



HAL
open science

Unique Continuation for the unperturbed discrete Maxwell operator

Olivier Poisson

► **To cite this version:**

Olivier Poisson. Unique Continuation for the unperturbed discrete Maxwell operator. 2021. hal-03276994

HAL Id: hal-03276994

<https://hal.science/hal-03276994>

Preprint submitted on 2 Jul 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Unique Continuation for the unperturbed discrete Maxwell operator

O. Poisson*

July 2, 2021

1 Introduction

1.1 The Discrete-Maxwell Operator

Let \mathbb{T}^d be the d -dimensional real torus, let U be the unitary transform between $L^2(\mathbb{T}^3)$ and $l^2(\mathbb{Z}^3)$:

$$\begin{aligned}\hat{f}(n) &= (2\pi)^{-\frac{3}{2}} \int_{\mathbb{T}^3} e^{inx} f(x) dx \equiv (Uf)(n), \quad n \in \mathbb{Z}^3, \\ f(x) &= (2\pi)^{-\frac{3}{2}} \sum_{n \in \mathbb{Z}^3} e^{-inx} \hat{f}(n) \equiv (U^* \hat{f})(x), \quad x \in \mathbb{T}^3.\end{aligned}$$

Position: $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, Momentum: $x = (x_1, x_2, x_3) \in \mathbb{T}^3 \approx (\mathbb{R}/(2\pi\mathbb{Z}))^3$.
We also write

$$\hat{f} = Ff.$$

We consider the unperturbed Maxwell operator $\hat{H}_0 = UH_0U^*$ as bounded in $(L^2(\mathbb{Z}^3))^6$ and defined from the bounded operator H_0 in $(L^2(\mathbb{T}^3))^6$, such that

$$(H_0u)(x) = H_0(x)u(x), \quad x \in \mathbb{T}^3, \quad u \in (L^2(\mathbb{T}^3))^6,$$

where

$$H_0(x) = \tilde{H}_0(\xi) \equiv \begin{pmatrix} 0 & M(\xi) \\ -M(\xi) & 0 \end{pmatrix}, \quad x \in \mathbb{T}^3,$$

with

$$\xi = (\xi_1, \xi_2, \xi_3), \quad \xi_j = \sin x_j,$$

* Aix Marseille Université, I2M, UMR CNRS 6632, France
(olivier.poisson@univ-amu.fr).

and $M(\xi)$ is the following real anti-symmetrical 3×3 matrix:

$$M(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}.$$

Then the operator H_0 acts on $u = (E, H) \in (L^2(\mathbb{T}^3))^3 \times (L^2(\mathbb{T}^3))^3$:

$$H_0(x)u = (M(\xi)H, -M(\xi)E).$$

Let $\mathcal{H} = L^2(\mathbb{T}^3, dx; \mathbb{R}^6)$ which can be written as the hilbertian sum

$$\mathcal{H} = \int_{\mathbb{T}^3}^{\oplus} \mathbb{R}^6 dx.$$

We then have

$$H_0 = \int_{\mathbb{T}^3}^{\oplus} H_0(x) dx.$$

It is clear that H_0 is a bounded self-adjoint operator on \mathcal{H} .

We are concerned with the unique continuation property (UCP) for \hat{H}_0 .

1.2 Spectrum of \hat{H}_0

We denote by $\sigma(A)$ the spectrum of an operator A . We introduce the following notations:

$$\xi = \sin x : \quad \xi_j = \sin x_j. \quad (1.1)$$

Let A be the following real symmetrical matrix:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (1.2)$$

Lemma 1.1. *Putting $p(x; \lambda) = \det(H(x) - \lambda)$, then*

$$p(x; \lambda) = \tilde{p}(\xi; \lambda) \equiv \det(\lambda^2 + M(\xi)^2) = \lambda^2(\lambda^2 - \xi^2)^2, \quad (1.3)$$

with the eigenvalue

$$\tau(\xi) = \xi^2, \quad \text{multiplicity} = 2,$$

associated with the eigenvectors (when they don't vanish)

$$w_1 = (-\xi_2, \xi_1, 0), \quad w_2 = \xi \times w_1 = (-\xi_1\xi_3, -\xi_2\xi_3, \xi_1^2 + \xi_2^2).$$

Proof: exercise.

We put

$$\lambda_+ = \max_{[-1,1]^3} \sqrt{\tau(\xi)} = \sqrt{3}.$$

Theorem 1.2. 1) $\sigma(H_0) = [-\lambda_+, \lambda_+]$.

2) The purely point spectrum is $\sigma_{pp}(\hat{H}_0) = \{0\}$.

3) The limiting absorption principle holds on $[-\lambda_+, \lambda_+] \setminus \{0\}$: the limits $\lim_{\varepsilon \rightarrow \pm 0} (H - \lambda + i\varepsilon)^{-1}$ exist in $B(\mathcal{B}_s, \mathcal{B}_s^*)$ for $s > \frac{1}{2}$, (where \mathcal{B}_s are Besov spaces).

4) The singular continuous spectrum of \hat{H}_0 is empty.

1.3 UCP

We analyse the UCP for the discrete unperturbed Maxwell system. Let $\lambda \in (0, \sqrt{3})$.

We denote $\hat{u} = (\hat{u}^E, \hat{u}^H) = (Fu^E, Fu^H)$. Let $\Omega \subset \mathbb{Z}^3$ a bounded set of the form

$$\Omega = \prod_{j=1}^3 \Omega_j, \quad \Omega_j = [a_j, b_j] \cap \mathbb{Z}.$$

Let us consider the following relation:

$$(\hat{H}_0 - \lambda)\hat{u}(n) = 0 \quad \text{in} \quad \{n \in \mathbb{Z}^3 \mid n \notin \Omega\}. \quad (1.4)$$

We write $u \sim 0$ at infinity if the following holds:

$$\lim_{R \rightarrow \infty} \frac{1}{R} \sum_{n: R_0 < |n| < R} |\hat{u}(n)|^2 = 0. \quad (1.5)$$

We claim:

Theorem 1.3. Assume that $(\hat{H}_0 - \lambda)\hat{u}$ has support in Ω , and that $\hat{u} \sim 0$ at infinity. Then $\text{supp } \hat{u} \subset \Omega$.

Proof. From the Rellich property, we can assume that \hat{u} has compact support in some bounded Ω' containing Ω .

Let $P(x)$ be the orthogonal projection of \mathbb{R}^3 onto ξ , for $\xi \neq 0$:

$$P(x)u^E = \frac{1}{\xi^2} \langle u^E, \xi \rangle_{\mathbb{R}^6} \xi, \quad u^E \in \mathbb{R}^3.$$

We then define the operator $P \in \mathcal{B}(\mathcal{H})$ which is multiplication by $P(x)$, and so, it is the orthogonal projection on $\xi(\cdot) \in \mathcal{H}$. We then write:

$$u = (u^E, u^H), \quad u^{E/H} = v^{E/H} + w^{E/H},$$

$$v^{E/H} := Pu^{E/H}, \quad \text{so that } w^{E/H}(x) \perp \xi,$$

$$v = (v^E, v^H), \quad w = (w^E, w^H),$$

(We also set $v(0) = 0$, $w(0) = u(0)$ for example).

Put $\hat{f} = (\hat{H}_0 - \lambda)\hat{u}$. Since $H_0(x)$ is self-adjoint, its range is ξ^\perp for $\xi \neq 0$. We define by $H_0^\perp(x)$ the restriction of $H_0(x)$ to the stable subspace $(\xi \otimes \xi)^\perp$.

Let us observe that $H_0P = PH_0 = 0$. Then $v, w \in \mathcal{H}$ satisfy:

$$-\lambda v^{E/H} + ((H_0 - \lambda)w)^{E/H} = f^{E/H} \quad \text{in } \mathbb{T}^3.$$

Writing $f^{E/H} = a^{E/H} + b^{E/H}$, $a^{E/H} := Pf^{E/H}$, $a = (a^E, a^H)$, $b = (b^E, b^H)$, we obtain

$$-\lambda \hat{v} = \hat{a} \quad \text{in } \mathbb{Z}^3, \quad (1.6)$$

$$(\hat{H}_0^\perp - \lambda)\hat{w} = \hat{b} \quad \text{in } \mathbb{Z}^3. \quad (1.7)$$

We observe that $\xi^2 a^{E/F}$, $\xi^2 b^{E/F}$, $\xi^2 v^{E/F}$ and $\xi^2 w^{E/F}$ are trigonometric polynomials, and $F(\xi^2 a^{E/F})$, $F(\xi^2 b^{E/F})$ have support in $D_2\Omega$ where we put

$$D_1\Omega = \{n \in \mathbb{Z}^3, \quad n \sim \Omega\} \cup \Omega, \quad D_k = D_1^k.$$

Here, $n \sim \Omega$ iff there exists $m \in \Omega$ such that $n - m \in \{\pm e_j, j = 1, 2, 3\}$. Equation (1.6) gives

$$F(\xi^2 v) = \frac{-1}{\lambda} F(\langle \xi, f \rangle \xi) \quad \text{in } \mathbb{Z}^3.$$

Then $F(\xi^2 v^{E/H}) = F(\langle \xi, u^{E/H} \rangle \xi)$ and $F(\xi^2 a^{E/H}) = F(\langle \xi, f^{E/H} \rangle \xi)$ have same support, included in $D_2\Omega$. Put

$$g = \xi^2 b = \xi^2 f - \langle \xi, f \rangle \xi, \quad q = \xi^2 w = \xi^2 u - \langle \xi, u \rangle \xi.$$

Equation (1.7) is then written

$$(H_0^\perp - \lambda)q = g \quad \text{in } \mathbb{T}^3,$$

that is,

$$M(\xi)q^H(\xi) - \lambda q^E(\xi) = g^E(\xi), \quad M(\xi)q^E(\xi) + \lambda q^H(\xi) = -g^H(\xi) \quad \text{in } \mathbb{T}^3.$$

We obtain:

$$q^E(\xi) = \frac{1}{\lambda} M(\xi)q^H(\xi) - \frac{1}{\lambda} g^E(\xi), \quad (1.8)$$

$$(M(\xi)^2 + \lambda^2)q^H(\xi) = M(\xi)g^E(\xi) - \lambda g^H(\xi) \quad \text{in } \mathbb{T}^3. \quad (1.9)$$

Since the non-zero eigenvalue of $M(\xi)^2$ are $-\xi^2$ with multiplicity 2, then the restriction of $M(\xi)^2$ to ξ^\perp is $-\xi^2 I_d$, and (1.9) provides

$$q^H(\xi) = \frac{1}{\lambda^2 - \xi^2} (M(\xi)g^E(\xi) - \lambda g^H(\xi)) \quad \text{in } \mathbb{T}^3.$$

Let us use the following

Lemma 1.4. 1) Let $r \in \mathbb{R}$, \hat{q} with compact support in \mathbb{Z}^3 , $j \in \{1, 2, 3\}$, and $m_j \in \mathbb{Z}$ such that $F((\xi^2 + r)q)(n) = 0$ for all n such that $n_j > m_j$. Then $\hat{q}(n) = 0$ for all n such that $n_j > m_j - 2$.

2) Let \hat{q} with compact support in \mathbb{Z}^3 such that $\text{supp } F((\xi^2 + r)q) \subset \Pi_j[c_j, d_j]$, where $d_j \geq c_j$ for all j . If some $d_j < c_j + 4$, then $\hat{q} = 0$. If $d_j \geq c_j + 4$ for all j , then $\text{supp } \hat{q} \subset \Pi_j[c_j + 2, d_j - 2]$ and $\text{supp } F((\xi^2 + r)q) \subset D_2(\Pi_j[c_j + 2, d_j - 2])$.

Proof. 1) The polynomial $(\xi_1^2 - \lambda^2)q$ in the variable e^{ix_j} has degree m_j at most, and so q is polynomial in the variable e^{ix_j} with degree $m_j - 2$ at most. The conclusion follows.

2) is straight consequence of 1). □

We apply Lemma 1.4 to Equation 1.9. Remember that Ω is the rectangle $\Pi_j[a_j, b_j]$. The support of \hat{g} is included in $D_2\Omega$, so the support of $F(M(\xi)g^E(\xi) - \lambda g^H(\xi))$ is included in $D_3\Omega \subset \Pi_j[a_j - 3, b_j + 3]$. Consequently, Fq^H has support in $\Pi_j[a_j - 1, b_j + 1]$. Similarly, Fq^E has support in $\Pi_j[a_j - 1, b_j + 1]$. Thus, $F(\xi^2 v + \xi^2 w)$ has compact support in $D_2\Omega$. From Lemma 1.4 again, we have $\text{supp } F(\xi^2 v + q) \subset \Omega$, and so, since $\hat{u} = F(\frac{1}{\xi^2}(\xi^2 v + q))$ has compact support, then \hat{u} has support in Ω . □