, Z_r^{\mathfrak{a}}\big) dr - \int_s^T Z_r^{\mathfrak{a}} dW_r, \ s \in [t,T].$$

It follows then by standard stability result for BSDEs (see e.g. [12]) that

$$\sup_{\mathfrak{a}\in\mathcal{A}} \big|Y_t^{n,\mathfrak{a}} - Y_t^{\mathfrak{a}}\big| \ \longrightarrow \ 0.$$

Therefore, one has

$$v^n(t,\mathrm{x}) = v^n(t,\mathrm{x},\mathrm{x}_t) = \sup_{\mathfrak{a}\in\mathcal{A}} Y_t^{n,\mathfrak{a}} \;\longrightarrow\; \mathrm{v}(t,\mathrm{x}) := \sup_{\mathfrak{a}\in\mathcal{A}} Y_t^{\mathfrak{a}},$$

and hence v is a π-approximate viscosity solution of (2.4)-(2.5) by Definition 2.6. □

Remark 2.9. (i) *Similarly, one can consider Hamilton-Jacobi-Bellman-Isaac equations associated to zero-sum games. The critical point is the uniform convergence of the approximation of the controlled diffusion processes as well as the stability of the related BSDEs.*

(ii) *Notice that, in the proof of Proposition 2.8, convergence holds (to the same function) for any sequence π of time grids, and the limit does not depends on π. In other words, the exact sequence π of time grids entering in Definition 2.6 does not play any role. This will also be the case for our general existence result in Theorem 2.11 below.*

2.3.3 Existence and stability for fully nonlinear equations

In this part, we prove the existence of a unique π-approximate viscosity solution as well as a stability result for a general class of fully nonlinear equations satisfying the following structure condition.

Assumption 2.10. *There exists a continuous non-anticipative function $H : \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \longrightarrow \mathbb{R}$, together with non-anticipative maps $r : \Theta \longrightarrow \mathbb{R}$, $\mu : \Theta \longrightarrow \mathbb{R}$ and $\sigma : \Theta \longrightarrow \mathbb{S}^d$, such that $\gamma \longmapsto H(\cdot,\gamma)$ is increasing, and for all $(t,\mathrm{x},y,z,\gamma) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$,*

$$F(t,\mathrm{x},y,z,\gamma) \;=\; H(t,\mathrm{x},y,z,\gamma) + r(t,\mathrm{x})y + \mu(t,\mathrm{x})\cdot z + \frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top(t,\mathrm{x})\gamma\big].$$

Moreover, there is a constant $L>0$ such that for all $t\in[0,T]$ and $(\mathrm{x},y,z,\gamma),(\mathrm{x}',y',z',\gamma') \in \Omega\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$, one has $|r(t,\mathrm{x})| \le L$ and

$$\big|g(\mathrm{x}) - g(\mathrm{x}')\big| \;\le\; L\rho(\mathrm{x},\mathrm{x}'), \tag{2.11}$$

$$\big|r(t,\mathrm{x}) - r(t,\mathrm{x}')\big| + \big|\mu(t,\mathrm{x}) - \mu(t,\mathrm{x}')\big| + \big|\sigma(t,\mathrm{x}) - \sigma(t,\mathrm{x}')\big| \;\le\; L\min(\rho_t(\mathrm{x},\mathrm{x}'), \|\mathrm{x}-\mathrm{x}'\|), \tag{2.12}$$

$$\big|H(t,\mathrm{x},y,z,\gamma) - H(t,\mathrm{x}',y',z',\gamma')\big| \;\le\; L\Big(\rho_t(\mathrm{x},\mathrm{x}') + |y-y'| + |z-z'| + |\gamma-\gamma'|\Big). \tag{2.13}$$

The above turns out to be enough not only to prove existence and uniqueness but also that the solution does not depend on the particular sequence $\pi = (\pi^n)_{n\ge1}$ of time grids used in Definition 2.6 (see also Remark 2.9). Recall that $\varpi_\circ : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a modulus of continuity, which is concave and satisfies $\varpi_\circ(x) \ge cx$, and $\varpi'_\circ(x) := \sqrt{\varpi_\circ(x^2)}$, for all $x \ge 0$.

Theorem 2.11. *Let Assumptions 2.3 and 2.10 hold true. Then:*

(i) *PPDE* (2.4)-(2.5) *has a unique π-approximate viscosity solution* v.

(ii) *For all $K>0$, there exists a constant $C_K>0$, that depends only on K and L, such that*

$$|\mathrm{v}(t,\mathrm{x}') - \mathrm{v}(t,\mathrm{x})| \le C_K\ \varpi'_\circ\big(\rho_t(\mathrm{x},\mathrm{x}')\big) \quad \text{and} \quad |\mathrm{v}(t',\mathrm{x}_{t\wedge}) - \mathrm{v}(t,\mathrm{x})| \le C_K\varpi'_\circ\big(\varpi'_\circ\big(|t'-t|^{\frac12}\big)\big), \tag{2.14}$$

for all $0 \le t \le t' \le T$ *and* $\mathrm{x}, \mathrm{x}' \in \Omega$ *satisfying* $\|\mathrm{x}\| \vee \|\mathrm{x}'\| \le K$.

(iii) *If* π' *is another increasing sequence of discrete time grids and* v' *is the* π'*-approximate viscosity solution of* (2.4)-(2.5)*, then* $\mathrm{v}' = \mathrm{v}$ *on* $[0,T] \times \Omega$.

Remark 2.12. *Let the conditions in Theorem 2.11 hold. As a by-product of* (2.34) *in the proof below, for all* $K > 0$*, there exists a constant* $C_K > 0$ *such that*

$$\big|v^n(t,\mathrm{x},\mathrm{x}_t) - \mathrm{v}(t,\mathrm{x})\big| \;\le\; C_K\; \varpi'_\circ\Big(\rho_t\big(\Pi^n_t[\mathrm{x}],\mathrm{x}_{t\wedge}\big) + |\pi^n|^{\frac{1}{4}}\Big), \tag{2.15}$$

whenever $\|\mathrm{x}\| \le K$.

Before to prove the above, let us state immediately the following stability result.

Proposition 2.13. *Let* $(F_k, g_k)_{k\ge 0}$ *be a sequence satisfying the assumptions in Theorem 2.11, uniformly in* $k \ge 0$*, and such that*

$$\big(F_k(t_k,\mathrm{x}_k,y_k,z_k,\gamma_k), g_k(\mathrm{x}_k)\big) \;\longrightarrow\; \big(F_0(t,\mathrm{x},y,z,\gamma), g_0(\mathrm{x})\big) \quad \textit{as } k \longrightarrow \infty,$$

whenever $(t_k,\mathrm{x}_k,y_k,z_k,\gamma_k) \longrightarrow (t,\mathrm{x},y,z,\gamma) \in \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$*. Let* v_k *be a* π*-approximate viscosity solution associated to* (F_k, g_k) *for each* $k \ge 0$*. Then, there exists a subsequence* $(k_m)_{m\ge 1}$ *such that* $(\mathrm{v}_{k_m})_{m\ge 1}$ *converges pointwise to* v_0.

Proof. For each $k \ge 0$, let v^n_k be the π_n-viscosity solution associated to (F_k, g_k). Then, for each fixed $n \ge 1$, it follows by stability of the viscosity solutions of classical PDEs that $v^n_k \longrightarrow v^n_0$ pointwise as $k \longrightarrow \infty$.

In view of Remark 2.1, one can find a countable subset $\Theta_\circ$ of Θ such that, for all $(t,\mathrm{x}) \in \Theta$, there exists a sequence $(t^i,\mathrm{x}^i)_{i\ge 1} \subset \Theta_\circ$ satisfying

$$\rho_{t_i}\big(\mathrm{x}^i,\mathrm{x}\big) + |t^i - t|^{1/2} \;\longrightarrow\; 0. \tag{2.16}$$

Then, by a standard diagonalization argument, we can find a subsequence $(k_n)_{n\ge 1}$ such that $|v^n_{k_n} - v^n_0| \longrightarrow 0$ pointwise on the countable set $\Theta_\circ$. On the other hand, (2.15) implies that, for each $(t,\mathrm{x}) \in \Theta_\circ$, there is some constant $C > 0$ such that

$$|\mathrm{v}_{k_n} - \mathrm{v}_0|(t,\mathrm{x}) \;\le\; |v^n_{k_n} - v^n_0|(t,\mathrm{x},\mathrm{x}_t) + 2C\varpi'_\circ\Big(\rho_t\big(\Pi^n_t[\mathrm{x}],\mathrm{x}\big) + |\pi^n|^{\frac{1}{4}}\Big), \text{ for all } n \ge 1.$$

Therefore, by Remark 2.1, one has $\mathrm{v}_{k_n} \longrightarrow \mathrm{v}$ pointwise on $\Theta_\circ$. Finally, by (2.14) and (2.16), it follows that $\mathrm{v}_{k_n} \longrightarrow \mathrm{v}_0$ pointwise on Θ. □

We now complete the proof of Theorem 2.11. The key ingredient is to provide a uniform estimate on the difference $v^m - v^n$ when $m, n \longrightarrow \infty$, where v^m (resp. v^n) is the π^m-(resp. π^n-)viscosity solution of (2.4) with terminal condition (2.9).

By (2.6) and Definition 2.2, it is clear that $v^n(t,\mathrm{x},x)$ depends only on $(\mathrm{x}_{t^n_j})_{j=0,\cdots,i}$ and x when $t \in [t^n_i, t^n_{i+1})$. For this reason, for each $n \ge 1$ and $t \in [t^n_i, t^n_{i+1})$, we introduce $v^n_i : [t^n_i, t^n_{i+1}) \times (\mathbb{R}^d)^{i+1} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ defined by

$$v^n_i\big(t,[\mathrm{x}]^n_i,x\big) \;:=\; v^n\big(t,\Pi^n[\mathrm{x}]_{t^n_i\wedge},x\big), \tag{2.17}$$

and $F_i^n : [t_i^n, t_{i+1}^n) \times (\mathbb{R}^d)^{i+1} \times \mathbb{R}^d \to \mathbb{R}$ as well as $g^n : (\mathbb{R}^d)^{n+1} \to \mathbb{R}$ defined as

$$F_i^n\big(t, [\mathrm{x}]_i^n, \cdot\big) \;:=\; F\big(t, \Pi^n[\mathrm{x}]_{t_i^n\wedge}, \cdot\big), \quad g^n\big([\mathrm{x}]_n^n\big) \;:=\; g\big(\Pi^n[\mathrm{x}]\big). \tag{2.18}$$

We similarly define r_i^n, μ_i^n, σ_i^n, i.e.

$$r_i^n\big(t, [\mathrm{x}]_i^n\big) \;:=\; r\big(t, \Pi^n[\mathrm{x}]_{t_i^n\wedge}\big), \;\; \mu_i^n\big(t, [\mathrm{x}]_i^n\big) \;:=\; \mu\big(t, \Pi^n[\mathrm{x}]_{t_i^n\wedge}\big), \;\; \sigma_i^n\big(t, [\mathrm{x}]_i^n\big) \;:=\; \sigma\big(t, \Pi^n[\mathrm{x}]_{t_i^n\wedge}\big).$$

Then, by Definition 2.2, $(t,x) \longmapsto v_i^n(t, [\mathrm{x}]_i^n, x)$ is the unique viscosity solution of the classical PDE

$$-\partial_t v_i^n\big(t, [\mathrm{x}]_i^n, x\big) - F_i^n\big(t, [\mathrm{x}]_i^n, v_i^n, Dv_i^n, D^2 v_i^n\big) = 0, \quad (t,x) \in [t_i^n, t_{i+1}^n) \in \mathbb{R}^d, \tag{2.19}$$

with terminal condition

$$v_i^n(t_{i+1}^n, [\mathrm{x}]_i^n, x) = v_{i+1}^n(t_{i+1}^n, ([\mathrm{x}]_i^n, x), x), \;\; i = 0, \cdots, n-2, \quad v_n^n\big(t_n^n, [\mathrm{x}]_n^n\big) = g^n\big([\mathrm{x}]_n^n\big),$$

where Dv_i^n (resp. $D^2 v_i^n$) represents the gradient (resp. Hessian matrix) of $x \longmapsto v_i^n(t, [\mathrm{x}]_i^n, x)$. Given $1 \le m \le n$, let us denote

$$d(\pi^m, \pi^n) \;:=\; \max_{t \in \pi^n} \; \min_{t' \in \pi^m \cap [t,T]} \; |t - t'|.$$

We first study the stability of $(v^n)_{n \ge 1}$ with respect to n and its space arguments.

Lemma 2.14. *Let the conditions of Theorem 2.11 hold, and let v^n be the π^n-viscosity solution of (2.4) satisfying the terminal condition (2.9) for each $n \ge 1$. Then, for all $K > 0$, there exists a constant $C_K > 0$, such that, for all $m, n \ge 1$, $t \in [0,T]$, and $\mathrm{x}, \mathrm{x}' \in \Omega$ satisfying $\|\mathrm{x}_{t\wedge}\| + \|\mathrm{x}'_{t\wedge}\| \le K$,*

$$\big|v^m(t, \mathrm{x}, \mathrm{x}_t) - v^n(t, \mathrm{x}', \mathrm{x}'_t)\big| \;\le\; C_K \varpi'_\circ\Big(\rho_t\big(\Pi^m[\mathrm{x}], \Pi^n[\mathrm{x}']\big) + |\mathrm{x}_t - \mathrm{x}'_t| + d(\pi^m, \pi^n)^{\frac{1}{4}}\Big). \tag{2.20}$$

Proof. We restrict to the one dimensional case $d = 1$ for ease of notations. Given $m \le n$, as $\pi^m \subset \pi^n$, the vector $[\mathrm{x}]_n^n$ contains all the information of $[\mathrm{x}]_m^m$. We will then consider a functional of $[\mathrm{x}]_\cdot^m$ as a functional of $[\mathrm{x}]_\cdot^n$. For this purpose, let us introduce, for all $i \le n-1$, $t \in [t_i^n, t_{i+1}^n)$, $\mathrm{x} \in \Omega$ and $x \in \mathbb{R}$,

$$v_i^{m,n}\big(t, [\mathrm{x}]_i^n, x\big) \;:=\; v_{I_m^n(i)}^m\big(t, [\mathrm{x}]_{I_m^n(i)}^m, x\big), \tag{2.21}$$

where $I_m^n(i) := \max\{j \;\; : t_j^m \le t_i^n\}$ for all $i = 0, 1, \cdots, n$. Similarly, one defines the functionals

$$F_i^{m,n}, \; H_i^{m,n}, \; r_i^{m,n}, \; \mu_i^{m,n}, \; \sigma_i^{m,n}, \; g^{m,n}.$$

(i) Given $i < n$, $(t, x, x') \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^2$ and $\mathrm{x}, \mathrm{x}' \in \Omega$, set

$$w_i^{m,n}\big(t, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, x, x'\big) := v_i^n\big(t, [\mathrm{x}]_i^n, x\big) - v_i^{m,n}\big(t, [\mathrm{x}']_i^n, x'\big).$$

We first show that $w_i^{m,n}(\cdot, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, \cdot, \cdot)$ is a viscosity subsolution of a HJB equation. Let $\phi : [t_i^n, t_{i+1}^n] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a smooth function and $(\hat{t}, \hat{x}, \hat{x}') \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^2$ be such that

$$\begin{aligned} 0 \;&=\; \max_{(t,x,x') \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^2} \big(w_i^{m,n}\big(t, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, x, x'\big) - \phi(t, x, x')\big) \\ &=\; w_i^{m,n}\big(\hat{t}, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, \hat{x}, \hat{x}'\big) - \phi(\hat{t}, \hat{x}, \hat{x}'). \end{aligned}$$

Recall that v_i^n is a viscosity (super-)solution of (2.8) with generator F^n and $v_i^{m,n}$ is a viscosity (sub-)solution of (2.8) with generator $F^{m,n}$. Then, it follows from Ishii's lemma (see e.g. [6, Theorem 3.2]) that, for all $\varepsilon > 0$, one can find $\gamma_\varepsilon, \gamma'_\varepsilon \in \mathbb{R}$ such that

$$\begin{pmatrix} \gamma_\varepsilon & 0 \\ 0 & -\gamma'_\varepsilon \end{pmatrix} \le D^2\phi(\hat{t},\hat{x},\hat{x}') + \varepsilon \left(D^2\phi(\hat{t},\hat{x},\hat{x}')\right)^2, \tag{2.22}$$

and

$$\begin{aligned} 0 \;\ge\; & - \partial_t\phi(\hat{t},\hat{x},\hat{x}') - F^n\big(\hat{t},[\mathrm{x}]_i^n, v_i^n\big(\hat{t},[\mathrm{x}]_i^n,\hat{x}\big), \partial_x\phi(\hat{t},\hat{x},\hat{x}'),\gamma_\varepsilon\big) \\ & + F^{m,n}\big(\hat{t},[\mathrm{x}]_i^n, v_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n,\hat{x}'\big), -\partial_{x'}\phi(\hat{t},\hat{x},\hat{x}'),\gamma'_\varepsilon\big). \end{aligned} \tag{2.23}$$

We now estimate the r.h.s. of (2.23). By (2.13) and (2.22), using the notation $\theta = (\hat{t},[\mathrm{x}]_i^n,\hat{x},\hat{x}',\gamma_\varepsilon,\gamma'_\varepsilon)$,

$$\begin{aligned} \Delta H_\varepsilon^{m,n} \;:=\; & - H^n\big(\hat{t},[\mathrm{x}]_i^n, v_i^n\big(\hat{t},[\mathrm{x}]_i^n,\hat{x}\big), \partial_x\phi(\hat{t},\hat{x},\hat{x}'),\gamma_\varepsilon\big) \\ & + H^{m,n}\big(\hat{t},[\mathrm{x}']_i^n, v_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n,\hat{x}'\big), -\partial_{x'}\phi(\hat{t},\hat{x},\hat{x}'),\gamma'_\varepsilon\big) \\ \ge\; & - L\Big(\rho_t(\Pi^n_{t_i^n}(\mathrm{x}),\Pi^m_{t_i^n}(\mathrm{x}')) + |\phi(\hat{t},\hat{x},\hat{x}')| + |\partial_x\phi(\hat{t},\hat{x},\hat{x}') + \partial_{x'}\phi(\hat{t},\hat{x},\hat{x}')|\Big) \\ & - \partial_\gamma H^n(\theta)\Big(\Delta\phi(\hat{t},\hat{x},\hat{x}') + \varepsilon\Delta^2\phi(\hat{t},\hat{x},\hat{x}')\Big), \end{aligned}$$

for some functional $\partial_\gamma H^n \ge 0$ bounded by L, and where

$$\Delta\phi := \partial^2_{xx}\phi + 2\partial^2_{xx'}\phi + \partial^2_{x'x'}\phi \text{ and } \Delta^2\phi := (1,1)\left(D^2\phi\right)^2\begin{pmatrix}1\\1\end{pmatrix}.$$

This implies that

$$\begin{aligned} & \liminf_{\varepsilon\downarrow 0} \Delta H_\varepsilon^{m,n} \\ \ge\; & \min_{(b^1,b^2,b^3)\in B}\Big(-b^1\phi(\hat{t},\hat{x},\hat{x}') - b^2\big(\partial_x\phi(\hat{t},\hat{x},\hat{x}') + \partial_{x'}\phi(\hat{t},\hat{x},\hat{x}')\big) - |b^3|^2\Delta\phi(\hat{t},\hat{x},\hat{x}')\Big) \\ & - L\rho_t\big(\Pi^n_{t_i^n}(\mathrm{x}),\Pi^m_{t_i^n}(\mathrm{x}')\big), \end{aligned} \tag{2.24}$$

with $B := [-L,L]^2 \times [0,\sqrt{L}]$.

Similarly, by (2.10) and (2.12),

$$\begin{aligned} & - r_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)v_i^n\big(\hat{t},[\mathrm{x}]_i^n,\hat{x}\big) + r_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n\big)v_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n,\hat{x}'\big) \\ \ge\; & - r_i^n\big(\hat{t},[\mathrm{x}']_i^n\big)\phi\big(\hat{t},\hat{x},\hat{y}\big) \;-\; \Delta r_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)v_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n,\hat{x}'\big), \end{aligned}$$

where

$$\Delta r_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n) \;:=\; r_i^{m,n}(\hat{t},[\mathrm{x}']_i^n) - r_i^n(\hat{t},[\mathrm{x}]_i^n)$$

satisfies

$$|\Delta r_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)| \;\le\; C\delta_{\hat{t}}^{m,n}(1 + \|\mathrm{x}'\| + |\hat{x}'|), \tag{2.25}$$

for some constant $C > 0$ that depends only on L, with

$$\delta_{\hat{t}}^{m,n} := L\min\big(\rho_{\hat{t}}\big(\Pi^n[\mathrm{x}],\Pi^m[\mathrm{x}']\big), \big\|\Pi^n[\mathrm{x}]_{\hat{t}\wedge} - \Pi^m[\mathrm{x}']_{\hat{t}\wedge}\big\|\big). \tag{2.26}$$

Further, one has

$$\begin{aligned} & -\mu_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)\partial_x\phi\big(\hat{t},\hat{x},\hat{x}'\big)+\mu_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n\big)\big(-\partial_{x'}\phi\big(\hat{t},\hat{x},\hat{x}'\big)\big) \\ = \; & -\mu_i^n(\hat{t},[\mathrm{x}]_i^n)\big[\partial_x\phi\big(\hat{t},\hat{x},\hat{x}'\big)+\partial_{x'}\phi\big(\hat{t},\hat{x},\hat{x}'\big)\big]-\Delta\mu_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\partial_{x'}\phi\big(\hat{t},\hat{x},\hat{x}'\big), \end{aligned}$$

where

$$\Delta\mu_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n) \;:=\; \mu_i^{m,n}(\hat{t},[\mathrm{x}']_i^n)-\mu_i^n(\hat{t},[\mathrm{x}]_i^n),$$

satisfies

$$\big|\Delta\mu_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n))\big| \;\le\; \delta_{\hat{t}}^{m,n}. \tag{2.27}$$

Finally, by (2.22),

$$\begin{aligned} & \liminf_{\varepsilon\downarrow 0}\big(-\sigma_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)^2\gamma_\varepsilon+\sigma_i^{m,n}\big(\hat{t},[\mathrm{x}']_i^n\big)\big)^2(-\gamma'_\varepsilon)\big) \\ \ge \; & -\sigma_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)^2\partial_{xx}^2\phi\big(\hat{t},\hat{x},\hat{x}'\big)-\big[\sigma_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)+\Delta\sigma_i^{m,n}(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\big]^2\partial_{x'x'}^2\phi\big(\hat{t},\hat{x},\hat{x}'\big) \\ & -2\sigma_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)\big[\sigma_i^n\big(\hat{t},[\mathrm{x}]_i^n\big)+\Delta\sigma_i^{m,n}(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\big]\partial_{xx'}^2\phi\big(\hat{t},\hat{x},\hat{x}'\big), \end{aligned}$$

where

$$\Delta\sigma_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n) \;:=\; \sigma_i^{m,n}(\hat{t},[\mathrm{x}']_i^n)-\sigma_i^n(\hat{t},[\mathrm{x}]_i^n),$$

so that

$$\big|\Delta\sigma_i^{m,n}(\hat{t},[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\big| \;\le\; \delta_{\hat{t}}^{m,n}. \tag{2.28}$$

By arbitrariness of $\varepsilon>0$, it follows that $w_i^{m,n}\big(\cdot,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n,\cdot,\cdot\big)$ is a viscosity subsolution on $[t_i^n,t_{i+1}^n)\times\mathbb{R}^2$ of

$$0 \;\ge\; -\;\partial_t\phi(t,x,x') \;-\; \max_{b\in B}\mathcal{L}_i^b[t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n]\phi(t,x,y)-f_i^n(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n,x),$$

in which

$$f_i^n(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n,x) \;:=\; (C+1)\big(1+\|\mathrm{x}'\|+|x'|\big)\delta_t^{m,n}, \tag{2.29}$$

and

$$\begin{aligned} & \mathcal{L}_i^b[t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n]\phi(t,x,x') \\ & :=(b^1+r_i^n)\big(t,[\mathrm{x}]_i^n\big)\phi\big(t,x,x'\big)+\big(b^2+\mu_i^n(t,[\mathrm{x}]_i^n)\big)\big(\partial_x\phi(t,x,x')+\partial_{x'}\phi(t,x,x')\big) \\ & +\Delta\mu_i^{m,n}(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\partial_x\phi\big(t,x,x'\big)+\frac{1}{2}|\sqrt{2}b^3|^2\Delta\phi(t,x,x')+\frac{1}{2}\sigma_i^n\big(t,[\mathrm{x}]_i^n\big)^2\partial_{xx}^2\phi\big(t,x,x'\big) \\ & +\frac{1}{2}\big[\sigma_i^n\big(t,[\mathrm{x}]_i^n\big)+\Delta\sigma_i^{m,n}(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\big]^2\partial_{x'x'}^2\phi\big(t,x,x'\big) \\ & +\sigma_i^n\big(t,[\mathrm{x}]_i^n\big)\big[\sigma_i^n\big(t,[\mathrm{x}]_i^n\big)+\Delta\sigma_i^{m,n}(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n)\big]\partial_{xx'}^2\phi\big(t,x,x'\big). \end{aligned}$$

Moreover, it satisfies the boundary condition

$$\lim_{t\uparrow t_{i+1}^n} w_i^{m,n}\big(t,[\mathrm{x}]_i^n,[\mathrm{x}']_i^n,x,x'\big) \;=\; w_{i+1}^{m,n}\big(t_{i+1}^n,([\mathrm{x}]_i^n,x),([\mathrm{x}']_i^n,x'),x,x'\big).$$

(ii) Given $(t,x,x')\in[t_i^n,t_{i+1}^n)\times\mathbb{R}^2$, $i<n$, and $\beta\in\mathcal{B}$, the collection of predictable B-valued processes, on some probability space endowed with the augmented filtration of a

two dimensional Brownian motion $W = (W^1, W^2)$, let us now define the two processes X^β and X'^β by

$$\begin{aligned} X^\beta =& [\mathrm{x} \boxplus_t x]_{t\wedge} + \int_t^\cdot [\beta_s^2 + \mu(s, \Pi^n[X^\beta])]ds + \int_t^\cdot \sqrt{2}\beta_s^3 dW_s^1 + \int_t^\cdot \sigma(s, \Pi^n[X^\beta])dW_s^2 \\ X'^\beta =& [\mathrm{x}' \boxplus_t x']_{t\wedge} + \int_t^\cdot dX_r^\beta + \int_t^\cdot \Delta\mu(s, \Pi^n[X^\beta], \Pi^m[X'^\beta])ds \\ &+ \int_t^\cdot \Delta\sigma(s, \Pi^n[X^\beta], \Pi^m[X'^\beta])dW_s^2. \end{aligned}$$

Assume that $\|\mathrm{x}\| \vee \|\mathrm{x}'\| \vee |x| \vee |x'| \leq K$. Then, using (2.12), (2.26), (2.27), (2.28), we can find C_K, that only depends on L and K, such that

$$\mathbb{E}\Big[\big\|\Pi^n\big[X^\beta\big]_{t\vee} - \Pi^m\big[X^\beta\big]_{t\vee}\big\|^2\Big] \;\leq\; C_K d(\pi^m, \pi^n)^{\frac{1}{2}}, \;\; \mathbb{E}\Big[\big\|X^\beta\big\|^2 + \big\|X'^\beta\big\|^2\Big] \;\leq\; C_K$$

and

$$\mathbb{E}\big[\big\|X^\beta_{t\vee} - X'^\beta_{t\vee}\big\|^2\big]^{\frac{1}{2}} \;\leq\; C_K\big(\rho_t\big(\Pi^n[\mathrm{x}], \Pi^m[\mathrm{x}']\big) + |x - x'| + d(\pi^m, \pi^n)^{\frac{1}{4}}\big). \tag{2.30}$$

By the Feynman-Kac formula, the above viscosity subsolution property, together with (2.29), (2.25), (2.11) and the fact that r is bounded by Assumption 2.10, we deduce that

$$w_i^{m,n}\big(t, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, x, x'\big) \;\leq\; \sup_{\beta\in\mathcal{B}} \mathbb{E}\left[e^{\int_t^T k_s^\beta ds} L\rho(\Pi^n_{t_n^n}[X^\beta], \Pi^m_{t_n^m}[X'^\beta]) + \int_t^T e^{\int_t^r k_s^\beta ds}\zeta_r^\beta dr\right],$$

where, for all $\beta \in \mathcal{B}$, k^β and ζ^β are predictable processes such that $|k^\beta| \leq L$ and

$$|\zeta_r^\beta| \;\leq\; L(1 + C_K)\; \rho_r\big(\Pi^n[X^\beta], \Pi^m[X'^\beta]\big)\big(1 + 2\|X'^\beta\|\big).$$

Recalling (2.30) and that the function $\varpi_\circ$ entering the definition of ρ is concave, one can check that, for $r \geq t$,

$$\rho_r\big(\Pi^n[X^\beta], \Pi^m[X'^\beta]\big) \leq 2\rho_t\big(\Pi^n[\mathrm{x}], \Pi^m[\mathrm{x}']\big) + 2\rho_r\big(\Pi^n[X^\beta]_{t\vee}, \Pi^m[X'^\beta]_{t\vee}\big),$$

and, by (2.30) and the concavity of ϖ_0,

$$\begin{aligned} \mathbb{E}\Big[\rho_r\big(\Pi^n[X^\beta]_{t\vee}, \Pi^m[X'^\beta]_{t\vee}\big)^2\Big] &\leq C\varpi_\circ\big(\mathbb{E}\big[\big\|\Pi^n[X^\beta]_{t\vee} - \Pi^m[X'^\beta]_{t\vee}\big\|^2\big]\big) \\ &\leq C\varpi'_\circ\Big(\rho_t\big(\Pi^n[\mathrm{x}], \Pi^m[\mathrm{x}']\big) + |x - x'| + d(\pi^m, \pi^n)^{\frac{1}{4}}\Big)^2. \end{aligned}$$

It follows that

$$w_i^{m,n}\big(t, [\mathrm{x}]_i^n, [\mathrm{x}']_i^n, x, x'\big) \leq C'_K\varpi'_\circ\Big(\rho_t\big(\Pi^n[\mathrm{x}], \Pi^m[\mathrm{x}']\big) + |x - x'| + d(\pi^m, \pi^n)^{\frac{1}{4}}\Big),$$

for some $C'_K > 0$ that only depends on K.

(iii) Finally, the corresponding lower-bound is derived similarly after exhibiting a viscosity supersolution property by the same arguments. □

We now discuss the stability with respect to the time argument.

Lemma 2.15. *Let the conditions of Theorem 2.11 hold and v^n be the π^n-viscosity solution of (2.4) satisfying the terminal condition (2.9) for each $n \geq 1$. Then, for all $K > 0$, there exists a constant $C_K > 0$, such that, for all $n \geq 1$, $0 \leq t \leq t' \leq T$ and $\|\mathrm{x}\| \leq K$,*

$$\big|v^n(t', \mathrm{x}_{t\wedge}, \mathrm{x}_t) - v^n(t, \mathrm{x}, \mathrm{x}_t)\big| \;\leq\; C_K \varpi'_\circ\big(\varpi'_\circ\big(|t'-t|^{\frac{1}{2}}\big)\big). \tag{2.31}$$

Proof. We use the notations introduced in the proof of Proposition 2.5. Fix $t' \leq T$. Given $\theta := (t, \mathrm{x}, x) \in [0, t'] \times \Omega \times \mathbb{R}^d$, and $\mathfrak{a} \in \mathcal{A}$, let $(X^{\theta,\mathfrak{a}}, Y^{\theta,\mathfrak{a}}, Z^{\theta,\mathfrak{a}})$ be the unique triplet of $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$-adapted processes satisfying

$$\mathbb{E}\Big[\|X^{\theta,\mathfrak{a}}\|^2 + \|Y^{\theta,\mathfrak{a}}\|^2 + \int_t^{t'} |Z_s^{\theta,\mathfrak{a}}|^2 ds\Big] < \infty$$

and that solves the (decoupled) forward-backward SDE

$$\begin{aligned} X_s^{\theta,\mathfrak{a}} &= \mathbf{1}_{\{s\in[0,t)\}}\mathrm{x}_s + \mathbf{1}_{\{s\in[t,T]\}}\Big(x + \int_t^s \underline{\sigma}(r, \Pi^n[X^{\theta,\mathfrak{a}}], \mathfrak{a}_r) dW_r\Big), \\ Y_s^{\theta,\mathfrak{a}} &= v^n\big(t', X^{\theta,\mathfrak{a}}\big) + \int_s^{t'} \underline{F}(r, \Pi^n[X^{\theta,\mathfrak{a}}], Y_r^{\theta,\mathfrak{a}}, Z_r^{\theta,\mathfrak{a}}) dr - \int_s^{t'} Z_r^{\theta,\mathfrak{a}} dW_r, \end{aligned}$$

for $t \leq s \leq t'$. Then, Item (iv) of Assumption 2.3 implies that

$$v^n(t, \mathrm{x}) \geq \inf_{\mathfrak{a}\in\mathcal{A}} Y^{\theta,\mathfrak{a}}, \tag{2.32}$$

see the proof of Proposition 2.5. By the Lipschitz property of $\underline{F}$, one can find a predictable process $(k^{\mathfrak{a}}, \zeta^{\mathfrak{a}})$ with values in $[-L, L]^2$ such that

$$\underline{F}(r, \Pi^n[X^{\theta,\mathfrak{a}}], Y_r^{\theta,\mathfrak{a}}, Z_r^{\theta,\mathfrak{a}}) \;=\; \underline{F}(r, \Pi^n[X^{\theta,\mathfrak{a}}], 0, 0) + k_r^{\mathfrak{a}} Y_r^{\theta,\mathfrak{a}} + \zeta_r^{\mathfrak{a}} Z_r^{\theta,\mathfrak{a}}, \quad t \leq r \leq t'.$$

Let $\mathbb{Q}^{\mathfrak{a}} \sim \mathbb{P}$ be such that $W^{\mathfrak{a}} = W - \int_0^\cdot \zeta_r^{\mathfrak{a}} dr$ is a $\mathbb{Q}^{\mathfrak{a}}$-Brownian motion. Then,

$$Y_t^{\theta,\mathfrak{a}} = \mathbb{E}^{\mathbb{Q}^{\mathfrak{a}}}\left[e^{\int_t^{t'} k_s^{\mathfrak{a}} ds} v^n\big(t', X^{\theta,\mathfrak{a}}\big) + \int_t^{t'} e^{\int_t^r k_s^{\mathfrak{a}} ds} \underline{F}(r, \Pi^n[X^{\theta,\mathfrak{a}}], 0, 0) dr\right].$$

We can then find $C'_K > 0$, that depends only on L and K, such that, for all $\|\mathrm{x}\| \leq K$, one has $\mathbb{E}^{\mathbb{Q}^{\mathfrak{a}}}[\|X^{\theta,\mathfrak{a}}_{(t'\vee t)\wedge} - \mathrm{x}_t\|^2]^{\frac{1}{2}} \leq C'_K (t'-t)^{\frac{1}{2}}$ and

$$Y_t^{\theta,\mathfrak{a}} - v^n(t', \mathrm{x}_{t\wedge}) \geq \mathbb{E}^{\mathbb{Q}^{\mathfrak{a}}}\left[e^{\int_t^{t'} k_s^{\mathfrak{a}} ds} v^n\big(t', X^{\theta,\mathfrak{a}}\big) - v^n(t', \mathrm{x}_{t\wedge}) - (t'-t) C'_K (1 + \|\mathrm{x}\|)\right].$$

Recall that the function $\varpi_\circ$ (used in the definition of $\varpi'_\circ$ in (2.1)) is concave, and that, for some constant $c > 0$, one has $\varpi_\circ(x) \geq cx$ for all $x \geq 0$. One can then apply Lemma 2.14 and (2.10) to deduce that there exists a constant $C''_K > 0$, that depends only on L and K, such that, for all $\|\mathrm{x}\| \leq K$,

$$Y_t^{\theta,\mathfrak{a}} - v^n(t', \mathrm{x}_{t\wedge}) \;\geq\; -C''_K\Big(\big|t'-t\big| + e^{C''_K|t'-t|} - 1 + \varpi'_\circ\big(\varpi'_\circ(|t'-t|^{\frac{1}{2}})\big)\Big).$$

Then by (2.32), there exists a constant $C_K > 0$ such that

$$v^n(t, \mathrm{x}) - v^n(t', \mathrm{x}_{t\wedge}) \;\geq\; -C_K \varpi'_\circ\big(\varpi'_\circ\big(|t'-t|^{\frac{1}{2}}\big)\big), \quad \text{for all } \|x\| \leq K.$$

A similar upper-bound is obtained by using the second inequality in Item (iv)-(b) of Assumption 2.3. □

Remark 2.16. *If we can choose $\varpi_\circ(x) := x$ for all $x \geq 0$, the estimations in Lemmas 2.14 and 2.15 imply that the function v^n is locally Lipschitz in $\Pi^n_t[\mathrm{x}]$ and $1/2$-Hölder in t.*

We can now conclude the proof of Theorem 2.11.

Proof of Theorem 2.11. (i) Recall that $(\pi^n)_{n\geq 1}$ is a sequence of discrete time grid on $[0,T]$, let v^n be a π^n-viscosity solution of (2.4) satisfying the terminal condition (2.9). Notice that, by the definition of ρ_t, it is easy to check that

$$\rho_t\big(\Pi^m[\mathrm{x}],\Pi^n[\mathrm{x}']\big) + |\mathrm{x}_t - \mathrm{x}'_t| \;\leq\; C\rho_t\big(\Pi^m_t[\mathrm{x}],\Pi^n_t[\mathrm{x}]\big), \tag{2.33}$$

for some constant $C > 0$ independent of $(t,\mathrm{x},\mathrm{x}')$. Then, for each $(t,\mathrm{x}) \in [0,T]\times\Omega$, it follows by Lemma 2.14 that, for some constant $C > 0$,

$$\big|v^m(t,\mathrm{x},\mathrm{x}_t) - v^n(t,\mathrm{x},\mathrm{x}_t)\big| \;\leq\; C\varpi'_\circ\Big(\rho_t\big(\Pi^m_t[\mathrm{x}],\Pi^n_t[\mathrm{x}]\big) + d(\pi^m,\pi^n)^{\frac{1}{4}}\Big), \quad \text{for all } n \geq m \geq 1.$$

Therefore, $(v^n(t,\mathrm{x},\mathrm{x}_t))_{n\geq 1}$ is a Cauchy sequence so that $v^n(t,\mathrm{x},\mathrm{x}_t) \longrightarrow \mathrm{v}(t,\mathrm{x})$ pointwise for some nonanticipative functional $\mathrm{v} : [0,T]\times\Omega \longrightarrow \mathbb{R}$. By Definition 2.6, v is a π-approximate viscosity solution of (2.4)-(2.5). Further, it is in fact the unique π-approximate viscosity solution, by the comparison principle in Proposition 2.7.

(ii) By the estimation in (2.20) with $m = n$, (2.33), and the estimation in (2.31), the limit function v satisfies (2.14).

(iii) We finally prove that the notion of approximate viscosity solution does not depend on the special choice of the increasing sequence of time grids. Let us fix two increasing sequences of time grids $\pi = (\pi^n)_{n\geq 1}$ and $\pi' = (\pi'^n)_{n\geq 1}$, and let v^n be the π^n-viscosity solution, v'^n be the π'^n-viscosity solution. We then define $\tilde\pi = (\tilde\pi^n)_{n\geq 1}$ by $\tilde\pi^k := \pi^k$ for $k \leq n$, and $\tilde\pi^k := \pi^n \cup \pi'^{k-n}$ for $k > n$, and let $\tilde v^k$ be the corresponding $\tilde\pi^k$-viscosity solution. In particular, one has $\tilde\pi^{2n} := \pi^n \cup \pi'^n$, $\tilde v^n = v^n$, and that $\tilde v^{2n}$ is the $\pi^n \cup \pi'^n$-viscosity solution. Then, it follows from (2.20) and (2.33) that, for some $C > 0$, independent on n,

$$\big|v^n - \tilde v^{2n}\big|(t,\mathrm{x},\mathrm{x}_t) \;=\; \big|\tilde v^n - \tilde v^{2n}\big|(t,\mathrm{x},\mathrm{x}_t) \;\leq\; C\varpi'_\circ\Big(\rho_t\big(\Pi^n_t[\mathrm{x}],\widetilde\Pi^{2n}_t[\mathrm{x}]\big) + d(\pi^n,\tilde\pi^{2n})^{\frac{1}{4}}\Big), \tag{2.34}$$

where $\widetilde\Pi^{2n}$ is defined as Π^{2n} but with respect to $\widetilde\pi^{2n}$. By the same arguments, one has

$$\big|v'^n - \tilde v^{2n}\big|(\cdot,\mathrm{x},\mathrm{x}_\cdot) \;\leq\; C\varpi'_\circ\Big(\rho_t\big(\Pi'^n_t[\mathrm{x}],\widetilde\Pi^{2n}_t[\mathrm{x}]\big) + d(\pi'^n,\tilde\pi^{2n})^{\frac{1}{4}}\Big).$$

where Π'^n is defined as Π^n but with respect to π'^n. Therefore, the sequences $(v^n)_{n\geq 1}$ and $(v'^n)_{n\geq 1}$ have the same limit as $n \to \infty$. □

2.4 A classical solution is an approximate viscosity solution

We prove in this section that our notion of approximate solution is consistent with the notion of smooth solution (in the sense of Dupire). For this, we will have to restrict to the case where either $d \leq 2$ or $\gamma \in \mathbb{S}^d \mapsto F(\cdot,\gamma)$ is concave (or convex, by a change of variables). Note that Proposition 2.17 below combined with Remark 2.12 actually provides a numerical algorithm for computing the smooth solution of a PPDE, if it exists.

Let us set for this subsection

$$\Omega = D([0,T]) \text{ so that } \Theta = [0,T] \times D([0,T]),$$

and let us recall the definition of derivatives of path-dependent functionals in the sense of Dupire [7]. Let $u : \Theta \longrightarrow \mathbb{R}$ be a non-anticipative function, and let us write

$$u(t, \mathrm{x}, x) \ := \ u(t, \mathrm{x} \boxplus_t x).$$

The function $u : \Theta \longrightarrow \mathbb{R}$ is said to be horizontally differentiable if, for all $(t, \mathrm{x}) \in [0,T) \times D([0,T])$, the horizontal derivative

$$\partial_t u(t, \mathrm{x}) \ := \ \lim_{h \searrow 0} \frac{u(t+h, \mathrm{x}_{t\wedge}) - u(t, \mathrm{x}_{t\wedge})}{h} \quad \text{is well-defined.}$$

Next, u is said to be vertically differentiable if, for all $(t, \mathrm{x}) \in \Theta$, the function

$$y \in \mathbb{R}^d \longmapsto u(t, \mathrm{x}, y) = u(t, \mathrm{x} \boxplus_t y) \text{ is differentiable at } \mathrm{x}_t, \tag{2.35}$$

whose derivative at $y = \mathrm{x}_t$ is called the vertical derivative of u at (t, x), denoted by $\nabla_{\mathrm{x}} u(t, \mathrm{x})$. Similarly, one can define the second-order vertical derivative $\nabla^2_{\mathrm{x}} u(t, \mathrm{x})$.

We say that u is locally bounded if $\sup_{t \in [0,T], \|\mathrm{x}\| \le K} |u(t, \mathrm{x})| < \infty$ for all $K > 0$. We denote by $\mathbb{C}^{\mathrm{loc}}_b(\Theta)$ the class of all non-anticipative, continuous, and locally bounded functions on Θ, and by $\mathbb{C}^{1,2}(\Theta)$ the collection of elements $u \in \mathbb{C}^{\mathrm{loc}}_b(\Theta)$ such that $\partial_t u$, $\nabla_{\mathrm{x}} u$ and $\nabla^2_{\mathrm{x}} u$ are all well defined, continuous and locally bounded on $[0,T) \times D([0,T])$.

We then say that $u \in \mathbb{C}^{1,2}(\Theta)$ is a classical solution of (2.4)-(2.5) if

$$-\partial_t u - F(\cdot, u, \nabla_{\mathrm{x}} u, \nabla^2_{\mathrm{x}} u) = 0 \text{ on } [0,T) \times D([0,T]) \text{ and } u(T, \cdot) = g \text{ on } D([0,T]).$$

Proposition 2.17. *Let Assumptions 2.3 and 2.10 hold true, let* w *be a classical solution to the PPDE* (2.4)-(2.5) *such that*

$$|\mathrm{w}(t, \mathrm{x})| \le C(1 + \|\mathrm{x}\|^p), \text{ for all } (t, \mathrm{x}) \in \Theta,$$

for some $C > 0$ *and* $p \in \mathbb{N}$. *Assume in addition that either* $d \le 2$ *or* $\gamma \in \mathbb{S}^d \mapsto F(\cdot, \gamma)$ *is concave. Then* w *is the approximate viscosity solution of* (2.4)-(2.5).

Proof. Let v^n be the π^n-viscosity solution of (2.4), or, equivalently, a viscosity solution of (2.8). It is enough to prove that $v^n \longrightarrow \mathrm{w}$ pointwisely as $n \to \infty$.

(i) In a first step, we notice that F^n can be considered as a function of a $(\mathrm{x}_{t^n_0}, \cdots, \mathrm{x}_{t^n_n}, y, z, \gamma)$, and that one can regularize it (and possibly add a small Laplacian term to guarantee uniform ellipticity) so that there is a sequence $(F^{n,k})_{k \ge 1}$ satisfying

$$F^{n,k}(\cdot, y, z, \gamma) = H^{n,k}(\cdot, y, z, \gamma) + r^{n,k}(\cdot) y + \mu^{n,k}(\cdot) \cdot z + \frac{1}{2} \mathrm{Tr}\big[\sigma^{n,k} \sigma^{n,k\top}(\cdot) \gamma\big],$$

for some smooth functions $(H^{n,k}, r^{n,k}, \mu^{n,k}, \sigma^{n,k})$ which converge to $(H^n, r^n, \mu^n, \sigma^n)$ uniformly. Moreover, when $d \le 2$ or F is concave in its last argument, the corresponding PDE (2.8), with generator $F^{n,k}$, has a classical solution $v^{n,k} \in C^{1,3}_b$ which converges locally

uniformly to v^n, see e.g. [13, Theorem 14.15] and [16, Appendix D]. In view of this, we can assume w.l.o.g. that v^n is a smooth solution of (2.8) with generator F^n.

(ii) Next, we will consider v^n as a path-dependent functional and essentially argue as in the proof of Lemma 2.14 with the additional property (recalling the convention (2.35)) that

$$\phi^n(t,\mathrm{x},\mathrm{x}',\mathrm{x}_t,\mathrm{x}'_t) = \phi^n(t,\mathrm{x},\mathrm{x}') := \mathrm{w}(t,\mathrm{x}) - v^n(t,\mathrm{x}')$$

is a smooth functional. We therefore only sketch the proof. For later use, note that

$$\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') = \nabla_{\mathrm{x}}\mathrm{w}(t,\mathrm{x}) \ \ , \ \ \nabla_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}') = -\nabla_{\mathrm{x}'}v^n(t,\mathrm{x}') \ \text{ and } \ \nabla_{\mathrm{x}'}\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') = 0.$$

For ease of notations, we pursue the proof with $d = 1$. We first compute that

$$\begin{aligned}
&- H\big(t,\mathrm{x},(\mathrm{w},\nabla_{\mathrm{x}}\mathrm{w},\nabla^2_{\mathrm{x}}\mathrm{w})(t,\mathrm{x})\big) + H^n\big(t,\mathrm{x}',(v^n,\nabla_{\mathrm{x}}v^n,\nabla^2_{\mathrm{x}}v^n)(t,\mathrm{x}')\big)\\
&= -f(t,\mathrm{x},\Pi^n(\mathrm{x}')) - \kappa^n(t,\mathrm{x},\mathrm{x}')\phi^n(t,\mathrm{x},\mathrm{x}') - b^n(t,\mathrm{x},\mathrm{x}')(\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') + \nabla_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}'))\\
&\quad - a^n(t,\mathrm{x},\mathrm{x}')\big(\nabla^2_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') + \nabla^2_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}')\big)
\end{aligned}$$

for some continuous map a^n that is non-negative and bounded by L, a continuous map (κ^n, b^n) bounded by $2L$, and where f is continuous and satisfies

$$|f(t,\mathrm{x},\Pi^n(\mathrm{x}'))| \le L\rho_t(\mathrm{x},\Pi^n(\mathrm{x}')),$$

recall Assumption 2.10. Note that a^n, b^n, κ^n depend on (t,x) through w and v^n possibly. Let us set

$$r^n(t,\mathrm{x}) := r\big(t,\Pi^n[\mathrm{x}]_{t_i^n\wedge}\big) \text{ if } t\in[t_i^n,t_{i+1}^n),\ i<n,$$

and define μ^n and σ^n similarly with respect to μ and σ. Next, we compute that

$$\begin{aligned}
&- \Big(r(t,\mathrm{x})\mathrm{w}(t,\mathrm{x}) + \mu(t,\mathrm{x})\nabla_{\mathrm{x}}\mathrm{w}(t,\mathrm{x}) + \frac{1}{2}\sigma(t,\mathrm{x})^2\nabla^2_{\mathrm{x}}\mathrm{w}(t,\mathrm{x})\Big)\\
&+ \Big(r^n(t,\mathrm{x}')v^n(t,\mathrm{x}') + \mu^n(t,\mathrm{x}')\nabla_{\mathrm{x}'}v^n(t,\mathrm{x}') + \frac{1}{2}\sigma^n(t,\mathrm{x}')^2\nabla^2_{\mathrm{x}'}v^n(t,\mathrm{x}')\Big)\\
=\ &- r^n(t,\mathrm{x}')\phi^n(t,\mathrm{x},\mathrm{x}') - \Delta r^n(t,\mathrm{x},\mathrm{x}')\mathrm{w}(t,\mathrm{x})\\
&- \mu^n(t,\mathrm{x}')\big(\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') + \nabla_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}')\big) - \Delta\mu^n(t,\mathrm{x},\mathrm{x}')\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}')\\
&- \frac{1}{2}\sigma^n(t,\mathrm{x}')^2\nabla^2_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}')\\
&- \frac{1}{2}(\sigma^n+\Delta\sigma^n)(t,\mathrm{x},\mathrm{x}')^2\nabla^2_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}'),
\end{aligned}$$

where $\Delta r^n(t,\mathrm{x},\mathrm{x}') := r(t,\mathrm{x}) - r^n(t,\mathrm{x}')$ satisfies

$$\big|\Delta r^n(t,\mathrm{x},\mathrm{x}')\big| \ \le\ L\rho_t\big(\mathrm{x},\Pi^n[\mathrm{x}']\big),$$

and $\Delta\mu^n(t,\mathrm{x},\mathrm{x}') := \mu(t,\mathrm{x}) - \mu^n(t,\mathrm{x}')$, $\Delta\sigma^n(t,\mathrm{x},\mathrm{x}') := \sigma(t,\mathrm{x}) - \sigma^n(t,\mathrm{x}')$ satisfy

$$\big|\Delta\mu^n(t,\mathrm{x},\mathrm{x}')\big| + \big|\Delta\sigma^n(t,\mathrm{x},\mathrm{x}')\big| \ \le\ 2L\min\big(\rho_t\big(\mathrm{x},\Pi^n[\mathrm{x}']\big), \|\mathrm{x}-\Pi^n[\mathrm{x}']\|\big).$$

Therefore, $(t,\mathrm{x},\mathrm{x}') \mapsto \phi^n(t,\mathrm{x},\mathrm{x}')$ satisfies on $[0,T) \times D([0,T])^2$

$$
\begin{aligned}
0 = &- (\kappa^n(t,\mathrm{x},\mathrm{x}') + r^n(t,\mathrm{x}'))\phi^n(t,\mathrm{x},\mathrm{x}') - \partial_t\phi^n(t,\mathrm{x},\mathrm{x}') \\
&- (b^n(t,\mathrm{x},\mathrm{x}') + \mu^n(t,\mathrm{x}'))(\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') + \nabla_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}')) - \Delta\mu^n(t,\mathrm{x},\mathrm{x}')\nabla_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') \\
&- a^n(t,\mathrm{x},\mathrm{x}')\big(\nabla^2_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') + \nabla^2_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}')\big) \\
&- \frac{1}{2}\sigma^n(t,\mathrm{x}')^2\nabla^2_{\mathrm{x}'}\phi^n(t,\mathrm{x},\mathrm{x}') - \frac{1}{2}(\sigma^n + \Delta\sigma^n)(t,\mathrm{x},\mathrm{x}')^2\nabla^2_{\mathrm{x}}\phi^n(t,\mathrm{x},\mathrm{x}') \\
&- (\Delta r^n(t,\mathrm{x},\mathrm{x}')\mathrm{w}(t,\mathrm{x}) + f(t,\mathrm{x},\Pi^n(\mathrm{x}')))
\end{aligned}
$$

with

$$\phi^n(T,\mathrm{x},\mathrm{x}') = g(\mathrm{x}) - g(\Pi^n(\mathrm{x}')), \text{ for } \mathrm{x},\mathrm{x}' \in \Omega.$$

Since $\nabla_{\mathrm{x}}\nabla_{\mathrm{x}'}\phi^n(\cdot,\mathrm{x},\mathrm{x}') = 0$, the fact that $\lim_{n\to\infty}\phi^n(t,\mathrm{x},\mathrm{x}) = 0$ for all $(t,\mathrm{x}) \in \Theta$ then follows from the same arguments as in (ii) of the proof of Lemma 2.14 by using [17, Theorem 3.16], the functional Itô's formula of [3] and the corresponding Feynman-Kac's formula, recall that v^n and w have polynomial growth. □

3 Regularity of approximate viscosity solutions

In this section, we discuss the regularity of π-approximate viscosity solutions for a fully nonlinear PPDE and a semi-linear PPDE. Namely, we provide in this section some conditions under which the Dupire's vertical derivative exists and is continuous. One of the motivations to study such a regularity property is the fact that it allows to apply the functional Itô formula in [1]. The study of further regularity of the solution, in particular the potential regularization effect of an uniform ellipticity condition, is left for future researches.

All over this section, we set

$$\Omega = D([0,T]) \text{ so that } \Theta = [0,T] \times D([0,T]).$$

We also recall that, for a functional defined on Θ, its derivatives in the sense of Dupire is given in Section 2.4.

3.1 A fully nonlinear case

A particular fully nonlinear operator has already been studied in [1, Section 3.2] in the case $d = 1$ and for a PPDE of the form $-\partial_t v - H(\nabla^2_{\mathrm{x}} v) = 0$ with a convex function H. We provide here a generalization which is proved by an appropriate modification of the arguments contained in the proof of [1, Proposition 3.13] and by appealing to the a-priori estimate of Lemma 2.15 and Remark 2.16. In particular, we allow for more general nonlinearities when the terminal condition is C^{1+1} (in the sense of Assumption 3.1 below when $\alpha = 1$). It fully uses the piecewise constant approximation of the path, embedded in the definition of approximate viscosity solutions, to reduce the analysis to the finite dimensional case which provides uniform estimates on the corresponding sequence of π^n-viscosity solutions, for all $n \geq 1$.

Assumption 3.1. (i) *The function $g \in \mathcal{C}_L$ and there exists $\alpha \in (0,1]$ and a finite positive measure λ on $[0,T]$ such that, for all $\mathrm{x}, \mathrm{x}' \in D([0,T])$, $B = [s,t) \subset [0,T]$ and $\delta \in \mathbb{R}^d$,*

$$|g(\mathrm{x}) - g(\mathrm{x}')| \le \int_0^T |\mathrm{x}_s - \mathrm{x}'_s| \lambda(ds), \tag{3.1}$$

$\delta' \in \mathbb{R}^d \longmapsto g(\mathrm{x} + \delta' \mathbf{1}_B)$ is differentiable and[3]

$$\Big| \frac{dg(\mathrm{x} + \delta \mathbf{1}_B + \mathrm{x}')}{d\delta} - \frac{dg(\mathrm{x} + \delta \mathbf{1}_B)}{d\delta} \Big| \le \Big(\int_0^T \big| \mathrm{x}'_s \big| \lambda(ds) \Big)^{\alpha} \lambda(B). \tag{3.2}$$

(ii) *For any increasing sequence $0 = t_0 < t_1 < \cdots < t_n = T$ with $\max_{i<n} |t_{i+1} - t_i|$ small enough, for all $1 \le i < j < n$ there exists a diagonal matrix $p^{i,j} \in \mathbb{S}^d$ with non-zero diagonal terms such that, for all $\delta \in \mathbb{R}^d$, and $(x_\ell)_{0 \le \ell \le n} \subset (\mathbb{R}^d)^{n+1}$,*

$$\begin{aligned} & g\left(\sum_{\ell=0}^{n-1} (x_\ell + \delta \mathbf{1}_{\{\ell = i\}}) \mathbf{1}_{[t_\ell, t_{\ell+1})} + \mathbf{1}_{\{T\}} x_n \right) \\ & = g\left(\sum_{\ell=0}^{n-1} (x_\ell + p^{i,j} \delta \mathbf{1}_{\{\ell \ge j\}}) \mathbf{1}_{[t_\ell, t_{\ell+1})} + \mathbf{1}_{\{T\}} (x_n + p^{i,j} \delta) \right). \end{aligned} \tag{3.3}$$

(iii) *F satisfies the conditions of Theorem 2.11. Moreover, F is concave*[4] *in γ or $d \le 2$. Further, one of the following holds:*

(a) *Either $\alpha \in (0,1)$ and*

$$F(t, \mathrm{x}, y, z, \gamma) = F_1(t) y + F_2(t) z + F_3(t, \gamma), \ (t, \mathrm{x}, y, z, \gamma) \in \Theta \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d, \tag{3.4}$$

for some continuous maps F_i, $i = 1, \dots, 3$.

(b) *Or $\alpha = 1$ and*

$$F(t, \mathrm{x}, y, z, \gamma) = F_1(t, y, \gamma) + F_2(t) z \tag{3.5}$$

for some continuous maps F_i, $i = 1, 2$, such that $y \in \mathbb{R} \longmapsto F_1(t, y, \gamma)$ is C^1 with bounded and Lipschitz first order derivative, uniformly in $\gamma \in \mathbb{R}^d$ and $t \le T$.

Example 3.2. *Let $d = 1$, $g_0 : \mathbb{R} \longrightarrow \mathbb{R}$ be in $C^{1+\alpha}$, and λ be a finite positive measure on $[0,T]$ such that $\lambda([T-\varepsilon, T]) > 0$ for any $\varepsilon > 0$. Then the following $g : D([0,T]) \longrightarrow \mathbb{R}$ defined by*

$$g(\mathrm{x}) \ := \ g_0\Big(\int_0^T \mathrm{x}_t \lambda(dt) \Big)$$

satisfies the conditions of Assumption 3.1.

Let us comment on the conditions in Assumption 3.1. In [1], we essentially "differentiate" twice regularized versions of the PDE's associated to the π^n-viscosity solutions v^n, and then derive the uniform boundedness and Hölder type estimates on the corresponding gradients

[3] We abuse notations to write $dg(\cdot + \delta \mathbf{1}_B + \cdot)/d\delta$ for the corresponding Jacobian with respect to δ.

[4] or convex, which is equivalent up to replacing $F(\mathrm{x}, \cdot)$ by $-F(\mathrm{x}, -\cdot)$.

of v^n. For this, we interpret the differentiated PDEs in terms of flow equations to propagate backward the regularity of the terminal condition g, see (3.1)-(3.2). The main difficulty is that a bump at time $t \in [t_i^n, t_{i+1}^n)$ on x is on $[t_{i+1}^n, t_{i+2}^n)$ a bump on the $i+1$-th element of $[\mathrm{x} \boxplus_{t_{i+1}^n} x]_{i+1}^n$ because of the boundary condition

$$\lim_{t \nearrow t_{i+1}^n, x' \to x} v^n(t, \mathrm{x}, x') = v^n\big(t_{i+1}^n, \Pi^n[\mathrm{x} \boxplus_{t_{i+1}^n} x], x\big).$$

Therefore, it turns, from the next term interval on, into a bump on a parameter of the PDEs. See also (3.10) in the proof below. The structure condition (3.3) allows one to transform a derivative with respect to a past value of the path into a derivative with respect to the end of the path. More precisely, we have the following.

Remark 3.3. *Assume that v^n is a smooth π^n-viscosity solution of* (2.4)-(2.5) *and that Assumption 3.1 holds. Then, a backward induction argument based on* (ii) *of Assumption 3.1 and the fact that F does not depend on the path* x *imply that $v^n(t, \cdot + \delta \mathbf{1}_{[t_\ell, t_{\ell+1})}, \cdot) = v^n(t, \cdot, \cdot + p^{\ell, i+1}\delta)$ for $t \in [t_i^n, t_{i+1}^n)$, $\ell < i+1 \le n$, for some $p^{\ell,i+1}$. Letting v_i^n be defined as in* (2.17) *and denoting by $\partial_\ell v_i^n(t, [\mathrm{x}]_i^n, x)$ the gradient of $v_i^n(t, [\mathrm{x}]_i^n, x)$ with respect to the ℓ-th component of $[\mathrm{x}]_i^n$, we deduce that*

$$\partial_\ell v_i^n(t, [\mathrm{x}]_i^n, x) = p^{\ell, i+1} D v_i^n(t, [\mathrm{x}]_i^n, x).$$

Note that this is no more possible if F depends on x.

As for (3.1)-(3.2), it is used to absorb the cumulation of derivatives on different time steps generated by the above mentioned phenomena, see (3.8)-(3.9) below.

Note that (3.1)-(3.2) is a $C^{1+\alpha}$-type regularity assumption on g. It actually turns into a $C^{1+\alpha}$-regularity in space on the solution v of the PPDE. This translates, as usual, into an Hölder continuity in time, up to an additional term associated to λ. The interior regularity that could be obtained under a uniform ellipticity condition is by far not obvious and remains an open question.

Theorem 3.4. *Let Assumption 3.1 hold true. Let* v *be the unique π-approximate viscosity solution of* (2.4)-(2.5). *Then, the vertical derivative $\nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x})$ is well defined for all $(t, \mathrm{x}) \in [0,T] \times D([0,T])$, and there exists $C > 0$ such that $|\nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x})| \le C$,*

$$|\nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x}') - \nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x})| \;\le\; C\Big(\Big| \int_0^t |\mathrm{x}'_s - \mathrm{x}_s| \lambda(ds) \Big|^\alpha + |\mathrm{x}'_t - \mathrm{x}_t|^\alpha \Big), \tag{3.6}$$

for all $(t, \mathrm{x}, \mathrm{x}') \in [0,T] \times D([0,T]) \times D([0,T])$. Moreover, for all $K > 0$, there exists a constant $C_K > 0$ such that

$$|\nabla_{\mathrm{x}} \mathrm{v}(t', \mathrm{x}_{t\wedge}) - \nabla_{\mathrm{x}} \mathrm{v}(t, \mathrm{x})| \;\le\; C_K \Big(|t' - t|^{\frac{\alpha}{2+2\alpha}} + \limsup_{\varepsilon \downarrow 0} \lambda([t-\varepsilon, t'+\varepsilon)) \Big), \tag{3.7}$$

for all $t \le t' \le T$ and $\mathrm{x} \in D([0,T])$ satisfying $\|\mathrm{x}\| \le K$.

Proof. For ease of notations, we restrict to $d = 1$. When $d > 1$, it suffices to derive the following estimates for each dimension separately. By (i) of Assumption 3.1, upon smoothing

$$g^n : (x_0, \dots, x_n) \in \mathbb{R}^{n+1} \longmapsto g\Big(\sum_{i=0}^{n-1} x_i \mathbf{1}_{[t_i^n, t_{i+1}^n)} + x_n \mathbf{1}_{\{T\}} \Big),$$

we can assume that it is in C^∞ and that

$$|\partial_{x_i} g^n(x_0,\dots,x_n)| \le \lambda([t_i^n, t_{i+1}^n)),\ i,j \le n,\ \text{if } \alpha \in (0,1], \tag{3.8}$$

$$|\partial^2_{x_i x_j} g^n(x_0,\dots,x_n)| \le \lambda([t_i^n, t_{i+1}^n))\lambda([t_j^n, t_{j+1}^n)),\ i,j \le n,\ \text{if } \alpha = 1, \tag{3.9}$$

with the convention $\lambda([t_n^n, t_{n+1}^n)) := \lambda(\{t_n^n\})$. Let V^n be defined by

$$V^n(t, [\mathrm{x}]_i^n, x) = v^n(t, \mathrm{x}, x),\ (\mathrm{x}, x) \in D([0,T]) \times \mathbb{R}^d,\ t \in [t_i^n, t_{i+1}^n),\ i < n.$$

(i) Upon smoothing also F and adding a uniform elliptic term, we can assume that V^n is $C_b^{1,3}$ on $[0,T)$, see e.g. [13, Theorem 14.15] and [16, Appendix D]. The estimates below will not depend upon this smoothing procedure. Let $\phi^{n,j}$ denote the gradient of V^n with respect to the j-th space component, and denote by $D\phi^{n,j}$ and $D^2\phi^{n,j}$ the first and second order derivatives of $\phi^{n,j}$ with respect to its last space variable. On each $[t_i^n, t_{i+1}^n)$, $\phi^{n,j}$, $j \le i+1$, solves

$$0 = -\partial_t \phi^{n,j} - \partial_y F(\Xi)\phi^{n,j} - \partial_z F(\Xi) D\phi^{n,j} - \partial_\gamma F(\Xi) D^2\phi^{n,j}$$

where $\Xi := (\cdot, V^n, DV^n, D^2V^n)$, with the boundary condition

$$\lim_{t \uparrow t_{i+1}^n} \phi^{n,j}(t,y,x) = \phi^{n,j}(t_{i+1}^n, (y,x), x) + \mathbf{1}_{\{j=i+1\}} \phi^{n,i+2}(t_{i+1}^n, (y,x), x),\ (y,x) \in \mathbb{R}^i \times \mathbb{R}. \tag{3.10}$$

By Feynman-Kac's formula and Assumption 3.1, for each $(t,z) \in [t_i^n, t_{i+1}^n) \times \mathbb{R}^{i+1}$, $j \le i+1 \le n$, we can then find a process X and a bounded random variable β, with bound $C_\beta > 0$ depending only on F, such that

$$\phi^{n,j}(t,z) = \mathbb{E}\left[\beta\left(\partial_j g^n(X_{t_0^n},\dots,X_{t_n^n}) + \mathbf{1}_{\{j=i+1\}} \sum_{j'=i+2}^n \partial_{j'} g^n(X_{t_0^n},\dots,X_{t_n^n})\right)\right].$$

By (3.8), it follows that

$$|\phi^{n,j}(t,\cdot)| \le C_\beta\left(\lambda([t_j^n, t_{j+1}^n)) + \mathbf{1}_{\{j=i+1\}}\lambda([t_{i+2}^n, T])\right),\ \text{on } [t_i^n, t_{i+1}^n),\ j \le i+1 \le n. \tag{3.11}$$

(ii) In the case $\alpha < 1$, the rest of the proof is exactly the same as the proof of [1, Proposition 3.13] up to some obvious modifications. In fact, it suffices to replace their two first estimates at the very beginning of part 4.c. of their proof by (3.11) and (2.31) (together with Remark 2.16).

(iii) In the following, we consider the case $\alpha = 1$, where the argument is slightly different and we only sketch the proof.

(a) Let us set $\psi^{n,j,\ell} := \partial_{x_\ell} \phi^{n,j}$, which solves the PDE

$$\begin{aligned} 0 = &- \partial_t \psi^{n,j,\ell} - \partial_y F(\Xi)\psi^{n,j,\ell} - \partial_z F(\Xi) D\psi^{n,j,\ell} - \partial_\gamma F(\Xi) D^2\psi^{n,j,\ell} \\ &- \phi^{n,j}\left(\partial^2_{yy} F(\Xi)\phi^{n,\ell} + \partial^2_{yz} F(\Xi) D\phi^{n,\ell} + \partial^2_{y\gamma} F(\Xi) D^2\phi^{n,\ell}\right) \\ &- D\phi^{n,j}\left(\partial^2_{zy} F(\Xi)\phi^{n,\ell} + \partial^2_{zz} F(\Xi) D\phi^{n,\ell} + \partial^2_{z\gamma} F(\Xi) D^2\phi^{n,\ell}\right) \\ &- D^2\phi^{n,j}\left(\partial^2_{\gamma y} F(\Xi)\phi^{n,\ell} + \partial^2_{\gamma z} F(\Xi) D\phi^{n,\ell} + \partial^2_{\gamma\gamma} F(\Xi) D^2\phi^{n,\ell}\right). \end{aligned} \tag{3.12}$$

By (3.5) and Remark 3.3, we can then find $q^{\ell,i+1} \in \mathbb{R}$, that depends on n, such that

$$\begin{aligned}
0 =& - \partial_t \psi^{n,j,\ell} - \partial_y F(\Xi)\psi^{n,j,\ell} \\
& - \Big(\partial_z F(\Xi) + q^{\ell,i+1}\phi^{n,j}\partial^2_{y\gamma}F(\Xi) + q^{\ell,i+1}D\phi^{n,j}\partial^2_{z\gamma}F(\Xi)\Big) D\psi^{n,j,\ell} \\
& - q^{\ell,i+1}\Big(\partial^2_{\gamma y}F(\Xi)\phi^{n,\ell} + \partial^2_{\gamma z}F(\Xi)D\phi^{n,\ell} + \partial^2_{\gamma\gamma}F(\Xi)D^2\phi^{n,\ell}\Big) D\psi^{n,j,\ell} \\
& - \partial_\gamma F(\Xi)D^2\psi^{n,j,\ell} - \phi^{n,j}\partial^2_{yy}F(\Xi)\phi^{n,\ell}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\lim_{t\uparrow t^n_{i+1}} \psi^{n,j,\ell}(t,y,x) =& \mathbf{1}_{\{\ell<i+1\}}\Big(\psi^{n,j,\ell} + \mathbf{1}_{\{j=i+1\}}\psi^{n,i+2,\ell}\Big)(t^n_{i+1},(y,x),x) \\
& + \mathbf{1}_{\{\ell=i+1\}}\big(\psi^{n,j,i+2} + \mathbf{1}_{\{j=i+1\}}\psi^{n,i+2,i+2}\big)(t^n_{i+1},(y,x),x),
\end{aligned}$$

for $(y,x) \in \mathbb{R}^i \times \mathbb{R}$. Then, by (iii) of Assumption 3.1 and (3.11), one can apply the Feynman-Kac's formula again to find, for given $(t,z) \in [t^n_i, t^n_{i+1}) \times \mathbb{R}^{i+1}$, $i < n$, and $j, \ell \le i+1$, adapted processes X, β^1 and β^2, that depend on n but such that $|(\beta^1,\beta^2)| \le \tilde{C}_\beta$ for some $\tilde{C}_\beta > 0$ that only depend on F and g, for which

$$\begin{aligned}
\psi^{n,j,\ell}(t,z) =& \mathbf{1}_{\{\ell<i+1\}}\mathbb{E}\Big[\beta^1_T\Big(\partial^2_{x_jx_\ell}g^n + \mathbf{1}_{\{j=i+1\}}\sum_{j'=i+2}^{n}\partial^2_{x_{j'}x_\ell}g^n\Big)\big(\Pi^n[X]\big)\Big] \\
& + \mathbf{1}_{\{\ell=i+1\}}\sum_{k=\ell+2}^{n}\mathbb{E}\Big[\beta^1_T\Big(\partial^2_{x_jx_k}g^n + \mathbf{1}_{\{j=i+1\}}\sum_{j'=i+2}^{n-1}\partial^2_{x_{j'}x_k}g^n\Big)\big(\Pi^n[X]\big)\Big] \\
& + \mathbb{E}\Big[\int_t^T \beta^2_s ds\Big].
\end{aligned}$$

Thus, by (3.9),

$$|\psi^{n,j,\ell}(t,z)| \le \tilde{C}_\beta\lambda([0,T])^2 + \tilde{C}_\beta T. \tag{3.13}$$

(b) Recall that $\alpha = 1$. It follows from (2.31) (by setting $\varpi_\circ(x) := x$ for all $x \ge 0$, see Remark 2.16), (3.11) and (3.13) that we can find C_K and C'_K, independent of n, such that

$$\begin{aligned}
&\big|Dv^n(t+h^4, \mathrm{x}_{t\wedge}, \mathrm{x}_t) - Dv^n(t,\mathrm{x},\mathrm{x}_t)\big| \\
&\le h^{-1}\big|v^n(t+h^4,\mathrm{x}_{t\wedge},\mathrm{x}_t+h) - v^n(t+h^4,\mathrm{x}_{t\wedge},\mathrm{x}_t) - v^n(t,\mathrm{x},\mathrm{x}_t+h) + v^n(t,\mathrm{x},\mathrm{x}_t)\big| \\
&\quad + 2C_Kh \\
&\le h^{-1}\big|v^n(t+h^4,(\mathrm{x}+\mathbf{1}_{\{t\}}h)_{t\wedge},\mathrm{x}_t+h) - v^n(t,\mathrm{x},\mathrm{x}_t+h)\big| \\
&\quad + h^{-1}\big|v^n(t,\mathrm{x},\mathrm{x}_t) - v^n(t+h^4,\mathrm{x}_{t\wedge},\mathrm{x}_t)\big| \\
&\quad + h^{-1}\big|v^n(t+h^4,\mathrm{x}_{t\wedge},\mathrm{x}_t+h) - v^n(t+h^4,(\mathrm{x}+\mathbf{1}_{\{t\}}h)_{t\wedge},\mathrm{x}_t+h)\big| \\
&\quad + 2C_Kh \\
&\le 2C'_Kh + C_\beta\lambda([t-|\pi^n|, t+h^4+|\pi^n|)),
\end{aligned}$$

for all $\mathrm{x} \in D([0,T])$ and $t < t+h \le T$.

The rest of the proof is similar to the one of [1, Proposition 3.13]: (i) implies that $\nabla_{\mathrm{x}}v$ is well-defined on a dense subset of Θ, including the (countably many) atoms of λ, while (a) and (b) allow to extend it in a continuous way and imply that the extension is actually the pointwise limit of Dv^n on Θ. The required estimates are deduced from (a) and (b) . □

3.2 A semi-linear case

We now consider a semi-linear PPDE with generator of the form

$$F(t,\mathrm{x},y,z,\gamma) \;=\; f(t,\mathrm{x},y,\sigma^\top z)+\mu(t,\mathrm{x})\cdot z+\frac{1}{2}\mathrm{Tr}\big[\sigma\sigma^\top\gamma\big], \tag{3.14}$$

for some non-anticipative functionals $f:\Theta\times\mathbb{R}\times\mathbb{R}^d\longrightarrow\mathbb{R}$ and $(\mu,\sigma):\Theta\longrightarrow\mathbb{R}^d\times\mathbb{S}^d$. In this case, we can remove the structure condition (3.3). We make the following assumption on the coefficients.

Assumption 3.5. (i) *The functionals μ and σ are continuous. For each $t\in[0,T]$, the map $\mathrm{x}\in D([0,T])\longmapsto(\mu(t,\mathrm{x}),\sigma(t,\mathrm{x}))$ is Fréchet differentiable with derivative*[5] *$(\lambda_\mu,\lambda_\sigma)(\cdot;t,\mathrm{x})$ at $\mathrm{x}\in D([0,T])$.*

(ii) *The functional g is Fréchet differentiable with derivative $\lambda_g(\cdot;\mathrm{x})$ at $\mathrm{x}\in D([0,T])$. The functional f is continuous and non-anticipative, and for each $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$, the map $\mathrm{x}\longmapsto f(t,\mathrm{x},y,z)$ is Fréchet differentiable with derivative $\lambda_f(\cdot;t,\mathrm{x},y,z)$ at $\mathrm{x}\in D([0,T])$. Further, $(y,z)\in\mathbb{R}\times\mathbb{R}^d\longmapsto f(t,\mathrm{x},y,z)$ is differentiable with derivatives $(\partial_y f,\partial_z f)$ which are bounded and continuous in (t,x,y,z).*

(iii) *There exists some finite positive measure λ on $[0,T]$, such that, for all $(t,\mathrm{x},y,z)\in[0,T]\times D([0,T])\times\mathbb{R}\times\mathbb{R}^d$,*

$$\big|\lambda_\mu(\cdot;t,\mathrm{x})\big|+\big|\lambda_\sigma(\cdot;t,\mathrm{x})\big|+\big|\lambda_f(\cdot;t,\mathrm{x},y,z)\big|+\big|\lambda_g(\cdot;\mathrm{x})\big|\ll\lambda(\cdot).$$

(iv) *There exist constants $\alpha\in[0,1]$ and $C_\alpha>0$ such that*

$$\begin{aligned}
&\left|\int_0^T\tilde{\mathrm{x}}_r\lambda_g(dr;\mathrm{x})-\int_0^T\tilde{\mathrm{x}}_r\lambda_g(dr;\mathrm{x}')\right|\le C_\alpha\|\mathrm{x}-\mathrm{x}'\|^\alpha\|\tilde{\mathrm{x}}\|,\\
&\left|\int_0^T\tilde{\mathrm{x}}_r\lambda_\ell(dr;s,\mathrm{x})-\int_0^T\tilde{\mathrm{x}}_r\lambda_\ell(dr;s,\mathrm{x}')\right|\le C_\alpha\|\mathrm{x}-\mathrm{x}'\|^\alpha\|\tilde{\mathrm{x}}\|,\ \ell\in\{b,\sigma\},\\
&\left|\int_0^T\tilde{\mathrm{x}}_r\lambda_f(dr;s,\mathrm{x},y,z)-\int_0^T\tilde{\mathrm{x}}_r\lambda_f(dr;s,\mathrm{x}',y',z')\right|\le C_\alpha\left(\|\mathrm{x}-\mathrm{x}'\|+|y-y'|+|z-z'|\right)^\alpha\|\tilde{\mathrm{x}}\|,\\
&\big|(\partial_y,\partial_z)f(s,\mathrm{x},y,z)-(\partial_y,\partial_z)f(s,\mathrm{x}',y',z')\big|\le C_\alpha\left(\|\mathrm{x}-\mathrm{x}'\|+|y-y'|+|z-z'|\right)^\alpha,
\end{aligned}$$

for all $\mathrm{x},\mathrm{x}',\tilde{\mathrm{x}}\in D([0,T])$ and $(s,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^d$.

Under Assumption 3.5, (2.4)-(2.5) has a unique π-approximate viscosity solution v (see Proposition 2.8), which corresponds to the solution of a BSDE. We will then show that v has a Dupire's vertical derivative which enjoys further continuity conditions. In the following, we assume that we are given a fixed complete probability space, equipped with a Brownian motion W together with the Brownian filtration. Let $S^2([t,T])$ denote the space of $\mathbb{R}^d$-valued predictable processes $X=(X_s)_{s\in[t,T]}$ such that $\mathbb{E}\big[\sup_{t\le s\le T}|X_s|^2\big]<\infty$, and $H^2([t,T])$ denote the space of $\mathbb{R}^d$-valued predictable processes $Z=(Z_s)_{s\in[t,T]}$ such that $\mathbb{E}\big[\int_t^T|Z_s|^2ds\big]<\infty$.

[5] We identify the Fréchet derivatives with the associated measures on $[0,T]$.

Theorem 3.6. (i) *Let F be given by* (3.14)*, and let Assumption 3.5 hold true. Let* v *be the unique π-approximate viscosity solution of* (2.4)-(2.5)*. Then, it admits a Dupire's vertical derivative $\nabla_{\rm x}{\rm v}$ on $[0,T)\times D([0,T])$ and there exists $C>0$ such that*

$$|\nabla_{\rm x}{\rm v}(t,{\rm x}')-\nabla_{\rm x}{\rm v}(t,{\rm x})| \;\le\; C\|{\rm x}'-{\rm x}\|^{\alpha},$$

and

$$|\nabla_{\rm x}{\rm v}(t',{\rm x}_{t\wedge})-\nabla_{\rm x}{\rm v}(t,{\rm x})| \;\le\; C\big(|t'-t|^{\frac{\alpha}{2+2\alpha}}(1+\|{\rm x}\|)+\lambda([t,t'))\big),$$

for all $t\le t'\le T$ and ${\rm x},{\rm x}'\in D([0,T])$.

(ii) *Moreover, if λ has finitely many atoms, then* v *solves the stochastic path-dependent equation*

$${\rm v}(\cdot,X^{t,{\rm x}})=g\big(X^{t,{\rm x}}\big)+\int_{\cdot}^{T}f(u,X^{t,{\rm x}},{\rm v}(u,X^{t,{\rm x}}),[\sigma^{\top}\nabla_{\rm x}{\rm v}](u,X^{t,{\rm x}}))du-\int_{\cdot}^{T}[\nabla_{\rm x}{\rm v}\sigma](u,X^{t,{\rm x}})dW_u,$$

in which $X^{t,{\rm x}}\in S^2([0,T])$ is the unique solution of

$$X_s^{t,{\rm x}}={\rm x}_{t\wedge s}+\int_t^{t\vee s}\mu(u,X^{t,{\rm x}})du+\int_t^{t\vee s}\sigma(u,X^{t,{\rm x}})dW_u,\;t\le s\le T.$$

We split the proof in different steps. First, we will use the probabilistic representation of the PPDE (with generator F in (3.14)) in terms of forward-backward SDE to deduce the regularity of the π-approximate viscosity solution. Namely, under Assumption 3.5, it follows from Proposition 2.8 that the approximate viscosity solution v is given by

$${\rm v}(t,{\rm x}) \;=\; Y_t^{t,{\rm x}}, \tag{3.15}$$

where $(Y^{t,{\rm x}},Z^{t,{\rm x}})\in S^2([t,T])\times H^2([t,T])$ solves the BSDE

$$Y_s^{t,{\rm x}}=g\big(X^{t,{\rm x}}\big)+\int_{t\vee s}^{T}f(u,X^{t,{\rm x}},Y_u^{t,{\rm x}},Z_u^{t,{\rm x}})du-\int_{t\vee s}^{T}Z_u^{t,{\rm x}}\cdot dW_u,\quad s\in[t,T].$$

Next, let us fix a path $\hat{\rm x}\in D([0,T])$, and then define a process $\big(\nabla_{\rm x}^tX_s^{t,{\rm x}}\big)_{s\in[0,T]}$ as the solution to the SDE

$$\nabla_{\rm x}^tX_s^{t,{\rm x}}=\hat{\rm x}_{s\wedge t}+\int_t^{s\vee t}\int_t^u\nabla_{\rm x}^tX_r^{t,{\rm x}}\lambda_\mu(dr;u,X^{t,{\rm x}})du+\int_t^s\int_t^u\nabla_{\rm x}^tX_r^{t,{\rm x}}\lambda_\sigma(dr;u,X^{t,{\rm x}})dW_u.$$

Namely, $\nabla_{\rm x}^tX_s^{t,{\rm x}}$ is the tangent process of $X^{t,{\rm x}}$ in the direction of $\hat{\rm x}$. We also define the processes $\big(\nabla_{\rm x}^tY_s^{t,{\rm x}},\nabla_{\rm x}^tZ_s^{t,{\rm x}}\big)_{s\in[t,T]}$ as the solution to the linear BSDE

$$\begin{aligned}\nabla_{\rm x}^tY_s^{t,{\rm x}}=&\int_0^T\nabla_{\rm x}^tX_r^{t,{\rm x}}\lambda_g(dr;X^{t,{\rm x}})+\int_s^T\Big(\int_t^u\nabla_{\rm x}^tX_r^{t,{\rm x}}\lambda_f(dr;\Theta_u^{t,{\rm x}})+A_u^{t,{\rm x}}\cdot(\nabla_{\rm x}^tY_u^{t,{\rm x}},\nabla_{\rm x}^tZ_u^{t,{\rm x}})\Big)du\\&-\int_s^T\nabla_{\rm x}^tZ_u^{t,{\rm x}}dW_u,\end{aligned}$$

in which

$$\Theta_s^{t,{\rm x}}:=\big(s,X^{t,{\rm x}},Y_s^{t,{\rm x}},Z_s^{t,{\rm x}}\big),\;\text{ and }A_s^{t,{\rm x}}:=\big(\partial_yf(\Theta_s^{t,{\rm x}}),\partial_zf(\Theta_s^{t,{\rm x}})\big). \tag{3.16}$$

Under Assumption 3.5, and by standard arguments (based on Burkholder-Davis-Gundy's inequality and Gronwall's lemma, see e.g. [12]), one has the following estimates on the processes $\big(X^{t,{\rm x}},Y^{t,{\rm x}},Z^{t,{\rm x}}\big)$ and $\big(\nabla_{\rm x}^tX^{t,{\rm x}},\nabla_{\rm x}^tY^{t,{\rm x}},\nabla_{\rm x}^tZ^{t,{\rm x}}\big)$, whose proof is ommited.

Lemma 3.7. *Let Assumption 3.5 hold true. Then, for all $p \geq 1$, there exists a constant $C_p > 0$, such that, for all $t, t' \in [0,T]$ and $\mathrm{x}, \mathrm{x}' \in D([0,T])$,*

$$\mathbb{E}\big[\|X^{t,\mathrm{x}}\|^p\big] + \sup_{s\in[0,T]} \mathbb{E}\big[|Y_s^{t,\mathrm{x}}|^p\big] + \mathbb{E}\Big[\Big(\int_0^T |Z_s^{t,\mathrm{x}}|^2\Big)^{\frac{p}{2}} ds\Big] \;\leq\; C_p\big(1 + \|\mathrm{x}_{t\wedge}\|^p\big), \tag{3.17}$$

$$\begin{aligned} &\mathbb{E}\Big[\big\|X^{t,\mathrm{x}} - X^{t',\mathrm{x}'}\big\|^p\Big]^{\frac{1}{p}} + \mathbb{E}\Big[\big\|Y^{t,\mathrm{x}} - Y^{t',\mathrm{x}'}\big\|^p\Big]^{\frac{1}{p}} + \mathbb{E}\Big[\Big(\int_0^T \big|Z_s^{t,\mathrm{x}} - Z_s^{t',\mathrm{x}'}\big|^2 ds\Big)^{\frac{p}{2}}\Big]^{\frac{1}{p}} \\ \leq\; & C_p\Big(\int_0^{t\vee t'} |\mathrm{x}_s - \mathrm{x}'_s|\lambda(ds) + (1 + \|\mathrm{x}\| + \|\mathrm{x}'\|)|t - t'|^{\frac{1}{2}}\Big), \end{aligned} \tag{3.18}$$

and

$$\mathbb{E}\Big[\big\|\nabla_{\mathrm{x}}^t X^{t,\mathrm{x}}\big\|^p\Big]^{\frac{1}{p}} + \mathbb{E}\Big[\big\|\nabla_{\mathrm{x}}^t Y^{t,\mathrm{x}}\big\|^p\Big]^{\frac{1}{p}} + \mathbb{E}\Big[\Big(\int_0^T \big|\nabla_{\mathrm{x}}^t Z_s^{t,\mathrm{x}}\big|^2 ds\Big)^{\frac{p}{2}}\Big] \leq C_p. \tag{3.19}$$

Next, for each $\delta > 0$, we denote

$$\big(\nabla_{\mathrm{x}}^{t,\delta} X_s^{t,\mathrm{x}}, \nabla_{\mathrm{x}}^{t,\delta} Y^{t,\mathrm{x}}, \nabla_{\mathrm{x}}^{t,\delta} Z^{t,\mathrm{x}}\big) := \delta^{-1}\Big(X_s^{t,\mathrm{x}+\delta\hat{\mathrm{x}}} - X_s^{t,\mathrm{x}}, Y^{t,\mathrm{x}+\delta\hat{\mathrm{x}}} - Y^{t,\mathrm{x}}, Z^{t,\mathrm{x}+\delta\hat{\mathrm{x}}} - Z^{t,\mathrm{x}}\Big).$$

Lemma 3.8. *Let Assumption 3.5 hold true. Then, for all $(t,\mathrm{x}) \in [0,T] \times C([0,T])$,*

$$\lim_{\delta\searrow 0} \mathbb{E}\Big[\big\|\nabla_{\mathrm{x}}^{t,\delta} X^{t,\mathrm{x}} - \nabla_{\mathrm{x}}^t X^{t,\mathrm{x}}\big\|^2\Big] = 0 \quad \textit{and} \quad \lim_{\delta\searrow 0} \Big|\nabla_{\mathrm{x}}^{t,\delta} Y_t^{t,\mathrm{x}} - \nabla_{\mathrm{x}}^t Y_t^{t,\mathrm{x}}\Big| = 0. \tag{3.20}$$

Proof. For ease of notation, let us omit the superscript (t,x) in the notations of the processes $\big(X^{t,\mathrm{x}+\delta\hat{\mathrm{x}}}, X^{t,\mathrm{x}}, \nabla_{\mathrm{x}}^{t,\delta} X^{t,\mathrm{x}}, \nabla_{\mathrm{x}}^t X^{t,\mathrm{x}}\big)$ and write them as $(X^\delta, X, \nabla_{\mathrm{x}}^{t,\delta} X, \nabla_{\mathrm{x}}^t X)$.

(i) We first notice that $(\nabla_{\mathrm{x}}^{t,\delta} X_s)_{s\in[t,T]}$ satisfies

$$\begin{aligned} \nabla_{\mathrm{x}}^{t,\delta} X_s &= I_d + \int_t^s \big(\mu(u, X^\delta) - \mu(u, X)\big)du + \int_t^{t\vee s} \big(\sigma(u, X^\delta) - \sigma(u, X)\big)dW_u \\ &= I_d + \int_t^s \int_t^u \big(\nabla_{\mathrm{x}}^{t,\delta} X_r\big)\lambda_\mu^\delta(dr;u)du + \int_t^s \int_t^u \big(\nabla_{\mathrm{x}}^{t,\delta} X_r\big)\lambda_\sigma^\delta(dr;u)dW_u, \end{aligned}$$

where I_d is the identity matrix and, as $\delta \searrow 0$,

$$\big(\lambda_\mu^\delta(\cdot,u), \lambda_\sigma^\delta(\cdot,u)\big) \longrightarrow \big(\lambda_\mu(dr;u), \lambda_\sigma(dr;u)\big) := \big(\lambda_\mu(dr;u,X), \lambda_\sigma(dr;u,X)\big). \tag{3.21}$$

Applying Itô formula on $\big(\nabla_{\mathrm{x}}^{t,\delta} X_s - \nabla_{\mathrm{x}}^t X_s\big)^2$ and then taking expectation, it follows by standard arguments that, for some constant $C > 0$ independent of δ, and a constant $C_\delta > 0$,

$$\mathbb{E}\Big[\big\|\nabla_{\mathrm{x}}^{t,\delta} X_{s\wedge} - \nabla_{\mathrm{x}}^t X_{s\wedge}\big\|^2\Big] \;\leq\; C\int_t^s \mathbb{E}\Big[\big\|\nabla_{\mathrm{x}}^{t,\delta} X_{r\wedge} - \nabla_{\mathrm{x}}^t X_{r\wedge}\big\|^2\Big]dr + C_\delta,$$

where, by the moment estimation in (3.17) and the convergence in (3.21),

$$\begin{aligned} C_\delta \;\leq\; & C\int_t^T \mathbb{E}\Big[\big|\nabla_{\mathrm{x}}^{t,\delta} X_u - \nabla_{\mathrm{x}}^t X_u\big| \int_t^T |\nabla_{\mathrm{x}}^t X_r|\big(\lambda_\mu^\delta - \lambda_\mu\big)\Big|(dr,u)\Big]du \\ & + C\int_t^T \mathbb{E}\Big[\Big(\int_t^T \nabla_{\mathrm{x}}^t X_r\big(\lambda_\mu^\delta - \lambda_\mu\big)(dr,u)\Big)^2\Big]du \;\longrightarrow\; 0. \end{aligned}$$

By Gronwall's lemma, it follows that the first convergence result in (3.20) holds true.

(ii) We notice that $\big(\nabla_{\rm x}^{t,\delta}Y, \nabla_{\rm x}^{t,\delta}Z\big)$ satisfies

$$\begin{aligned}\nabla_{\rm x}^{t,\delta}Y_s =\ & g(X^\delta) - g(X) + \int_s^T \Big(f(u,X^\delta,Y_u^\delta,Z_u^\delta) - f(u,X,Y_u,Z_u)\Big)du - \int_s^T \nabla_{\rm x}^{t,\delta}Z_u dW_u \\ =\ & \int_t^T \big(\nabla_{\rm x}^{t,\delta}X_r\big)\ \lambda_g^\delta(dr) + \int_s^T \Big(\int_t^u \big(\nabla_{\rm x}^{t,\delta}X_r\big)\lambda_f^\delta(dr;u) + A_u^\delta\cdot\big(\nabla_{\rm x}^{t,\delta}Y_u, \nabla_{\rm x}^{t,\delta}Z_u\big)\Big)du \\ & - \int_s^T \nabla_{\rm x}^{t,\delta}Z_u dW_u,\end{aligned}$$

where

$$\big(\lambda_f^\delta(\cdot,u),\lambda_g^\delta(\cdot)\big) \longrightarrow \big(\lambda_f(dr;u),\lambda_g(dr)\big) := \big(\lambda_f(dr;u,X,Y,Z),\lambda_g(dr;X)\big),\ \text{as } \delta\searrow 0,$$

and

$$A_u^\delta\cdot\big(\nabla_{\rm x}^{t,\delta}Y_u, \nabla_{\rm x}^{t,\delta}Z_u\big)\ :=\ \delta^{-1}\Big(f\big(u,X^\delta,Y_u^\delta,Z_u^\delta\big) - f\big(u,X^\delta,Y,Z\big)\Big).$$

Recalling (3.16), it follows that, for all $p\geq 1$,

$$\mathbb{E}\Big[\int_t^T |A_u^\delta - A_u|^p du\Big]\ \longrightarrow\ 0,\ \text{as } \delta\searrow 0.$$

One can then conclude by the stability result for BSDEs (see e.g. [12, Proposition 2.1] or [2, Theorem 4.1 and Remark 4.1]). □

Proof of Theorem 3.6. (i) Let us first fix $\hat{\rm x} := \mathbf{1}_{\{[t,T]\}}$. By (3.15) and (3.18), we observe that v is locally 1/2-Hölder in time and Lipschitz in space. Moreover, by Lemma 3.8, $\nabla_{\rm x}{\rm v}(t,{\rm x}) = \nabla_{\rm x}^t Y_t^{t,{\rm x}}$.

Thus, it is enough to study the regularity of $\nabla_{\rm x}^t Y_t^{t,{\rm x}}$ in $(t,{\rm x})$. By direct computation, one has (using the notation (3.16))

$$\begin{aligned}\Delta(t,{\rm x},{\rm x}') :=\ & \big|\nabla_{\rm x}^t Y_t^{t,{\rm x}} - \nabla_{\rm x}^t Y_t^{t,{\rm x}'}\big|^2 \\ \leq\ & \mathbb{E}\Big[\Big|\int_t^T \nabla_{\rm x}^t X_r^{t,{\rm x}}\lambda_g(dr;X^{t,{\rm x}}) - \int_t^T \nabla_{\rm x}^t X_r^{t,{\rm x}'}\lambda_g(dr;X^{t,{\rm x}'})\Big|^2\Big] \\ & +\ \mathbb{E}\Big[\int_t^T\Big|\int_t^s \nabla_{\rm x}^t X_r^{t,{\rm x}}\lambda_f(dr;\Theta_s^{t,{\rm x}}) - \int_t^s \nabla_{\rm x}^t X_r^{t,{\rm x}'}\lambda_f(dr;\Theta_s^{t,{\rm x}'})\Big|^2 ds\Big] \\ & +\ \mathbb{E}\Big[\int_t^T\Big|(A_s^{t,{\rm x}} - A_s^{t,{\rm x}'})\cdot(\nabla_{\rm x}^t Y_s^{t,{\rm x}}, \nabla_{\rm x}^t Z_s^{t,{\rm x}})\Big|^2 ds\Big].\end{aligned}$$

Further, using Assumption 3.5 and Grownwall's lemma, one can find a constant $C>0$ such that

$$\mathbb{E}\Big[\big\|\nabla_{\rm x}^t X^{t,{\rm x}} - \nabla_{\rm x}^t X^{t,{\rm x}'}\big\|^2\Big]^{\frac12} + \Delta(t,{\rm x},{\rm x}')^{\frac12}\ \leq\ C\|{\rm x}-{\rm x}'\|^\alpha.$$

It follows that $\nabla_{\rm x}{\rm v}$ is α-Hölder in space.

We now repeat the argument used at the end of the proof of Theorem 3.4. In view of (3.18), we can find C, C' such that

$$\begin{aligned}
&\big|\nabla_{\rm x} {\rm v}(t+h^{2+2\alpha}, {\rm x}_{t\wedge}) - \nabla_{\rm x} {\rm v}(t,{\rm x},{\rm x}_t)\big| \\
\le\ & h^{-1}\big|{\rm v}(t+h^{2+2\alpha}, {\rm x}_{t\wedge} + \mathbf{1}_{\{t+h^{2+2\alpha}\}} h) - {\rm v}(t+h^{2+2\alpha}, {\rm x}_{t\wedge}) - {\rm v}(t, {\rm x}\oplus_t h) + {\rm v}(t,{\rm x})\big| + 2Ch^\alpha \\
\le\ & h^{-1}\big|{\rm v}(t+h^{2+2\alpha}, ({\rm x}+\mathbf{1}_{\{t\}}h)_{t\wedge}) - {\rm v}(t,{\rm x}\oplus_t h)\big| + h^{-1}\big|{\rm v}(t,{\rm x}) - {\rm v}(t+h^{2+2\alpha}, {\rm x}_{t\wedge})\big| \\
&\quad + h^{-1}\big|{\rm v}(t+h^{2+2\alpha}, {\rm x}_{t\wedge} + \mathbf{1}_{\{t+h^{2+2\alpha}\}} h) - {\rm v}(t+h^{2+2\alpha}, ({\rm x}+\mathbf{1}_{\{t\}}h)_{t\wedge})\big| + 2Ch^\alpha \\
\le\ & C'\left(h^\alpha(1+\|{\rm x}\|+h) + \lambda([t,t+h))\right),
\end{aligned}$$

for all ${\rm x}\in D([0,T])$ and $t < t+h \le T$.

(ii) Assume now that λ has finitely many atoms at times $(s_i)_{i\le I} \subset [0,T]$ with $s_i < s_{i+1}$ for $i+1 \le I$. Then, [1, Theorem 2.5] can be applied on each time interval $[s_i, s_{i+1})$. Since ${\rm v}(\cdot, X^{t,{\rm x}})$ and $\int_t^\cdot [\nabla_{\rm x}{\rm v}\sigma](s, X^{t,{\rm x}})dW_s$ are continuous, this implies that we can find a continuous (and therefore predictable) orthogonal process A, see [1, Definition 2.3] such that

$$ {\rm v}(\cdot, X^{t,{\rm x}}) = {\rm v}(t, X^{t,{\rm x}}) + \int_t^\cdot [\nabla_{\rm x}{\rm v}\sigma](s, X^{t,{\rm x}})dW_s + A\ . $$

In particular $[A,W]=0$. Thus, the fact that $Y^{t,{\rm x}} = {\rm v}(\cdot, X^{t,{\rm x}})$ implies that $(Z^{t,{\rm x}})^\top = [\nabla_{\rm x}{\rm v}\sigma](\cdot, X^{t,{\rm x}})\ dt\times d\mathbb{P}$ on $[t,T]$ by uniqueness of the co-quadratic variation of semimartingales. □

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