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► **To cite this version:**

Titus Lupu, Hao Wu. A level line of the Gaussian free field with measure-valued boundary conditions. Science China Mathematics, 2023, 67 (2), pp.367-404. 10.1007/s11425-021-2081-8 . hal-03277685

**HAL Id: hal-03277685**

**<https://hal.science/hal-03277685v1>**

Submitted on 4 Jul 2021

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# A level line of the Gaussian free field with measure-valued boundary conditions

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## Abstract

In this article, we construct samples of SLE-like curves out of samples of CLE and Poisson point process of Brownian excursions. We show that the law of these curves depends continuously on the intensity measure of the Brownian excursions. Using such construction of curves, we extend the notion of level lines of GFF to the case when the boundary condition is measure-valued.

**Keywords:** Schramm Loewner evolution (SLE), conformal loop ensemble (CLE), Brownian excursion, Gaussian free field (GFF).

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Continuity of the envelop . . . . .	3
1.2	Identification of the envelop when $\kappa = 4$ . . . . .	4
<b>2</b>	<b>Preliminaries</b>	<b>6</b>
2.1	Local connectedness and cut points . . . . .	6
2.2	Poisson point processes of boundary to boundary excursions . . . . .	9
2.3	Loewner chain and SLE . . . . .	12
2.4	Gaussian free field and level lines . . . . .	13
<b>3</b>	<b>Construction of chordal curves</b>	<b>14</b>
3.1	Proof of Propositions 1.1 and 1.3 . . . . .	14
3.2	Local absolute continuity with respect to $SLE_\kappa$ away from the boundary . . . . .	18
3.3	Curves hitting the boundary with positive measure . . . . .	20
<b>4</b>	<b>Continuous dependence on boundary conditions</b>	<b>22</b>
4.1	Continuous dependence of the Poisson point process of excursions . . . . .	22
4.2	Continuous dependence of the curve $\eta_{\kappa,\nu}$ and proof of Theorem 1.2 . . . . .	27
4.3	Continuous dependence of the driving functions and proof of Proposition 1.4 . . . . .	30

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\*Email: titus.lupu@upmc.fr. Funded by the French National Research Agency (ANR) grant within the project MALIN (ANR-16-CE93-0003).

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<b>5</b>	<b>Identification with level lines of the GFF for <math>\kappa = 4</math></b>	<b>33</b>
5.1	Proof of Theorem 1.7 . . . . .	33
5.2	Proof of Theorem 1.8 . . . . .	35
5.3	An equation for the driving function . . . . .	37
<b>6</b>	<b>Some open questions</b>	<b>40</b>
<b>A</b>	<b>Appendix: Non-negative harmonic functions</b>	<b>40</b>

## 1 Introduction

The goal of the present paper is to construct samples of variants of Schramm Loewner Evolution (SLE) curves out of samples of Conformal Loop Ensemble (CLE) and Poisson point process of Brownian excursions. The intensity of Brownian excursions is parametrized by a non-negative Radon measure  $\nu$  supported on a boundary arc. Our construction generalizes that of [WW13], where instead of the measure  $\nu$  one had a constant. We further show the continuity in law of the curve with respect to the boundary measure  $\nu$ . For the parameter  $\kappa = 4$  of the CLE, we show that the curve we construct is actually distributed as a level line of a Gaussian free field (GFF) with boundary condition  $\nu$ . This is related to random walk/Brownian motion representations of the GFF, known as isomorphism theorems, and in particular relies on a construction in [ALS20]. In [PW17], the authors constructed the level lines of a GFF with regulated boundary conditions and satisfying an additional condition related to the continuation threshold. The construction of our paper allows both to go beyond the regulated setting and to drop the continuation threshold condition.

In order to state our main conclusions properly, let us first introduce the CLE and the Brownian excursions. We denote by  $\mathbb{D}$  the unit disc and by  $\mathbb{H}$  the upper half-plane:

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}, \quad \mathbb{H} := \{z \in \mathbb{C} : \text{Im}(z) > 0\}.$$

We first introduce the CLE. Consider probability measures on collections of countably many continuous simple loops in simply connected domains such that these loops are two-by-two disjoint and non-surrounding. In [SW12], the authors prove that there exists a one-parameter family of such probability measures satisfying conformal invariance, a certain domain Markov property and an extra regularity assumption “local finiteness”. This family is denoted by  $\text{CLE}_\kappa$  with  $\kappa \in (8/3, 4]$ . In a  $\text{CLE}_\kappa$  sample, the loops all locally look like  $\text{SLE}_\kappa$  curves.

We next introduce the Brownian excursion measure. In this paper, “Brownian” will refer to the standard Brownian motion in  $\mathbb{C}$ . Given  $x, y \in \partial\mathbb{D}$ , denote by  $\mu_{x,y}^{\mathbb{D}}$  the non-normalized measure on Brownian excursions from  $x$  to  $y$  in  $\mathbb{D}$  (see Section 2.2). Let  $\sigma_{\partial\mathbb{D}}$  denote the arc-length measure on  $\partial\mathbb{D}$ , with  $\sigma_{\partial\mathbb{D}}(\partial\mathbb{D}) = 2\pi$ . Let  $A_L$  and  $A_R$  denote the left and right half-circles in  $\partial\mathbb{D}$ :

$$A_L = \{z \in \partial\mathbb{D} : \text{Re}(z) < 0\}, \quad A_R = \{z \in \partial\mathbb{D} : \text{Re}(z) > 0\}. \tag{1.1}$$

Let  $\nu$  be a finite non-negative Radon measure on  $\overline{A_L}$ . The Brownian excursion measure  $\mu_\nu^{\mathbb{D}}$  is defined as follows:

$$\mu_\nu^{\mathbb{D}}(\cdot) = \frac{1}{2} \iint_{\overline{A_L} \times \overline{A_L}} d(\nu \otimes \nu)(x, y) \mu_{x,y}^{\mathbb{D}}(\cdot). \tag{1.2}$$

Now, we are ready to state our construction. Fix  $\kappa \in (8/3, 4]$  and let  $\mathfrak{C}_\kappa$  be a  $\text{CLE}_\kappa$  loop ensemble in  $\mathbb{D}$ . Fix  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$  and let  $\Xi_\nu$  be a Poisson point process of intensity  $\mu_\nu^{\mathbb{D}}$ , independent of  $\mathfrak{C}_\kappa$ . For  $\gamma \in \Xi_\nu$ , let  $\mathcal{S}_\kappa(\gamma)$  be the union of  $\gamma$  and all loops in  $\mathfrak{C}_\kappa$  intersecting  $\gamma$ . Let  $\mathcal{S}_{\kappa,\nu}$  be the union of all  $\mathcal{S}_\kappa(\gamma)$  with  $\gamma \in \Xi_\nu$ . In the limit case  $\kappa = 8/3$ , we just set  $\mathcal{S}_{8/3,\nu}$  to be the union of all  $\gamma \in \Xi_\nu$ . By construction,  $\mathcal{S}_{\kappa,\nu} \cap A_R = \emptyset$ . Let  $\mathcal{D}_{\mathbb{R},\kappa,\nu}$  be the connected component of  $\mathbb{D} \setminus (\mathcal{S}_{\kappa,\nu} \cup \overline{A_L})$

that contains  $A_{\mathbb{R}}$ . The set  $\mathcal{D}_{\mathbb{R},\kappa,\nu}$  is of form  $\mathcal{O}_{\kappa,\nu} \cup A_{\mathbb{R}}$ , where  $\mathcal{O}_{\kappa,\nu}$  is an open simply connected domain. Let  $\psi_{\kappa,\nu}$  be the conformal transformation from  $\mathbb{D}$  to  $\mathcal{O}_{\kappa,\nu}$ , uniquely defined by the normalization

$$\psi_{\kappa,\nu}(-i) = -i, \quad \psi_{\kappa,\nu}(1) = 1, \quad \psi_{\kappa,\nu}(i) = i \quad \text{and} \quad \psi_{\kappa,\nu}(A_{\mathbb{R}}) = A_{\mathbb{R}}. \quad (1.3)$$

Set

$$\eta_{\kappa,\nu} := (\partial\mathcal{D}_{\mathbb{R},\kappa,\nu}) \setminus A_{\mathbb{R}}. \quad (1.4)$$

Informally,  $\eta_{\kappa,\nu}$  is constructed as the envelop from the right of the set  $\mathcal{S}_{\kappa,\nu} \cup \overline{A_{\mathbb{L}}}$ . See Figure 1.1.

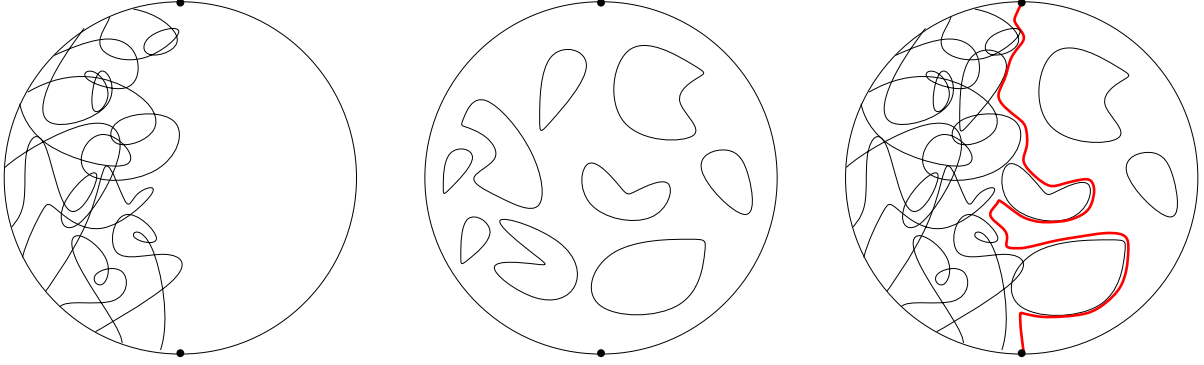


Figure 1.1: In the left panel, the curves indicate a Poisson point process of Brownian excursions with intensity  $\mu_{\nu}^{\mathbb{D}}$ . In the middle panel, the loops indicate a  $\text{CLE}_{\kappa}$  loop ensemble. In the right panel, the curve in red indicates  $\eta_{\kappa,\nu}$ .

## 1.1 Continuity of the envelop

The first goal of this paper is to derive continuity properties of  $\eta_{\kappa,\nu}$ .

**Proposition 1.1.** *Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_{\mathbb{L}}}$ .*

- (1) *The law of  $\eta_{\kappa,\nu}$  is conformal covariant in the following sense: given  $\psi$  a Möbius transformation of  $\mathbb{D}$ , with  $\psi(-i) = -i$  and  $\psi(i) = i$ , the set  $\psi(\eta_{\kappa,\nu})$  is distributed as  $\eta_{\kappa,\nu_{\psi}}$ , where*

$$d\nu_{\psi}(x) = |\psi' \circ \psi^{-1}(x)| d(\psi_{*}\nu)(x).$$

- (2) *The set  $\mathbb{C} \setminus \mathcal{O}_{\kappa,\nu}$  is locally connected. The conformal map  $\psi_{\kappa,\nu}$  extends continuously to  $\overline{\mathbb{D}}$  and  $(\psi_{\kappa,\nu}(x))_{x \in \overline{A_{\mathbb{L}}}}$  parameterizes  $\eta_{\kappa,\nu}$  as a continuous curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$ .*

**Theorem 1.2.** *Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_{\mathbb{L}}}$ . Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_{\mathbb{L}}}$ , converging weakly to  $\nu$ . Then the sequence of curves  $\left( (\psi_{\kappa,\nu_n}(x))_{x \in \overline{A_{\mathbb{L}}}} \right)_{n \geq 0}$  converges in law to  $(\psi_{\kappa,\nu}(x))_{x \in \overline{A_{\mathbb{L}}}}$  for the uniform topology.*

Next, we will consider  $\eta_{\kappa,\nu}$  as a Loewner chain. To this end, it is more convenient to work in  $\mathbb{H}$ . Let  $\psi_0$  be the Möbius transformation from  $\mathbb{D}$  to  $\mathbb{H}$  with  $\psi_0(0) = i$ ,  $\psi_0(-i) = 0$ , and hence  $\psi_0(i) = \infty$ :

$$\psi_0(z) = -i \frac{z+i}{z-i}. \quad (1.5)$$

Define

$$\tilde{\eta}_{\kappa,\nu} := \psi_0(\eta_{\kappa,\nu}).$$

**Proposition 1.3.** Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Suppose the support of  $\nu$  equals  $\overline{A_L}$ . Then we can parameterize  $\tilde{\eta}_{\kappa, \nu}$  (up to its first hitting time of  $\infty$ ) by the half-plane capacity  $(\tilde{\eta}_{\kappa, \nu}(t))_{0 \leq t < T_{\max}}$ , with  $T_{\max} \in (0, +\infty]$ , such that

$$\tilde{\eta}_{\kappa, \nu}(0) = 0, \quad \lim_{t \rightarrow T_{\max}} \tilde{\eta}_{\kappa, \nu}(t) = \infty; \quad \text{hcap}(\tilde{\eta}_{\kappa, \nu}([0, t])) = 2t, \quad \forall t \in [0, T_{\max}). \quad (1.6)$$

When parameterized by the half-plane capacity,  $\tilde{\eta}_{\kappa, \nu}$  is a continuous curve with continuous driving function.

**Proposition 1.4.** Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_L}$ , converging weakly to  $\nu$ . Suppose the supports of  $\nu$  and of  $\nu_n$  equal  $\overline{A_L}$ . When parameterized by the half-plane capacity,  $\tilde{\eta}_{\kappa, \nu_n}$  converges in law to  $\tilde{\eta}_{\kappa, \nu}$  and the driving function of  $\tilde{\eta}_{\kappa, \nu_n}$  converges in law to the driving function of  $\tilde{\eta}_{\kappa, \nu}$  for the local uniform topology.

We will complete the proof of Propositions 1.1 and 1.3 in Section 3. We will complete the proof of Theorem 1.2 and Proposition 1.4 in Section 4. For the proof of Theorem 1.2 we rely on a strong coupling between the Poisson point processes with intensity  $\mu_\nu^{\mathbb{D}}$ , respectively  $\mu_{\nu_n}^{\mathbb{D}}$  (see Proposition 4.1) and on the notion of uniform local connectedness (see Definition 2.1).

In [WW13], the authors construct the same process  $\eta_{\kappa, \nu}$  as in our construction except that they focus on the case when  $\nu$  is a constant  $a > 0$  times the arc-length measure  $\sigma_{\partial\mathbb{D}}$  restricted to  $\overline{A_L}$ . In such case, they prove that  $\eta_{\kappa, \nu}$  has the same law as  $\text{SLE}_\kappa(\rho)$  process where  $\rho > -2$  is uniquely determined by  $\kappa$  and  $a > 0$ , see Theorem 3.7. Readers may wonder whether the law of  $\eta_{\kappa, \nu}$  for general  $\nu$  is the same as  $\text{SLE}_\kappa(\rho)$  process with multiple force points. We will show that  $\eta_{\kappa, \nu}$  is absolutely continuous with respect to  $\text{SLE}_\kappa$  process away from the boundary, see Proposition 3.9. However,  $\eta_{\kappa, \nu}$  is not in the family of  $\text{SLE}_\kappa(\rho)$  processes with multiple force points in general. Here is a preciser answer.

- The answer is negative for  $\kappa \in [8/3, 4)$ . By construction,  $\eta_{\kappa, \nu}$  enjoys “reversibility”: the time-reversal of  $\eta_{\kappa, \nu}$  has the same law as  $\eta_{\kappa, \bar{\nu}}$ , where  $\bar{\nu}$  is the image of  $\nu$  by the reflection  $z \mapsto \bar{z}$ . However,  $\text{SLE}_\kappa(\rho)$  with multiple force points does not have such reversibility in general, see discussion in [Dub07, MS16b].
- The answer is positive for  $\kappa = 4$  and this is related to the second goal of this paper, see Section 5.3.

## 1.2 Identification of the envelop when $\kappa = 4$

The (zero-boundary) Gaussian free field (GFF) in the unit disc  $\mathbb{D}$  is a centered Gaussian process  $\Phi$  indexed by the set of smooth functions with compact support in  $\mathbb{D}$  where the covariance is given by the Green’s function (see Section 2.2):

$$\mathbb{E}[(\Phi, f)(\Phi, g)] = \iint_{\mathbb{D} \times \mathbb{D}} f(z)G_{\mathbb{D}}(z, w)g(w)dzdw.$$

Suppose  $\nu$  is a finite Radon measure on  $\partial\mathbb{D}$ . We will again denote by  $\nu$  the harmonic extension of  $\nu$  in  $\mathbb{D}$ :

$$\nu(z) := \int_{\partial\mathbb{D}} H_{\mathbb{D}}(z, x)d\nu(x), \quad \forall z \in \mathbb{D},$$

where  $H_{\mathbb{D}}(z, x)$  is the Poisson kernel (see Section 2.2). Note also that any non-negative harmonic function on  $\mathbb{D}$  is a harmonic extension of a finite Radon measure on  $\partial\mathbb{D}$ ; see Appendix A. The GFF in  $\mathbb{D}$  with boundary condition  $\nu$  is  $\Phi + \nu$  where  $\Phi$  is a zero-boundary GFF. The definition for GFF in a general simply connected domain can be passed by conformal invariance.

Next, we introduce level lines of GFF. Fix  $\lambda = \sqrt{\pi/8}$ . Suppose  $(\eta(t))_{t \geq 0}$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  and  $\psi_0(\eta)$  has continuous driving function. Define  $\nu_t$  to be the harmonic extension of

the following boundary data on  $\mathbb{D} \setminus \eta[0, t]$ : it is  $2\lambda$  on the left side of  $\eta[0, t]$  and is 0 on the right side of  $\eta[0, t]$ , and it coincides with  $\nu$  on  $\partial\mathbb{D}$ . That is

$$\nu_t(z) = \int_{\partial\mathbb{D}} H_{\mathbb{D} \setminus \eta[0, t]}(z, x) d\nu(x) + 2\lambda H_{\mathbb{D} \setminus \eta[0, t]}(z, \mathcal{L}(\eta[0, t])), \quad z \in \mathbb{D} \setminus \eta[0, t], \quad (1.7)$$

where  $\mathcal{L}(\eta[0, t])$  denotes the left side of  $\eta[0, t]$ .

**Definition 1.5.** *Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{D}$  and  $\nu$  is a finite non-negative Radon measure on  $A_L$ . Suppose  $\eta$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$ . We say that  $\eta$  is a level line of  $\Phi + \nu$  if there exists a coupling  $(\Phi, \eta)$  such that the following domain Markov property holds: for any finite  $\eta$ -stopping time  $\tau$ , the conditional law of  $(\Phi + \nu)|_{\mathbb{D} \setminus \eta[0, \tau]}$  given  $\eta[0, \tau]$  is equal to the law of GFF in  $\mathbb{D} \setminus \eta[0, \tau]$  with boundary condition  $\nu_\tau$  as defined in (1.7).*

The definition in simply connected domains is given via conformal image.

The notion of level lines of GFF originally appears in [Dub09, SS09, SS13]. In particular, the authors of [SS13] prove that, in  $\mathbb{D}$ , the coupling exists when  $\nu = 2\lambda \mathbb{1}_{A_L} \sigma_{\partial\mathbb{D}}$ , and the law of the level line is an SLE<sub>4</sub> in  $\mathbb{D}$  from  $-i$  to  $+i$ . In [WW17], the authors give a survey on level lines of GFF when the boundary condition is piecewise constant; later, in [PW17], the authors construct level lines of GFF when the boundary condition is regulated. In this article, we provide a more general coupling.

First, we recall the result of [ALS20] which relates some level lines of the GFF to envelopes  $\eta_{4, \nu}$  in the case of  $\nu$  being a piecewise constant function; see [ALS20, Proposition 5.11].

**Theorem 1.6** (Aru-Lupu-Sepúlveda [ALS20]). *Fix  $\kappa = 4$ . Let  $u$  be a strictly positive piecewise constant function on  $A_L$  assuming finitely many values. Denote  $\eta_{4, u} := \eta_{4, u \mathbb{1}_{A_L} \sigma_{\partial\mathbb{D}}}$ . In this case, there exists a coupling  $(\Phi, \eta_{4, u})$  such that  $\eta_{4, u}$  is a level line of  $\Phi + u$ .*

We extend the result of [ALS20] beyond the piecewise constant case. Suppose  $\nu$  is a finite non-negative Radon measure on  $\overline{A_L}$ . Denote by  $\text{Atom}(\nu)$  the set of atoms of  $\nu$ , if any. Denote

$$\begin{aligned} \text{Atom}_{\text{conv}}^l(\nu) &:= \left\{ x \in \text{Atom}(\nu) : \int_{\substack{y \in \overline{A_L} \\ \text{Im}(y) > \text{Im}(x)}} \frac{1}{|y - x|^2} d\nu(y) < +\infty \right\}, \\ \text{Atom}_{\text{conv}}^r(\nu) &:= \left\{ x \in \text{Atom}(\nu) : \int_{\substack{y \in \overline{A_L} \\ \text{Im}(y) < \text{Im}(x)}} \frac{1}{|y - x|^2} d\nu(y) < +\infty \right\}. \end{aligned} \quad (1.8)$$

Set

$$\text{Atom}_{\text{conv}}^*(\nu) := \text{Atom}_{\text{conv}}^l(\nu) \cup \text{Atom}_{\text{conv}}^r(\nu). \quad (1.9)$$

**Theorem 1.7.** *Fix  $\kappa = 4$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$  such that the support of  $\nu$  equals  $\overline{A_L}$  and  $\text{Atom}_{\text{conv}}^*(\nu) = \emptyset$ . Then we have the followings.*

- *The curve  $\eta_{4, \nu}$  is a continuous simple curve with continuous driving function.*
- *Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{D}$ . There exists a coupling  $(\Phi, \eta_{4, \nu})$  such that  $\eta_{4, \nu}$  is a level line of  $\Phi + \nu$ .*

The condition  $\text{Atom}_{\text{conv}}^*(\nu) = \emptyset$  above is only to ensure that  $\eta_{4, \nu}$  a.s. does not hit an atom of  $\nu$ . See Lemma 3.6.

**Theorem 1.8.** *If  $(\Phi, \eta_{4, \nu})$  are coupled as in Theorem 1.7, then  $\eta_{4, \nu}$  is almost surely determined by  $\Phi$ .*

We will complete the proof of Theorems 1.7 and 1.8 in Section 5. These two theorems are generalizations of [PW17, Theorems 1.2 and 1.3] where the authors prove the same conclusion under the assumption that  $\nu$  is a regulated function and is bounded away from 0, i.e. the assumption (2.7). Let

us briefly summarize the proof in [PW17]. The idea is to approximate uniformly the regulated function by piecewise constant functions and then show that the level lines corresponding to piecewise constant boundary functions are convergent. For that one shows that the limit of the level lines gives the desired coupling. However, the limiting process is not automatically a continuous curve with continuous driving function. In order to guarantee that the limiting process can be encoded as a Loewner chain with continuous driving function, the authors use the conclusions from [KS17]. The technical assumption from [KS17], related to the probabilities of crossings of quadrilaterals, restricts the method in [PW17] and the authors are only able to show the above conclusion under the assumption (2.7). We will see in Section 5 that our construction improves the conclusion from [PW17]. The continuity result of Theorem 1.2, which replaces the technical tool from [KS17], allows to replace the uniform convergence of the boundary data by the weak convergence of measures, and does not require the assumption (2.7). The above method works for any finite non-negative Radon measure  $\nu$  whose support equals  $\overline{A_L}$  and such that  $\text{Atom}_{\text{conv}}^*(\nu) = \emptyset$ .

We end the introduction with a few interesting open questions. Our construction of level lines of the GFF is a significant generalization of previous constructions, but there is still an interesting scenario that we do not understand. For instance, we do not know what happens to the coupling between the GFF and  $\eta_{4,\nu}$  when the curve hits an atom of  $\nu$  with positive probability, see open questions in Section 6.

Let us come back to the question at the end of Section 1.1. Fix  $\kappa = 4$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$  as in Theorem 1.7. We parameterize the curve  $\eta_{4,\nu}$  by the half-plane capacity. Then  $\eta_{4,\nu}$  can be encoded as a generalization of  $\text{SLE}_4(\rho)$  process on the time interval when it is away from the boundary, see Section 5.3. However, as it is possible for  $\eta_{4,\nu}$  to intersect the boundary with a positive Lebesgue measure (see Section 3.3), we do not know what happens to the driving function when the curve hits the boundary, see open questions in Section 6.

## 2 Preliminaries

### 2.1 Local connectedness and cut points

We will introduce the notions of *local connectedness* and *uniform local connectedness*, and cite and derive a few elementary properties which will be useful later. We refer interested readers to [Pom92, Section 2.2] for more detail.

**Definition 2.1.** *Given  $C$  a closed non-empty subset of  $\mathbb{C}$ , and  $z, z' \in C$ , we say that  $z$  and  $z'$  are  $\varepsilon$ -connected in  $C$  if there is  $K$  a compact connected subset of  $C$  with  $\text{diam}(K) < \varepsilon$  such that  $z, z' \in K$ .*

*A closed non-empty subset  $C \subset \mathbb{C}$  is locally connected if for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that for every  $z, z' \in C$  with  $|z' - z| < \delta$ , the points  $z$  and  $z'$  are  $\varepsilon$ -connected in  $C$ .*

*A family of closed non-empty subsets  $(C_n)_{n \geq 0}$  of  $\mathbb{C}$  is uniformly locally connected if for every  $\varepsilon > 0$ , there is  $\delta > 0$  such that for every  $n \geq 0$  and every  $z, z' \in C_n$  with  $|z' - z| < \delta$ , the points  $z$  and  $z'$  are  $\varepsilon$ -connected in  $C_n$ .*

**Lemma 2.2.** (1) *If  $\gamma : [0, 1] \rightarrow \mathbb{C}$  is a continuous parametrized curve, then  $\text{Range}(\gamma)$  is locally connected.*

(2) *If  $K_1$  and  $K_2$  are two locally connected compact subsets of  $\mathbb{C}$ , then  $K_1 \cup K_2$  is locally connected.*

*Proof.* For (1),  $\text{Range}(\gamma)$  is locally connected since it is the image of the compact locally connected set  $[0, 1]$  by a continuous map; see [New64, Theorem 8.2, Chapter IV] and [Pom92, Section 2.2]. For (2), see [New64, Theorem 8.1, Chapter IV] and the subsequent corollary, and [Pom92, Section 2.2].  $\square$

**Lemma 2.3.** *Let  $(K_n)_{n \geq 0}$  be a sequence of non-empty compact subsets of  $\mathbb{C}$ . Assume that the following four conditions hold:*

(a) *For each  $n \geq 0$ ,  $K_n$  is locally connected.*

(b) For each  $n \geq 1$ ,  $K_n$  is connected.

(c) For each  $n \geq 1$ ,  $K_n \cap K_0 \neq \emptyset$ .

(d)  $\text{diam}(K_n) \rightarrow 0$  as  $n \rightarrow +\infty$ .

Then the union  $\bigcup_{n \geq 0} K_n$  is compact and locally connected.

*Proof.* The compactness of  $\bigcup_{n \geq 0} K_n$  is ensured by the compactness of each  $K_n$  and the conditions (c) and (d). It remains to check the local connectedness.

For  $N \geq 1$ , denote

$$\tilde{K}_N := \bigcup_{n=0}^{N-1} K_n.$$

The condition (a) and Lemma 2.2 (2) ensure that the compact sets  $\tilde{K}_N$  are locally connected. Fix  $\varepsilon > 0$ . Since  $K_0$  is locally connected (the condition (a)), there is  $\delta_0 > 0$  such that for every  $z, z' \in K_0$  with  $|z' - z| < \delta_0$ ,  $z$  and  $z'$  are  $\varepsilon/3$ -connected in  $K_0$ . The condition (d) ensures that there is  $N_0 \geq 1$  such that for every  $n \geq N_0$ ,  $\text{diam}(K_n) < (\varepsilon \wedge \delta_0)/3$ . Further, there is  $\delta_1 > 0$  such that for every  $z, z' \in \tilde{K}_{N_0}$  with  $|z' - z| < \delta_1$ ,  $z$  and  $z'$  are  $\varepsilon/2$ -connected in  $\tilde{K}_{N_0}$ . Then there is  $N_1 \geq N_0$  such that for every  $n \geq N_1$ ,  $\text{diam}(K_n) < (\varepsilon \wedge \delta_1)/2$ . Finally, there is  $\delta_2 > 0$  such that for every  $z, z' \in \tilde{K}_{N_1}$  with  $|z' - z| < \delta_2$ ,  $z$  and  $z'$  are  $\varepsilon$ -connected in  $\tilde{K}_{N_1}$ . Set

$$\delta := \frac{\delta_0}{3} \wedge \frac{\delta_1}{2} \wedge \delta_2 \wedge \frac{\varepsilon}{3}.$$

Take  $z, z' \in \bigcup_{j \geq 0} K_j$  with  $|z' - z| < \delta$ . Since  $z$  and  $z'$  play symmetric roles, there are three cases to consider:

- Case 1:  $z \in K_n, z' \in K_{n'}$ , with  $n, n' \geq N_0$ .
- Case 2:  $z \in \tilde{K}_{N_0}$  and  $z' \in K_{n'}$  with  $n' \leq N_1 - 1$ .
- Case 3:  $z \in \tilde{K}_{N_0}$  and  $z' \in K_{n'}$  with  $n' \geq N_1$ .

In Case 1, the condition (c) ensures that one can take points  $\tilde{z} \in K_n \cap K_0$  and  $\tilde{z}' \in K_{n'} \cap K_0$ . Then

$$|\tilde{z}' - \tilde{z}| < |z' - z| + \frac{2}{3}\delta_0 < \delta_0.$$

So there is  $K$  a compact connected subset of  $K_0$  with  $\text{diam}(K) < \varepsilon/3$  such that  $\tilde{z}, \tilde{z}' \in K$ . Then  $K \cup K_n \cup K_{n'}$  is a compact subset of  $\bigcup_{j \geq 0} K_j$ , by the condition (b), it is connected, it contains  $z$  and  $z'$ , and

$$\text{diam}(K \cup K_n \cup K_{n'}) \leq \text{diam}(K) + \text{diam}(K_n) + \text{diam}(K_{n'}) < \varepsilon.$$

In Case 2, we have  $z, z' \in \tilde{K}_{N_1}$  and  $|z' - z| < \delta_2$ . So  $z'$  and  $z$  are  $\varepsilon$ -connected in  $\tilde{K}_{N_1}$ , and thus in  $\bigcup_{j \geq 0} K_j$ .

In Case 3, consider  $\tilde{z}' \in K_{n'} \cap K_0$ . Then

$$|\tilde{z}' - z| < |z' - z| + \frac{1}{2}\delta_1 < \delta_1.$$

Thus, there is  $K$  a compact connected subset of  $\tilde{K}_{N_0}$  with  $\text{diam}(K) < \varepsilon/2$  such that  $z, \tilde{z}' \in K$ . Then  $K \cup K_{n'}$  is a compact connected subset of  $\bigcup_{j \geq 0} K_j$  containing  $z$  and  $z'$ , and  $\text{diam}(K \cup K_{n'}) < \varepsilon$ .  $\square$

**Lemma 2.4.** (1) Let  $C$  and  $\tilde{C}$  be closed non-empty subsets of  $\mathbb{C}$ , with  $\tilde{C} \subset C$ . Assume that  $C \setminus \tilde{C}$  is an open subset of  $\mathbb{C}$ . Then, if  $\tilde{C}$  is locally connected, then so is  $C$ .



(2) Let  $(C_n)_{n \geq 0}$  and  $(\tilde{C}_n)_{n \geq 0}$  be two families of closed non-empty subsets of  $\mathbb{C}$ . Assume that for every  $n \geq 0$ ,  $\tilde{C}_n \subset C_n$  and  $C_n \setminus \tilde{C}_n$  is an open subset of  $\mathbb{C}$ . Then, if the family  $(\tilde{C}_n)_{n \geq 0}$  is uniformly locally connected, then so is the family  $(C_n)_{n \geq 0}$ .

*Proof.* Since (2) clearly implies (1), it suffices to show (2). Fix  $\varepsilon > 0$ ,  $n \geq 0$  and  $z, z' \in C_n$  with  $|z' - z| < \varepsilon/2$ . One can consider the straight line segment  $I_{z,z'}$  joining  $z$  and  $z'$ . If  $I_{z,z'} \cap \tilde{C}_n = \emptyset$ , then necessarily  $I_{z,z'} \subset C_n \setminus \tilde{C}_n \subset C_n$ , because  $C_n \setminus \tilde{C}_n$  is open. Otherwise, one can consider  $\tilde{z}$  the point of  $I_{z,z'} \cap \tilde{C}_n$  which is the closest to  $z$ , and  $\tilde{z}'$  the point of  $I_{z,z'} \cap \tilde{C}_n$  which is the closest to  $z'$ . By construction,  $|\tilde{z}' - \tilde{z}| \leq |z' - z|$ . Let  $I_{z,\tilde{z}}$ , respectively  $I_{\tilde{z}',z'}$ , be the subsegment joining  $z$  and  $\tilde{z}$ , respectively  $z'$  and  $\tilde{z}'$ . We have that  $I_{z,\tilde{z}} \cup I_{\tilde{z}',z'} \subset C_n$ . Thus,  $z$  and  $z'$  are  $\varepsilon$ -connected in  $C_n$  as soon as  $\tilde{z}$  and  $\tilde{z}'$  are  $\varepsilon/2$ -connected in  $\tilde{C}_n$ .  $\square$

Next, we will introduce the notion of *cut points*, and cite a few elementary properties which will be useful later. We refer interested readers to [Pom92, Section 2.3] for more detail.

**Definition 2.5.** Given  $C$  a closed connected non-empty subset of  $\mathbb{C}$ , a point  $z \in C$  is said to be a *cut point* of  $C$  if  $C \setminus \{z\}$  is not connected.

Next lemma is standard and we state it without proof.

**Lemma 2.6.** Let  $K$  be a compact connected subset of the Riemann sphere  $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \cong \mathbb{S}^2$ , such that  $K$  and  $\widehat{\mathbb{C}} \setminus K$  are both non-empty. Then for every  $\mathcal{O}$  connected component of  $\widehat{\mathbb{C}} \setminus K$ ,  $\mathcal{O}$  is simply connected, and in particular there is a conformal transformation from  $\mathbb{D}$  to  $\mathcal{O}$ . The boundary  $\partial\mathcal{O}$  is connected.

**Lemma 2.7.** Let  $K$  be a compact connected non-empty subset of  $\mathbb{C}$ . Assume that  $K$  has no cut points. Then for every  $\mathcal{O}$  connected component of  $\mathbb{C} \setminus K$ ,  $\partial\mathcal{O}$  has no cut points.

*Proof.* Since we can always consider the Riemann sphere  $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ , we assume without loss of generality that  $\mathcal{O}$  is bounded. Assume that  $z$  is a cut point of  $\partial\mathcal{O}$ . It follows from the proof of [Pom92, Proposition 2.5] that there are two points  $z_1, z_2 \in \partial\mathcal{O}$  that are in two distinct connected components of  $(\mathbb{C} \setminus \mathcal{O}) \setminus \{z\}$ . Since  $K \subset \mathbb{C} \setminus \mathcal{O}$ , the points  $z_1$  and  $z_2$  are also in two distinct connected components of  $K \setminus \{z\}$ . This contradicts the assumptions.  $\square$

Next we recall Carathéodory's theorem on the extension of conformal maps to the boundary; see [Pom92, Theorem 2.1, Theorem 2.6, Corollary 2.8].

**Theorem 2.8.** Let  $D$  be an open bounded simply connected domain in  $\mathbb{C}$ . Let  $\psi$  be a conformal map from  $\mathbb{D}$  to  $D$ .

- (1) If  $\mathbb{C} \setminus D$  is locally connected, then  $\psi$  extends continuously to  $\overline{\mathbb{D}}$ . In particular  $\partial D$  can be parametrized as a continuous closed curve.
- (2) If on top of that,  $\partial D$  has no cut points, then  $\partial D$  is a Jordan curve, i.e. continuous closed simple curve, and  $\psi$  extends to a homeomorphism from  $\overline{\mathbb{D}}$  to  $\overline{D}$ .

Next we recall the notion of *Carathéodory convergence*. See [Pom92, Section 1.4].

**Definition 2.9.** Let  $D$  and  $(D_n)_{n \geq 0}$  be open non-empty simply connected domains in  $\mathbb{C}$ , different from  $\mathbb{C}$ . Let  $w \in D$ , respectively  $w_n \in D_n$ . The sequence of marked domains  $((D_n, w_n))_{n \geq 0}$  is said to converge to  $(D, w)$  in the Carathéodory sense if the following holds:

- (1)  $w_n \rightarrow w$ ;

(2) for every  $z \in D$ , there is a neighborhood  $U$  of  $z$  in  $D$  such that

$$U \subset \bigcap_{n \geq m} D_n$$

for  $m$  large enough.

(3) for every  $z \in \partial D$ , there exist  $z_n \in D_n$  such that  $z_n \rightarrow z$  as  $n \rightarrow +\infty$ .

Note that the Carathéodory convergence does not imply that  $D_n$  converges  $D$  for the Hausdorff distance, even for  $D$  bounded.

## 2.2 Poisson point processes of boundary to boundary excursions

Recall that  $\mathbb{D}$  denotes the unit disc and  $A_L, A_R$  denote the left and right half-circles in  $\partial\mathbb{D}$  as in (1.1). We first introduce Green's function and Poisson kernel. Denote by  $G_{\mathbb{D}}(z, w)$  the Green's function on  $\mathbb{D}$  with Dirichlet 0 boundary conditions:

$$G_{\mathbb{D}}(z, w) = \frac{1}{2\pi} \log \left| \frac{1 - \bar{z}w}{z - w} \right|, \quad z \neq w \in \mathbb{D}.$$

For any simply connected domain  $D$ , we define Green's function via conformal image. Let  $\psi : \mathbb{D} \rightarrow D$  be any conformal map, we have

$$G_D(z, w) = G_{\mathbb{D}}(\psi(z), \psi(w)), \quad z \neq w \in D.$$

Denote by  $H_{\mathbb{D}}(z, x)$  the Poisson kernel on  $\mathbb{D}$ :

$$H_{\mathbb{D}}(z, x) = \frac{1}{2\pi} \frac{1 - |z|^2}{|x - z|^2}, \quad z \in \mathbb{D}, x \in \partial\mathbb{D}. \quad (2.1)$$

For any simply connected domain  $D$  with a boundary point  $x \in \partial D$  such that  $\partial D$  is analytic in neighborhood of  $x$ , we define Poisson kernel via conformal image. Let  $\psi : \mathbb{D} \rightarrow D$  be any conformal map, we have

$$H_D(z, x) = |\psi' \circ \psi^{-1}(x)|^{-1} H_{\mathbb{D}}(\psi^{-1}(z), \psi^{-1}(x)), \quad z \in D, x \in \partial D.$$

Denote by  $H_{\mathbb{D}}(x, y)$  the boundary Poisson kernel on  $\partial\mathbb{D}$  (see [Law05, Section 5.2]):

$$H_{\mathbb{D}}(x, y) = \frac{1}{\pi|y - x|^2}, \quad x \neq y \in \partial\mathbb{D}.$$

For any simply connected domain  $D$  with two boundary points  $x, y$  such that  $\partial D$  is analytic in neighborhoods of  $x$  and  $y$ , we define the boundary Poisson kernel via conformal image. Let  $\psi : \mathbb{D} \rightarrow D$  be any conformal map, we have

$$H_D(x, y) = |\psi' \circ \psi^{-1}(x)|^{-1} |\psi' \circ \psi^{-1}(y)|^{-1} H_{\mathbb{D}}(\psi^{-1}(x), \psi^{-1}(y)), \quad x \neq y \in \partial D.$$

Next, we describe the measures on Brownian excursions. Given  $x \neq y \in \partial\mathbb{D}$ , denote by  $\mu_{x,y}^{\mathbb{D},\#}$  the normalized probability measure on Brownian excursions from  $x$  to  $y$  in  $\mathbb{D}$ ; see [Law05, Section 5.2]. Denote by  $\mu_{x,y}^{\mathbb{D}}$  the non-normalized measure

$$\mu_{x,y}^{\mathbb{D}} := H_{\mathbb{D}}(x, y) \mu_{x,y}^{\mathbb{D},\#}.$$

For  $x \in \partial\mathbb{D}$ , let  $\mu_{x,x}^{\mathbb{D}}$  denote the measure on Brownian excursions from  $x$  to  $x$  in  $\mathbb{D}$

$$\mu_{x,x}^{\mathbb{D}} = \lim_{y \rightarrow x} \lim_{y \in \partial\mathbb{D} \setminus \{x\}} \mu_{x,y}^{\mathbb{D}}.$$

Note that  $\mu_{x,x}^{\mathbb{D}}$  is up to a constant the Brownian bubble measure of [Law05, Section 5.5]. It has infinite total mass. However, for every  $\varepsilon > 0$ ,

$$\mu_{x,x}^{\mathbb{D}}(\{\gamma : \text{diam Range}(\gamma) > \varepsilon\}) < +\infty.$$

For a general simply connected domain  $D$  with two boundary points  $x, y$  such that  $\partial D$  is analytic in neighborhoods of  $x, y$ , we may extend the definition of Brownian excursion measure via conformal image: Let  $\psi : \mathbb{D} \rightarrow D$  be any conformal map,

$$\mu_{x,y}^D = |\psi' \circ \psi^{-1}(x)|^{-1} |\psi' \circ \psi^{-1}(y)|^{-1} \psi_* \mu_{\psi^{-1}(x), \psi^{-1}(y)}^{\mathbb{D}}.$$

The total mass of  $\mu_{x,y}^D$  is given by  $H_D(x, y)$ .

Suppose  $\nu$  is a finite non-negative Radon measure on  $\overline{A_L}$ , we separate its atomic and non-atomic parts:

$$\text{Atom}(\nu) := \{x \in \overline{A_L} : \nu(\{x\}) > 0\}, \quad \hat{\nu} := \nu - \sum_{x \in \text{Atom}(\nu)} \nu(\{x\}) \delta_x.$$

Note that  $\text{Atom}(\nu)$  is at most countable. We define  $\mu_{\nu}^{\mathbb{D}}$  as in (1.2) and we see that

$$\begin{aligned} \mu_{\nu}^{\mathbb{D}}(\cdot) &= \frac{1}{2} \iint_{\overline{A_L} \times \overline{A_L}} d(\nu \otimes \nu)(x, y) \mu_{x,y}^{\mathbb{D}}(\cdot) \\ &= \frac{1}{2} \iint_{A_L \times A_L} d\hat{\nu}(x) d\hat{\nu}(y) \mu_{x,y}^{\mathbb{D}}(\cdot) + \frac{1}{2} \sum_{x \in \text{Atom}(\nu)} \nu(\{x\}) \int_{A_L} d\hat{\nu}(y) \mu_{x,y}^{\mathbb{D}}(\cdot) \\ &\quad + \frac{1}{2} \sum_{y \in \text{Atom}(\nu)} \nu(\{y\}) \int_{A_L} d\hat{\nu}(x) \mu_{x,y}^{\mathbb{D}}(\cdot) + \frac{1}{2} \sum_{(x,y) \in \text{Atom}(\nu)^2} \nu(\{x\}) \nu(\{y\}) \mu_{x,y}^{\mathbb{D}}(\cdot). \end{aligned}$$

Note that the last of the four terms above also involves measures  $\mu_{x,x}^{\mathbb{D}}$ . All other terms only involve measures  $\mu_{x,y}^{\mathbb{D}}$  for  $y \neq x$ . The measure  $\mu_{\nu}^{\mathbb{D}}$  is conformally covariant in the following sense. If  $D \subsetneq \mathbb{C}$  is an open simply connected domain with piecewise analytic boundary and  $\psi$  is a conformal transformation from  $\mathbb{D}$  to  $D$ , then the image of  $\mu_{\nu}^{\mathbb{D}}$  by  $\psi$  is the measure

$$\iint_{\psi(A_L) \times \psi(A_L)} d((\psi_* \nu) \otimes (\psi_* \nu))(x, y) |\psi' \circ \psi^{-1}(x)| |\psi' \circ \psi^{-1}(y)| \mu_{x,y}^D(d\gamma)$$

up to a change of time in excursions  $ds = |(\psi^{-1})'(\gamma(t))|^2 dt$ .

Denote by  $\Xi_{\nu}$  the Poisson point process of intensity  $\mu_{\nu}^{\mathbb{D}}$ . We see it as a random at most countable collection of time-parametrized Brownian boundary-to-boundary excursions in  $\mathbb{D}$ . Given  $\varepsilon > 0$ , denote

$$\Xi_{\nu, \varepsilon} := \{\gamma \in \Xi_{\nu} : \text{diam Range}(\gamma) > \varepsilon\}. \quad (2.2)$$

**Lemma 2.10.** *Let  $\nu$  be a finite non-negative non-zero Radon measure on  $\overline{A_L}$ . Then  $\Xi_{\nu}$  satisfies the following.*

- (1) *A.s. for every  $\varepsilon > 0$ , the subset  $\Xi_{\nu, \varepsilon}$  is finite.*
- (2) *A.s. for every subarc  $A$  of  $\overline{A_L}$  such that  $\nu(A) > 0$ , the subset  $\{\gamma \in \Xi_{\nu} : \gamma \text{ has both ends in } A\}$  is infinite.*

*Proof.* The first point comes from that

$$\sup_{x,y \in \partial \mathbb{D}} \mu_{x,y}^{\mathbb{D}}(\{\gamma : \text{diam Range}(\gamma) > \varepsilon\}) < +\infty.$$

See [Law05, Section 5.2].

For the second point it is enough to restrict to a countable collection of subarcs. If such a subarc  $A$  contains an atom  $x_0$  of  $\nu$ , then  $\mu_{\nu}^{\mathbb{D}} \geq \frac{1}{2}\nu\{x_0\}^2\mu_{x_0,x_0}^{\mathbb{D}}$ , and the measure  $\mu_{x_0,x_0}^{\mathbb{D}}$  on excursions with both endpoints in  $x_0$  has infinite total mass. If  $A \cap \text{Atom}(\nu) = \emptyset$ , then one needs to check that

$$\iint_{A \times A} \frac{1}{|y-x|^2} d\nu(x) d\nu(y) = +\infty.$$

The integral above is the two-dimensional energy of the measure  $\mathbb{1}_A \nu$ ; see [BP16, Definition 3.4.1]. Since the Hausdorff dimension of  $A$  is 1, and in particular smaller than 2, the two-dimensional energy equals  $+\infty$ ; see [BP16, Theorem 3.4.2].  $\square$

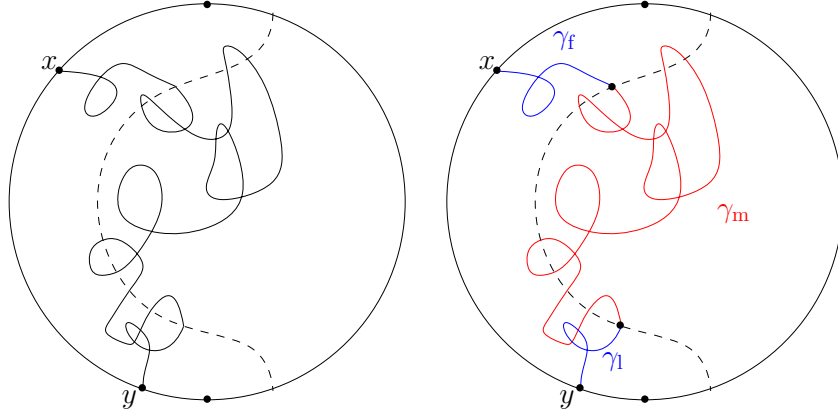


Figure 2.1: In the left panel, the dashed line indicates  $\partial \mathbb{D}_\varepsilon \cap \mathbb{D}$  and the curve indicates a continuous path  $\gamma$  intersecting  $\mathbb{D}_\varepsilon$ . In the right panel, the path  $\gamma$  is decomposed into three pieces. The first piece is  $\gamma_f$  (in blue): it is the part of  $\gamma$  from 0 to  $T_{\gamma,\varepsilon}^f$ . The second piece is  $\gamma_m$  (in red): it is the part of  $\gamma$  from  $T_{\gamma,\varepsilon}^f$  to  $T_{\gamma,\varepsilon}^l$ . The last piece is  $\gamma_l$  (in blue): it is the part of  $\gamma$  from  $T_{\gamma,\varepsilon}^l$  to  $T_\gamma$ .

Next we describe the Markovian decomposition of the measures on Brownian excursions. Given  $z \neq w \in \mathbb{D}$ , denote by  $\mu_{z,w}^{\mathbb{D},\#}$  the normalized probability measure on Brownian excursions from  $z$  to  $w$  in  $\mathbb{D}$ ; see [Law05, Section 5.2]. Denote by  $\mu_{z,w}^{\mathbb{D}}$  the non-normalized measure

$$\mu_{z,w}^{\mathbb{D}} := G_{\mathbb{D}}(z,w) \mu_{z,w}^{\mathbb{D},\#}.$$

For  $\varepsilon \in (0,1)$ , denote

$$\mathbb{D}_\varepsilon := \{z \in \mathbb{D} : \text{dist}(z, A_L) > \varepsilon\}, \quad \widehat{\mathbb{D}}_\varepsilon := \{z \in \mathbb{D} : \text{dist}(z, A_L) < \varepsilon\}. \quad (2.3)$$

The domain  $\widehat{\mathbb{D}}_\varepsilon$  is open and simply connected, with piecewise analytic boundary. Recall that  $\mu_{x,y}^{\widehat{\mathbb{D}}_\varepsilon}$  denotes the non-normalized measure on Brownian excursions from  $x$  to  $y$  in  $\widehat{\mathbb{D}}_\varepsilon$ . Denote by  $\sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}$  the arc-length measure on  $\partial \widehat{\mathbb{D}}_\varepsilon$ . Denote by  $T_\gamma$  the total duration of a generic element  $\gamma$  of  $\Xi_\nu$ . Given  $(\gamma(t))_{0 \leq t \leq T_\gamma}$  a continuous path intersecting  $\mathbb{D}_\varepsilon$ , denote

$$T_{\gamma,\varepsilon}^f := \inf\{t \in (0, T_\gamma) : \gamma(t) \in \mathbb{D}_\varepsilon\}, \quad T_{\gamma,\varepsilon}^l := \sup\{t \in (0, T_\gamma) : \gamma(t) \in \mathbb{D}_\varepsilon\}. \quad (2.4)$$

The Markovian decomposition is as follows. For details, see [Law05, Section 5.2] and [ALS20, Proposition 3.7]. See also Figure 2.1 for illustration.

**Proposition 2.11.** Fix  $\varepsilon \in (0, 1)$ . We will denote by  $F$  an arbitrary bounded measurable functional on the appropriate space. For  $x, y \in \overline{A_L}$ ,

$$\begin{aligned} & \int_{\substack{\gamma \text{ s.t.} \\ \text{Range}(\gamma) \cap \mathbb{D}_\varepsilon \neq \emptyset}} F(\gamma(T_{\gamma,\varepsilon}^f), \gamma(T_{\gamma,\varepsilon}^l), (\gamma(t))_{0 \leq t \leq T_{\gamma,\varepsilon}^f}, (\gamma(T_{\gamma,\varepsilon}^f + t))_{0 \leq t \leq T_{\gamma,\varepsilon}^l - T_{\gamma,\varepsilon}^f}, (\gamma(T_{\gamma,\varepsilon}^l + t))_{0 \leq t \leq T_{\gamma,\varepsilon}^l - T_{\gamma,\varepsilon}^f}) \mu_{x,y}^{\mathbb{D}}(d\gamma) \\ = & \iint_{(\partial \widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D})^2} \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dz) \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dw) \iiint \mu_{x,z}^{\widehat{\mathbb{D}}_\varepsilon}(d\gamma_f) \mu_{z,w}^{\mathbb{D}}(d\gamma_m) \mu_{w,y}^{\widehat{\mathbb{D}}_\varepsilon}(d\gamma_l) F(z, w, \gamma_f, \gamma_m, \gamma_l). \end{aligned}$$

In particular,

$$\begin{aligned} & \int_{\substack{\gamma \text{ s.t.} \\ \text{Range}(\gamma) \cap \mathbb{D}_\varepsilon \neq \emptyset}} F(\gamma(T_{\gamma,\varepsilon}^f), \gamma(T_{\gamma,\varepsilon}^l), (\gamma(T_{\gamma,\varepsilon}^f + t))_{0 \leq t \leq T_{\gamma,\varepsilon}^l - T_{\gamma,\varepsilon}^f}) \mu_{x,y}^{\mathbb{D}}(d\gamma) \\ = & \iint_{(\partial \widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D})^2} \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dz) \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dw) H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) H_{\widehat{\mathbb{D}}_\varepsilon}(w, y) \int \mu_{z,w}^{\mathbb{D}}(d\gamma_m) F(z, w, \gamma_m). \end{aligned}$$

### 2.3 Loewner chain and SLE

Recall that  $\psi_0$  is the Möbius transformation from  $\mathbb{D}$  to  $\mathbb{H}$  as in (1.5). Since we will deal with conformally invariant objects, working in  $\mathbb{D}$  or in  $\mathbb{H}$  will be equivalent, but it will be more convenient to handle Brownian excursions in  $\mathbb{D}$ , and to work with Loewner chains in  $\mathbb{H}$ .

An  $\mathbb{H}$ -hull is a compact subset  $K$  of  $\overline{\mathbb{H}}$  such that  $\mathbb{H} \setminus K$  is simply connected. By Riemann's mapping theorem, there exists a unique conformal map  $g_K$  from  $\mathbb{H} \setminus K$  onto  $\mathbb{H}$  with the hydrodynamic normalization  $\lim_{z \rightarrow \infty} |g_K(z) - z| = 0$ . The quantity

$$\text{hcap}(K) := \lim_{z \rightarrow \infty} z(g_K(z) - z)$$

is non-negative and we call it the half-plane capacity of  $K$  (seen from  $\infty$ ). For background on the half-plane capacity, see [Law05, Section 3.4] and [BN16, Section 6.2].

Loewner chain is a collection of  $\mathbb{H}$ -hulls  $(K_t)_{t \geq 0}$  associated to the family of conformal maps  $(g_t)_{t \geq 0}$  which solves the following Loewner equation: for each  $z \in \mathbb{H}$ ,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}, \quad g_0(z) = z, \quad (2.5)$$

where  $(\xi_t)_{t \geq 0}$  is a one-dimensional continuous function which we call the driving function. For  $z \in \overline{\mathbb{H}}$ , the swallowing time of  $z$  is defined to be  $\sup \{t \geq 0 : \min_{s \in [0, t]} |g_s(z) - \xi_s| > 0\}$ . Let  $K_t$  be the closure of  $\{z \in \mathbb{H} : T_z \leq t\}$ . It turns out that  $g_t$  is the unique conformal map from  $\mathbb{H} \setminus K_t$  onto  $\mathbb{H}$  with normalization  $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ . Since  $\text{hcap}(K_t) = \lim_{z \rightarrow \infty} z(g_t(z) - z) = 2t$ , we say that the process  $(K_t)_{t \geq 0}$  is parameterized by the half-plane capacity. We say that  $(K_t)_{t \geq 0}$  can be generated by continuous curve  $(\eta(t))_{t \geq 0}$  if, for any  $t$ , the unbounded connected component of  $\mathbb{H} \setminus \eta[0, t]$  is the same as  $\mathbb{H} \setminus K_t$ .

The following proposition explains which kind of continuous curve enjoys continuous driving function.

**Proposition 2.12.** Suppose  $T \in (0, \infty]$ . Let  $\eta : [0, T) \rightarrow \overline{\mathbb{H}}$  be a continuous curve with  $\eta(0) = 0$ . Assume the following hold: for every  $t \in (0, T)$ ,

- (a)  $\eta(t, T)$  is contained in the closure of the unbounded connected component of  $\mathbb{H} \setminus \eta[0, t]$ ,
- (b)  $\eta^{-1}(\eta[0, t] \cup \mathbb{R})$  has empty interior in  $(t, T)$ .

For each  $t > 0$ , let  $g_t$  be the conformal map from the unbounded connected component of  $\mathbb{H} \setminus \eta[0, t]$  onto  $\mathbb{H}$  with normalization  $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ . After reparameterization,  $(g_t)_{t \geq 0}$  solves (2.5) with continuous driving function  $(\xi_t)_{t \geq 0}$ .

*Proof.* See [Pom66], [Kin15, Theorem 1.2], [Law05, Section 4] and [MS16a, Proposition 6.12].  $\square$

Schramm Loewner evolution (SLE) is a Loewner chain with driving function equal a multiple of Brownian motion. For  $\kappa > 0$ ,  $\text{SLE}_\kappa$  is the Loewner chain with driving function  $\xi_t = \sqrt{\kappa} B_t$  where  $(B_t)_{t \geq 0}$  is a standard one-dimensional Brownian motion. It is known that  $\text{SLE}_\kappa$  is almost surely generated by a continuous curve for all  $\kappa$ ; see [RS05]. In particular, when  $\kappa \in (0, 4]$ , it is a simple curve.

## 2.4 Gaussian free field and level lines

In this section, we will collect some known results on level lines of GFF from [Dub09, SS09, SS13, WW17, PW17] and relate the level lines to variants of  $\text{SLE}_4$  process. To this end, it is more convenient to work in  $\mathbb{H}$ .

We first consider the case when GFF has piecewise constant boundary data. Suppose  $x_n < \dots < x_1 < 0$  and  $\rho_n, \dots, \rho_1 \in \mathbb{R}$ . Denote

$$\bar{\rho}_k := \sum_{j=1}^k \rho_j, \quad \text{for all } k \in \{1, \dots, n\}.$$

Consider GFF on  $\mathbb{H}$  with the following boundary data:

$$\zeta(x) = 2\lambda \mathbb{1}_{(x_1, 0)}(x) + \sum_{k=1}^n \lambda(2 + \bar{\rho}_k) \mathbb{1}_{(x_{k+1}, x_k]}(x), \quad x < 0; \quad \zeta(x) = 0, \quad x > 0, \quad (2.6)$$

where we use the convention that  $x_{n+1} = -\infty$ . Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{H}$  and suppose  $\bar{\rho}_k > -2$  for all  $k \in \{1, \dots, n\}$ . Then the level line of  $\Phi + \zeta$  exists and is uniquely determined by  $\Phi$ . Furthermore, it is a continuous curve with continuous driving function  $(\xi_t)_{t \geq 0}$  which is the solution to the following SDE:

$$d\xi_t = 2dB_t + \sum_{j=1}^n \frac{\rho_j dt}{\xi_t - V_t^j}, \quad dV_t^k = \frac{2dt}{V_t^k - \xi_t}, \quad \text{for } k \in \{1, \dots, n\},$$

with initial values  $\xi(0) = 0$  and  $V_0^k = x_k$  for  $k \in \{1, \dots, n\}$ . Note that the Loewner chain with the above driving function is called  $\text{SLE}_4(\rho_n, \dots, \rho_1)$  process with force points  $(x_n, \dots, x_1)$ . For more detail on  $\text{SLE}_\kappa(\rho)$  with multiple force points, see [MS16a, Section 2.2].

Next, we consider GFF with regulated boundary conditions. Suppose the boundary condition is a regulated function  $\zeta$  on  $\mathbb{R}$ . Assume that there exists  $\epsilon > 0$  such that

$$\zeta(x) \geq \epsilon, \quad x < 0; \quad \zeta(x) = 0, \quad x > 0. \quad (2.7)$$

The authors in [PW17] prove that there exists a coupling  $(\Phi, \eta)$  as in Definition 1.5 with boundary data  $\zeta$  and  $\eta$  is a continuous simple curve with continuous driving function. Furthermore, they identify the law of  $\eta$  when  $\zeta$  is of bounded variation.

Suppose  $\zeta$  is of bounded variation and  $\zeta = 0$  on  $\mathbb{R}_+$ . Such function can be described almost every as the integral of a finite signed Radon measure  $\rho$  on  $(-\infty, 0]$ :

$$\zeta(x^+) = 2\lambda + \lambda\rho((x, 0]), \quad x < 0. \quad (2.8)$$

Suppose that there exists  $\epsilon > 0$  such that

$$\rho((x, 0]) \geq -2 + \epsilon/\lambda, \quad x < 0. \quad (2.9)$$

Under the assumption (2.9), the authors in [PW17] prove that the law of  $\eta$  is an  $\text{SLE}_4(\rho)$  process defined as follows.

**Definition 2.13.** Suppose  $(B_t)_{t \geq 0}$  is one-dimensional Brownian motion. We say that the process

$$(\xi_t, (V_t(x))_{x \leq 0})_{t \geq 0}$$

describes an  $\text{SLE}_4(\rho)$  process if it is adapted to the filtration of  $B$  and the following hold:

- We have  $\xi_0 = 0$  and  $V_0(x) = x$  for  $x \leq 0$ .
- The processes  $B_t, \xi_t, (V_t(x))_{x \leq 0}$  satisfy the following SDE on time intervals where  $\xi_t$  does not collide with any of the  $V_t(x)$ :

$$d\xi_t = 2dB_t + \left( \int_{(-\infty, 0]} \frac{d\rho(x)}{\xi_t - V_t(x)} \right) dt, \quad dV_t(x) = \frac{2dt}{V_t(x) - \xi_t}, \quad x \leq 0. \quad (2.10)$$

- We have instantaneous reflection of  $\xi_t$  off the  $V_t(x)$ , i.e. it is almost surely the case that for Lebesgue almost all times  $t$  we have that  $\xi_t \neq V_t(x)$  for each  $x \leq 0$ .

The  $\text{SLE}_4(\rho)$  process is then defined to be the Loewner chain with driving function  $(\xi_t)_{t \geq 0}$ .

Note that the existence of  $\text{SLE}_4(\rho)$  is not clear from the above definition. It is part of the conclusion from [PW17] that there exists an  $\text{SLE}_4(\rho)$  process under the assumption (2.9) and it is a continuous simple curve with continuous driving function. We emphasize that [PW17] only provides the existence of  $\text{SLE}_4(\rho)$ , and it does not give the uniqueness in law.

### 3 Construction of chordal curves

#### 3.1 Proof of Propositions 1.1 and 1.3

Let us recall the construction of  $\eta_{\kappa, \nu}$  given in the introduction and provide more detail. Our construction of chordal curves in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  involves two ingredients: Brownian excursions introduced in Section 2.2 and conformal loop ensembles  $\text{CLE}_\kappa$  with  $\kappa \in (8/3, 4]$ . For the construction of the CLE, see [She09, SW12]. Note that according to [SW12], a  $\text{CLE}_\kappa$  is also the set of outermost boundaries of clusters in Brownian loop-soups that were introduced in [LW04]. Here we emphasize that  $\text{CLE}_\kappa$  satisfies a local finiteness property: a.s., for every  $\varepsilon > 0$ , there are only finitely many loops of diameter greater than  $\varepsilon$ .

Here is our construction. Fix  $\kappa \in (8/3, 4]$  and let  $\mathfrak{C}_\kappa$  denote a  $\text{CLE}_\kappa$  loop ensemble. Fix  $\nu$  a finite non-negative Radon measure on  $\overline{A}_L$  and let  $\Xi_\nu$  be a Poisson point process of excursions of intensity  $\mu_\nu^\mathbb{D}$ , independent of  $\mathfrak{C}_\kappa$ . For  $\gamma \in \Xi_\nu$ , denote

$$\tilde{\mathfrak{C}}_\kappa(\gamma) := \{\tilde{\gamma} \in \mathfrak{C}_\kappa : \text{Range}(\tilde{\gamma}) \cap \text{Range}(\gamma) \neq \emptyset\}, \quad \mathcal{S}_\kappa(\gamma) := \left( \bigcup_{\tilde{\gamma} \in \tilde{\mathfrak{C}}_\kappa(\gamma)} \text{Range}(\tilde{\gamma}) \right) \cup \text{Range}(\gamma). \quad (3.1)$$

Define

$$\tilde{\mathfrak{C}}_{\kappa, \nu} := \bigcup_{\gamma \in \Xi_\nu} \tilde{\mathfrak{C}}_\kappa(\gamma) = \{\tilde{\gamma} \in \mathfrak{C}_\kappa : \exists \gamma \in \Xi_\nu, \text{Range}(\tilde{\gamma}) \cap \text{Range}(\gamma) \neq \emptyset\}. \quad (3.2)$$

Let  $\mathcal{S}_{\kappa, \nu}$  be the following random subset of  $\overline{\mathbb{D}}$ :

$$\mathcal{S}_{\kappa, \nu} := \bigcup_{\gamma \in \Xi_\nu} \mathcal{S}_\kappa(\gamma) = \left( \bigcup_{\tilde{\gamma} \in \tilde{\mathfrak{C}}_{\kappa, \nu}} \text{Range}(\tilde{\gamma}) \right) \cup \left( \bigcup_{\gamma \in \Xi_\nu} \text{Range}(\gamma) \right).$$

In the limit case  $\kappa = 8/3$ , we set

$$\mathcal{S}_{8/3, \nu} := \bigcup_{\gamma \in \Xi_\nu} \text{Range}(\gamma).$$

By construction,  $\mathcal{S}_{\kappa,\nu} \cap A_R = \emptyset$ . Let  $\mathcal{D}_{R,\kappa,\nu}$  be the connected component of  $\overline{\mathbb{D}} \setminus (\mathcal{S}_{\kappa,\nu} \cup \overline{A_L})$  that contains  $A_R$ . Then  $\mathcal{D}_{R,\kappa,\nu}$  is of form  $\mathcal{O}_{\kappa,\nu} \cup A_R$ , where  $\mathcal{O}_{\kappa,\nu}$  is an open simply connected domain. Set

$$\eta_{\kappa,\nu} := (\partial\mathcal{D}_{R,\kappa,\nu}) \setminus A_R.$$

Informally,  $\eta_{\kappa,\nu}$  is constructed as the envelop from the right of the set  $\mathcal{S}_{\kappa,\nu} \cup \overline{A_L}$ . First of all, we will show that  $\eta_{\kappa,\nu}$  is a continuous curve and satisfies conformal covariance.

*Proof of Proposition 1.1.* The conformal covariance in law of  $\eta_{\kappa,\nu}$  follows from the conformal invariance in law of the  $\text{CLE}_\kappa$  and the conformal covariance in law of  $\Xi_\nu$ . It remains to show Proposition 1.1 (2).

For the continuity of the envelop we will use a somewhat different argument from [WW13, Section 2.4], relying on Lemma 2.3. If one shows that  $\partial\mathcal{O}_{\kappa,\nu}$  is a continuous closed curve, then one gets that  $\eta_{\kappa,\nu} = \partial\mathcal{O}_{\kappa,\nu} \setminus A_R$  is a continuous curve. Its endpoints are  $-i$  and  $i$  since these are also the endpoints of  $A_R$ . According to Theorem 2.8, to show that  $\partial\mathcal{O}_{\kappa,\nu}$  is a continuous closed curve one needs to check that  $\mathbb{C} \setminus \mathcal{O}_{\kappa,\nu}$  is locally connected. According to Lemma 2.4, it is enough to check that  $\mathcal{S}_{\kappa,\nu} \cup \partial\mathbb{D}$  is locally connected. To this end, we will apply Lemma 2.3 twice. The first time, we apply it to  $K_0 = \partial\mathbb{D}$  and  $(K_n)_{n \geq 1} = \Xi_\nu$ . We get that  $\partial\mathbb{D} \cup \bigcup_{\gamma \in \Xi_\nu} \text{Range}(\gamma)$  is locally connected. The second time, we apply it to  $K_0 = \partial\mathbb{D} \cup \bigcup_{\gamma \in \Xi_\nu} \text{Range}(\gamma)$  and  $(K_n)_{n \geq 1} = \tilde{\mathfrak{C}}_{\kappa,\nu}$  and get that  $\mathcal{S}_{\kappa,\nu} \cup \partial\mathbb{D}$  is locally connected. This completes the proof.  $\square$

From the construction, the curves  $\eta_{\kappa,\nu}$  satisfy an obvious monotonicity in  $\nu$ . Indeed, if  $\nu_1 \leq \nu_2$ ,  $\Xi_{\nu_1}$  can be realized as a subset of  $\Xi_{\nu_2}$ .

**Proposition 3.1.** *Fix  $\kappa \in [8/3, 4]$ . Let  $\nu_1$  and  $\nu_2$  be two finite non-negative Radon measures on  $\overline{A_L}$  such that  $\nu_1 \leq \nu_2$ , i.e.  $\nu_2 - \nu_1$  is a non-negative measure. Then  $\eta_{\kappa,\nu_1}$  and  $\eta_{\kappa,\nu_2}$  can be coupled on the same probability space such that a.s.,  $\eta_{\kappa,\nu_1}$  is contained between  $\overline{A_L}$  and  $\eta_{\kappa,\nu_2}$ , and in particular,*

$$(\eta_{\kappa,\nu_2} \cap A_L) \subset (\eta_{\kappa,\nu_1} \cap A_L).$$

By construction,  $\eta_{\kappa,\nu} \cap A_R = \emptyset$ . However,  $\eta_{\kappa,\nu}$  may intersect  $A_L$ . Next we give a condition under which  $\eta_{\kappa,\nu}$  may contain a whole subarc of  $A_L$ .

**Proposition 3.2.** *Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $A$  be a non-empty open subarc of  $A_L$  and let  $A_1$  and  $A_2$  denote the two connected components of  $\overline{A_L} \setminus A$ . Then  $\mathbb{P}(\overline{A} \subset \eta_{\kappa,\nu}) > 0$  if and only if  $\nu(A) = 0$ . Moreover, in the latter case, the event  $\overline{A} \subset \eta_{\kappa,\nu}$  coincides a.s. with the event defined by the following two conditions (and only the first one if  $\kappa = 8/3$ ).*

- (1) *There is no excursion in  $\Xi_\nu$  joining  $A_1$  and  $A_2$ .*
- (2) *The process  $\Xi_\nu$  does not contain an excursion from  $A_1$  to  $A_1$  and an excursion from  $A_2$  to  $A_2$  intersecting the same  $\text{CLE}_\kappa$  loop in  $\mathfrak{C}_\kappa$  (provided  $\kappa \neq 8/3$ ).*

*On the complementary event, again in the case  $\nu(A) = 0$ , we have a.s.  $A \cap \eta_{\kappa,\nu} = \emptyset$ .*

*Proof.* First assume that  $\nu(A) > 0$ . According to Lemma 2.10, the process  $\Xi_\nu$  contains a.s. an excursion  $\gamma$  with both endpoints  $x, y$  in  $A$ . This implies that  $A(x, y) \cap \eta_{\kappa,\nu} = \emptyset$ , where  $A(x, y)$  is the open subarc of  $A$  with endpoints  $x, y$ . In particular, a.s.  $A \not\subset \eta_{\kappa,\nu}$ .

Next assume that  $\nu(A) = 0$ . Also take  $\kappa \neq 8/3$ . The case  $\kappa = 8/3$  is actually simpler. Since  $\nu(A) = 0$ , the process  $\Xi_\nu$  does not contain excursions with one or both endpoints in  $A$ .

- Let  $E_0$  denote the event that  $\overline{A} \subset \eta_{\kappa,\nu}$ . Let  $\tilde{E}_0$  be the event that  $A \cap \eta_{\kappa,\nu} = \emptyset$ . Clearly,  $\tilde{E}_0 \subset E_0^c$ .
- Let  $E_1$  denote the event that there is an excursion in  $\Xi_\nu$  joining  $A_1$  and  $A_2$ . If  $\gamma$  is such an excursion, its range disconnects in  $\overline{\mathbb{D}}$  the arc  $A$  from  $A_R$ . Thus,  $E_1 \subset \tilde{E}_0$ .



- Let  $E_2$  denote the event that there is an excursion from  $A_1$  to  $A_1$  and an excursion from  $A_2$  to  $A_2$  intersecting the same  $\text{CLE}_\kappa$  loop in  $\mathfrak{C}_\kappa$ . If  $\gamma_1$ , respectively  $\gamma_2$ , are such excursion from  $A_1$ , respectively  $A_2$ , and  $\tilde{\gamma}$  is the common loop in  $\mathfrak{C}_\kappa$  they intersect, then  $\text{Range}(\gamma_1) \cup \text{Range}(\gamma_2) \cup \text{Range}(\tilde{\gamma})$  disconnects in  $\overline{\mathbb{D}}$  the arc  $A$  from  $A_R$ . Thus,  $E_2 \subset \tilde{E}_0$ .

Let us further show that  $E_1^c \cap E_2^c \subset E_0$ , which will establish  $E_0 = \tilde{E}_0^c = E_1^c \cap E_2^c$ . Set

$$\Xi_1 := \{\gamma \in \Xi_\nu \mid \gamma \text{ has both endpoints in } A_1\}, \quad \Xi_2 := \{\gamma \in \Xi_\nu \mid \gamma \text{ has both endpoints in } A_2\},$$

$$\tilde{\mathfrak{C}}_{\kappa,1} := \{\tilde{\gamma} \in \mathfrak{C}_\kappa : \exists \gamma \in \Xi_1, \text{Range}(\tilde{\gamma}) \cap \text{Range}(\gamma) \neq \emptyset\},$$

$$\tilde{\mathfrak{C}}_{\kappa,2} := \{\tilde{\gamma} \in \mathfrak{C}_\kappa : \exists \gamma \in \Xi_2, \text{Range}(\tilde{\gamma}) \cap \text{Range}(\gamma) \neq \emptyset\},$$

$$\mathcal{S}_1 := \left( \bigcup_{\tilde{\gamma} \in \tilde{\mathfrak{C}}_{\kappa,1}} \text{Range}(\tilde{\gamma}) \right) \cup \left( \bigcup_{\gamma \in \Xi_1} \text{Range}(\gamma) \right), \quad \mathcal{S}_2 := \left( \bigcup_{\tilde{\gamma} \in \tilde{\mathfrak{C}}_{\kappa,2}} \text{Range}(\tilde{\gamma}) \right) \cup \left( \bigcup_{\gamma \in \Xi_2} \text{Range}(\gamma) \right).$$

The local finiteness of the CLE ensures that  $\mathcal{S}_1 \cup A_1$  and  $\mathcal{S}_2 \cup A_2$  are closed subsets of  $\overline{\mathbb{D}}$  and that  $\partial\mathbb{D} \cap \mathcal{S}_1 \subset A_1$ ,  $\partial\mathbb{D} \cap \mathcal{S}_2 \subset A_2$ . Thus, neither  $\mathcal{S}_1 \cup A_1$  nor  $\mathcal{S}_2 \cup A_2$  disconnect  $A$  from  $A_R$  in  $\overline{\mathbb{D}}$ . Moreover, on the event  $E_2^c$ , we have

$$(\mathcal{S}_1 \cup A_1) \cap (\mathcal{S}_2 \cup A_2) = \emptyset.$$

It follows from the Janiszewski's theorem, on the event  $E_2^c$ , we have that  $(\mathcal{S}_1 \cup A_1) \cup (\mathcal{S}_2 \cup A_2)$  does not disconnect  $A$  from  $A_R$  in  $\overline{\mathbb{D}}$ ; see [Pom92, Section 1.1]. Moreover, on the event  $E_1^c$ , we have  $\mathcal{S}_{\kappa,\nu} = \mathcal{S}_1 \cup \mathcal{S}_2$ . So on the event  $E_1^c \cap E_2^c$ , the set  $\mathcal{S}_{\kappa,\nu}$  does not disconnect  $A$  from  $A_R$  in  $\overline{\mathbb{D}}$  and thus,  $\overline{A} \subset \eta_{\kappa,\nu}$ .

Finally, let us check that  $\mathbb{P}(E_1^c \cap E_2^c) > 0$ . Actually, the two events are independent and it suffices to show  $\mathbb{P}(E_1^c) > 0$  and  $\mathbb{P}(E_2^c) > 0$ .

- We have  $\mathbb{P}(E_1) < 1$  because the intensity measure of excursions from  $A_1$  to  $A_2$  is finite.
- Let  $U_1$  be an open neighborhood of  $A_1$  such that  $\overline{U_1} \cap A_2 = \emptyset$ . The local finiteness and the fact that the CLE loops do not hit the boundary guarantee that there is  $U_2$  an open neighborhoods of  $A_2$ , such that  $\overline{U_1} \cap \overline{U_2} = \emptyset$  and such that with positive probability no loop in  $\mathfrak{C}_\kappa$  intersects both  $U_1$  and  $U_2$ . With positive probability, all the excursions of  $\Xi_1$  are contained in  $U_1$ , and all the excursions of  $\Xi_2$  are contained in  $U_2$ , and these are independent events; see Lemma 2.10. Thus,  $\mathbb{P}(E_2^c) > 0$ .

These complete the proof. □

Now, we are ready to complete the proof of Proposition 1.3.

*Proof of Proposition 1.3.* Recall that  $\psi_0$  is the conformal transformation from  $\mathbb{D}$  to  $\mathbb{H}$  given by (1.5) and  $\tilde{\eta}_{\kappa,\nu} := \psi_0(\eta_{\kappa,\nu})$ . It suffices to check that  $\eta_{\kappa,\nu}$  satisfies the conditions in Proposition 2.12.

- From the construction,  $\eta_{\kappa,\nu}$  clearly satisfies Proposition 2.12 (a).
- As the support of  $\nu$  equals  $\overline{A_L}$ ,  $\eta_{\kappa,\nu}$  satisfies Proposition 2.12 (b) due to Proposition 3.2.

Therefore,  $\tilde{\eta}_{\kappa,\nu}$  is a continuous curve in  $\overline{\mathbb{H}}$  from 0 to  $\infty$  with continuous driving function. The half-plane capacity is a continuous strictly increasing function on  $\tilde{\eta}_{\kappa,\nu}$ . So one can parametrize  $\tilde{\eta}_{\kappa,\nu}$  as  $(\tilde{\eta}_{\kappa,\nu}(t))_{0 \leq t < T_{\max}}$ , with  $T_{\max} \in (0, +\infty]$ , such that (1.6) holds. Note that one does not necessarily have  $T_{\max} = +\infty$ . □

Next, we consider the simplicity of  $\eta_{\kappa,\nu}$ .

**Lemma 3.3.** *Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Assume further that  $\nu$  has no atoms. Then the curve  $\eta_{\kappa,\nu}$  is a.s. simple.*

*Proof.* According to Theorem 2.8, to show that  $\partial\mathcal{O}_{\kappa,\nu}$  is a Jordan curve one additionally needs to check that  $\partial\mathcal{O}_{\kappa,\nu}$  has no cut points. Since  $\nu$  has no atoms, for every  $\gamma \in \Xi_\nu$ , the two endpoints of  $\gamma$  are distinct. Denote by  $\mathcal{R}(\gamma)$  the right boundary of the excursion  $\gamma$ , defined rigorously as a portion of the boundary of the connected component of  $\overline{\mathbb{D}} \setminus \text{Range}(\gamma)$  that contains  $A_R$ . According to [LSW03, Corollary 8.5],  $\mathcal{R}(\gamma)$  is a continuous simple curve joining the two endpoints of  $\gamma$ , more specifically distributed as a chordal  $\text{SLE}_{8/3}(\rho)$  process, with  $\rho = 2/3$ . Denote

$$\widehat{\mathfrak{C}}_{\kappa,\nu} := \{\tilde{\gamma} \in \mathfrak{C}_\kappa : \exists \gamma \in \Xi_\nu, \text{Range}(\tilde{\gamma}) \cap \mathcal{R}(\gamma) \neq \emptyset\}, \quad \widehat{\mathfrak{S}}_{\kappa,\nu} := \left( \bigcup_{\tilde{\gamma} \in \widehat{\mathfrak{C}}_{\kappa,\nu}} \text{Range}(\tilde{\gamma}) \right) \cup \left( \bigcup_{\gamma \in \Xi_\nu} \mathcal{R}(\gamma) \right).$$

Then  $\mathcal{O}_{\kappa,\nu}$  is a connected component of  $\mathbb{C} \setminus (\widehat{\mathfrak{S}}_{\kappa,\nu} \cup \partial\mathbb{D})$ . According to Lemma 2.7, it is enough to check that  $\widehat{\mathfrak{S}}_{\kappa,\nu} \cup \partial\mathbb{D}$  has no cut points. To this end, we classify the points of  $\widehat{\mathfrak{S}}_{\kappa,\nu} \cup \partial\mathbb{D}$  as follows:

- (1) The points of  $\partial\mathbb{D}$  that are not an endpoints of an excursion  $\gamma \in \Xi_\nu$ .
- (2) The endpoints of excursions  $\gamma \in \Xi_\nu$ .
- (3) The points on  $\mathcal{R}(\gamma)$ , for  $\gamma \in \Xi_\nu$ , that are not endpoints and do not lie on  $\text{Range}(\tilde{\gamma})$  for  $\tilde{\gamma} \in \widehat{\mathfrak{C}}_{\kappa,\nu}$ .
- (4) The points on  $\text{Range}(\tilde{\gamma})$ , for  $\tilde{\gamma} \in \widehat{\mathfrak{C}}_{\kappa,\nu}$ , that do not lie on  $\mathcal{R}(\gamma)$  for  $\gamma \in \Xi_\nu$ .
- (5) The points that belong to an intersection  $\mathcal{R}(\gamma) \cap \text{Range}(\tilde{\gamma})$  for  $\gamma \in \Xi_\nu$  and  $\tilde{\gamma} \in \widehat{\mathfrak{C}}_{\kappa,\nu}$ .

It is clear that the points of type (1) cannot be cut points. The points of type (2) cannot be cut points because each  $\mathcal{R}(\gamma)$  has two distinct endpoints. The points of type (3) cannot be cut points because the curve  $\mathcal{R}(\gamma)$  is simple. The points of type (4) cannot be cut points because  $\text{CLE}_\kappa$  loops are Jordan curves. Regarding the points of type (5), they cannot be cut points because for every  $\gamma \in \Xi_\nu$  and  $\tilde{\gamma} \in \widehat{\mathfrak{C}}_{\kappa,\nu}$ , the intersection  $\mathcal{R}(\gamma) \cap \text{Range}(\tilde{\gamma})$  is either empty or contains at least two points. Indeed, given an independent Brownian motion that hits a  $\text{CLE}_\kappa$  loop, it will a.s. enter the interior surrounded by the loop.  $\square$

Let us explore more the multiple points of the curve  $\eta_{\kappa,\nu}$  in the case the measure  $\nu$  has atoms. Denote

$$\text{Atom}_{\text{isol}}(\nu) := \{x \in \text{Atom}(\nu) : \exists A \subset \partial\mathbb{D}, A \text{ open arc}, x \in A, \mathbb{1}_A \nu = \nu(\{x\})\delta_x\},$$

$$\text{Atom}_{\text{conv}}(\nu) := \left\{ x \in \text{Atom}(\nu) : \int_{A_L \setminus \{x\}} \frac{1}{|y-x|^2} d\nu(y) < +\infty \right\}.$$

Note that  $\text{Atom}_{\text{isol}}(\nu) \subset \text{Atom}_{\text{conv}}(\nu)$ .

**Proposition 3.4.** *Fix  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $\psi$  be an uniformizing map from  $\mathbb{D}$  to  $\mathcal{O}_{\kappa,\nu}$ . Then a.s., for every  $x \in \eta_{\kappa,\nu}$ , the number  $\text{Card}(\psi^{-1}(\{x\}))$  (which does not depend on the choice of  $\psi$ ) is either 1 or 2. Moreover, a.s.,*

$$\{x \in \eta_{\kappa,\nu} : \text{Card}(\psi^{-1}(\{x\})) = 2\} \subset \text{Atom}_{\text{conv}}(\nu) \subset \overline{A_L}. \quad (3.3)$$

*In particular, if  $\text{Atom}_{\text{conv}}(\nu) = \emptyset$ , then the curve  $\eta_{\kappa,\nu}$  is a.s. simple. As a partial converse, we have that for every  $x \in \text{Atom}_{\text{isol}}(\nu)$ ,*

$$\mathbb{P}(\text{Card}(\psi^{-1}(\{x\})) = 2) > 0. \quad (3.4)$$

*Proof.* According to [Pom92, Proposition 2.5] and the corresponding proof, for every  $x \in \eta_{\kappa,\nu}$ , the number  $\text{Card}(\psi^{-1}(\{x\}))$  equals the number of connected components in  $\mathbb{C} \setminus (\mathcal{O}_{\kappa,\nu} \cup \{x\})$ . Similarly to Lemma 3.3, one can show that  $\mathbb{C} \setminus \mathcal{O}_{\kappa,\nu}$  does not have cut points in  $\eta_{\kappa,\nu} \setminus \text{Atom}(\nu)$ . Thus  $\text{Card}(\psi^{-1}(\{x\}))$  is either 1 or 2 and it can be 2 only for points  $x \in \text{Atom}(\nu)$ .

Then we show (3.3). From Lemma 3.5 below and an absolute continuity argument near an endpoint applied to Brownian excursions, it follows that for every  $\gamma, \gamma' \in \Xi_\nu$ , such that  $\text{Range}(\gamma) \cap \text{Range}(\gamma') \neq \emptyset$ ,

one also has  $(\text{Range}(\gamma) \cap \text{Range}(\gamma')) \setminus \partial\mathbb{D} \neq \emptyset$ . If  $x \in \text{Atom}(\nu) \setminus \text{Atom}_{\text{conv}}(\nu)$ , then a.s. there is an excursion  $\gamma \in \Xi_\nu$  with one endpoint  $x$  and the other endpoint different from  $x$ . For every  $\gamma'$  excursion in  $\Xi_\nu$  with both endpoints in  $x$ , we have  $(\text{Range}(\gamma) \cap \text{Range}(\gamma')) \setminus \{x\} \neq \emptyset$ . This implies that  $\mathcal{S}_{\kappa,\nu} \setminus \{x\}$  is connected a.s. Thus, for every  $x \in \eta_{\kappa,\nu} \setminus \text{Atom}_{\text{conv}}(\nu)$ , we have  $\text{Card}(\psi^{-1}(\{x\})) = 1$ . This completes the proof for (3.3).

Now, consider  $x \in \text{Atom}_{\text{isol}}(\nu)$ . Let  $\Xi_x$  be the subset of  $\Xi_\nu$  made of all the excursions with both endpoints in  $x$ . Denote

$$\mathcal{S}_x := \bigcup_{\gamma \in \Xi_x} \mathcal{S}_\kappa(\gamma).$$

Since for all  $\gamma, \gamma' \in \Xi_x$ , we have  $(\text{Range}(\gamma) \cap \text{Range}(\gamma')) \setminus \{x\} \neq \emptyset$ . Thus  $\mathcal{S}_x \setminus \{x\}$  is connected a.s. Define the event

$$E_x := \{\text{for every } \gamma \in \Xi_\nu \setminus \Xi_x, \text{ we have } \text{Range}(\gamma) \cap \mathcal{S}_x = \emptyset\}.$$

On the event  $E_x$ , the set  $\mathbb{C} \setminus (\mathcal{O}_{\kappa,\nu} \cup \{x\})$  has exactly two connected components, one containing  $\mathcal{S}_x \setminus \{x\}$ , and the other  $\mathbb{C} \setminus \bar{\mathbb{D}}$ . So, on the event  $E_x$ , we have  $\text{Card}(\psi^{-1}(\{x\})) = 2$ . On the complementary event  $E_x^c$ , the set  $\mathbb{C} \setminus (\mathcal{O}_{\kappa,\nu} \cup \{x\})$  is connected and  $\text{Card}(\psi^{-1}(\{x\})) = 1$ . Further, it is easy to see that for  $x \in \text{Atom}_{\text{isol}}(\nu)$ , we have  $\mathbb{P}(E_x) > 0$ . This completes the proof for (3.4).  $\square$

**Lemma 3.5.** *Let  $\wp_1$  and  $\wp_2$  be two i.i.d. Brownian excursions from  $-i$  to  $i$  in  $\mathbb{D}$ , sampled according to  $\mu_{-i,i}^{\mathbb{D},\#}$ . Then*

$$\text{dist}(-i, (\text{Range}(\wp_1) \cap \text{Range}(\wp_2)) \setminus \{-i\}) = 0 \quad \text{a.s.}$$

*Proof.* Because of the conformal covariance, one can consider  $\tilde{\wp}_1$  and  $\tilde{\wp}_2$  two i.i.d. Brownian excursions from 0 to  $\infty$  in the upper half-plane  $\mathbb{H}$ . Define the random variable

$$\tilde{\delta} := \text{dist}(0, (\text{Range}(\tilde{\wp}_1) \cap \text{Range}(\tilde{\wp}_2)) \setminus \{0\})$$

with values in  $[0, +\infty]$ . It is enough to show that  $\tilde{\delta} = 0$  a.s. Since the law of  $\tilde{\wp}_1$  and  $\tilde{\wp}_2$  is invariant under Brownian scaling, we have that for every  $c > 0$ ,  $c\tilde{\delta}$  is distributed as  $\tilde{\delta}$ . This in turn implies that  $\tilde{\delta} \in \{0, +\infty\}$  a.s. So we have only to show that  $\mathbb{P}(\tilde{\delta} = +\infty) = 0$ . On the event that  $\tilde{\delta} = +\infty$ , we in particular have that  $\tilde{\wp}_1([0, 1]) \cap \tilde{\wp}_2([0, 1]) = \{0\}$ . However,

$$\mathbb{P}(\tilde{\wp}_1([1, +\infty)) \cap \tilde{\wp}_2([1, +\infty)) \neq \emptyset | (\tilde{\wp}_1(t), \tilde{\wp}_2(t))_{0 \leq t \leq 1}) > 0 \quad \text{a.s.}$$

So, if the probability  $\mathbb{P}(\tilde{\delta} = +\infty)$  were positive, then we would have that  $\mathbb{P}(\tilde{\delta} \in (0, +\infty)) > 0$ , which is a contradiction. Thus,  $\mathbb{P}(\tilde{\delta} = +\infty) = 0$ .  $\square$

Next we complement Proposition 3.4 and give a partial results on atoms of  $\nu$  which cannot be hit by  $\eta_{\kappa,\nu}$ . We will need this result in Section 5. Recall that  $\text{Atom}_{\text{conv}}^l(\nu)$ ,  $\text{Atom}_{\text{conv}}^r(\nu)$  and  $\text{Atom}_{\text{conv}}^*(\nu)$  are defined in (1.8) and (1.9). Note that by construction,  $\text{Atom}(\nu) \cap \{-i, i\} \subset \text{Atom}_{\text{conv}}^*(\nu)$ .

**Lemma 3.6.** *For every  $x \in \text{Atom}(\nu) \setminus \text{Atom}_{\text{conv}}^*(\nu)$ ,  $\mathbb{P}(x \in \eta_{\kappa,\nu}) = 0$ .*

*Proof.* Let  $x \in \text{Atom}(\nu) \setminus \text{Atom}_{\text{conv}}^*(\nu)$ . Then a.s., there is an excursion  $\gamma \in \Xi_\nu$  with one endpoint  $x$  and the other endpoint strictly to the left of  $x$ , and an other excursion  $\gamma' \in \Xi_\nu$  with one endpoint  $x$  and the other endpoint strictly to the right of  $x$ . Moreover, by Lemma 3.5,  $\gamma$  and  $\gamma'$  a.s. intersect near  $x$ . Thus,  $\text{Range}(\gamma) \cup \text{Range}(\gamma')$  disconnects a neighborhood of  $x$  in  $\bar{\mathbb{D}}$  from  $A_{\mathbb{R}}$ .  $\square$

### 3.2 Local absolute continuity with respect to $\text{SLE}_\kappa$ away from the boundary

In this section we will show that for  $\nu \neq 0$ , the curve  $\eta_{\kappa,\nu}$  is in some sense absolutely continuous with respect to a chordal  $\text{SLE}_\kappa$  away from the boundary. First we recall the result of [WW13] which identifies the law of  $\eta_{\kappa,\nu}$  when  $\nu$  is a constant on  $A_{\mathbb{L}}$ .

**Theorem 3.7** (Werner-Wu [WW13]). *Let  $\kappa \in [8/3, 4]$ . Assume that  $\nu = a\mathbb{1}_{A_L}\sigma_{\partial\mathbb{D}}$ , with  $a > 0$  a constant. Let  $\tilde{\eta}_{\kappa,a}$  denote  $\tilde{\eta}_{\kappa,\nu} = \psi_0(\eta_{\kappa,\nu})$  in this case. Then  $\tilde{\eta}_{\kappa,a}$  is distributed as a chordal  $\text{SLE}_\kappa(\rho)$  curve in  $\mathbb{H}$  from 0 to  $\infty$ , with one force point at  $0^-$ , with  $\rho$  be the unique real in  $(-2, +\infty)$  satisfying*

$$a^2 = \frac{\pi(\rho+2)(\rho+6-\kappa)}{2\kappa}.$$

*In particular, if  $a = \sqrt{\pi(6-\kappa)/\kappa}$ , then  $\tilde{\eta}_{\kappa,a}$  is distributed as a chordal  $\text{SLE}_\kappa$ .*

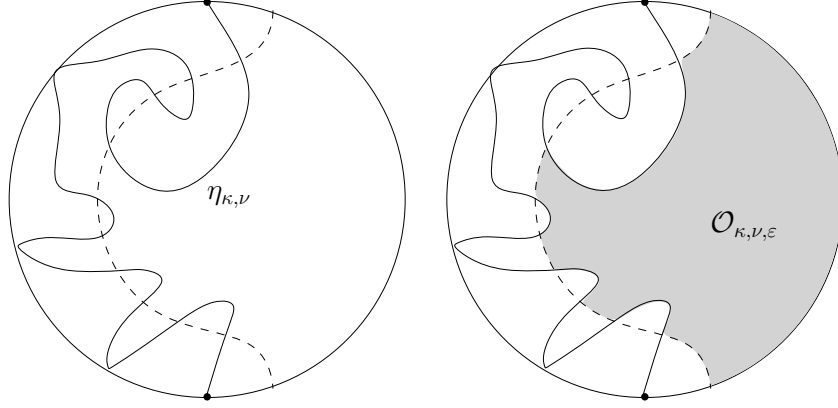


Figure 3.1: In the left panel, the dashed line indicates  $\partial\mathbb{D}_\varepsilon \cap \mathbb{D}$  and the curve indicates  $\eta_{\kappa,\nu}$  intersecting  $\mathbb{D}_\varepsilon$ . The set  $\mathcal{O}_{\kappa,\nu}$  is the connected component of  $\mathbb{D} \setminus \eta_{\kappa,\nu}$  adjacent to  $A_R$ . Consider  $\mathcal{O}_{\kappa,\nu} \cap \mathbb{D}_\varepsilon$ , there are many connected components, and  $\mathcal{O}_{\kappa,\nu,\varepsilon}$  is the one adjacent to  $A_R \cap \overline{\mathbb{D}_\varepsilon}$ . In the right panel, the region in gray indicates  $\mathcal{O}_{\kappa,\nu,\varepsilon}$ .

For  $\varepsilon \in (0, 1)$ , let  $\mathcal{O}_{\kappa,\nu,\varepsilon}$  denote the connected component of  $\mathcal{O}_{\kappa,\nu} \cap \mathbb{D}_\varepsilon$  (see (2.3)) adjacent to  $A_R \cap \overline{\mathbb{D}_\varepsilon}$ ; see Figure 3.1. To motivate what will follow, we state the next proposition.

**Proposition 3.8.** *Let be  $\kappa \in [8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Assume that  $\nu \neq 0$ . Then a.s., for every  $z \in \eta_{\kappa,\nu} \setminus \overline{A_L}$ , there is  $U$  a neighborhood of  $z$  in  $\eta_{\kappa,\nu} \setminus \overline{A_L}$  and  $\varepsilon \in (0, 1)$  such that  $U \subset \partial\mathcal{O}_{\kappa,\nu,\varepsilon}$ .*

*Proof.* For  $w \in \eta_{\kappa,\nu} \setminus \overline{A_L}$ , let  $I_w$  denote the straight line segment in  $\overline{\mathbb{D}}$  with endpoints 1 and  $\psi_{\kappa,\nu}^{-1}(w)$ , where  $\psi_{\kappa,\nu}$  is the conformal map from  $\mathbb{D}$  to  $\mathcal{O}_{\kappa,\nu}$  defined in Section 1. Take  $z \in \eta_{\kappa,\nu} \setminus \overline{A_L}$ . Let  $A$  be an open subarc of  $A_L$  containing  $\psi_{\kappa,\nu}^{-1}(z)$ , such that  $\psi_{\kappa,\nu}(A) \subset \eta_{\kappa,\nu} \setminus \overline{A_L}$ . Let  $K$  be

$$K := \bigcup_{w \in A} \psi_{\kappa,\nu}(I_w).$$

$K$  is a connected compact subset of  $\overline{\mathcal{O}_{\kappa,\nu}}$  containing 1,  $z$  and a neighborhood of  $z$  in  $\eta_{\kappa,\nu} \setminus \overline{A_L}$ . By construction,  $K \cap \overline{A_L} = \emptyset$ . Thus, for  $\varepsilon \in (0, \text{dist}(K, A_L) \wedge 1)$ ,  $K \subset \overline{\mathcal{O}_{\kappa,\nu,\varepsilon}}$ .  $\square$

Next we state the absolute continuity result.

**Proposition 3.9.** *Let  $\kappa \in [8/3, 4]$  and  $\nu_1, \nu_2$  be two finite non-negative Radon measure on  $\overline{A_L}$ . Assume that both  $\nu_1$  and  $\nu_2$  are non-zero. Then, for every  $\varepsilon \in (0, 1)$ , the laws of  $\mathcal{O}_{\kappa,\nu_1,\varepsilon}$  and  $\mathcal{O}_{\kappa,\nu_2,\varepsilon}$  are mutually absolutely continuous. In particular, if  $\eta_\kappa$  denotes a chordal  $\text{SLE}_\kappa$  curve in  $\mathbb{D}$  from  $-i$  to  $i$  and  $\mathcal{O}_{\kappa,\varepsilon}$  denotes the connected component of  $\mathbb{D} \setminus \eta_\kappa$  adjacent to  $A_R \cap \overline{\mathbb{D}_\varepsilon}$ , then for every  $\nu$  non-negative non-zero Radon measure on  $\overline{A_L}$ , the law of  $\mathcal{O}_{\kappa,\nu,\varepsilon}$  is absolutely continuous with respect to that of  $\mathcal{O}_{\kappa,\varepsilon}$ .*

*Proof.* The second part of the statement follows from the first part and Theorem 3.7.

For the first part, fix  $\varepsilon \in (0, 1)$ . We assume that  $\kappa \neq 8/3$ , the case  $\kappa = 8/3$  being simpler. Let  $\mathfrak{C}_{\kappa, \varepsilon}$  be the subset of  $\mathfrak{C}_\kappa$  made of CLE loops intersecting  $\mathbb{D}_\varepsilon$ . Let  $\tilde{\varepsilon}$  be the following r.v.:

$$\tilde{\varepsilon} := \inf_{\tilde{\gamma} \in \mathfrak{C}_{\kappa, \varepsilon}} \text{dist}(\text{Range}(\tilde{\gamma}), A_L).$$

We have that  $\tilde{\varepsilon} \in (0, \varepsilon)$  a.s. Define

$$\tilde{\Xi}_{\nu_1, \tilde{\varepsilon}} := \left\{ (\gamma(T_{\gamma, \tilde{\varepsilon}}^f + t))_{0 \leq t \leq T_{\gamma, \tilde{\varepsilon}}^l - T_{\gamma, \tilde{\varepsilon}}^f} : \gamma \in \Xi_{\nu_1}, \text{Range}(\gamma) \cap \mathbb{D}_{\tilde{\varepsilon}} \neq \emptyset \right\},$$

where  $T_{\gamma, \tilde{\varepsilon}}^f$  and  $T_{\gamma, \tilde{\varepsilon}}^l$  are given by (2.4), and where  $\Xi_{\nu_1}$  is independent from  $\mathfrak{C}_\kappa$ . Similarly define  $\tilde{\Xi}_{\nu_2, \tilde{\varepsilon}}$ . We have that the law of  $(\mathfrak{C}_\kappa, \tilde{\Xi}_{\nu_2, \tilde{\varepsilon}})$  is absolutely continuous with respect to that of  $(\mathfrak{C}_\kappa, \tilde{\Xi}_{\nu_1, \tilde{\varepsilon}})$ . This follows from the Markovian decomposition of Proposition 2.11. Further,  $\mathcal{O}_{\kappa, \nu_1, \varepsilon}$ , respectively  $\mathcal{O}_{\kappa, \nu_2, \varepsilon}$ , is measurable with respect to  $(\mathfrak{C}_\kappa, \tilde{\Xi}_{\nu_1, \tilde{\varepsilon}})$ , respectively  $(\mathfrak{C}_\kappa, \tilde{\Xi}_{\nu_2, \tilde{\varepsilon}})$ . This concludes the proof.  $\square$

### 3.3 Curves hitting the boundary with positive measure

Fix  $\kappa \in [8/3, 4]$ . Assume that the measure  $\nu$  has full support on  $\overline{A_L}$ . By Proposition 3.2,  $\eta_{\kappa, \nu} \cap \overline{A_L}$  has a.s. empty interior. However,  $\eta_{\kappa, \nu}$  may still hit  $\overline{A_L}$ , depending on  $\nu$ . Here we will show that actually for some  $\nu$ -s,  $\eta_{\kappa, \nu} \cap \overline{A_L}$  may have a.s. empty interior, yet have, with positive probability, a positive mass for the arc-length measure  $\sigma_{\partial \mathbb{D}}$ . We will construct examples with  $\nu$  actually being a continuous function  $u : \overline{A_L} \rightarrow [0, +\infty)$ . We will write  $\eta_{\kappa, u}$  in this case.

For  $k \geq 0$ , denote

$$Q_k := \left\{ \frac{\pi}{2} + (2j+1)\frac{\pi}{2^{k+1}} : 0 \leq j \leq 2^k - 1 \right\}.$$

Set

$$Q := \bigcup_{k \geq 0} Q_k.$$

Note that  $Q_k \cap Q_{k'} = \emptyset$  for  $k \neq k'$  and that  $Q$  is everywhere dense in  $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ . Given  $\varepsilon \in (0, 1)$ , denote  $f_{k, \varepsilon} : [\frac{1}{2}\pi, \frac{3}{2}\pi] \rightarrow [0, +\infty)$  the following function:

$$f_{k, \varepsilon}(x) := (2^{-(k+1)}\pi - \varepsilon^{-1} \text{dist}(x, Q_k)) \vee 0.$$

The function  $f_{k, \varepsilon}$  is continuous and bounded from above by  $2^{-(k+1)}\pi$ . Moreover, given  $\varepsilon \leq \varepsilon' \in (0, 1)$ , we have that  $f_{k, \varepsilon} \leq f_{k, \varepsilon'}$ .

Given a sequence  $(\varepsilon_k)_{k \geq 0}$  in  $(0, 1)$ , let  $f = f_{(\varepsilon_k)_{k \geq 0}}$  be the following function on  $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ :

$$f = \sum_{k \geq 0} f_{k, \varepsilon_k}.$$

The function  $f$  is non-negative, continuous, and positive on  $Q$  whatever the choice of  $(\varepsilon_k)_{k \geq 0}$ . Moreover,  $f \leq \pi$ . Let  $u = u_{(\varepsilon_k)_{k \geq 0}}$  be the function on  $\overline{A_L}$  defined by  $u(e^{i\theta}) = f(\theta)$  for  $\theta \in [\frac{1}{2}\pi, \frac{3}{2}\pi]$ .

**Proposition 3.10.** *Fix  $\kappa \in [8/3, 4]$ . There is a sequence  $(\varepsilon_k)_{k \geq 0}$  in  $(0, 1)$  such that*

$$\mathbb{P}(\sigma_{\partial \mathbb{D}}(\eta_{\kappa, u} \cap \overline{A_L}) > 0) > 0,$$

where  $u = u_{(\varepsilon_k)_{k \geq 0}}$ .

*Proof.* First note that given the measurable functions  $v : \overline{A_L} \rightarrow [0, \pi]$ , the Poisson point processes of excursions  $\Xi_v$  are all naturally coupled on the same probability space. First one takes  $\Xi_\pi$ , which contains countably many excursions. For each  $\gamma \in \Xi_\pi$ , one takes two i.i.d. random variables  $U_\gamma^f$  and  $U_\gamma^l$ , uniform in  $(0, 1)$ . Given  $v : \overline{A_L} \rightarrow [0, \pi]$ , one gets  $\Xi_v$  by keeping an excursion  $\gamma \in \Xi_\pi$  if

$$\frac{1}{\pi}v(\gamma(0)) > U_\gamma^f \quad \text{and} \quad \frac{1}{\pi}v(\gamma(T_\gamma)) > U_\gamma^l.$$

In this way,  $\Xi_v \subset \Xi_\pi$  a.s. This coupling of the  $\Xi_v$ -s induces a coupling of the curves  $\eta_{\kappa, v}$ , by taking the same  $\text{CLE}_\kappa$  for different  $v$ -s. We will further consider this coupling.

For  $k \geq 0$  and  $\varepsilon_0, \dots, \varepsilon_k \in (0, 1)$ , denote

$$f_{\varepsilon_0, \dots, \varepsilon_k} := \sum_{j=0}^k f_{k, \varepsilon_j},$$

and define  $u_{\varepsilon_0, \dots, \varepsilon_k}$  by  $u_{\varepsilon_0, \dots, \varepsilon_k}(e^{i\theta}) = f_{\varepsilon_0, \dots, \varepsilon_k}(\theta)$ .

Given a sequence  $(\varepsilon_k)_{k \geq 0}$  in  $(0, 1)$ , we have that

$$\Xi_{u_{(\varepsilon_k)_{k \geq 0}}} = \bigcup_{k \geq 0} \Xi_{u_{\varepsilon_0, \dots, \varepsilon_k}} \quad \text{a.s.},$$

and

$$\eta_{\kappa, u_{(\varepsilon_k)_{k \geq 0}}} \cap \overline{A_L} = \bigcap_{k \geq 0} (\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_k}} \cap \overline{A_L}) \quad \text{a.s.},$$

where the last intersection is non-increasing. In particular,

$$\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{(\varepsilon_k)_{k \geq 0}}} \cap \overline{A_L}) = \lim_{k \rightarrow +\infty} \sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_k}} \cap \overline{A_L}) \quad \text{a.s.}$$

In particular, for any  $\delta > 0$ ,

$$\mathbb{P}(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{(\varepsilon_k)_{k \geq 0}}} \cap \overline{A_L}) \geq \delta) = \lim_{k \rightarrow +\infty} \mathbb{P}(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_k}} \cap \overline{A_L}) \geq \delta).$$

Further, fix  $k \geq 1$ . Consider the values of  $\varepsilon_k$  of form  $2^{-n}$ . We have that

$$\bigcup_{n \geq 1} (\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}, 2^{-n}}} \cap \overline{A_L}) = (\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}}} \cap \overline{A_L}) \setminus Q_k \quad \text{a.s.}$$

Therefore,

$$\lim_{\varepsilon_k \rightarrow 0} \sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}, \varepsilon_k}} \cap \overline{A_L}) = \sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}}} \cap \overline{A_L}) \quad \text{a.s.}$$

Thus, for every  $\delta > 0$ ,

$$\lim_{\varepsilon_k \rightarrow 0} \mathbb{P}(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}, \varepsilon_k}} \cap \overline{A_L}) > \delta) = \mathbb{P}(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_{k-1}}} \cap \overline{A_L}) > \delta). \quad (3.5)$$

Similarly,

$$\lim_{\varepsilon_0 \rightarrow 0} \sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0}} \cap \overline{A_L}) = \sigma_{\partial\mathbb{D}}(\overline{A_L}) = \pi \quad \text{a.s.},$$

and for every  $\delta \in (0, \pi)$ ,

$$\lim_{\varepsilon_0 \rightarrow 0} \mathbb{P}(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0}} \cap \overline{A_L}) > \delta) = 1. \quad (3.6)$$

Therefore, (3.5) and (3.6) ensure that one can choose the sequence  $(\varepsilon_k)_{k \geq 0}$  in  $(0, 1)$  such that for every  $k \geq 0$ ,

$$\mathbb{P}\left(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{\varepsilon_0, \dots, \varepsilon_k}} \cap \overline{A_L}) \geq \left(\frac{1}{2} + \frac{1}{2^{k+1}}\right)\pi\right) \geq \frac{1}{2} + \frac{1}{2^{k+1}}.$$

For such a sequence,

$$\mathbb{P}\left(\sigma_{\partial\mathbb{D}}(\eta_{\kappa, u_{(\varepsilon_k)_{k \geq 0}}} \cap \overline{A_L}) \geq \frac{\pi}{2}\right) \geq \frac{1}{2}. \quad \square$$

## 4 Continuous dependence on boundary conditions

### 4.1 Continuous dependence of the Poisson point process of excursions

In this section, we deal with the continuity in  $\nu$  of the Poisson point process  $\Xi_\nu$ . Suppose  $S_1$  and  $S_2$  are two finite sets of continuous paths  $(\gamma(t))_{0 \leq t \leq T_\gamma}$  in  $\mathbb{C}$  with  $T_\gamma < +\infty$ . We define

$$d_{\text{curves}}(S_1, S_2) := \begin{cases} \min_{\sigma \in \text{Bij}(S_1, S_2)} \sum_{\gamma \in S_1} (|T_\gamma - T_{\sigma(\gamma)}| + \max_{s \in [0, 1]} |\gamma(sT_\gamma) - \sigma(\gamma)(sT_{\sigma(\gamma)})|), & \text{if } \text{Card}(S_1) = \text{Card}(S_2), \\ +\infty, & \text{if } \text{Card}(S_1) \neq \text{Card}(S_2). \end{cases}$$

Note that  $d_{\text{curves}}$  is a distance. By definition, the distance of the empty set to any non-empty set is  $+\infty$ .

In the following, we will consider distance between Poisson point processes. Although the Poisson point process  $\Xi_\nu$  contains infinitely many excursions, its cutoff is finite (2.2). In this section, we will consider the following three types of cutoff. Recall from (2.2) that

$$\Xi_{\nu, \varepsilon} := \{\gamma \in \Xi_\nu : \text{diam Range}(\gamma) > \varepsilon\}.$$

We also define, for  $\varepsilon > 0$ ,

$$\widehat{\Xi}_{\nu, \varepsilon} := \{\gamma \in \Xi_\nu : \text{Range}(\gamma) \cap \mathbb{D}_\varepsilon \neq \emptyset\}, \quad \widetilde{\Xi}_{\nu, \varepsilon} := \left\{ (\gamma(T_{\gamma, \varepsilon}^f + t))_{0 \leq t \leq T_{\gamma, \varepsilon}^l - T_{\gamma, \varepsilon}^f} : \gamma \in \widehat{\Xi}_{\nu, \varepsilon} \right\},$$

where  $\mathbb{D}_\varepsilon$  is given by (2.3), and  $T_{\gamma, \varepsilon}^f$  and  $T_{\gamma, \varepsilon}^l$  by (2.4).

**Proposition 4.1.** *Fix  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_L}$ , converging weakly to  $\nu$ . Then, for every  $\varepsilon > 0$ ,  $(\Xi_{\nu_n, \varepsilon})_{n \geq 0}$  converges in law to  $\Xi_{\nu, \varepsilon}$  for  $d_{\text{curves}}$ . Moreover, it is possible to couple on the same probability space all the processes  $(\Xi_{\nu_n})_{n \geq 0}$  and  $\Xi_\nu$  such that the following two conditions hold a.s.*

(1) For every  $\varepsilon \in (0, 1)$ ,  $\lim_{n \rightarrow +\infty} d_{\text{curves}}(\Xi_{\nu_n, \varepsilon}, \Xi_{\nu, \varepsilon}) = 0$ .

(2) For every  $\varepsilon \in (0, 1)$ , there is  $n_\varepsilon \in \mathbb{N}$ , such that  $\widetilde{\Xi}_{\nu_n, \varepsilon} = \widetilde{\Xi}_{\nu, \varepsilon}$  for every  $n \geq n_\varepsilon$ .

The proof of Proposition 4.1 will be split into several lemmas. In the rest of this section, we fix the following assumptions: Fix  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_L}$ , converging weakly to  $\nu$ .

**Lemma 4.2.** *Fix  $\varepsilon \in (0, 1)$ .*

(1) If  $\nu \neq 0$ , then for every  $n \geq 0$ , the law of  $\widetilde{\Xi}_{\nu_n, \varepsilon}$  is absolutely continuous with respect to that of  $\widetilde{\Xi}_{\nu, \varepsilon}$ . Moreover, the corresponding density  $Y_{\varepsilon, \nu, \nu_n}$ , converges a.s. to 1 as  $n \rightarrow +\infty$ .

(2) If  $\nu = 0$ , then

$$\lim_{n \rightarrow +\infty} \mathbb{P}(\widetilde{\Xi}_{\nu_n, \varepsilon} = \emptyset) = 1. \quad (4.1)$$

*Proof.* Both  $\widetilde{\Xi}_{\nu_n, \varepsilon}$  and  $\widetilde{\Xi}_{\nu, \varepsilon}$  are a.s. finite Poisson point processes. According to Proposition 2.11, the intensity measure of  $\widetilde{\Xi}_{\nu, \varepsilon}$  is

$$\frac{1}{2} \iint_{(\partial \widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D})^2} \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dz) \sigma_{\partial \widehat{\mathbb{D}}_\varepsilon}(dw) \mu_{z, w}^{\mathbb{D}} \iint_{\overline{A_L} \times \overline{A_L}} H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) H_{\widehat{\mathbb{D}}_\varepsilon}(w, y) d(\nu \otimes \nu)(x, y).$$

The intensity measure for  $\widetilde{\Xi}_{\nu_n, \varepsilon}$  has same expression, with  $\nu_n$  instead of  $\nu$ .

If  $\nu \neq 0$ , then the intensity measure for  $\nu_n$  is absolutely continuous with respect to that for  $\nu$ , both being absolutely continuous with respect to

$$\iint_{(\widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D})^2} \sigma_{\widehat{\mathbb{D}}_\varepsilon}(dz) \sigma_{\widehat{\mathbb{D}}_\varepsilon}(dw) \mu_{z,w}^{\mathbb{D}}. \quad (4.2)$$

The density from  $\nu$  to  $\nu_n$  is

$$\frac{\iint_{A_L \times A_L} H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) H_{\widehat{\mathbb{D}}_\varepsilon}(w, y) d(\nu_n \otimes \nu_n)(x, y)}{\iint_{A_L \times A_L} H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) H_{\widehat{\mathbb{D}}_\varepsilon}(w, y) d(\nu \otimes \nu)(x, y)},$$

where  $z, w \in \widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D}$  are the two endpoints of the path. This density converges to 1 almost everywhere and in  $\mathbb{L}^1$ . This follows from the weak convergence of  $(\nu_n)_{n \geq 0}$  to  $\nu$  together with the continuity of the boundary Poisson kernel  $H_{\widehat{\mathbb{D}}_\varepsilon}$ . Since the Poisson point processes are a.s. finite, this implies the absolute continuity of Poisson point processes and the a.s. convergence of the density to 1.

If  $\nu = 0$ , then the total mass of the intensity for  $\nu_n$  converges to 0, which implies (4.1).  $\square$

**Lemma 4.3.** *Let  $S$  be an abstract Polish space and  $\mathcal{B}$  be its Borel  $\sigma$ -algebra. Let  $X$  and  $X_n$ , for  $n \geq 0$ , be random variables taking values in  $(S, \mathcal{B})$ . Assume that for every  $n \geq 0$ , the law of  $X_n$  is absolutely continuous with respect to that of  $X$ , with density denoted by  $Y_n$ . Assume moreover that  $(Y_n)_{n \geq 0}$  converges  $d\mathbb{P}_X$ -a.s. to 1 as  $n \rightarrow +\infty$ . Then it is possible to couple  $X$  and all  $X_n$  for  $n \geq 0$  on the same probability space such that a.s.  $X_n = X$  for every  $n$  large enough.*

*Proof.* Note that the sequence  $(Y_n)_{n \geq 0}$  is naturally defined on the same probability space as  $X$  and is measurable with respect to  $X$ . For  $n \geq 0$  such that  $\mathbb{P}(Y_n > 1) > 0$ , let  $\widetilde{X}_n$  be a random variable taking values in  $(S, \mathcal{B})$ , with density

$$\frac{(Y_n - 1)_+}{\mathbb{E}[(Y_n - 1)_+]}$$

with respect to  $X$ . We also take  $X$  and all the  $\widetilde{X}_n$  to be independent. Let  $U$  be a uniform random variable on  $(0, 1)$ , independent from  $(X, (\widetilde{X}_n)_{n \geq 0})$ . We construct the sequence  $(\widehat{X}_n)_{n \geq 0}$  as follows. On the event  $\{Y_n \geq U\}$ , we set  $\widehat{X}_n = X$ . On the event  $\{Y_n < U\}$ , we set  $\widehat{X}_n = \widetilde{X}_n$ . It is easy to check that for every  $n \geq 0$ ,  $\widehat{X}_n$  has same distribution as  $X_n$ . Moreover, a.s. for every  $n$  large enough,  $Y_n \geq U$  and  $\widehat{X}_n = X$ .  $\square$

**Lemma 4.4.** *It is possible to couple  $(\widehat{\Xi}_{\nu_n, \varepsilon})_{n \geq 0}$  and  $\widehat{\Xi}_{\nu, \varepsilon}$  on the same probability space such that the following conditions hold a.s.*

- (1)  $\lim_{n \rightarrow +\infty} d_{\text{curves}}(\widehat{\Xi}_{\nu_n, \varepsilon}, \widehat{\Xi}_{\nu, \varepsilon}) = 0$ .
- (2) For every  $n$  large enough,  $\widetilde{\Xi}_{\nu_n, \varepsilon} = \widetilde{\Xi}_{\nu, \varepsilon}$ .

*Proof.* We can assume that  $\nu \neq 0$ . The case  $\nu = 0$  is trivial by (4.1). The fact that there is a coupling such that the condition (2) is satisfied follows from Lemmas 4.2 and 4.3. It remains to couple the slices

$$\left\{ (\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f} : \gamma \in \widehat{\Xi}_{\nu_n, \varepsilon} \right\}, \quad \left\{ (\gamma(T_{\gamma, \varepsilon}^l + t))_{0 \leq t \leq T_\gamma - T_{\gamma, \varepsilon}^l} : \gamma \in \widehat{\Xi}_{\nu_n, \varepsilon} \right\},$$

and

$$\left\{ (\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f} : \gamma \in \widehat{\Xi}_{\nu, \varepsilon} \right\}, \quad \left\{ (\gamma(T_{\gamma, \varepsilon}^l + t))_{0 \leq t \leq T_\gamma - T_{\gamma, \varepsilon}^l} : \gamma \in \widehat{\Xi}_{\nu, \varepsilon} \right\},$$

in a way that the condition (1) holds. We only construct coupling of the slices  $(\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f}$ . The coupling for the slices  $(\gamma(T_{\gamma, \varepsilon}^l + t))_{0 \leq t \leq T_\gamma - T_{\gamma, \varepsilon}^l}$  can be obtained similarly.



According to Proposition 2.11, given  $\gamma \in \widehat{\Xi}_{\nu, \varepsilon}$ , conditionally on  $(\gamma(T_{\gamma, \varepsilon}^f), \gamma(T_{\gamma, \varepsilon}^l))$ , the three slices  $(\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f}$ ,  $(\gamma(T_{\gamma, \varepsilon}^f + t))_{0 \leq t \leq T_{\gamma, \varepsilon}^l - T_{\gamma, \varepsilon}^f}$  and  $(\gamma(T_{\gamma, \varepsilon}^l + t))_{0 \leq t \leq T_{\gamma, \varepsilon} - T_{\gamma, \varepsilon}^l}$  are independent. The conditional distribution of  $(\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f}$  is

$$\left( \int_{\overline{A_L}} d\nu(x) H_{\widehat{\mathbb{D}}_\varepsilon}(x, \gamma(T_{\gamma, \varepsilon}^f)) \right)^{-1} \int_{\overline{A_L}} d\nu(x) \mu_{x, \gamma(T_{\gamma, \varepsilon}^f)}^{\widehat{\mathbb{D}}_\varepsilon}.$$

In the case of  $\widehat{\Xi}_{\nu_n, \varepsilon}$  the distribution is the same, with  $\nu_n$  instead of  $\nu$ .

Given  $\theta \in [\frac{1}{2}\pi, \frac{3}{2}\pi]$ , let  $A[i, e^{i\theta}]$  denote the closed subarc of  $\overline{A_L}$  with endpoints  $i$  and  $e^{i\theta}$ . Given  $z \in \partial\widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D}$ , let  $\vartheta_{z, \nu}$  be the following function from  $[0, 1]$  to  $[\frac{1}{2}\pi, \frac{3}{2}\pi]$ :

$$\vartheta_{z, \nu}(u) := \inf \left\{ \theta \in \left[ \frac{1}{2}\pi, \frac{3}{2}\pi \right] : \int_{A[i, e^{i\theta}]} d\nu(x) H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) \geq u \int_{\overline{A_L}} d\nu(x) H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) \right\}.$$

Suppose  $U$  is a uniform random variable on  $(0, 1)$ , then  $e^{i\vartheta_{z, \nu}(U)}$  has the distribution

$$\left( \int_{\overline{A_L}} d\nu(\tilde{x}) H_{\widehat{\mathbb{D}}_\varepsilon}(\tilde{x}, z) \right)^{-1} H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) d\nu(x).$$

The functions  $\vartheta_{z, \nu_n}$  are defined similarly, with  $\nu_n$  instead of  $\nu$ . We have that  $(\vartheta_{z, \nu_n}(U))_{n \geq 0}$  converges a.s. to  $\vartheta_{z, \nu}(U)$ .

Given  $x \in \overline{A_L}$  and  $z \in \partial\widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D}$ , let  $\psi_{x, z}$  be the conformal map from  $\mathbb{D}$  to  $\widehat{\mathbb{D}}_\varepsilon$ , uniquely defined by

$$\psi_{x, z}(-i) = x, \quad \psi_{x, z}(i) = z, \quad |\psi'_{x, z}(-i)| = 1.$$

Given  $(\varphi(t))_{0 \leq t \leq T_\varphi}$  a continuous curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$ , let  $\mathcal{T}_{x, z}(\varphi)$  denote the continuous curve in  $\widehat{\mathbb{D}}_\varepsilon$  from  $x$  to  $z$  obtained by applying to the curve  $\varphi$  the conformal map  $\psi_{x, z}$  and the change of time  $ds = |\psi'_{x, z}(\varphi(t))|^2 dt$ . The image of the normalized excursion probability measure  $\mu_{-i, i}^{\mathbb{D}, \#}$  under the map  $\mathcal{T}_{x, z}$  is the normalized excursion probability measure  $\mu_{x, z}^{\widehat{\mathbb{D}}_\varepsilon, \#}$ .

Now fix  $z \in \partial\widehat{\mathbb{D}}_\varepsilon \cap \mathbb{D}$ . Let  $\varphi$  be a Brownian excursion from  $-i$  to  $i$  in  $\mathbb{D}$ , sampled according to  $\mu_{-i, i}^{\mathbb{D}, \#}$ , and let  $U$  be an independent random variable uniform on  $(0, 1)$ . Then the random curve  $\mathcal{T}_{e^{i\vartheta_{z, \nu}(U)}, z}(\varphi)$  is distributed according to the probability measure

$$\left( \int_{\overline{A_L}} d\nu(x) H_{\widehat{\mathbb{D}}_\varepsilon}(x, z) \right)^{-1} \int_{\overline{A_L}} d\nu(x) \mu_{x, z}^{\widehat{\mathbb{D}}_\varepsilon}.$$

The curve  $\mathcal{T}_{e^{i\vartheta_{z, \nu_n}(U)}, z}(\varphi)$  has a similar distribution, with  $\nu_n$  instead of  $\nu$ . Moreover, as  $n \rightarrow +\infty$ , the sequence  $(\mathcal{T}_{e^{i\vartheta_{z, \nu_n}(U)}, z}(\varphi))_{n \geq 0}$  converges a.s. to  $\mathcal{T}_{e^{i\vartheta_{z, \nu}(U)}, z}(\varphi)$ . So, this construction provides a way to couple the slices  $(\gamma(t))_{0 \leq t \leq T_{\gamma, \varepsilon}^f}$  for  $\gamma \in \widehat{\Xi}_{\nu, \varepsilon}$ , respectively  $\gamma \in \widehat{\Xi}_{\nu_n, \varepsilon}$ , so that the a.s. convergence holds.  $\square$

**Lemma 4.5.** *Fix  $\tilde{\varepsilon} > 0$ . Then*

$$\lim_{\varepsilon \rightarrow 0} \mathbb{P}(\Xi_{\nu, \tilde{\varepsilon}} \setminus \widehat{\Xi}_{\nu, \varepsilon} \neq \emptyset) = 0, \quad \limsup_{\varepsilon \rightarrow 0} \mathbb{P}(\Xi_{\nu_n, \tilde{\varepsilon}} \setminus \widehat{\Xi}_{\nu_n, \varepsilon} \neq \emptyset) = 0.$$

*Proof.* The Poisson point processes  $\Xi_{\nu, \tilde{\varepsilon}} \setminus \widehat{\Xi}_{\nu, \varepsilon}$  and  $\Xi_{\nu_n, \tilde{\varepsilon}} \setminus \widehat{\Xi}_{\nu_n, \varepsilon}$  consist precisely of excursions of diameter greater than  $\tilde{\varepsilon}$ , but that do not visit  $\mathbb{D}_\varepsilon$ . It is easy to see that

$$\lim_{\varepsilon \rightarrow 0} \sup_{x, y \in \overline{A_L}} \mu_{x, y}^{\mathbb{D}}(\{\gamma : \text{diam}(\gamma) > \tilde{\varepsilon}, \text{Range}(\gamma) \cap \mathbb{D}_\varepsilon = \emptyset\}) = 0.$$

The conclusion follows.  $\square$

Now we are ready to complete the proof of Proposition 4.1.

*Proof of Proposition 4.1.* According to Lemma 4.5, for every  $k \geq 1$ , there is  $\varepsilon_k \in (0, 2^{-k}]$  such that

$$\mathbb{P}\left(\Xi_{\nu, 2^{-k}} \setminus \widehat{\Xi}_{\nu, \varepsilon_k} \neq \emptyset\right) \leq 2^{-k}, \quad \sup_{n \geq 0} \mathbb{P}\left(\Xi_{\nu_n, 2^{-k}} \setminus \widehat{\Xi}_{\nu_n, \varepsilon_k} \neq \emptyset\right) \leq 2^{-k}.$$

We may also take the sequence  $(\varepsilon_k)_{k \geq 1}$  to be non-increasing.

According to Lemma 4.4, for every  $k \geq 1$ , there is a coupling on the same probability space of  $\Xi_{\nu}^{(k)}$  and  $\Xi_{\nu_n}^{(k)}$  for  $n \geq 0$ , with  $\Xi_{\nu}^{(k)}$  distributed as  $\Xi_{\nu}$  and  $\Xi_{\nu_n}^{(k)}$  distributed as  $\Xi_{\nu_n}$ , such that a.s.,  $\lim_{n \rightarrow +\infty} d_{\text{curves}}(\widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)}, \widehat{\Xi}_{\nu, \varepsilon_k}^{(k)}) = 0$  and for every  $n$  large enough,  $\widetilde{\Xi}_{\nu_n, \varepsilon_k}^{(k)} = \widetilde{\Xi}_{\nu, \varepsilon_k}^{(k)}$ . Moreover, since  $\Xi_{\nu}^{(k)} \setminus \widehat{\Xi}_{\nu, \varepsilon_k}^{(k)}$  is independent from  $\widehat{\Xi}_{\nu, \varepsilon_k}^{(k)}$  and  $\Xi_{\nu_n}^{(k)} \setminus \widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)}$  is independent from  $\widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)}$ , one can further require that for every  $k \geq 1$ ,

$$\begin{aligned} & \mathbb{P}\left(\Xi_{\nu, 2^{-k}}^{(k)} \setminus \widehat{\Xi}_{\nu, \varepsilon_k}^{(k)} \neq \emptyset \text{ or } \exists n \geq 0, \Xi_{\nu_n, 2^{-k}}^{(k)} \setminus \widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)} \neq \emptyset\right) \\ & \leq \mathbb{P}\left(\Xi_{\nu, 2^{-k}}^{(k)} \setminus \widehat{\Xi}_{\nu, \varepsilon_k}^{(k)} \neq \emptyset\right) \vee \sup_{n \geq 0} \mathbb{P}\left(\Xi_{\nu_n, 2^{-k}}^{(k)} \setminus \widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)} \neq \emptyset\right) \leq 2^{-k}. \end{aligned}$$

By considering the conditional law of the sequence  $(\Xi_{\nu_n}^{(k)})_{n \geq 0}$  given  $\Xi_{\nu}^{(k)}$ , one can further couple all the  $\Xi_{\nu}^{(k)}$  and  $\Xi_{\nu_n}^{(k)}$  for  $n \geq 0$  and  $k \geq 1$  on the same probability space such that the Poisson point processes  $\Xi_{\nu}^{(k)}$  are a.s. all the same for different values of  $k$ . Will denote by  $\Xi_{\nu}^*$  their common value. It is distributed as  $\Xi_{\nu}$ .

Set  $N_1 := 0$ , and for  $k \geq 2$ ,

$$N_k := \min \left\{ N > N_{k-1} : \mathbb{P}\left(\exists n \geq N, d_{\text{curves}}(\widehat{\Xi}_{\nu_n, \varepsilon_k}^{(k)}, \widehat{\Xi}_{\nu, \varepsilon_k}^{(k)}) > 2^{-k} \text{ or } \widetilde{\Xi}_{\nu_n, \varepsilon_k}^{(k)} \neq \widetilde{\Xi}_{\nu, \varepsilon_k}^* \right) \leq 2^{-k} \right\}.$$

We would like to emphasize that the sequence  $(N_k)_{k \geq 1}$  is deterministic. We define the sequence  $(\Xi_{\nu_n}^*)_{n \geq 1}$  as follows. Given  $n \geq 0$ , there is a unique  $k \geq 1$  such that  $N_k \leq n < N_{k+1}$ , and we set  $\Xi_{\nu_n}^* = \Xi_{\nu_n}^{(k)}$ . For every  $n \geq 0$ ,  $\Xi_{\nu_n}^*$  is distributed as  $\Xi_{\nu_n}$ .

For  $j \geq 1$  and  $k \geq j$ , let  $E_{j,k}$  denote the event that there is  $n \in \{N_k, \dots, N_{k+1} - 1\}$  such that  $\widetilde{\Xi}_{\nu_n, \varepsilon_j}^{(k)} \neq \widetilde{\Xi}_{\nu, \varepsilon_j}^*$ . By construction, for every  $k \geq 1$   $\mathbb{P}(E_{j,k}) \leq 2^{-k}$ . Moreover,  $E_{j,k} \subset E_{k,k}$  for  $j \leq k$ . Thus, for every  $j \geq 1$ ,

$$\sum_{k \geq j} \mathbb{P}(E_{j,k}) < +\infty.$$

By Borel-Cantelli lemma, this means that a.s., the events  $E_{j,k}$  occur for only finitely many values of  $k$ . Thus, a.s. for every  $n$  large enough,  $\widetilde{\Xi}_{\nu_n, \varepsilon_j}^* = \widetilde{\Xi}_{\nu, \varepsilon_j}^*$ .

Similarly, by using the Borel-Cantelli lemma, we get that for every  $j \geq 1$ , a.s.

$$\lim_{n \rightarrow +\infty} d_{\text{curves}}\left(\widehat{\Xi}_{\nu_n, \varepsilon_j}^*, \widehat{\Xi}_{\nu, \varepsilon_j}^*\right) = 0.$$

By applying the Borel-Cantelli lemma once more, we get that a.s., for every  $j \geq 1$ , there is  $k \geq j$  such that for every  $n \geq N_k$ ,  $\Xi_{\nu_n, 2^{-j}}^* \subset \widehat{\Xi}_{\nu_n, \varepsilon_k}^*$ . This concludes the proof.  $\square$

We end this section with the following lemma which will be useful for the proof of Theorem 1.2.

**Lemma 4.6.** *Assume  $(\Xi_{\nu_n})_{n \geq 0}$  and  $\Xi_{\nu}$  are coupled on the same probability space as in Proposition 4.1. Then a.s. the family*

$$(\text{Range}(\gamma) \cup \partial \mathbb{D})_{\gamma \in \Xi_{\nu_n}, n \geq 0}$$

*is uniformly locally connected; see Definition 2.1. Furthermore, a.s. the family*

$$(\text{Range}(\gamma) \cup \text{Range}(\gamma') \cup \partial \mathbb{D})_{\gamma, \gamma' \in \Xi_{\nu_n}, n \geq 0}$$

*is uniformly locally connected, too.*

*Proof.* We will only prove the first point. The proof of the second point is similar and we omit it.

First note that for any fixed  $n \geq 0$ , the family  $(\text{Range}(\gamma) \cup \partial\mathbb{D})_{\gamma \in \Xi_{\nu_n}}$  is uniformly locally connected. Indeed, each  $\text{Range}(\gamma) \cup \partial\mathbb{D}$ , is compact, connected, and locally connected due to Lemma 2.2. Moreover, for every  $\varepsilon > 0$ , there are only finitely many  $\gamma \in \Xi_{\nu_n}$  such that  $\text{diam Range}(\gamma) \geq \varepsilon$ . So, if  $(\text{Range}(\gamma) \cup \partial\mathbb{D})_{\gamma \in \Xi_{\nu_n}, n \geq 0}$  were not uniformly locally connected, one of the two cases would occur:

- Case 1: there are  $\varepsilon > 0$ , a subsequence  $(n_j)_{j \geq 0}$ , with  $n_j \rightarrow +\infty$  as  $j \rightarrow +\infty$ , excursions  $\gamma_{n_j} \in \Xi_{\nu_{n_j}}$ , and points  $z_{n_j}, z'_{n_j} \in \text{Range}(\gamma_{n_j})$ , such that  $|z'_{n_j} - z_{n_j}| \rightarrow 0$  and  $z_{n_j}$  and  $z'_{n_j}$  are not  $\varepsilon$ -connected in  $\text{Range}(\gamma_{n_j})$ .
- Case 2: there are  $\varepsilon > 0$ , a subsequence  $(n_j)_{j \geq 0}$ , with  $n_j \rightarrow +\infty$  as  $j \rightarrow +\infty$ , excursions  $\gamma_{n_j} \in \Xi_{\nu_{n_j}}$ , and points  $z_{n_j} \in \text{Range}(\gamma_{n_j})$ ,  $z'_{n_j} \in \partial\mathbb{D}$ , such that  $|z'_{n_j} - z_{n_j}| \rightarrow 0$  and  $z_{n_j}$  and  $z'_{n_j}$  are not  $\varepsilon$ -connected in  $\text{Range}(\gamma_{n_j})$ .

In both cases, necessarily

$$\inf_{j \geq 0} \text{diam Range}(\gamma_{n_j}) > 0.$$

So, up to further extracting a subsequence, one can assume that  $\gamma_{n_j}$  converges to an excursion  $\gamma_\infty \in \Xi_\nu$  for  $d_{\text{curves}}$ , and that  $z_{n_j}$  and  $z'_{n_j}$  converge to  $z_\infty$ .

In Case 1,  $z_\infty \in \text{Range}(\gamma_\infty)$ . One can further distinguish the following subcases:

- Case 1a:  $z_\infty \in \text{Range}(\gamma_\infty) \cap \mathbb{D}$ .
- Case 1b:  $z_\infty \in \text{Range}(\gamma_\infty) \cap \overline{A_L}$ .

In Case 1a, consider  $\varepsilon' \in (0, \text{dist}(z_\infty, \overline{A_L}) \wedge 1)$ . For  $j$  large enough, we have that

$$(\gamma_{n_j}(T_{\gamma_{n_j}, \varepsilon'}^f + t))_{0 \leq t \leq T_{\gamma_{n_j}, \varepsilon'}^l - T_{\gamma_{n_j}, \varepsilon'}^f} = (\gamma_\infty(T_{\gamma_\infty, \varepsilon'}^f + t))_{0 \leq t \leq T_{\gamma_\infty, \varepsilon'}^l - T_{\gamma_\infty, \varepsilon'}^f},$$

and

$$z_{n_j}, z'_{n_j} \in \{\gamma_\infty(T_{\gamma_\infty, \varepsilon'}^f + t) : 0 \leq t \leq T_{\gamma_\infty, \varepsilon'}^l - T_{\gamma_\infty, \varepsilon'}^f\} \subset \text{Range}(\gamma_{n_j}).$$

However, since the set  $\{\gamma_\infty(T_{\gamma_\infty, \varepsilon'}^f + t) : 0 \leq t \leq T_{\gamma_\infty, \varepsilon'}^l - T_{\gamma_\infty, \varepsilon'}^f\}$  is locally connected, we get a contradiction. So Case 1a cannot occur.

In Case 1b, there is a sequence of positive times  $(t_{n_j})_{j \geq 0}$  converging to 0 such that for every  $j \geq 0$ ,

$$z_{n_j}, z'_{n_j} \in \{\gamma_{n_j}(t) : 0 \leq t \leq t_{n_j}\} \cup \{\gamma_{n_j}(t) : T_{\gamma_{n_j}} - t_{n_j} \leq t \leq T_{\gamma_{n_j}}\}.$$

We have that

$$\lim_{j \rightarrow +\infty} \text{diam}\{\gamma_{n_j}(t) : 0 \leq t \leq t_{n_j}\} = \lim_{j \rightarrow +\infty} \text{diam}\{\gamma_{n_j}(t) : T_{\gamma_{n_j}} - t_{n_j} \leq t \leq T_{\gamma_{n_j}}\} = 0,$$

and that the set  $\{\gamma_{n_j}(t) : 0 \leq t \leq t_{n_j}\} \cup \{\gamma_{n_j}(t) : T_{\gamma_{n_j}} - t_{n_j} \leq t \leq T_{\gamma_{n_j}}\} \cup \partial\mathbb{D}$  is closed and connected. So we get that the family

$$(\{\gamma_{n_j}(t) : 0 \leq t \leq t_{n_j}\} \cup \{\gamma_{n_j}(t) : T_{\gamma_{n_j}} - t_{n_j} \leq t \leq T_{\gamma_{n_j}}\} \cup \partial\mathbb{D})_{j \geq 0}$$

is uniformly locally connected. So Case 1b cannot occur.

In Case 2, again  $z_\infty \in \text{Range}(\gamma_\infty) \cap \overline{A_L}$ . So Case 2 can be ruled out by an argument very similar to that used for Case 1b.  $\square$

## 4.2 Continuous dependence of the curve $\eta_{\kappa,\nu}$ and proof of Theorem 1.2

In this section we deal with the dependence of the curve  $\eta_{\kappa,\nu}$  on the measure  $\nu$ . Recall that  $\mathcal{O}_{\kappa,\nu}$  is an open simply connected subset of  $\mathbb{D}$ , and  $\partial\mathcal{O}_{\kappa,\nu} = \eta_{\kappa,\nu} \cup A_R$ . Recall that  $\psi_{\kappa,\nu}$  is the conformal transformation from  $\mathbb{D}$  to  $\mathcal{O}_{\kappa,\nu}$  uniquely defined by the normalization  $\psi_{\kappa,\nu}(-i) = -i$ ,  $\psi_{\kappa,\nu}(1) = 1$ ,  $\psi_{\kappa,\nu}(i) = i$  and  $\psi_{\kappa,\nu}(A_R) = A_R$ . According to Theorem 2.8,  $\psi_{\kappa,\nu}$  extends continuously to  $\overline{\mathbb{D}}$ . In case the curve  $\eta_{\kappa,\nu}$  is simple (see Proposition 3.4),  $\psi_{\kappa,\nu}$  induces a homeomorphism from  $\overline{\mathbb{D}}$  to  $\overline{\mathcal{O}_{\kappa,\nu}}$ . In general,  $\psi_{\kappa,\nu}$  induces a homeomorphism from  $\mathbb{D} \cup \overline{A_R}$  to  $\mathcal{O}_{\kappa,\nu} \cup \overline{A_R}$ ; see e.g. [Pom92, Theorem 2.15]. By construction,  $\eta_{\kappa,\nu} = \psi_{\kappa,\nu}(\overline{A_L})$ . The goal of this section is to complete the proof of Theorem 1.2.

We will restrict to the case  $\kappa \neq 8/3$ , as the case  $\kappa = 8/3$  is simpler. Note that all the probabilistic content of our proof is already contained in Proposition 4.1. We will additionally rely on deterministic geometrical arguments and some a.s. properties of Brownian excursions and CLE. In the rest of this section, we fix the following assumptions: Fix  $\kappa \in (8/3, 4]$  and  $\nu$  a finite non-negative Radon measure on  $\overline{A_L}$ . Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_L}$ , converging weakly to  $\nu$ . Assume  $(\Xi_{\nu_n})_{n \geq 0}$  and  $\Xi_\nu$  are coupled on the same probability space as in Proposition 4.1. Let  $\mathfrak{C}_\kappa$  be a  $\text{CLE}_\kappa$  in  $\mathbb{D}$  independent from  $((\Xi_{\nu_n})_{n \geq 0}, \Xi_\nu)$ .

**Lemma 4.7.** Denote by  $\tilde{\mathfrak{C}}_{\kappa,\nu}$  the set constructed from  $\mathfrak{C}_\kappa$  and  $\Xi_\nu$  as in (3.2). Recall that  $\mathcal{O}_{\kappa,\nu,\varepsilon}$  denotes the connected component of  $\mathcal{O}_{\kappa,\nu} \cap \mathbb{D}_\varepsilon$  (see (2.3)) adjacent to  $A_R \cap \overline{\mathbb{D}_\varepsilon}$ . Define  $\tilde{\mathfrak{C}}_{\kappa,\nu_n}$  and  $\mathcal{O}_{\kappa,\nu_n,\varepsilon}$  for  $\mathfrak{C}_\kappa$  and  $\Xi_{\nu_n}$  similarly. Then a.s., for every  $\varepsilon \in (0, 1)$ , there is  $n'_\varepsilon \in \mathbb{N}$ , such that, for every  $n \geq n'_\varepsilon$ ,

$$\mathcal{O}_{\kappa,\nu_n,\varepsilon} = \mathcal{O}_{\kappa,\nu,\varepsilon}.$$

*Proof.* It is enough to check this for fixed  $\varepsilon$ , and then consider a sequence of  $\varepsilon$  converging to 0. Fix  $\varepsilon \in (0, 1)$ . The local finiteness of the CLE ensures that a.s. there is  $\varepsilon' \in (0, \varepsilon)$  such that all the  $\text{CLE}_\kappa$  loops in  $\mathfrak{C}_\kappa$  intersecting  $\mathbb{D}_\varepsilon$  are at distance greater than  $\varepsilon'$  from  $A_L$ . The condition (2) in Proposition 4.1 ensures that one can take  $n'_\varepsilon = n_{\varepsilon'}$ .  $\square$

**Lemma 4.8.** A.s., for every  $w \in \mathcal{O}_{\kappa,\nu}$ , the point  $w$  belongs to  $\mathcal{O}_{\kappa,\nu_n}$  for every  $n$  large enough and  $(\mathcal{O}_{\kappa,\nu_n}, w)$  converges to  $(\mathcal{O}_{\kappa,\nu}, w)$  in the Carathéodory sense as  $n \rightarrow +\infty$ ; see Definition 2.9.

*Proof.* The condition (1) in Definition 2.9 is automatic. We then check the condition (2) in Definition 2.9. Given  $z \in \overline{\mathcal{O}_{\kappa,\nu}}$ , let  $I_z$  denote the straight line segment in  $\overline{\mathbb{D}}$  with endpoints 1 and  $\psi_{\kappa,\nu}^{-1}(z)$ . If  $z$  is a multiple point on  $\partial\mathcal{O}_{\kappa,\nu}$ , then  $\psi_{\kappa,\nu}^{-1}(z)$  will denote an arbitrary choice of a preimage of  $z$ . Let  $J_z$  denote  $\psi_{\kappa,\nu}(I_z)$ . It is a continuous curve in  $\overline{\mathcal{O}_{\kappa,\nu}}$  from 1 to  $z$ . If  $z \in \mathcal{O}_{\kappa,\nu}$ , then  $\text{dist}(I_z, A_L) > 0$ , and thus

$$\text{dist}(J_z, A_L) \geq \text{dist}(J_z, \eta_{\kappa,\nu}) > 0.$$

So, for  $\varepsilon \in (0, \text{dist}(J_z, A_L)/2)$ , we have  $J_z \subset \mathcal{O}_{\kappa,\nu} \cap \mathbb{D}_\varepsilon$ . Then necessarily  $J_z \subset \mathcal{O}_{\kappa,\nu,\varepsilon}$ . According to Lemma 4.7, for  $n \geq n'_\varepsilon$ , we have  $\mathcal{O}_{\kappa,\nu,\varepsilon} = \mathcal{O}_{\kappa,\nu_n,\varepsilon}$ , and thus  $\mathcal{O}_{\kappa,\nu_n,\varepsilon}$  is a neighborhood of  $z$  in  $\mathcal{O}_{\kappa,\nu_n}$ . So we get that for every  $w \in \mathcal{O}_{\kappa,\nu}$ , the point  $w$  belongs to  $\mathcal{O}_{\kappa,\nu_n}$  for every  $n$  large enough. This guarantees the condition (2) in Definition 2.9. It remains to check the condition (3).

So consider  $z \in \partial\mathcal{O}_{\kappa,\nu}$ . There are two cases, either  $z \in \mathbb{D}$  or  $z \in \overline{A_L}$ . In the first case,  $z \in (\partial\mathcal{O}_{\kappa,\nu}) \cap \mathbb{D}$ , we still have  $\text{dist}(J_z, A_L) > 0$ . For  $\varepsilon \in (0, \text{dist}(J_z, A_L)/2)$ , one has that  $z \in (\partial\mathcal{O}_{\kappa,\nu,\varepsilon}) \cap \mathbb{D}_\varepsilon$ . It follows that for  $n \geq n'_\varepsilon$ ,  $z \in \overline{\mathcal{O}_{\kappa,\nu_n,\varepsilon}}$  and also  $z \notin \mathcal{O}_{\kappa,\nu_n}$ , since otherwise one would have  $z \in \mathcal{O}_{\kappa,\nu,\varepsilon}$ . Thus, for  $n \geq n'_\varepsilon$ ,  $z \in \partial\mathcal{O}_{\kappa,\nu_n}$ .

Now consider the second case,  $z \in (\partial\mathcal{O}_{\kappa,\nu}) \cap \overline{A_L}$ . For  $j \geq 1$ , let  $\tilde{z}_j$  be the point

$$\tilde{z}_j := \psi_{\kappa,\nu} \left( \left( 1 - \frac{1}{j} \right) \psi_{\kappa,\nu}^{-1}(z) + \frac{1}{j} \right) \in \mathcal{O}_{\kappa,\nu}.$$

We have that  $\tilde{z}_j \rightarrow z$  as  $j \rightarrow +\infty$ . Let  $J_{z,j}$  be the part of the curve  $J_z$  running from 1 to  $\tilde{z}_j$ . Then for every  $j \geq 1$ ,

$$\text{dist}(J_{z,j}, A_L) \geq \text{dist}(J_{z,j}, \eta_{\kappa,\nu}) > 0.$$

Thus, for  $\varepsilon_j \in (0, \text{dist}(J_{z,j}, A_L)/2)$ , we have  $\tilde{z}_j \in \mathcal{O}_{\kappa, \nu, \varepsilon_j}$ , and moreover, for  $n \geq n'_{\varepsilon_j}$ , we have  $\tilde{z}_j \in \mathcal{O}_{\kappa, \nu_n, \varepsilon_j}$ . Using a diagonal extraction, we get a subsequence  $(\tilde{z}_{j_n})_{n \geq n_0}$ , with for every  $n \geq n_0$ ,  $\tilde{z}_{j_n} \in \mathcal{O}_{\kappa, \nu_n}$ , and

$$\lim_{n \rightarrow +\infty} \tilde{z}_{j_n} = z.$$

Since  $z \in \partial\mathbb{D}$ , we have that  $z \notin \mathcal{O}_{\kappa, \nu_n}$ . Thus, the straight line segment from  $\tilde{z}_{j_n}$  to  $z$  contains a point  $z_n \in \partial\mathcal{O}_{\kappa, \nu_n}$ . Moreover, by construction,  $|z - z_n| < |z - \tilde{z}_{j_n}|$ , and so  $z_n \rightarrow z$  as  $n \rightarrow +\infty$ . So one gets the condition (3) of Definition 2.9.  $\square$

**Lemma 4.9.** *A.s. the family  $(\mathbb{C} \setminus \mathcal{O}_{\kappa, \nu_n})_{n \geq 0}$  is uniformly locally connected; see Definition 2.1.*

*Proof.* According to Lemma 2.4, it is enough to check that the family  $(\mathcal{S}_{\kappa, \nu_n} \cup \partial\mathbb{D})_{n \geq 0}$  is uniformly locally connected. If this is not the case, then at least one of the following happens:

- Case 1: there are  $\varepsilon > 0$ , a subsequence  $(n_j)_{j \geq 0}$ , excursions  $\gamma_{n_j} \in \Xi_{\nu_{n_j}}$ , and points  $z_{n_j}, z'_{n_j} \in \mathcal{S}_{\kappa}(\gamma_{n_j})$  (see (3.1)) such that  $|z'_{n_j} - z_{n_j}| \rightarrow 0$  and  $z_{n_j}$  and  $z'_{n_j}$  are not  $\varepsilon$ -connected in  $\mathcal{S}_{\kappa, \nu_{n_j}} \cup \partial\mathbb{D}$ .
- Case 2: there are  $\varepsilon > 0$ , a subsequence  $(n_j)_{j \geq 0}$ , excursions  $\gamma_{n_j} \in \Xi_{\nu_{n_j}}$ , and points  $z_{n_j} \in \mathcal{S}_{\kappa}(\gamma_{n_j})$ ,  $z'_{n_j} \in \partial\mathbb{D}$ , such that  $|z'_{n_j} - z_{n_j}| \rightarrow 0$  and  $z_{n_j}$  and  $z'_{n_j}$  are not  $\varepsilon$ -connected in  $\mathcal{S}_{\kappa, \nu_{n_j}} \cup \partial\mathbb{D}$ .
- Case 3: there are  $\varepsilon > 0$ , a subsequence  $(n_j)_{j \geq 0}$ , excursions  $\gamma_{n_j}, \gamma'_{n_j} \in \Xi_{\nu_{n_j}}$ ,  $\gamma_{n_j} \neq \gamma'_{n_j}$ , and points  $z_{n_j} \in \mathcal{S}_{\kappa}(\gamma_{n_j})$ ,  $z'_{n_j} \in \mathcal{S}_{\kappa}(\gamma'_{n_j})$ , such that  $|z'_{n_j} - z_{n_j}| \rightarrow 0$  and  $z_{n_j}$  and  $z'_{n_j}$  are not  $\varepsilon$ -connected in  $\mathcal{S}_{\kappa, \nu_{n_j}} \cup \partial\mathbb{D}$ .

First consider Case 1. One can further distinguish between the case  $\inf_{j \geq 0} \text{diam Range}(\gamma_{n_j}) = 0$  and the case  $\inf_{j \geq 0} \text{diam Range}(\gamma_{n_j}) > 0$ . By extracting sub-subsequences, one can thus reduce Case 1 to the following two subcases:

- Case 1a: Case 1 with moreover  $\lim_{j \rightarrow +\infty} \text{diam Range}(\gamma_{n_j}) = 0$ .
- Case 1b: Case 1 with moreover  $\gamma_{n_j}$  converging to an excursion  $\gamma_{\infty} \in \Xi_{\nu}$  for  $d_{\text{curves}}$ .

Regarding Case 1a,  $\mathcal{S}_{\kappa}(\gamma_{n_j})$  is a connected compact subset connecting  $z_{n_j}$  and  $z'_{n_j}$ . Moreover, the fact that we use for every  $j \geq 0$  the same  $\text{CLE}_{\kappa}$ , together with the local finiteness of the  $\text{CLE}_{\kappa}$ , ensures that

$$\lim_{j \rightarrow +\infty} \text{diam}(\mathcal{S}_{\kappa}(\gamma_{n_j})) = 0. \quad (4.3)$$

So Case 1a cannot occur.

Regarding Case 1b, by considering sub-subsequences, one can further reduce it to the following subcases:

- Case 1ba: Case 1b with moreover  $z_{n_j}, z'_{n_j} \in \text{Range}(\gamma_{n_j})$ .
- Case 1bb: Case 1b with moreover  $z'_{n_j} \in \text{Range}(\gamma_{n_j})$  and  $z_{n_j} \in \text{Range}(\tilde{\gamma}_{n_j})$  for  $\tilde{\gamma}_{n_j} \in \tilde{\mathcal{C}}_{\kappa}(\gamma_{n_j})$  (see (3.1)), with  $\text{diam Range}(\tilde{\gamma}_{n_j}) \rightarrow 0$  as  $j \rightarrow +\infty$ .
- Case 1bc: Case 1b with moreover  $z'_{n_j} \in \text{Range}(\gamma_{n_j})$  and  $z_{n_j} \in \text{Range}(\tilde{\gamma})$  for  $\tilde{\gamma} \in \bigcap_{j \geq 0} \tilde{\mathcal{C}}_{\kappa}(\gamma_{n_j})$ .
- Case 1bd: Case 1b with moreover  $z_{n_j} \in \text{Range}(\tilde{\gamma}_{n_j})$  and  $z'_{n_j} \in \text{Range}(\tilde{\gamma}'_{n_j})$  for  $\tilde{\gamma}_{n_j}, \tilde{\gamma}'_{n_j} \in \tilde{\mathcal{C}}_{\kappa}(\gamma_{n_j})$ , with  $\text{diam Range}(\tilde{\gamma}_{n_j}) \rightarrow 0$  and  $\text{diam Range}(\tilde{\gamma}'_{n_j}) \rightarrow 0$  as  $j \rightarrow +\infty$ .
- Case 1be: Case 1b with moreover  $z_{n_j} \in \text{Range}(\tilde{\gamma}_{n_j})$  and  $z'_{n_j} \in \text{Range}(\tilde{\gamma}')$  for  $\tilde{\gamma}_{n_j} \in \tilde{\mathcal{C}}_{\kappa}(\gamma_{n_j})$  and  $\tilde{\gamma}' \in \bigcap_{j \geq 0} \tilde{\mathcal{C}}_{\kappa}(\gamma_{n_j})$ , with  $\text{diam Range}(\tilde{\gamma}_{n_j}) \rightarrow 0$  as  $j \rightarrow +\infty$ .

- Case 1bf: Case 1b with moreover  $z_{n_j}, z'_{n_j} \in \text{Range}(\tilde{\gamma})$  for  $\tilde{\gamma} \in \bigcap_{j \geq 0} \tilde{\mathfrak{C}}_\kappa(\gamma_{n_j})$ .

Lemma 4.6 ensures that Case 1ba cannot occur.

In Case 1bb, take points  $\tilde{z}_{n_j} \in \text{Range}(\tilde{\gamma}_{n_j}) \cap \text{Range}(\gamma_{n_j})$ . Then for  $j$  large enough,  $\text{diam Range}(\tilde{\gamma}_{n_j}) < \varepsilon/2$ , and thus  $\tilde{z}_{n_j}$  and  $z'_{n_j}$  cannot be  $\varepsilon/2$ -connected in  $\mathcal{S}_{\kappa, \nu_n} \cup \partial\mathbb{D}$ . So Case 1bb reduces to Case 1ba and cannot occur.

In Case 1bc, one can proceed similarly to the proof of Lemma 4.6. Indeed, away from  $\overline{A_L}$ ,  $\text{Range}(\gamma_{n_j})$  coincides with  $\text{Range}(\gamma_\infty)$  for  $j$  large enough, and by Lemma 2.2,  $\text{Range}(\gamma_\infty) \cup \text{Range}(\tilde{\gamma})$  is locally connected. This rules out Case 1bc.

Case 1bd reduces to Case 1bb by considering points  $\tilde{z}_{n_j} \in \text{Range}(\tilde{\gamma}_{n_j}) \cap \text{Range}(\gamma_{n_j})$ .

Similarly, Case 1be reduces to Case 1bc.

Case 1bf cannot occur because  $\text{Range}(\tilde{\gamma})$  is locally connected.

In Case 2 one can see that  $\text{dist}(z'_{n_j}, \text{Range}(\gamma_{n_j}) \cap \overline{A_L}) \rightarrow 0$  as  $j \rightarrow +\infty$ . Thus, Case 2 reduces to Case 1.

Case 3 can be reduced, by considering sub-subsequences, to the following two subcases:

- Case 3a: Case 3 with moreover  $\lim_{j \rightarrow +\infty} \text{diam Range}(\gamma_{n_j}) = 0$ .
- Case 3b: Case 3 with moreover  $\gamma_{n_j}$ , respectively  $\gamma'_{n_j}$ , converging to excursions  $\gamma_\infty$ , respectively  $\gamma'_\infty$  in  $\Xi_\nu$  for  $d_{\text{curves}}$ .

In Case 3a, since (4.3) holds, and in this way Case 3a reduces to Case 2.

Case 3b can be ruled out by arguments similar to those used for Case 1b.  $\square$

*Proof of Theorem 1.2.* As mentioned, we deal only with  $\kappa \in (8/3, 4]$ . Assume  $(\Xi_{\nu_n})_{n \geq 0}$  and  $\Xi_\nu$  are coupled on the same probability space as in Proposition 4.1, and that  $\mathfrak{C}_\kappa$  is sampled independent from  $((\Xi_{\nu_n})_{n \geq 0}, \Xi_\nu)$ . We will deduce the a.s. convergence of  $((\psi_{\kappa, \nu_n}(x))_{x \in \overline{A_L}})_{n \geq 0}$  to  $(\psi_{\kappa, \nu}(x))_{x \in \overline{A_L}}$  in this coupling.

Take  $w \in \mathcal{O}_{\kappa, \nu}$ . According to Lemma 4.8, there is  $n_w \geq 0$  such that for  $n \geq n_w$ , we have  $w \in \mathcal{O}_{\kappa, \nu_n}$ . Denote by  $\psi_w$  the conformal map from  $\mathbb{D}$  to  $\mathcal{O}_{\kappa, \nu}$  uniquely determined by the normalization:  $\psi_w(0) = w$  and  $\psi'_w(0) > 0$ . According to Theorem 2.8,  $\psi_w$  extends continuously from  $\overline{\mathbb{D}}$  to  $\overline{\mathcal{O}_{\kappa, \nu}}$ . Define  $\psi_{w, n}$  for  $\mathcal{O}_{\kappa, \nu_n}$  similarly. Since  $(\mathcal{O}_{\kappa, \nu_n}, w)$  converges to  $(\mathcal{O}_{\kappa, \nu}, w)$  in the Carathéodory sense (Lemma 4.8), it follows that  $\psi_{w, n}$  converges to  $\psi_w$  uniformly on compact subsets of  $\mathbb{D}$ ; see [Pom92, Theorem 1.8]. Since the family  $(\mathbb{C} \setminus \mathcal{O}_{\kappa, \nu_n})_{n \geq n_w}$  is uniformly locally connected (Lemma 4.9),  $\psi_{w, n}$  converges to  $\psi_w$  uniformly on  $\overline{\mathbb{D}}$ ; see [Pom92, Corollary 2.4].

Further, we write

$$\psi_{\kappa, \nu} = \psi_w \circ \tilde{\psi}_w,$$

where  $\tilde{\psi}_w$  is the Möbius transformation from  $\mathbb{D}$  to  $\mathbb{D}$  uniquely determined by the normalization

$$\tilde{\psi}_w(-i) = \psi_w^{-1}(-i), \quad \tilde{\psi}_w(i) = \psi_w^{-1}(i), \quad \tilde{\psi}_w(1) = \psi_w^{-1}(1).$$

In case  $-i$  or  $i$  are not simple points of  $\eta_{\kappa, \nu}$ , the notions  $\psi_w^{-1}(-i)$  and  $\psi_w^{-1}(i)$  are to be understood as

$$\psi_w^{-1}(-i) = \lim_{\substack{\theta \rightarrow -\frac{\pi}{2} \\ \theta > -\frac{\pi}{2}}} \psi_w^{-1}(e^{i\theta}), \quad \psi_w^{-1}(i) = \lim_{\substack{\theta \rightarrow \frac{\pi}{2} \\ \theta < \frac{\pi}{2}}} \psi_w^{-1}(e^{i\theta}).$$

Define  $\tilde{\psi}_{w, n}$  for  $\psi_{\kappa, \nu_n}$  and  $\psi_{w, n}$  as  $\psi_{\kappa, \nu_n} = \psi_{w, n} \circ \tilde{\psi}_{w, n}$  similarly. Since  $\psi_{w, n}^{-1}(-i)$ ,  $\psi_{w, n}^{-1}(i)$ , respectively  $\psi_{w, n}^{-1}(1)$ , converges to  $\psi_w^{-1}(-i)$ ,  $\psi_w^{-1}(i)$ , respectively  $\psi_w^{-1}(1)$ , we get that  $\tilde{\psi}_{w, n}$  converges to  $\tilde{\psi}_w$  uniformly on  $\overline{\mathbb{D}}$ , and  $\psi_{\kappa, \nu_n}$  converges to  $\psi_{\kappa, \nu}$  uniformly on  $\overline{\mathbb{D}}$ .  $\square$

### 4.3 Continuous dependence of the driving functions and proof of Proposition 1.4

Suppose  $\eta_n$  is a continuous curve with continuous driving function  $\xi^{(n)}$ . In the literature, one is always interested in the following question: whether the convergence of driving function  $\xi^{(n)}$  implies the convergence of curves  $\eta_n$ . See [SS12] and [KS17]. In this section, we are interested in the question in reverse direction and the goal is to show Proposition 1.4. We first give a general conclusion on convergence of driving functions out of convergence of curves: Proposition 4.10. Then Proposition 1.4 follows. Recall that  $\psi_0$  is the conformal map from  $\mathbb{D}$  to  $\mathbb{H}$  defined in (1.5).

**Proposition 4.10.** *Suppose  $(\eta_n(t))_{0 \leq t \leq 1}$  and  $(\eta(t))_{0 \leq t \leq 1}$  are parameterized continuous curves in  $\overline{\mathbb{D}}$  from  $-i$  to  $+i$ . Assume the following hold.*

(a) *The curves  $((\eta_n(t))_{0 \leq t \leq 1})_n$  converges to  $(\eta(t))_{0 \leq t \leq 1}$  for the uniform topology:*

$$\|\eta_n - \eta\|_\infty := \sup\{|\eta_n(t) - \eta(t)| : 0 \leq t \leq 1\} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.4)$$

(b) *The curves  $\psi_0(\eta_n)$  and  $\psi_0(\eta)$  satisfy assumptions in Proposition 2.12.*

For each  $t \in (0, 1)$ , let  $g_t$  be the conformal map from the unbounded connected component of  $\mathbb{H} \setminus \psi_0(\eta[0, t])$  onto  $\mathbb{H}$  with normalization  $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ . Denote by  $\xi_t = g_t(\psi_0(\eta(t)))$ . Define  $g_t^{(n)}$  and  $\xi_t^{(n)}$  for  $\psi_0(\eta_n)$  similarly. Then we have the following conclusions.

(1) *The half-plane capacity converges: for any  $t \in (0, 1)$ ,*

$$\sup\{|\text{hcap}(\psi_0(\eta_n[0, s])) - \text{hcap}(\psi_0(\eta[0, s]))| : 0 \leq s \leq t\} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.5)$$

(2) *When parameterized by the half-plane capacity, we have  $\psi_0(\eta_n) \rightarrow \psi_0(\eta)$  and  $\xi^{(n)} \rightarrow \xi$  for the local uniform topology.*

*Proof.* For each  $t \in (0, 1)$ , define  $f_t = \psi_0^{-1} \circ g_t \circ \psi_0$ . From Schwarz reflection principle,  $f_t$  can be extended analytically in a neighborhood of  $i$ . Denote by  $D_t$  the connected component of  $\mathbb{D} \setminus \eta[0, t]$  with  $i$  on the boundary. From elementary calculation, the function  $f_t$  is the conformal map from  $D_t$  onto  $\mathbb{D}$  with the normalization  $f_t(i) = i$ ,  $f_t'(i) = 1$ , and  $f_t''(i) = 0$ . Moreover, we have

$$\text{hcap}(\psi_0(\eta[0, t])) = \frac{2}{3} f_t'''(i).$$

We define  $f_t^{(n)} = \psi_0^{-1} \circ g_t^{(n)} \circ \psi_0$  for  $\eta_n$  similarly.

For each  $t \in (0, 1)$ , we have  $\eta_n[0, t] \rightarrow \eta[0, t]$  in Hausdorff metric. Consequently,  $f_t^{(n)} \rightarrow f_t$  uniformly when bounded away from  $\eta[0, t]$ . Therefore,

$$\text{hcap}(\psi_0(\eta_n[0, t])) = \frac{2}{3} (f_t^{(n)})'''(i) \rightarrow \frac{2}{3} f_t'''(i) = \text{hcap}(\psi_0(\eta[0, t])), \quad n \rightarrow \infty.$$

This gives the pointwise convergence of half-plane capacity. We will explain the uniform convergence below. Fix  $t \in (0, 1)$ . From [Law05, Lemma 4.1], one can show that, for any  $0 \leq s_1 < s_2 \leq t$ ,

$$\|f_{s_1} - f_{s_2}\|_\infty := \sup\{|f_{s_1}(z) - f_{s_2}(z)| : z \in D_t\} \leq C(\eta[0, t]) \sqrt{\text{osc}(\eta, s_2 - s_1, t)}, \quad (4.6)$$

where  $\text{osc}(\eta, \delta, t) := \sup\{|\eta(u) - \eta(v)| : 0 \leq u, v \leq t, |u - v| \leq \delta\}$  and  $C(\eta[0, t])$  is a constant depending on the diameter of  $\psi_0(\eta[0, t])$ . Similarly, we have

$$\|f_{s_1}^{(n)} - f_{s_2}^{(n)}\|_\infty \leq C(\eta_n[0, t]) \sqrt{\text{osc}(\eta_n, s_2 - s_1, t)}.$$

From (4.4), we may choose  $C_t^\eta$  large so that  $C(\eta_n[0, t]), C(\eta[0, t]) \leq C_t^\eta$ . Note that

$$\text{osc}(\eta_n, \delta, t) \leq 2\|\eta_n - \eta\|_\infty + \text{osc}(\eta, \delta, t).$$

Therefore, there exists a constant  $C_t^\eta \in (0, \infty)$  depending on  $\eta[0, t]$  such that

$$\|f_{s_1} - f_{s_2}\|_\infty \leq C_t^\eta \sqrt{\text{osc}(\eta, s_2 - s_1, t)}, \quad \|f_{s_1}^{(n)} - f_{s_2}^{(n)}\|_\infty \leq C_t^\eta \sqrt{2\|\eta_n - \eta\|_\infty + \text{osc}(\eta, s_2 - s_1, t)}. \quad (4.7)$$

Combining with (4.4) and the pointwise convergence, we obtain the uniform convergence of half-plane capacity (4.5). As a consequence, we have  $\psi_0(\eta_n) \rightarrow \psi_0(\eta)$  locally uniformly when parameterized by the half-plane capacity. It remains to show the convergence of  $\xi^{(n)}$ .

Pick  $x > 0$  large enough so that it has positive distance to  $\psi_0(\eta[0, t])$ . For  $t \in (0, 1)$ , define

$$h_t(x) = g_t(x) - \xi_t. \quad (4.8)$$

Define  $h_t^{(n)}$  for  $\eta_n$  similarly. From Lemma 4.11, we have  $\sup\{|h_s^{(n)}(x) - h_s(x)| : 0 \leq s \leq t\} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore,

$$\sup\{|g_s^{(n)}(x) - \xi_s^{(n)} - g_s(x) + \xi_s| : 0 \leq s \leq t\} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.9)$$

From (4.7) and pointwise convergence, we have  $\sup\{|g_s^{(n)}(x) - g_s(x)| : 0 \leq s \leq t\} \rightarrow 0$  as  $n \rightarrow \infty$ . Combining with (4.9), we have  $\sup\{|\xi_s^{(n)} - \xi_s| : 0 \leq s \leq t\} \rightarrow 0$  as  $n \rightarrow \infty$ . As the half-plane capacity also converge as in (4.5), we have that  $\xi^{(n)} \rightarrow \xi$  locally uniformly when parameterized by the half-plane capacity.  $\square$

**Lemma 4.11.** *Assume the same setup as in Proposition 4.10. Define  $h_t(x)$  for  $\eta$  and define  $h_t^{(n)}(x)$  for  $\eta_n$  as in (4.8). Then we have*

$$\sup\{|h_s^{(n)}(x) - h_s(x)| : 0 \leq s \leq t\} \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

*Proof.* The proof relies on a useful interpretation of the quantity  $h_t(x)$ . For  $\eta[0, t]$ , denote by  $\mathcal{R}(\psi_0(\eta[0, t]))$  the right-side of  $\psi_0(\eta[0, t])$  and by  $\mathcal{L}(\psi_0(\eta[0, t]))$  the left-side of  $\psi_0(\eta[0, t])$ . Denote by  $\mathcal{B} = (\mathcal{B}_t)_{t \geq 0}$  the Brownian motion in  $\mathbb{H}$  starting from  $yi$  with  $y > 0$  large. Define  $\tau$  to be the first time that it exits  $\mathbb{H} \setminus \psi_0(\eta[0, t])$ . From conformal invariance of Brownian motion, we have

$$h_t(x) = \lim_{y \rightarrow \infty} \pi y \mathbb{P}^{yi}(\mathcal{B}_\tau \in \mathcal{R}(\psi_0(\eta[0, t])) \cup (0, x)).$$

Similarly, define  $\tau^{(n)}$  to be the first time that  $\mathcal{B}$  exits  $\mathbb{H} \setminus \psi_0(\eta_n[0, t])$ . Then we have

$$h_t^{(n)}(x) = \lim_{y \rightarrow \infty} \pi y \mathbb{P}^{yi}(\mathcal{B}_{\tau^{(n)}} \in \mathcal{R}(\psi_0(\eta_n[0, t])) \cup (0, x)).$$

For  $\epsilon > 0$ , denote by  $\mathcal{V}_\epsilon(\psi_0(\eta[0, t]))$  the  $\epsilon$ -neighborhood of  $\psi_0(\eta[0, t])$  and define  $T^\epsilon$  to be the first time that  $\mathcal{B}$  hits  $\mathcal{V}_\epsilon(\psi_0(\eta[0, t]))$ . Choose  $n$  large enough so that

$$\sup\{|\psi_0(\eta_n(s)) - \psi_0(\eta(s))| : 0 \leq s \leq t\} \leq \epsilon/2. \quad (4.10)$$

Then  $\psi_0(\eta_n[0, t])$  is contained in  $\mathcal{V}_\epsilon(\psi_0(\eta[0, t]))$ . Denote by

$$V_t^R = \sup\{\psi_0(\eta[0, t]) \cap \mathbb{R}\}, \quad V_t^L = \inf\{\psi_0(\eta[0, t]) \cap \mathbb{R}\}.$$

If  $\tau < T^\epsilon$ , we have  $\mathcal{B}_{\tau^{(n)}} = \mathcal{B}_\tau \in (V_t^R + \epsilon, x)$ . Thus

$$h_t(x) - h_t^{(n)}(x) = \lim_{y \rightarrow \infty} \pi y \mathbb{P}^{yi}(R_t^{(n)}) - \lim_{y \rightarrow \infty} \pi y \mathbb{P}^{yi}(S_t^{(n)}), \quad (4.11)$$



where

$$\begin{aligned} R_t^{(n)} &= \{T^\epsilon \leq \tau, \mathcal{B}_\tau \in \mathcal{R}(\psi_0(\eta[0, t])) \cup (V_t^R, V_t^R + \epsilon), \mathcal{B}_{\tau(n)} \notin \mathcal{R}(\psi_0(\eta_n[0, t])) \cup (0, x)\}, \\ S_t^{(n)} &= \{T^\epsilon \leq \tau, \mathcal{B}_{\tau_n} \in \mathcal{R}(\psi_0(\eta_n[0, t])) \cup (V_t^R, V_t^R + \epsilon), \mathcal{B}_\tau \notin \mathcal{R}(\psi_0(\eta[0, t])) \cup (0, x)\}. \end{aligned}$$

Let us first estimate the probability of  $R_t^{(n)}$ . For  $\delta > 0$ , consider a path in  $\mathbb{H} \setminus \psi_0(\eta[0, t])$  starting from a point in

$$\mathcal{R}(\psi_0(\eta[0, t])) \setminus B(\psi_0(\eta(t)), \delta)$$

and terminating at a point in

$$\mathcal{L}(\psi_0(\eta[0, t])) \cup (-\infty, V_t^L) \cup (x, \infty).$$

Denote by  $r_t(\delta)$  the infimum of the length of such paths. Recall that  $\eta$  satisfies Proposition 2.12 (a). We have  $r_t(\delta) > 0$  and  $r_t(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . From (4.10), we see that any path in  $\mathbb{H} \setminus \psi_0(\eta[0, t])$  starting from a point in

$$\mathcal{R}(\psi_0(\eta[0, t])) \setminus B(\psi_0(\eta(t)), \delta)$$

and terminating at a point in

$$\mathcal{L}(\psi_0(\eta_n[0, t])) \cup (-\infty, V_t^L) \cup (x, \infty)$$

has length at least  $r_t(\delta) - \epsilon$ . From Beurling estimate, there exists a universal constant  $c \in (0, \infty)$  such that

$$\begin{aligned} \mathbb{P}^{yi} \left( R_t^{(n)} \right) &\leq \mathbb{P}^{yi} \left( R_t^{(n)} \cap \{ \mathcal{B}_\tau \notin B(\psi_0(\eta(t)), \delta) \} \right) + \mathbb{P}^{yi} (T^\epsilon \leq \tau, \mathcal{B}_\tau \in B(\psi_0(\eta(t)), \delta)) \\ &\leq c \sqrt{\frac{\epsilon}{r_t(\delta) - \epsilon}} \mathbb{P}^{yi} (T^\epsilon \leq \tau) + \mathbb{P}^{yi} (T^\epsilon \leq \tau, \mathcal{B}_\tau \in B(\psi_0(\eta(t)), \delta)). \end{aligned}$$

Next, we estimate  $\mathbb{P}(S_t^{(n)})$  in a similar way. For  $\delta > 0$ , consider a path in  $\mathbb{H} \setminus \psi_0(\eta[0, t])$  such that it starts from a point in

$$\mathcal{L}(\psi_0(\eta[0, t])) \cup (-\infty, V_t^L) \cup (x, \infty) \setminus B(\psi_0(\eta(t)), \delta)$$

and it terminates at a point in

$$\mathcal{R}(\psi_0(\eta[0, t])).$$

Denote by  $s_t(\delta)$  the infimum of the length of such paths. Similarly, we have  $s_t(\delta) > 0$  and  $s_t(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ , and

$$\begin{aligned} \mathbb{P}^{yi} \left( S_t^{(n)} \right) &\leq \mathbb{P}^{yi} \left( S_t^{(n)} \cap \{ \mathcal{B}_\tau \notin B(\psi_0(\eta(t)), \delta) \} \right) + \mathbb{P}^{yi} (T^\epsilon \leq \tau, \mathcal{B}_\tau \in B(\psi_0(\eta(t)), \delta)) \\ &\leq c \sqrt{\frac{\epsilon}{s_t(\delta) - \epsilon}} \mathbb{P}^{yi} (T^\epsilon \leq \tau) + \mathbb{P}^{yi} (T^\epsilon \leq \tau, \mathcal{B}_\tau \in B(\psi_0(\eta(t)), \delta)). \end{aligned}$$

Plugging these into (4.11), we have

$$|h_t(x) - h_t^{(n)}(x)| \leq C_t^\eta(\epsilon, \delta) + F_t^\eta(\delta),$$

where  $C_t^\eta(\epsilon, \delta) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $F_t^\eta(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$ . The same analysis applies for all  $s \in [0, t]$ . Thus, there exist  $\tilde{C}_t^\eta(\epsilon, \delta)$  and  $\tilde{F}_t^\eta(\delta)$  such that  $\tilde{C}_t^\eta(\epsilon, \delta) \rightarrow 0$  as  $\epsilon \rightarrow 0$  and  $\tilde{F}_t^\eta(\delta) \rightarrow 0$  as  $\delta \rightarrow 0$  and that

$$\sup\{|h_s(x) - h_s^{(n)}(x)| : s \in [0, t]\} \leq \tilde{C}_t^\eta(\epsilon, \delta) + \tilde{F}_t^\eta(\delta), \quad \text{as long as (4.10) holds.}$$

This gives the conclusion.  $\square$

*Proof of Proposition 1.4.* The curves  $\eta_{\kappa, \nu_n}$  and  $\eta_{\kappa, \nu}$  satisfy the conditions in Proposition 4.10.

- From Theorem 1.2,  $(\eta_{\kappa, \nu_n})_{n \geq 0}$  and  $\eta_{\kappa, \nu}$  satisfy Proposition 4.10 (a).
- From the proof of Proposition 1.3,  $(\eta_{\kappa, \nu_n})_{n \geq 0}$  and  $\eta_{\kappa, \nu}$  satisfy all the conditions in Proposition 2.12.

Thus the conclusion follows.  $\square$

## 5 Identification with level lines of the GFF for $\kappa = 4$

### 5.1 Proof of Theorem 1.7

In this section, we will construct a level line of GFF and complete the proof of Theorem 1.7. Recall that  $\Phi$  is a zero-boundary GFF in  $\mathbb{D}$  and  $\nu$  is a finite non-negative Radon measure on  $\overline{A_L}$  such that the support of  $\nu$  equals  $\overline{A_L}$  and that  $\pm i$  are not atoms of  $\nu$ . The goal of this section is to construct a level line of  $\Phi + \nu$  as in Definition 1.5. The strategy is to approximate such level line by the level line of GFF with piecewise constant boundary data which is already well understood.

Let us introduce approximations of the measure  $\nu$ . For  $n \geq 1$ , we first decompose the arc  $\overline{A_L}$  into subarcs of length  $\frac{\pi}{n+1}$ :

$$\theta_k^{(n)} = \frac{\pi}{2} + \frac{(n+1-k)\pi}{n+1}, \quad k \in \{0, 1, \dots, n+1\}.$$

Note that  $\theta_{n+1}^{(n)} = \frac{\pi}{2} < \theta_n^{(n)} < \dots < \theta_1^{(n)} < \theta_0^{(n)} = \frac{3\pi}{2}$ . Denote by  $A_k^{(n)}$  the subarc of  $\overline{A_L}$ :

$$A_k^{(n)} = \left\{ e^{i\theta} : \theta_{k+1}^{(n)} < \theta \leq \theta_k^{(n)} \right\}, \quad k \in \{0, 1, \dots, n\}.$$

Define the measure  $\nu_n$  on  $\overline{A_L}$ :

$$\nu_n = 2\lambda \mathbb{1}_{A_0^{(n)}} \sigma_{\partial\mathbb{D}} + \sum_{k=1}^n \frac{\nu(A_k^{(n)})}{\sigma_{\partial\mathbb{D}}(A_k^{(n)})} \mathbb{1}_{A_k^{(n)}} \sigma_{\partial\mathbb{D}}. \quad (5.1)$$

Denote by  $\eta^{(n)} = \eta_{4, \nu_n}$  the curve constructed from  $\text{CLE}_4$  and the Poisson point process of intensity  $\mu_{\nu_n}^{\mathbb{D}}$ . Then we have the following.

- From [ALS20],  $\eta^{(n)}$  can be coupled with  $\Phi$  as the level line of  $\Phi + \nu_n$ .
- From conclusions recalled in Section 2.4, in the above coupling,  $\eta^{(n)}$  is a.s. determined by  $\Phi$ .

In particular, the law of  $\eta^{(n)}$  is the same as  $\text{SLE}_4(\rho_n^{(n)}, \dots, \rho_1^{(n)})$  in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  with force points  $(e^{i\theta_n^{(n)}}, \dots, e^{i\theta_1^{(n)}})$ . Comparing with (2.6), the parameters  $(\rho_n^{(n)}, \dots, \rho_1^{(n)})$  are determined as follows:

$$\bar{\rho}_k^{(n)} = \frac{1}{\lambda} \frac{\nu(A_k^{(n)})}{\sigma_{\partial\mathbb{D}}(A_k^{(n)})} - 2, \quad k \in \{1, \dots, n\}; \quad \rho_k^{(n)} = \bar{\rho}_k^{(n)} - \bar{\rho}_{k-1}^{(n)}, \quad k \in \{1, \dots, n\};$$

with the convention that  $\rho_0^{(n)} = 0$ . From Proposition 1.4, the sequence  $\eta^{(n)}$  converges to  $\eta_{4, \nu}$ .

**Lemma 5.1.** *Suppose  $\nu$  is a finite non-negative Radon measure on  $\overline{A_L}$  such that the support of  $\nu$  equals  $\overline{A_L}$  and  $\text{Atom}_{\text{conv}}^*(\nu) = \emptyset$ . Then the sequence  $(\nu_n)_n$  converges weakly to  $\nu$ . Consequently, when parameterized by the half-plane capacity,  $\tilde{\eta}^{(n)} := \psi_0(\eta^{(n)})$  converges in law to  $\tilde{\eta}_{4, \nu} := \psi_0(\eta_{4, \nu})$  and the driving function of  $\tilde{\eta}^{(n)}$  converges in law to the driving function of  $\tilde{\eta}_{4, \nu}$  for the local uniform topology.*

We already know that  $\eta^{(n)}$  can be coupled with  $\Phi$  as a level line of  $\Phi + \nu_n$ . The goal is to argue that  $\eta$  can be coupled with  $\Phi$  as a level line of  $\Phi + \nu$ . To this end, the following lemma plays an essential role.

**Lemma 5.2.** *Suppose  $\eta$  is a continuous curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  such that  $\psi_0(\eta)$  has continuous driving function  $\xi$ . Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{D}$ . Then the pair  $(\Phi, \eta)$  can be coupled such that  $\eta$  is a level line of  $\Phi + \nu$  as in Definition 1.5 if and only if for every choice of  $z \in \mathbb{D}$ , the process  $(\nu_t(z))_{t \geq 0}$  is a Brownian motion with respect to the filtration generated by  $\xi$  when parameterized by minus log of the conformal radius:*

$$\log \text{CR}(z, \mathbb{D}) - \log \text{CR}(z, \mathbb{D} \setminus \eta[0, t]).$$

*Proof.* The proof follows [SS13, Section 2.2], see also [PW17, Lemma 2.16].  $\square$

*Proof of Theorem 1.7.* Suppose  $\eta^{(n)}$  is parameterized by the half-plane capacity of  $\psi_0(\eta^{(n)})$  and define  $\nu_{n,t}$  as in Definition 1.5. Applying Lemma 5.2 to  $\eta^{(n)}$ , for any  $z \in \mathbb{D}$ , the process  $(\nu_{n,t}(z))_{t \geq 0}$  is a Brownian motion when parameterized by

$$C_t^{(n)}(z) := \log \text{CR}(z, \mathbb{D}) - \log \text{CR}(z, \mathbb{D} \setminus \eta^{(n)}[0, t]).$$

Suppose  $\eta_{4,\nu}$  is parameterized by the half-plane capacity of  $\psi_0(\eta_{4,\nu})$  and define  $\nu_t$  as in Definition 1.5. We wish to argue that  $\eta_{4,\nu}$  can be coupled with  $\Phi$  as a level line of  $\Phi + \nu$ . Fix an arbitrary  $z \in \mathbb{D}$ . From Lemma 5.2, we need to argue that  $(\nu_t(z))_{t \geq 0}$  is a Brownian motion when parameterized by

$$C_t(z) = \log \text{CR}(z, \mathbb{D}) - \log \text{CR}(z, \mathbb{D} \setminus \eta[0, t]).$$

It suffices to show that  $(\nu_{n,t}(z))_{t \geq 0}$  converges to  $(\nu_t(z))_{t \geq 0}$  and  $(C_t^{(n)}(z))_{t \geq 0}$  converges to  $(C_t(z))_{t \geq 0}$  for the local uniform topology on processes parametrized by the time  $t$ . To this end, we use similar analysis as in Section 4.3.

We parameterize  $\psi_0(\eta_{4,\nu})$  by the half-plane capacity and denote by  $g_t$  be the conformal map from the unbounded connected component of  $\mathbb{H} \setminus \psi_0(\eta_{4,\nu}[0, t])$  with normalization  $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ . Set  $f_t = \psi_0^{-1} \circ g_t \circ \psi_0$ . Define  $g_t^{(n)}$  and  $f_t^{(n)}$  for  $\eta^{(n)}$  similarly.

From the conformal covariance of the conformal radius, we have

$$\begin{aligned} C_t(z) &= \log |f_t'(z)| - \log \text{CR}(f_t(z), \mathbb{D}) + \log \text{CR}(z, \mathbb{D}); \\ C_t^{(n)}(z) &= \log |(f_t^{(n)})'(z)| - \log \text{CR}(f_t^{(n)}(z), \mathbb{D}) + \log \text{CR}(z, \mathbb{D}). \end{aligned}$$

From the proof of Proposition 4.10,  $(f_t^{(n)}(z))_{t \geq 0}$  converges to  $(f_t(z))_{t \geq 0}$  and  $((f_t^{(n)})'(z))_{t \geq 0}$  converges to  $(f_t'(z))_{t \geq 0}$  for the local uniform topology, this implies that  $(C_t^{(n)}(z))_{t \geq 0}$  converges to  $(C_t(z))_{t \geq 0}$  for the local uniform topology.

We further claim that  $(\nu_{n,t}(z))_{t \geq 0}$  converges to  $(\nu_t(z))_{t \geq 0}$  for the local uniform topology. The convergence of  $2\lambda H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, \mathcal{L}(\eta^{(n)}[0, t]))$  towards  $2\lambda H_{\mathbb{D} \setminus \eta[0,t]}(z, \mathcal{L}(\eta[0, t]))$ , locally uniformly in  $t$ , is similar to Lemma 4.11. It remains to check the convergence of

$$\int_{\partial \mathbb{D}} H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, x) d\nu_n(x) \tag{5.2}$$

towards

$$\int_{\partial \mathbb{D}} H_{\mathbb{D} \setminus \eta[0,t]}(z, x) d\nu(x). \tag{5.3}$$

Without loss of generality, we assume that  $\eta$  and the  $\eta^{(n)}$  are defined on the same probability space such that the convergence of the  $\eta^{(n)}$  towards  $\eta$  is a.s. for the local uniform topology. Denote by  $A_t(z)$  the maximal open arc of  $\partial \mathbb{D}$  that can be accessed from  $z$  without hitting  $\eta[0, t]$ . Then the following holds a.s.:

- Both  $H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, x)$  and  $H_{\mathbb{D} \setminus \eta[0,t]}(z, x)$  are bounded from above by  $H_{\mathbb{D}}(z, x)$ .
- $H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, x)$  converges to  $H_{\mathbb{D} \setminus \eta[0,t]}(z, x)$  uniformly for  $x$  belonging to compact subsets of  $A_t(z)$ , and locally uniformly in  $t$ .
- $H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, x)$  converges to 0 uniformly for  $x$  belonging to compact subsets of  $\partial \mathbb{D} \setminus \overline{A_t(z)}$ , and locally uniformly in  $t$ .

Note that we do not claim that  $H_{\mathbb{D} \setminus \eta^{(n)}[0,t]}(z, x)$  converges to  $H_{\mathbb{D} \setminus \eta[0,t]}(z, x)$  for  $x \in \partial A_t(z)$ , and the convergence of  $\eta^{(n)}$  to  $\eta$  is not sufficient to ensure that. However, this is not needed, since by Lemma 3.6, a.s. for every  $t$ ,  $\nu(\partial A_t(z)) = 0$ . So the three points listed above are sufficient to ensure the convergence of (5.2) towards (5.3).  $\square$

## 5.2 Proof of Theorem 1.8

We may repeat the same argument in [PW17, Section 4] and arrive at the following conclusion.

**Proposition 5.3.** *Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{D}$ . Fix  $\epsilon > 0$ . Suppose  $\nu$  is a finite non-negative Radon measure on  $\overline{A_L}$  such that*

$$\nu \geq \epsilon \mathbb{1}_{\overline{A_L}} \sigma_{\partial\mathbb{D}}. \quad (5.4)$$

*Suppose that  $\eta_\nu$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  with continuous driving function. Assume that  $\eta_\nu$  is coupled with  $\Phi$  as a level line of  $\Phi + \nu$ . Then the level line coupling is unique and  $\eta_\nu$  is almost surely determined by  $\Phi$ .*

The proof of Proposition 5.3 follows the same argument in [PW17, Section 4]. To be self-contained, we briefly summarize the proof below. The proof relies on the following three lemmas 5.4, 5.5 and 5.6.

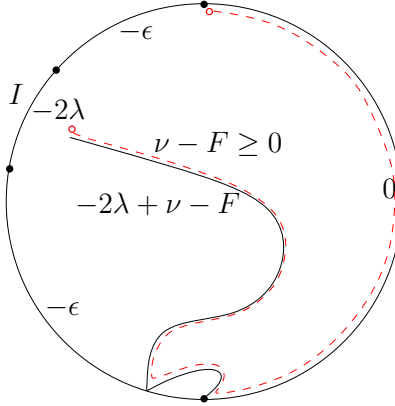


Figure 5.1: Given  $\eta_\nu[0, T_\delta]$ , let  $\tilde{\Phi}$  be  $\Phi$  restricted to the connected component of  $\mathbb{D} \setminus \eta_\nu[0, T_\delta]$  with  $i$  on the boundary. Then the boundary data for  $-\tilde{\Phi} - F$  is as follows: it equals  $F$  on  $\partial\mathbb{D}$ , it equals  $-2\lambda + \nu - F$  to the left side of  $\eta_\nu[0, T_\delta]$ , and it equals  $\nu - F$  to the right side of  $\eta_\nu[0, T_\delta]$ . Note that  $\nu - F \geq 0$ . Thus  $\eta'_F$  does not hit the union of the right side of  $\eta_\nu[0, T_\delta]$  and  $A_R$  (the dashed line marked in red).

**Lemma 5.4.** *Assume the same assumptions as in Proposition 5.3. Suppose there is an open arc  $I$  of  $A_L$  such that  $\nu \geq 2\lambda \mathbb{1}_I \sigma_{\partial\mathbb{D}}$ . Then  $\eta_\nu \cap I = \emptyset$  almost surely.*

*Proof.* We prove by contradiction. Suppose  $\eta_\nu$  does hit  $I$  with positive probability. Then there exists an open arc  $J \subset \bar{J} \subset I$  so that  $\eta_\nu$  hits  $J$  with positive probability. On this event, for  $\delta > 0$ , define  $T_\delta$  to the first time that  $\eta_\nu$  gets within  $\delta$  of  $J$ . Let  $F$  be the bounded harmonic function in  $\mathbb{D}$  with the following boundary data: it equals  $2\lambda$  on  $I$ , it equals  $\epsilon$  on  $A_L \setminus I$ , and it equals zero on  $A_R$ . Then it is clear that  $\nu(z) \geq F(z)$  for all  $z \in \mathbb{D}$ . Let  $\eta'_F$  be the level line of  $-\tilde{\Phi} - F$  in  $\mathbb{D}$  from  $i$  to  $-i$  and assume that the triple  $(\Phi, \eta_\nu, \eta'_F)$  is coupled so that  $\eta_\nu$  and  $\eta'_F$  are conditionally independent given  $\Phi$ . Note that  $F$  is piecewise constant on  $\partial\mathbb{D}$  and that  $\eta'_F$  does not hit  $I$  almost surely.

For any  $\delta > 0$ , given  $\eta_\nu[0, T_\delta]$ , let  $\tilde{\Phi}$  be  $\Phi$  restricted to the connected component of  $\mathbb{D} \setminus \eta_\nu[0, T_\delta]$  with  $i$  on the boundary. Then, given  $\eta_\nu[0, T_\delta]$ , the curve  $\eta'_F$  is coupled with  $-\tilde{\Phi}$  as the level line of  $-\tilde{\Phi} - F$  whose boundary data is shown in Figure 5.1 up to the first hitting time of  $\eta_\nu[0, T_\delta]$ . Note that such boundary data is regulated. From [PW17, Lemma 4.1], the curve  $\eta'_F$  does not hit the union of the right side of  $\eta_\nu[0, T_\delta]$  and  $A_R$ . This implies that  $\eta'_F$  has to get within  $\delta$  of  $J$ . This holds for any  $\delta > 0$ . Let  $\delta \rightarrow 0$ , it implies that  $\eta'_F$  hits  $\bar{J} \subset I$  with positive probability, contradiction.  $\square$

**Lemma 5.5.** *Assume the same assumptions as in Proposition 5.3. For any fixed  $x_0 \in A_L$ , the curve  $\eta_\nu$  does not hit  $\{x_0\}$  almost surely.*

*Proof.* Let  $F$  be the bounded harmonic function in  $\mathbb{D}$  with the following boundary data: it equals  $\epsilon$  on  $A_L$  and it equals zero on  $A_R$ . Then  $\nu(z) \geq F(z)$  for all  $z \in \mathbb{D}$ . Let  $\eta'_F$  be the level line of  $-\Phi - F$  in  $\overline{\mathbb{D}}$  from  $i$  to  $-i$  and assume that the triple  $(\Phi, \eta_\nu, \eta'_F)$  is coupled so that  $\eta_\nu$  and  $\eta'_F$  are conditionally independent given  $\Phi$ . Note that  $F$  is piecewise constant on  $\partial\mathbb{D}$  and that  $\eta'_F$  does not hit  $\{x_0\}$  almost surely. If  $\eta_\nu$  hits  $\{x_0\}$  with positive probability, by the same argument as in the proof of Lemma 5.4, it would imply that  $\eta'_F$  hits  $\{x_0\}$  with positive probability, contradiction.  $\square$

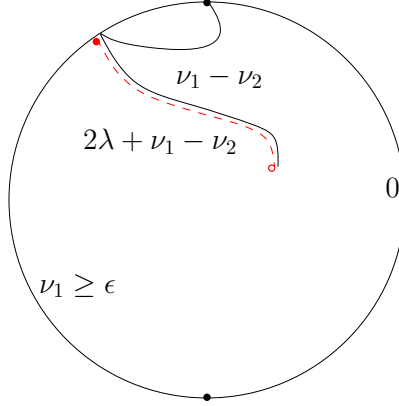


Figure 5.2: Given  $\eta'_{\nu_2}[0, \tau']$ , let  $\tilde{\Phi}$  be  $\Phi$  restricted to the connected component of  $\mathbb{D} \setminus \eta'_{\nu_2}[0, \tau']$  with  $-i$  on the boundary. Then the boundary data for  $\tilde{\Phi} + \nu_1$  is as follows: it equals  $\nu_1$  on  $\partial\mathbb{D}$ , it equals  $2\lambda + \nu_1 - \nu_2$  to the right side of  $\eta'_{\nu_2}[0, \tau']$ , and it equals  $\nu_1 - \nu_2$  to the left side of  $\eta'_{\nu_2}[0, \tau']$ . As  $2\lambda + \nu_1 - \nu_2 \geq 2\lambda$ , Lemma 5.4 and absolute continuity imply that  $\eta_{\nu_1}$  does not hit the right side of  $\eta'_{\nu_2}[0, \tau']$ . Let  $x'$  be the last point of  $\partial\mathbb{D} \cap \eta'_{\nu_2}[0, \tau']$  (the solid point marked in red). As the boundary data is greater than  $\epsilon$  in neighborhood of  $x'$ , Lemma 5.5 and absolute continuity imply that  $\eta_{\nu_1}$  does not hit the point  $x'$ . In summary,  $\eta_{\nu_1}$  does not hit the union of the right side of  $\eta'_{\nu_2}[0, \tau']$  and the point  $\{x'\}$  (the dash line marked in red). This implies that the point  $\eta'_{\nu_2}(\tau')$  stays to the left of  $\eta_{\nu_1}$ .

**Lemma 5.6.** Fix  $\epsilon > 0$ . Suppose  $\nu_1, \nu_2$  are finite non-negative Radon measures on  $\overline{A_L}$  such that

$$\nu_1 \geq \nu_2, \quad \text{and} \quad \nu_1 \geq \epsilon \mathbb{1}_{\overline{A_L}} \sigma_{\partial\mathbb{D}}.$$

Suppose that  $\eta_{\nu_1}$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  with continuous driving function and suppose  $\eta'_{\nu_2}$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $i$  to  $-i$  with continuous driving function. Assume that  $\eta_{\nu_1}$  is coupled with  $\Phi$  as a level line of  $\Phi + \nu_1$  from  $-i$  to  $i$  and  $\eta'_{\nu_2}$  is coupled with  $\Phi$  as a level line of  $-\Phi - \nu_2$  from  $i$  to  $-i$  and that the triple  $(\Phi, \eta_{\nu_1}, \eta'_{\nu_2})$  is coupled so that  $\eta_{\nu_1}$  and  $\eta'_{\nu_2}$  are conditionally independent given  $\Phi$ . Then  $\eta_{\nu_1}$  stays to the right of  $\eta'_{\nu_2}$  almost surely.

*Proof.* As both  $\eta_{\nu_1}$  and  $\eta'_{\nu_2}$  are simple, it suffices to show that, for any  $\eta'_{\nu_2}$ -stopping time  $\tau'$ , the point  $\eta'_{\nu_2}(\tau')$  is to the left of  $\eta_{\nu_1}$ . Given  $\eta'_{\nu_2}[0, \tau']$ , let  $\tilde{\Phi}$  be  $\Phi$  restricted to the connected component of  $\mathbb{D} \setminus \eta'_{\nu_2}[0, \tau']$  with  $-i$  on the boundary. Then, given  $\eta'_{\nu_2}[0, \tau']$ , the curve  $\eta_{\nu_1}$  is coupled with  $\tilde{\Phi}$  as a level line of  $\tilde{\Phi} + \nu_1$  whose boundary data is shown in Figure 5.2 up to the first hitting time of  $\eta'_{\nu_2}[0, \tau']$ . From Lemmas 5.4 and 5.5, the curve  $\eta_{\nu_1}$  does not hit the right side of  $\eta'_{\nu_2}[0, \tau']$ , see detail in Figure 5.2. This implies the point  $\eta'_{\nu_2}(\tau')$  is to the left of  $\eta_{\nu_1}$  as desired.  $\square$

Now, we are ready to complete the proof of Proposition 5.3.

*Proof of Proposition 5.3.* Suppose  $\eta'_\nu$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $i$  to  $-i$  with continuous driving function. Suppose that  $\eta'_\nu$  is coupled with  $\Phi$  as a level line of  $-\Phi - \nu$  from  $i$  to  $-i$  and that the

triple  $(\Phi, \eta_\nu, \eta'_\nu)$  is coupled so that  $\eta_\nu$  and  $\eta'_\nu$  are conditionally independent given  $\Phi$ . From Lemma 5.6, we know that  $\eta_\nu$  stays to the right of  $\eta'_\nu$  and that  $\eta_\nu$  stays to the left of  $\eta'_\nu$  almost surely. As both of them are simple, we have  $\eta_\nu = \eta'_\nu$  (viewed as sets) almost surely. Since  $\eta_\nu$  and  $\eta'_\nu$  are coupled with  $\Phi$  so that they are conditionally independent given  $\Phi$ , the fact  $\eta_\nu = \eta'_\nu$  implies that  $\eta_\nu$  is almost surely determined by  $\Phi$ .  $\square$

**Corollary 5.7.** *Suppose  $\Phi$  is zero-boundary GFF in  $\mathbb{D}$ . Fix  $\epsilon > 0$ . Suppose  $\nu_1, \nu_2$  are finite non-negative Radon measures on  $\overline{A_L}$  such that*

$$\nu_1 \geq \nu_2, \quad \text{and} \quad \nu_1 \geq \epsilon \mathbb{1}_{\overline{A_L} \sigma \partial \mathbb{D}}. \quad (5.5)$$

*Suppose that  $\eta_{\nu_1}, \eta_{\nu_2}$  are continuous simple curves in  $\overline{\mathbb{D}}$  from  $-i$  to  $i$  with continuous driving functions. Assume that  $\eta_{\nu_1}$  (resp.  $\eta_{\nu_2}$ ) is coupled with  $\Phi$  as a level line of  $\Phi + \nu_1$  (resp. of  $\Phi + \nu_2$ ), then  $\eta_{\nu_1}$  stays to the right of  $\eta_{\nu_2}$  almost surely.*

*Proof.* Suppose  $\eta'_{\nu_1}$  is a continuous simple curve in  $\overline{\mathbb{D}}$  from  $i$  to  $-i$  with continuous driving function. Suppose that  $\eta'_{\nu_1}$  is coupled with  $\Phi$  as a level line of  $-\Phi - \nu_1$  from  $i$  to  $-i$  and that  $(\Phi, \eta_{\nu_1}, \eta_{\nu_2}, \eta'_{\nu_1})$  is coupled so that  $\eta_{\nu_1}, \eta_{\nu_2}$  and  $\eta'_{\nu_1}$  are conditionally independent given  $\Phi$ . From Lemma 5.6, the curve  $\eta'_{\nu_1}$  stays to the right of  $\eta_{\nu_2}$  almost surely. From the proof of Proposition 5.3, we have  $\eta_{\nu_1} = \eta'_{\nu_1}$  (viewed as sets) almost surely. These imply that  $\eta_{\nu_1}$  stays to the right of  $\eta_{\nu_2}$  almost surely as desired.  $\square$

We emphasize that in the above proof of Proposition 5.3, we follow the method in [PW17] and the assumption (5.4) plays an essential role. In the following, we will remove the assumption and complete the proof of Theorem 1.8.

*Proof of Theorem 1.8.* Consider  $(\Phi, \eta_{4,\nu})$  to be coupled as in Theorem 1.7. For  $n \geq 1$ , denote  $\nu_n := \nu + 2^{-n}$ . Let  $\eta_{\nu_n}$  be the level line of  $\Phi + \nu_n$  from  $-i$  to  $i$ . The existence of  $\eta_{\nu_n}$  is ensured by Theorem 1.7. Its uniqueness and measurability with respect to  $\Phi$  is given by Proposition 5.3. Moreover, for every  $n \geq 1$ ,  $\eta_{\nu_n}$  is distributed as  $\eta_{4,\nu_n}$  given by (1.4). However, the sequence  $(\eta_{\nu_n})_{n \geq 1}$  is *a priori* not coupled in the same way as the sequence  $(\eta_{4,\nu_n})_{n \geq 1}$  in Section 4.2 in the proof of Theorem 1.2.

According to Corollary 5.7,  $\eta_{\nu_n}$  stays a.s. to the right of  $\eta_{4,\nu}$  and to the right of  $\eta_{\nu_{n+1}}$ , for every  $n \geq 1$ . Let  $\mathcal{O}_{4,\nu}$  denote the connected component of  $\mathbb{D} \setminus \eta_{4,\nu}$  to the right of  $\eta_{4,\nu}$ , and  $\mathcal{O}_n$  the connected component of  $\mathbb{D} \setminus \eta_{\nu_n}$  to the right of  $\eta_{\nu_n}$ . A.s.,  $\mathcal{O}_n \subset \mathcal{O}_{n+1}$ . Denote

$$\mathcal{O}_\infty := \bigcup_{n \geq 1} \mathcal{O}_n.$$

We have that  $\mathcal{O}_\infty \subset \mathcal{O}_{4,\nu}$  a.s. Moreover, Theorem 1.2 ensures that  $\mathcal{O}_\infty$  has the same distribution as  $\mathcal{O}_{4,\nu}$ . This implies that  $\mathcal{O}_\infty = \mathcal{O}_{4,\nu}$  a.s. Since the sequence  $(\eta_{\nu_n})_{n \geq 1}$  is measurable with respect to  $\Phi$ , we get that  $\mathcal{O}_{4,\nu}$ , and thus  $\eta_{4,\nu}$ , are measurable with respect to  $\Phi$ .  $\square$

### 5.3 An equation for the driving function

Let  $\nu$  be a finite non-negative Radon measure on  $\overline{A_L}$ . We assume that  $\nu$  has full support on  $\overline{A_L}$ . We also assume that a.s., the curve  $\eta_{4,\nu}$  does not hit  $\text{Atom}(\nu)$ . A sufficient condition for that is given by Lemma 3.6. Denote  $\zeta_\nu$  the following Radon measure on  $\mathbb{R}$ :

$$d\zeta_\nu(x) := \frac{1}{2}(1+x^2)d((\psi_0)_*\nu)(x),$$

where  $\psi_0$  is the Möbius transformation from  $\mathbb{D}$  to  $\mathbb{H}$  given by (1.5).  $\zeta_\nu$  is a non-negative Radon measure on  $(-\infty, 0]$  satisfying

$$\int_{(-\infty, 0]} \frac{1}{1+x^2} d\zeta_\nu(x) < +\infty.$$

On  $(0, +\infty)$ ,  $\zeta_\nu$  equals 0. We see  $\zeta_\nu$  as an analogue of the boundary condition (2.8). In particular, if  $\nu$  is of the form  $\nu = a\mathbb{1}_{A_L}\sigma_{\partial\mathbb{D}}$  with  $a > 0$  a constant, then  $\zeta_\nu$  is a piecewise constant function, equal to  $a$  on  $(-\infty, 0)$  and 0 on  $(0, +\infty)$ . Similarly to (2.8), we define  $\rho_\nu$  by

$$\rho_\nu := -\frac{1}{\lambda} \frac{d}{dx} \zeta_\nu - 2\delta_0,$$

where  $\delta_0$  is the Dirac measure at 0 and where the derivative  $\frac{d}{dx}$  is to be taken in the sense of generalized function. In general,  $\rho_\nu$  is an order 1 generalized function on  $\mathbb{R}$  which is 0 on  $(0, +\infty)$ . Given  $f \in \mathcal{C}^1(\mathbb{R})$  with compact support, by integration by parts, we have that

$$\int_{\mathbb{R}} f \rho_\nu dx = -2f(0) + \frac{1}{\lambda} \int_{\mathbb{R}} f'(x) d\zeta_\nu(x). \quad (5.6)$$

Consider now the curve  $\eta_{4,\nu}$ , parametrized by the half-plane capacity. Denote by  $g_t$  be the conformal map from the unbounded connected component of  $\mathbb{H} \setminus \psi_0(\eta_{4,\nu}[0, t])$  with normalization  $\lim_{z \rightarrow \infty} |g_t(z) - z| = 0$ , and  $\xi_t$  the driving function in the corresponding Loewner chain. Following (2.10), we are interested in giving a meaning to

$$\int_{(-\infty, 0]} \frac{\rho_\nu dx}{\xi_t - g_t(x)}. \quad (5.7)$$

Denote

$$x_L(t) := \min\{x \in (-\infty, 0] : x \in \psi_0(\eta_{4,\nu}[0, t])\}.$$

Given (5.6), we set

$$Z_t := -\frac{2}{\xi_t - g_t(0^-)} + \frac{1}{\lambda} \int_{(-\infty, x_L(t)]} \frac{g_t'(x) d\zeta_\nu(x)}{(\xi_t - g_t(x))^2}. \quad (5.8)$$

If  $\nu$  is actually a piecewise constant function on  $A_L$ , then  $Z_t$  coincides with (5.7). Denote  $f_t = \psi_0^{-1} \circ g_t \circ \psi_0$ , and let  $U_t \in (-\frac{3}{2}\pi, \frac{1}{2}\pi)$  such that  $\psi_0(e^{iU_t}) = \xi_t$ . Then  $Z_t$  can be expressed as

$$Z_t = \frac{(e^{iU_t} - i)(f_t((-i)^-) - i)}{e^{iU_t} - f_t((-i)^-)} - \frac{i(e^{iU_t} - i)^2}{2\lambda} \int_{A_L(t)} \frac{x f_t'(x)}{(e^{iU_t} - f_t(x))^2} d\nu(x), \quad (5.9)$$

where

$$f_t((-i)^-) = \lim_{\substack{\theta \rightarrow -\frac{\pi}{2} \\ \theta < -\frac{\pi}{2}}} f_t(e^{i\theta}),$$

and  $A_L(t)$  is the connected component of  $A_L \setminus \eta_{4,\nu}[0, t]$  adjacent to  $i$ .

**Proposition 5.8.** *Let  $\nu$  be a finite non-negative Radon measure on  $\overline{A_L}$  with full support and assume  $\eta_{4,\nu}$  is parametrized by the half-plane capacity. Also assume that a.s. the curve  $\eta_{4,\nu}$  does not hit  $\text{Atom}(\nu)$ .*

(1) *Let  $Z_t$  be given by (5.8). Then  $Z_t$  is well defined and continuous on the subset of times*

$$I_\nu := \{t \in [0, T_{\max}) : \eta_{4,\nu}(t) \notin \overline{A_L}\}. \quad (5.10)$$

(2) *Let  $(\nu_n)_{n \geq 0}$  be a sequence of finite non-negative Radon measures on  $\overline{A_L}$  with full support, converging weakly to  $\nu$ . Assume that for every  $n \geq 0$ , a.s.  $\eta_{4,\nu_n}$  does not hit  $\text{Atom}(\nu_n)$ . Assume that each  $\eta_{4,\nu_n}$  is parametrized by the half-plane capacity and that  $\eta_{4,\nu}$  and all the  $\eta_{4,\nu_n}$  are coupled on the same probability space such that the sequence  $(\eta_{4,\nu_n})_{n \geq 0}$  converges a.s. locally uniformly to  $\eta_{4,\nu}$ . Let  $Z_t^{(n)}$  be defined as  $Z_t$ , but with  $\nu_n$  and  $\eta_{4,\nu_n}$  instead of  $\nu$  and  $\eta_{4,\nu}$ . Then, as  $n \rightarrow +\infty$ ,  $Z_t^{(n)}$  converges a.s. to  $Z_t$  uniformly on compact subsets of  $I_\nu$ .*

*Proof.* We will use the expression (5.9). Observe that  $A_L(t)$  is constant on connected components of  $I_\nu$ . For the first point we use the following:

- $f_t$  is continuous on  $\overline{A_L(t)}$ , locally uniformly in  $t$  for  $t \in I_\nu$ ; see (4.6).
- $f'_t$  is bounded on  $A_L(t)$ , locally uniformly in  $t$  for  $t \in I_\nu$ . Indeed,  $t \in [0, T_{\max})$  and  $x \in A_L(t)$ ,

$$|f'_t(x)| = 2\pi H_{\mathbb{D} \setminus \eta_{4, \nu_n}[0, t]}((f_t)^{-1}(0), x) \leq 2\pi H_{\mathbb{D}}((f_t)^{-1}(0), x).$$

- $f'_t$  is continuous on compact subsets of  $A_L(t)$ , locally uniformly in  $t$  for  $t \in I_\nu$ . Indeed, one can use the Schwarz reflection principle so as to analytically extend  $f_t$  across  $A_L(t)$ , and then Cauchy's integral formula to express  $f'_t$  through  $f_t$ .
- For every  $t \in I_\nu$ ,  $e^{iU_t} \notin f_t(\overline{A_L(t)})$ .

Now let us check the second point. Let  $T_{\max}^{(n)} \in (0, +\infty]$  denote the maximal parameter in the parametrization of  $\eta_{4, \nu_n}$  by half-plane capacity. Denote

$$I_{\nu_n} := \{t \in [0, T_{\max}^{(n)}) : \eta_{4, \nu_n}(t) \not\subset \overline{A_L}\}.$$

Denote  $A_L^{(n)}(t)$  the connected component of  $A_L \setminus \eta_{4, \nu_n}[0, t]$  adjacent to  $i$ . We will also use the notations  $U_t^{(n)}$  and  $f_t^{(n)}$  in the case of  $\eta_{4, \nu_n}$ , with straightforward meaning. Every compact subset of  $I_\nu$  is contained in  $I_{\nu_n}$  for  $n$  large enough. Moreover, for every  $t \in I_\nu$ ,

$$A_L(t) \subset \liminf_{n \rightarrow +\infty} A_L^{(n)}(t).$$

The equality does not hold in general. The following holds.

- $U_t^{(n)}$  converges to  $U_t$ , locally uniformly in  $t$ ; see Proposition 1.4.
- For every  $t \in I_\nu$  and  $K$  compact subset of  $A_L^{(n)}(t) \cup \{i\}$ ,  $f_t^{(n)}$ , respectively  $(f_t^{(n)})'$  converges to  $f_t$ , respectively  $f'_t$ , uniformly on  $K$  and locally uniformly in  $t$ .
- For every  $n \geq 0$ ,  $t \in [0, T_{\max}^{(n)})$  and  $x \in A_L^{(n)}(t)$ ,

$$|(f_t^{(n)})'(x)| = 2\pi H_{\mathbb{D} \setminus \eta_{4, \nu_n}[0, t]}((f_t^{(n)})^{-1}(0), x) \leq 2\pi H_{\mathbb{D}}((f_t^{(n)})^{-1}(0), x).$$

In particular, for every  $t_0 \in [0, T_{\max})$ ,

$$\limsup_{n \rightarrow +\infty} \sup_{t \in [0, t_0]} \sup_{x \in A_L^{(n)}(t)} |(f_t^{(n)})'(x)| < +\infty.$$

- For every  $t_0 \in [0, T_{\max})$ ,

$$\lim_{n \rightarrow +\infty} \sup_{t \in [0, t_0]} \sup_{x \in A_L^{(n)}(t) \setminus A_L(t)} |(f_t^{(n)})'(x)| = 0.$$

This implies the convergence. □

The following proposition tells that the driving function  $\xi_t$  satisfies the SDE

$$d\xi_t = 2dB_t + Z_t dt$$

on the set of times (5.10), where  $(B_t)_{t \geq 0}$  is a standard Brownian motion.



**Proposition 5.9.** *Let  $\nu$  be a finite non-negative Radon measure on  $\overline{A_L}$  with full support and assume  $\eta_{4,\nu}$  is parametrized by the half-plane capacity. Also assume that a.s. the curve  $\eta_{4,\nu}$  does not hit  $\text{Atom}(\nu)$ . Let  $(\mathcal{F}_t)_{t \geq 0}$  be the natural filtration of  $\eta_{4,\nu}[0, t \wedge T_{\max}]$ . Fix  $t_0 > 0$ . Let  $E_{t_0}$  be the event defined by  $t_0 < T_{\max}$  and by  $\eta_{4,\nu}(t_0) \notin \overline{A_L}$ . Let  $\tau$  be the stopping time*

$$\tau(t_0) := \sup\{t \in [t_0, T_{\max}) : \eta_{4,\nu}[t_0, t] \cap \overline{A_L} = \emptyset\}.$$

*Then, conditionally on the event  $E_{t_0}$ , the stochastic process*

$$\left( \xi_{t \wedge \tau(t_0)} - \int_{t_0}^{t \wedge \tau(t_0)} Z_s ds \right)_{t \geq t_0}$$

*is a continuous martingale for the filtration  $(\mathcal{F}_t)_{t \geq t_0}$ , with quadratic variation given by*

$$4(t \wedge \tau(t_0) - t_0).$$

*Proof.* The result is true if  $\nu$  is a piecewise constant function. For general  $\nu$ , one takes an approximation of  $\nu$  for the weak topology on measures by piecewise constant functions. For instance, one can take (5.1). Then the result follows by convergence, by applying Proposition 5.8.  $\square$

## 6 Some open questions

Here we present some open questions related to this work:

1. In Proposition 3.4 we present a necessary and a sufficient condition for the presence of double points in  $\eta_{\kappa,\nu}$ , but the two do not match. What is the optimal criterion for the presence of double points?
2. Similarly, in Lemma 3.6 we give a necessary condition for  $\eta_{\kappa,\nu}$  hitting an atom of  $\nu$  with positive probability. But what is the optimal criterion for this?
3. If  $\nu$  is a Dirac measure at  $-i$ , then  $\eta_{\kappa,\nu}$  draws a bubble from  $-i$  to  $-i$  in  $\mathbb{D}$ . What is the distribution of this bubble? We believe that it is singular to the usual  $\text{SLE}_\kappa$  bubble measure [SW12, Section 4] because of the behaviour near  $-i$ .
4. If  $\nu$  is a Dirac measure at  $-i$  and  $\kappa = 4$ , what is the harmonic extension of  $\nu$  inside the bubble created by  $\eta_{4,\nu}$ ? Does an uniformizing map for this bubble actually admit a derivative at  $-i$ ?
5. In Proposition 5.9 we give an equation for the driving function of  $\eta_{4,\nu}$  when the curve is away from the boundary. But what happens when the curve hits the boundary? Is there an additional term accounting for the interaction with the boundary? This might be the case in some situations, given that  $\eta_{4,\nu}$  can actually intersect the boundary with a positive Lebesgue measure (Proposition 3.10).
6. What would be an equation for the driving function of  $\eta_{\kappa,\nu}$  for  $\kappa \neq 4$ ? This is not known even for  $\nu$  being a piecewise constant function, but it is known that the curve does not belong in general to the  $\text{SLE}_\kappa(\rho)$  family.

## A Appendix: Non-negative harmonic functions

Here we recall some classical properties of non-negative harmonic functions.

**Proposition A1.** *Let  $f$  be a non-negative harmonic function on the unit disk  $\mathbb{D}$ . Then there is a finite non-negative Radon measure  $\nu$  on  $\partial\mathbb{D}$ , such that for every  $z \in \mathbb{D}$ ,*

$$f(z) = \int_{\partial\mathbb{D}} H_{\mathbb{D}}(z, x) d\nu(x), \tag{A.1}$$

*where  $H_{\mathbb{D}}(z, x)$  is the Poisson kernel on  $\mathbb{D}$  (2.1). Moreover, the measure  $\nu$  is unique.*

*Proof.* Let us first prove the existence of  $\nu$ . For  $\varepsilon \in (0, 1)$ , denote the following absolutely continuous measure on  $\partial\mathbb{D}$ :

$$d\nu_\varepsilon(x) := f((1 - \varepsilon)x)\sigma_{\partial\mathbb{D}}(dx).$$

For every  $z \in \mathbb{D}$  with  $|z| < 1 - \varepsilon$ , we have that

$$f(z) = \int_{\partial\mathbb{D}} H_{\mathbb{D}}((1 - \varepsilon)^{-1}z, x)d\nu_\varepsilon(x),$$

In particular, the total mass of  $\nu_\varepsilon$  is always  $2\pi f(0)$ . Thus, the family  $(\nu_\varepsilon)_{\varepsilon \in (0, 1)}$  is relatively compact for the weak topology of measures, and admits subsequential limits as  $\varepsilon \rightarrow 0$ . Any such subsequential limit  $\nu$  satisfies (A.1).

Now let us show the uniqueness. Let  $\nu$  be such that (A.1) is satisfied. Let  $u$  be a continuous function on  $\partial\mathbb{D}$ . We have that

$$\int_{\partial\mathbb{D}} u(x)f((1 - \varepsilon)x)\sigma_{\partial\mathbb{D}}(dx) = \int_{\partial\mathbb{D}} d\nu(y) \left( \int_{\partial\mathbb{D}} u(x)H_{\mathbb{D}}((1 - \varepsilon)x, y)\sigma_{\partial\mathbb{D}}(dx) \right).$$

The function

$$y \mapsto \int_{\partial\mathbb{D}} u(x)H_{\mathbb{D}}((1 - \varepsilon)x, y)\sigma_{\partial\mathbb{D}}(dx)$$

converges uniformly to  $u$  as  $\varepsilon \rightarrow 0$ . Thus,

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial\mathbb{D}} u(x)f((1 - \varepsilon)x)\sigma_{\partial\mathbb{D}}(dx) = \int_{\partial\mathbb{D}} u(y)d\nu(y).$$

This characterizes  $\nu$ . □

**Corollary A2.** *A function  $f : \mathbb{D} \rightarrow \mathbb{R}$  is of form*

$$f(z) = \int_{A_{\mathbb{L}}} H_{\mathbb{D}}(z, x)d\nu(x), \quad z \in \mathbb{D},$$

where  $\nu$  is a finite non-negative Radon measure on  $\overline{A_{\mathbb{L}}}$  if and only if  $f$  is non-negative harmonic on  $\mathbb{D}$ , and for every  $x \in A_{\mathbb{R}}$ ,

$$\lim_{\substack{z \rightarrow x \\ z \in \mathbb{D}}} f(z) = 0.$$

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