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COHERENT PRESENTATIONS OF A CLASS OF MONOIDS ADMITTING A GARSIDE FAMILY

PIERRE-LOUIS CURIEN, ALEN ĐURIĆ, AND YVES GUIRAUD

ABSTRACT. This paper shows how to construct coherent presentations of a class of monoids, including left-cancellative noetherian monoids containing no nontrivial invertible element and admitting a Garside family. Thereby, it resolves the question of finding a unifying generalisation of the following two distinct extensions of Deligne’s original construction of coherent presentations for spherical Artin-Tits monoids: to general Artin-Tits monoids, and to Garside monoids. The result is applied to a dual braid monoid, and to some monoids which are neither Artin-Tits nor Garside. For the Artin-Tits monoid of type \tilde{A}_2 , a finite coherent presentation is given, having a finite Garside family as a generating set.

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INTRODUCTION

Coherent presentations of monoids. A coherent presentation of a monoid consists of a set of generators, a set of generating relations, and a set of generators of the relations among relations, described formally in Section 1. The notion of coherent presentation is closely related to other two known notions: cofibrant approximations in the canonical model structure on 2-categories, given by Lack in [19], [20]; and weak actions of Artin-Tits monoids upon categories, investigated by Deligne in [8]. This is briefly recalled in Subsection 1.3. Coherent presentations form the first dimensions of polygraphic resolutions of monoids [21], from which abelian resolutions can be deduced.

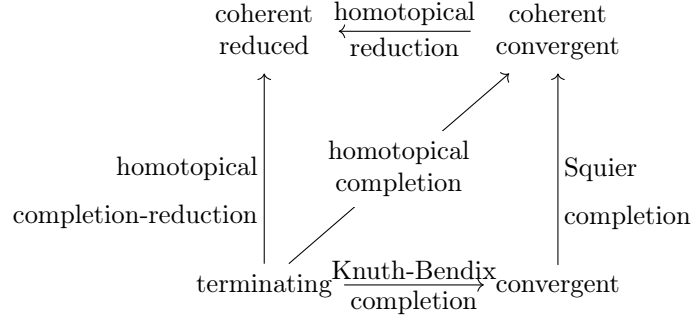
Deligne [8] studied weak actions of Artin-Tits monoids upon categories. In the case of M being a spherical Artin-Tits monoid $B^+(W)$, Deligne proved that to define a weak action, it suffices to consider endofunctors $F(w)$ for every w of W (i.e. it is not necessary to do it for every element of $B^+(W)$). This alternative definition is based on what is called Garside's presentation of Artin-Tits monoids and denoted $\text{Gar}_2(W)$ in [11].

It has been proved in [11] that Deligne's characterisation of the weak actions of spherical Artin monoids on categories is equivalent to saying that a coherent presentation, denoted $\text{Gar}_3(W)$ in [11], of a spherical Artin-Tits monoid $B^+(W)$ is given by adjoining suitable relations among relations. In [11], Deligne's result has been extended in two disjoint directions: to general Artin-Tits monoids, and to Garside monoids. This has been done using methods from rewriting theory and Squier's work.

Rewriting methods. A monoid can be presented by a set of generators and a set of directed relations called rewriting rules. A presentation is called terminating if there is no infinite rewriting sequence; it is called confluent if any two distinct rewriting sequences starting from the same word eventually lead to a common result; it is convergent if it is both terminating and confluent.

The homotopical completion-reduction procedure, developed in [11], enriches a terminating presentation to a coherent one. The main element is Squier's theorem, which allows one to simply compute generators of the relations among relations for a convergent presentation. The procedure has three stages. Firstly, the Knuth-Bendix completion procedure enriches a terminating presentation to a convergent one by adding a, not necessarily finite, number of relations. Secondly, the Squier completion procedure adjoins relations among relations, thus providing a coherent presentation of the monoid admitting the starting presentation. Thirdly, the homotopical reduction procedure removes redundant relations. These homotopical

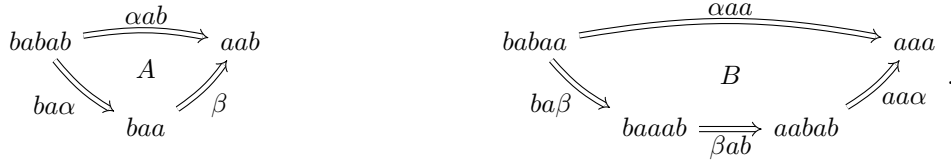
transformations of presentations having certain properties are illustrated by the following diagram and recollected in Section 2.



Let us also illustrate the first two stages by giving a preview of Example 2.2.5. Consider the following presentation of the Klein bottle monoid:

$$\langle a, b \mid bab \stackrel{\alpha}{\Rightarrow} a \rangle.$$

There is exactly one critical branching, i.e. minimal overlap of the rewriting steps: $\{\alpha ab, ba\alpha\}$. The Knuth-Bendix completion procedure adds the rewriting rule $\beta : baa \Rightarrow ab$. The Squier completion procedure adds the generator A of the relations among relations. The new rewriting rule β causes only one new critical branching $\{\alpha aa, ba\beta\}$, which is already confluent, hence only the generator B of the relations among relations is added. Here are the shapes of A and B :



In [11], Gaussent, the third author and Malbos have performed a homotopical completion-reduction procedure to compute coherent presentations of two disjoint generalisations of Artin-Tits monoids of spherical type: general Artin-Tits monoids, and Garside monoids. We recall those two generalisations in Section 3 as Theorems 3.4.1 and 3.6.1, respectively. In [17], the third author, Malbos and Mimram have computed coherent presentations of plactic and Chinese monoids by applying the homotopical completion-reduction procedure.

Garside families. A Garside family in a monoid is a generating family, not minimal in general, but ensuring some desirable properties. Namely, the notion of Garside family [4] is a result of successive generalisations to wider classes of monoids of a particular type of normal form defined in braid monoids, known as the greedy normal form. For a chronological overview of the development, see [3, Introduction]. This normal form is easily computed as it has very nice locality properties. These notions are recalled in Section 4.

Garside [10] investigated arithmetic properties of braid groups. He solved the word problem and the conjugacy problem in braid groups by introducing braid monoids. Among other things, he proved that the braid monoid B_n^+ is left-cancellative, and that any two elements of B_n^+ admit a least common multiple. He also

introduced the Garside element (he called it the fundamental word) of a braid monoid.

Garside's observations for braid monoids were generalised to spherical Artin-Tits monoids by Brieskorn and Saito [1], and by Deligne who later explicitly gave Garside's presentation for spherical Artin-Tits monoids in [8]. Michel [22] stated this presentation for all Artin-Tits monoids.

The greedy normal form was first introduced for braid monoids in the 1980s, based on Garside's observations (see [3, Introduction] for references). Dehornoy and Paris [7] introduced Garside monoids in order to abstract properties which establish the existence of the greedy normal form. Dehornoy, Digne and Michel [4] further generalised Garside monoids to categories admitting Garside families (as recalled for monoids in Subsection 4.2 here). A thorough development of the notion of a Garside family can be found in the book [3]. Dehornoy and the third author [6] introduced monoids admitting quadratic normalisations, thereby generalising monoids admitting Garside families. We refer the reader to the survey [2] for an overview of the successive extensions of the greedy normal form from braid monoids to monoids admitting left-weighted quadratic normalisations.

Contributions. The objective of the present paper is to unify the two above-mentioned results of [11] in the same generalisation. Namely, we apply the homotopical completion-reduction procedure to compute coherent presentations of a certain class of monoids admitting a Garside family. Our present contribution is motivated by the following two key observations.

- (1) Firstly, Garside's presentation $\text{Gar}_2(W)$ is a special case of a presentation, here denoted $\text{Gar}_2(S)$, provided by a result from [6]. We will recall this result as Proposition 4.2.7. The presentation $\text{Gar}_2(S)$ of a monoid admitting a Garside family S has a generating set $S \setminus \{1\}$ and relations α of the form $s|t = st$, for $s, t \in S \setminus \{1\}$ with $st \in S$. We interpret $\text{Gar}_2(W)$ as an output of Proposition 4.2.7 for some conveniently chosen input. This allows us to extend Theorem 3.4.1 to a wider class of monoids, including left-cancellative noetherian monoids containing no nontrivial invertible element, admitting a Garside family.
- (2) Secondly, although certain steps of the proof of Theorem 3.4.1 do rely on the arithmetic properties of Artin-Tits monoids (as the proof is devised having a particular input in mind, namely, Garside's presentation of an Artin-Tits monoid), the general structure of the proof mostly relies neither on the fact that the monoid considered is an Artin-Tits monoid nor on the fact that the starting presentation is Garside's presentation, but rather on the specific "shape" of the relations involved.

Building on those two observations, we apply a generalised Theorem 3.4.1 after applying Proposition 4.2.7. In Section 5, our main result (Theorem 5.1.4) is stated and proved. Here we give a weaker, yet simpler to state, version thereof (Corollary 5.5.1).

Theorem. *Assume that M is a left-cancellative noetherian monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family containing 1. Then M admits the coherent presentation $\text{Gar}_3(S)$.*

The generators of the relations among relations of $\text{Gar}_3(S)$ are

$$\begin{array}{ccccc}
 & & uv|w & & \\
 & \nearrow^{\alpha_{u,v}|w} & & \searrow_{\alpha_{uv,w}} & \\
 u|v|w & & A_{u,v,w} & & uvw \\
 & \searrow_{\alpha_{u,v,w}} & & \nearrow^{\alpha_{u,vw}} & \\
 u|\alpha_{v,w} & & u|vw & &
 \end{array}$$

4

for all $u, v, w \in S \setminus \{1\}$ such that $uv, vw, uvw \in S$. Note that $A_{u,v,w}$ can be read as a relation ensuring associativity. We shall reach $\text{Gar}_3(S)$ by applying the homotopical completion-reduction procedure to the presentation $\text{Gar}_2(S)$.

In Section 6, the result is used to compute coherent presentations of some monoids which are neither Artin-Tits nor Garside, and to construct a finite coherent presentation of the Artin-Tits monoid of type \tilde{A}_2 .

1. PRESENTATIONS OF CATEGORIES BY POLYGRAPHS

In this section, we briefly recall the notions concerning polygraphic presentations of categories (technical elaboration whereof can be found in [11] and [14]). Basic terminology is given in Subsection 1.1. The notion of polygraph and some basic notions of higher-dimensional rewriting theory are recalled in Subsections 1.2 and 1.4, respectively. Subsection 1.3 recalls motivation for studying coherent presentations.

1.1. Cellular extensions of higher categories. Throughout the present article, an n -category always means strict, globular n -category (see e.g. [14, Chapter 2]). An n -category is called an (n, p) -category if its k -cells are invertible for all $k > p$. For a k -cell f of an n -category \mathcal{C} , the identity $(k+1)$ -cell and the i -source and i -target of f are respectively denoted by 1_f and $s_i(f)$ and $t_i(f)$; the subscript i is dropped if $i = k - 1$. If $t_i(f) = s_i(g)$, that is if f and g are i -composable k -cells, their i -composite is denoted by $f \star_i g$; abbreviated to fg if $i = 0$. When 1_f (resp. the identity of 1_f , etc.) is composed with cells of dimension $k+1$ (resp. $k+2$, etc.), it is denoted by f , for simplicity. In diagrams, distinct arrows are used to denote k -cells for low k : $\rightarrow, \Rightarrow, \Rrightarrow$ for k equal to 1, 2 and 3, respectively.

For $k \geq 1$, two k -cells are called **parallel** if they have a common source and a common target. An (ordered) pair of parallel k -cells in an n -category is called a **k -sphere**. Every two 0-cells are considered parallel. The source and target of a k -sphere $\gamma = (f, g)$ are f and g , respectively, which is written as $s(\gamma) = f$ and $t(\gamma) = g$. For a k -cell γ of \mathcal{C} , the $(k-1)$ -sphere $(s(\gamma), t(\gamma))$ is called the **boundary** of γ .

For a set Γ and an n -category \mathcal{C} , a map from Γ to the set of n -spheres of \mathcal{C} is called a **cellular extension** of \mathcal{C} . Usually, we identify elements of Γ with their images and say that Γ is a cellular extension of \mathcal{C} . A cellular extension of \mathcal{C} can be interpreted as a set of directed relations between parallel n -cells. Identifying the n -cells $s(\gamma)$ and $t(\gamma)$ for every n -sphere γ of Γ , produces the quotient n -category \mathcal{C}/Γ . A cellular extension of \mathcal{C} can also be interpreted as a set of formal $(n+1)$ -cells, filling the n -spheres of \mathcal{C} . For an n -category \mathcal{C} and its cellular extension Γ , one constructs the **free $(n+1)$ -category generated by Γ over \mathcal{C}** , denoted by $\mathcal{C}[\Gamma]$, by adjoining to \mathcal{C} all the formal compositions of elements of Γ , treated as formal $(n+1)$ -cells. Similarly, for an $(n, 1)$ -category \mathcal{C} and its cellular extension Γ , one constructs the **free $(n+1, 1)$ -category generated by Γ over \mathcal{C}** , denoted by $\mathcal{C}(\Gamma)$, as the quotient $\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \tilde{\Gamma}] / \text{Inv}(\Gamma)$, where $\tilde{\Gamma}$ denotes the cellular extension obtained by reversing directions (switching components) of spheres in Γ , and $\text{Inv}(\Gamma)$ denotes the cellular extension of $\mathcal{C}[\Gamma, \tilde{\Gamma}]$ consisting of $(n+2)$ -cells $\tilde{\chi} \star_n \chi \rightarrow 1_{t(\chi)}$ and $\chi \star_n \tilde{\chi} \rightarrow 1_{s(\chi)}$ for every $(n+1)$ -cell χ in Γ .

An **acyclic cellular extension** of an n -category \mathcal{C} is a cellular extension Γ of \mathcal{C} such that all the n -spheres of the quotient n -category \mathcal{C}/Γ are of the form (f, f) or, equivalently, such that for every n -sphere γ of \mathcal{C} , there is an $(n+1)$ -cell whose boundary is γ in the $(n+1)$ -category $\mathcal{C}(\Gamma)$.

1.2. Polygraphs. Polygraphs provide a generalisation of a presentation of a monoid by generators and relations to the higher categories which are free up to codimension 1.

A polygraph is a higher-dimensional generalisation of a graph. Recall that a (directed) graph is a pair (X_0, X_1) of sets, together with two maps, called source and target, from X_1 to X_0 . Just like a graph generates a free category (or a free groupoid), an n -polygraph generates a free n -category (or a free n -groupoid). A 0-polygraph (X_0) is a set, a **1-polygraph** (X_0, X_1) is a graph. An **n -polygraph** is a $(n+1)$ -tuple $X = (X_0, \dots, X_{n-1}, X_n)$, where (X_0, \dots, X_{n-1}) is an n -polygraph and X_n is a cellular extension of the free category X_{n-1}^* generated by (X_0, \dots, X_{n-1}) (see [14, Section 2.4]). A polygraph is **finite** if it has finitely many generating cells in every dimension.

For a 2-polygraph X , the **category presented by X** , the **free 2-category over X** , and the **free $(2,1)$ -category over X** are defined respectively as $\overline{X} = X_1^*/X_2$, $X_2^* = X_1^*[X_2]$, and $X_2^\top = X_1^*(X_2)$. A **$(3,1)$ -polygraph** is a quadruple $X = (X_0, X_1, X_2, X_3)$, where (X_0, X_1, X_2) is a 2-polygraph and X_3 is a cellular extension of X_2^\top . Elements of X_k are called **generating k -cells**.

For a $(3,1)$ -polygraph X , the **free $(3,1)$ -category X_3^\top over X** and the **$(2,1)$ -category presented by X** are defined respectively as $X_3^\top = X_2^\top(X_3)$ and $\widetilde{X} = X_2^\top/X_3$. The **category presented by a $(3,1)$ -polygraph X** is again \overline{X} , the category presented by its underlying 2-polygraph. Formal definitions of general n -polygraphs and (n,p) -polygraphs can be found in [14, Section 2.4]. We say that a generating 1-cell (resp. 2-cell, resp. 3-cell) x occurs in a 1-cell (resp. 2-cell, resp. 3-cell) γ of X_3^\top if it occurs in a minimal-length decomposition of γ into generic 1-cells (resp. 2-cells, resp. 3-cells).

Assume \mathcal{C} is a category. A **presentation of \mathcal{C}** is a 2-polygraph X such that \mathcal{C} is isomorphic to \overline{X} . An **extended presentation of \mathcal{C}** is a $(3,1)$ -polygraph X such that \mathcal{C} is isomorphic to \overline{X} .

Definition 1.2.1. A **coherent presentation of \mathcal{C}** is an extended presentation (X_0, X_1, X_2, X_3) of \mathcal{C} such that X_3 is an acyclic cellular extension of X_2^\top .

Note that this definition, in particular, applies to monoids, seen as categories having a single object.

Example 1.2.2 (The standard coherent presentation). Let \mathcal{C} be a category. The **standard presentation** of \mathcal{C} is the 2-polygraph $\text{Std}_2(\mathcal{C})$ consisting of:

- a generating 0-cell x for every 0-cell x of \mathcal{C} ;
 - a generating 1-cell $\widehat{u} : x \rightarrow y$ for every 1-cell $u : x \rightarrow y$ of \mathcal{C} ;
 - a generating 2-cell $\gamma_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ for every pair of composable 1-cells u and v of \mathcal{C} ;
 - a generating 2-cell $\iota_x : 1_x \Rightarrow \widehat{1}_x$ for every 0-cell x of \mathcal{C} .
- Let us extend $\text{Std}_2(\mathcal{C})$ with the following 3-cells

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \gamma_{u,v}\widehat{w} & \xrightarrow{\quad} & \widehat{uv}\widehat{w} \\
 \widehat{uv}\widehat{w} & \xrightarrow{\quad} & \widehat{uvw} \\
 \widehat{u}\gamma_{v,w} & \xrightarrow{\quad} & \widehat{uv}\widehat{w} \\
 & \Downarrow A_{u,v,w} & \\
 & \widehat{uv}\widehat{w} & \\
 & \xrightarrow{\quad} & \widehat{uvw} \\
 & \xrightarrow{\quad} & \gamma_{u,vw}
 \end{array}
 &
 \begin{array}{ccc}
 & \widehat{1}_x\widehat{u} & \\
 \iota_x\widehat{u} & \xrightarrow{\quad} & \widehat{1}_x\widehat{u} \\
 \widehat{u} & \xrightarrow{\quad} & \widehat{u} \\
 & \Downarrow L_u & \\
 & \widehat{1}_x\widehat{u} & \\
 & \xrightarrow{\quad} & \widehat{u} \\
 & \xrightarrow{\quad} & \gamma_{1_x,u}
 \end{array}
 &
 \begin{array}{ccc}
 & \widehat{u}\widehat{1}_y & \\
 \widehat{u}\iota_y & \xrightarrow{\quad} & \widehat{u}\widehat{1}_y \\
 \widehat{u} & \xrightarrow{\quad} & \widehat{u} \\
 & \Downarrow R_u & \\
 & \widehat{u}\widehat{1}_y & \\
 & \xrightarrow{\quad} & \widehat{u} \\
 & \xrightarrow{\quad} & \gamma_{u,1_y}
 \end{array}
 \end{array}$$

for every triple of 1-cells u, v and w of \mathcal{C} such that the pairs u, v and v, w are composable. The resulting $(3,1)$ -polygraph, denoted by $\text{Std}_3(\mathcal{C})$, is called the **standard coherent presentation** of \mathcal{C} (see [14, Subsection 3.3.3] for the explanation why $\text{Std}_3(\mathcal{C})$ is, indeed, a coherent presentation).

1.3. Motivation. To motivate studying coherent presentations, let us recall their connection with cofibrant approximations of 2-categories, and with weak actions of Artin-Tits monoids upon categories.

Assume that \mathcal{C} and \mathcal{D} are 2-categories. A 2-functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called a weak equivalence if there exists a pseudofunctor (i.e. a 2-functor preserving composition and identities of 1-cells only up to 2-isomorphism; see [11, 5.1.1]) $G : \mathcal{D} \rightarrow \mathcal{C}$ such that GF is isomorphic to $1_{\mathcal{C}}$ and FG is isomorphic to $1_{\mathcal{D}}$. A 2-category \mathcal{C} is called cofibrant if its underlying 1-category is free, and \mathcal{D} is a cofibrant approximation of \mathcal{C} if \mathcal{D} is cofibrant and if there exists a weak equivalence $\mathcal{D} \rightarrow \mathcal{C}$. The next theorem displays the close relation between coherent presentations of categories and their cofibrant approximations. Recall that \tilde{X} denotes the $(2, 1)$ -category presented by X .

Theorem ([11, Theorem 1.3.1]). *Assume that \mathcal{C} is a category, and X is $(3, 1)$ -polygraph that presents \mathcal{C} . Then X is a coherent presentation of \mathcal{C} if, and only if, \tilde{X} is a cofibrant approximation of \mathcal{C} (viewed as a 2-category).*

A strict action of a monoid M upon a category \mathcal{C} is a morphism of monoids $F : M \rightarrow \text{End}(\mathcal{C})$. A weak action of M upon \mathcal{C} , replaces the morphism equalities $F(fg) = F(f)F(g)$ and $F(1) = 1_{\mathcal{C}}$ with natural isomorphisms which satisfy coherence relations. Pseudofunctors generalise weak actions of monoids on categories. A consequence of the next theorem is that the Deligne's result [8, Theorem 1.5] is equivalent to saying that Theorem 3.4.1 holds for spherical Artin-Tits monoids.

Theorem ([11, Theorem 5.1.6]). *Assume that \mathcal{C} is a category, and X is $(3, 1)$ -polygraph that presents \mathcal{C} . Then X is a coherent presentation of \mathcal{C} if, and only if, for every 2-category \mathcal{D} , the category of 2-functors from \mathcal{C} to \mathcal{D} and the category of pseudofunctors from \tilde{X} to \mathcal{D} are equivalent, and this equivalence is natural in \mathcal{D} .*

1.4. Rewriting properties of polygraphs. Let us adopt some basic terminology from string rewriting. If S is a set, S^* denotes the free monoid over S . Elements of S and S^* are respectively called letters and **words**. We write $u|v$ for the concatenation of two words u and v , sometimes omitting the separation symbol when that does not cause ambiguity. Let M be a monoid generated by a set S . A **normal form** for M with respect to S is a set-theoretic section of the canonical projection (evaluation map) $\text{ev} : S^* \rightarrow M$. In other words, a normal form maps elements of M to distinguished representative words. A word $s_1| \cdots | s_p$ is said to be a **decomposition** of an element f of M if the equality $s_1 \cdots s_p = f$ holds in M .

Standard notions from rewriting theory naturally translate into the framework of polygraphs. A **rewriting step** of a 2-polygraph X is a 2-cell of the free category X_2^* which contains a single generating 2-cell of X , here considered as a transformation of its source into its target. So, a rewriting step has a shape

$$\bullet \xrightarrow{w} \bullet \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} \bullet \xrightarrow{w'} \bullet ,$$

where $\alpha : u \Rightarrow v$ is a generating 2-cell of X , and w and w' are 1-cells of X_2^* , and all the 0-cells are denoted by \bullet .

Let u and v be 1-cells of X_2^* . It is said that u **rewrites** into v if there is a finite composable sequence of rewriting steps with source u and target v . A 1-cell u is **reduced** if there is no rewriting step whose source is u . A **normal form** of u , denoted by \hat{u} , is a reduced 1-cell into which u rewrites.

Let X be a 2-polygraph. A **termination order** on X is a well-founded order relation \leq on parallel 1-cells of X_2^* enjoying the following properties:

- the compositions by 1-cells of X_2^* are strictly monotone in both arguments, i.e. \leq is compatible with the composition of 1-cells;
- for every generating 2-cell α of X , the strict inequality $s(\alpha) > t(\alpha)$ holds.

A 2-polygraph X is **terminating** if it has no infinite sequence of rewriting steps. Admitting a termination order is equivalent to being terminating (in a terminating polygraph, a termination order is obtained by putting $u > v$ for 1-cells u and v if u rewrites into v).

A **branching** of a 2-polygraph X is an unordered pair $\{\alpha, \beta\}$ of sequences of rewriting steps of X_2^* having the same source, called the source of branching. If α and β are rewriting steps, a branching $\{\alpha, \beta\}$ is called **local**. A local branching is **trivial** if it has one of the following two shapes: $\{\alpha, \alpha\}$, or $\{\alpha v, u\beta\}$ for $u = s(\alpha)$ and $v = s(\beta)$. Local branchings can be compared by the order \preceq generated by the relations $\{\alpha, \beta\} \preceq \{u\alpha v, u\beta v\}$ given for every local branching $\{\alpha, \beta\}$ and all possible 1-cells u and v of X_2^* . A minimal non-trivial local branching is called **critical**. A branching $\{\alpha, \beta\}$ is confluent if α and β can be completed into sequences having the same target. A 2-polygraph X is **confluent** (resp. locally confluent, resp. critically confluent) if all its branchings (resp. local branchings, resp. critical branchings) are confluent. If X is terminating and confluent, it is called **convergent**. A convergent 2-polygraph X is called a **convergent presentation** of any category isomorphic to \bar{X} . In that case, every 1-cell of X^* has a unique normal form. A $(3, 1)$ -polygraph is **convergent** if its underlying 2-polygraph is.

The following two basic results of rewriting theory, called Newman's lemma and the critical branchings theorem respectively, concerning confluence, are also valid for polygraphs.

Theorem 1.4.1. *Let X be a 2-polygraph.*

- (1) *If X is terminating, then X is confluent if, and only if, it is locally confluent.*
- (2) *X is locally confluent if, and only if, it is critically confluent.*

Proofs of more general versions of these results can be found in [14, Section 3.2]. As a consequence of Theorem 1.4.1, a 2-polygraph is convergent if, and only if, it is terminating and its critical branchings are confluent.

Example 1.4.2. Consider the free abelian monoid:

$$(1.1) \quad \mathbb{N}^3 = \langle a, b, c \mid ba \xrightarrow{\alpha} ab, cb \xrightarrow{\beta} bc, ca \xrightarrow{\gamma} ac \rangle.$$

This presentation (1.1) admits the following termination order: comparing the lengths of words, then applying lexicographic order if words have the same length. Hence, it is terminating.

Let us illustrate confluence of (1.1) on the unique critical branching $\{\beta a, c\alpha\}$:

$$\begin{array}{ccccc}
 & & bca & \xrightarrow{b\gamma} & bac & & \\
 & \nearrow^{\beta a} & & & & \searrow^{\alpha c} & \\
 cba & & & & & & abc \\
 & \searrow_{c\alpha} & & & & \nearrow_{a\beta} & \\
 & & cab & \xrightarrow{\gamma b} & acb & &
 \end{array}$$

Thus, the presentation (1.1) is convergent, by Theorem 1.4.1.

2. HOMOTOPICAL TRANSFORMATIONS OF POLYGRAPHS

This section recalls the notion of homotopical completion-reduction, introduced in [11]. Subsection 2.1 recollects the basic transformations one can perform on a polygraph without changing the presented category. The homotopical completion and the homotopical reduction are described in Subsections 2.2 and 2.3, respectively.

2.1. Tietze transformations and Tietze equivalence. Polygraphs allow one to treat presentations of categories homotopically. This subsection recalls a framework for homotopical transformations of $(3, 1)$ -polygraphs. Let X be a $(3, 1)$ -polygraph. A 2-cell (resp. 3-cell, resp. 3-sphere) α of X is called **collapsible** if it meets the following two requirements:

- the target of α is a generating 1-cell (resp. 2-cell, resp. 3-cell) of X ,
- the source of α is a 1-cell (resp. 2-cell, resp. 3-cell) of the free $(3, 1)$ -category over $X \setminus \{t(\alpha)\}$.

The following six operations are called **elementary Tietze transformations** of X :

- simultaneous adjunction or elimination of a generating 1-cell v and a collapsible 2-cell $\alpha : u \Rightarrow v$,
- simultaneous adjunction or elimination of a generating 2-cell β and a collapsible 3-cell $A : \alpha \Rightarrow \beta$,
- simultaneous adjunction or elimination of a generating 3-cell B such that there is a collapsible 3-sphere whose target is B .

Formally, these transformations define 3-functors with domain X_3^\top (see [11, 2.1.1]). A **Tietze transformation** is a composition of elementary Tietze transformations.

We say that two $(3, 1)$ -polygraphs are **Tietze-equivalent** if they satisfy the following two conditions:

- the 1-categories they present are isomorphic,
- the $(2, 1)$ -categories they present are equivalent.

Consequently, Tietze-equivalent $(3, 1)$ -polygraphs have the same 0-cells (up to a bijection), and two coherent presentations of the same category are Tietze-equivalent.

It is easily shown that two $(3, 1)$ -polygraphs related by a Tietze transformation are Tietze-equivalent. Namely, it suffices to verify that every one of the six elementary Tietze transformations preserves Tietze equivalence. Furthermore, Tietze transformations preserve the property of being coherent, which we formally state for future reference.

Proposition 2.1.1 ([11, Theorem 2.1.3]). *If a $(3, 1)$ -polygraph X is a coherent presentation of a category \mathcal{C} and if there exists a Tietze transformation from X to a $(3, 1)$ -polygraph Y , then Y is a coherent presentation of \mathcal{C} .*

An **elementary Nielsen transformation** on a $(3, 1)$ -polygraph X is any of the following operations:

- replacement of a 2-cell or a 3-cell with its formal inverse;
- replacement of a 3-cell $A : \alpha \Rightarrow \beta$ with $\tilde{A} : \chi \star_1 \alpha \star_1 \chi' \Rightarrow \chi \star_1 \beta \star_1 \chi'$, where χ and χ' are 2-cells of X_3^\top .

Elementary Nielsen transformations are Tietze transformations (see [11, 2.1.4]). A **Nielsen transformation** is a composition of elementary ones. In a homotopical completion-reduction procedure, Nielsen transformations are performed implicitly for convenience.

2.2. Homotopical completion. Starting with a terminating 2-polygraph X , equipped with a total termination order \leq , the Knuth-Bendix completion procedure adjoins generating 2-cells aiming to produce a convergent 2-polygraph, which presents a category presented by X . It works by iteratively examining all the critical branchings and adjoining a new generating 2-cell whenever the branching is not already confluent. Namely, for a critical branching $\{\alpha, \beta\}$, if $t(\widehat{\alpha}) > t(\widehat{\beta})$ (resp. $t(\widehat{\beta}) > t(\widehat{\alpha})$), a generating 2-cell $\gamma : t(\widehat{\alpha}) \Rightarrow t(\widehat{\beta})$ (resp. $\gamma : t(\widehat{\beta}) \Rightarrow t(\widehat{\alpha})$) is

adjoined, thus forcing the confluence of the branching:

$$\begin{array}{ccccc}
 & \alpha & \longrightarrow & t(\alpha) & \Longrightarrow & t(\widehat{\alpha}) \\
 & \nearrow & & & & \vdots \\
 * & & & & & \gamma \\
 & \searrow & & & & \vdots \\
 & \beta & \longrightarrow & t(\beta) & \Longrightarrow & t(\widehat{\beta})
 \end{array}$$

If new critical branchings are created by adjoining additional generating 2-cells, confluence of such critical branchings is examined. For details, see [16, 3.2.1]. This procedure is not guaranteed to terminate. In fact, its termination depends on the chosen termination order (see [9, Example 6.3.1]). If it does terminate, the result is a convergent 2-polygraph. Otherwise, it produces an increasing sequence of 2-polygraphs, and the result is the union of this sequence. Either way, the result is called a **Knuth-Bendix completion** of X . Note that different orders of examining critical branchings may result in different 2-polygraphs.

Theorem 2.2.1 ([16, Theorem 3.2.2]). *Every Knuth-Bendix completion of a 2-polygraph X equipped with a total termination order is a convergent presentation of the category \overline{X} .*

Remark 2.2.2. The Knuth-Bendix completion procedure, as described above, requires not only termination, but also the presence of a total termination order, to be able to orient the generating 2-cells rules which are added, and to be able to maintain the termination during the completion. There is an alternative approach. Namely, we can orient the newly added generating 2-cells "by hand", according to our inspiration, and verify after each addition in an ad hoc manner whether we maintain a terminating presentation, without having defined a total order at the beginning (we shall do this in the proofs of Theorem 3.4.1 and Proposition 5.3.1). Therefore, we can invoke Theorem 2.2.1 even if we do not provide a total order, as long as we are able to ensure termination after each addition of a generating 2-cell (we shall do this in the proof of Corollary 5.3.2).

A **family of generating confluences** of a convergent 2-polygraph X is a cellular extension of X_2^\top containing, for every critical branching $\{\alpha, \beta\}$ of X , exactly one 3-cell $A : \alpha \star_1 \alpha' \cong \beta \star_1 \beta'$, where α' and β' are completing α and β , respectively, into sequences having the same target (such α' and β' exist by the assumption of confluence):

$$\begin{array}{ccccc}
 & \alpha & \longrightarrow & * & \xrightarrow{\alpha'} & * \\
 & \nearrow & & \Downarrow A & & \searrow \\
 * & & & & & * \\
 & \searrow & & & & \nearrow \\
 & \beta & \longrightarrow & * & \xrightarrow{\beta'} & *
 \end{array}$$

A **Squier completion** of a convergent 2-polygraph X is a $(3, 1)$ -polygraph, denoted by $\mathcal{S}(X)$, whose generating 3-cells form a family of generating confluences of X . The following result is due to Squier; we state its contemporary version in terms of polygraphs and higher-dimensional categories proved in [16].

Theorem 2.2.3 ([16, Theorem 4.3.2]). *Let X be a convergent 2-polygraph. Then every family of generating confluences of X is an acyclic cellular extension of X_2^\top .*

In other words, for every convergent presentation X of a category \mathcal{C} , a Squier completion of X is a coherent presentation of \mathcal{C} . Theorem 2.2.3 is extended to higher-dimensional polygraphs in [15, Proposition 4.3.4].

Let X be a terminating 2-polygraph equipped with a total termination order \leq . A homotopical completion of X is a Squier completion of a Knuth-Bendix completion of X . We have seen that the Knuth-Bendix completion procedure enriches a terminating 2-polygraph to a convergent one, and that the Squier completion of a convergent 2-polygraph X is a coherent presentation of \overline{X} . Those two transformations can be performed consecutively. They can also be performed simultaneously (see [11, 2.2.4]). The result is called a **homotopical completion** of X , here denoted by $\mathcal{H}(X)$. Theorem 2.2.3 has the following consequence.

Theorem 2.2.4. *Assume that a 2-polygraph X is a terminating presentation of a category \mathcal{C} . Then, every homotopical completion of X is a coherent convergent presentation of \mathcal{C} .*

Example 2.2.5 (Klein bottle monoid). We consider the Klein bottle monoid K^+ , as defined in [3, Subsection I.3.2]. It has the following presentation:

$$(2.1) \quad \langle a, b \mid bab = a \rangle.$$

The name comes from the fact that K^+ is the submonoid generated by a and b of the fundamental group of the Klein bottle generated by a and b subject to relation $bab = a$. Every element of K^+ admits a unique expression of the form $a^p b^q$ for $p, q \geq 0$ or $a^p b^q a$ for $p \geq 0$ and $q \geq 1$. That form is called canonical.

Let us apply the homotopical completion procedure to the presentation (2.1). We have the generating 1-cells a and b , and a single generating 2-cell $\alpha : bab \Rightarrow a$. Let us adopt the following termination order: comparing the lengths of words, then applying lexicographic order if words have the same length. For instance, $b < aa < ab$. The only critical branching is $\{\alpha ab, ba\alpha\}$, with source $babab$. The homotopical completion procedure adjoins the 2-cell $\beta : baa \Rightarrow aab$, and the 3-cell A for coherence. The 2-cell β causes only one new critical branching, namely $\{\alpha aa, ba\beta\}$ with source $babaa$, which is confluent, hence only the 3-cell B is adjoined. Diagrammatically, the 3-cells have the shapes as follows:

$$\begin{array}{ccc}
 \begin{array}{ccc}
 babab & \xrightarrow{\alpha ab} & aab \\
 & \searrow \alpha ab & \nearrow \beta \\
 & A & \\
 & \swarrow ba\alpha & \nearrow \beta \\
 & baa &
 \end{array}
 &
 &
 \begin{array}{ccc}
 babaa & \xrightarrow{\alpha aa} & aaa \\
 & \searrow ba\beta & \nearrow aa\alpha \\
 & B & \\
 & \swarrow baa\beta & \nearrow \beta ab \\
 & baa\beta & \xrightarrow{\beta ab} & aabab &
 \end{array}
 \end{array}$$

By Theorem 2.2.4, we have thus obtained a convergent coherent presentation $\mathcal{H}(K^+)$ of the Klein bottle monoid, consisting of two generating 1-cells, two generating 2-cells, and two generating 3-cells:

$$\left(a, b \mid bab \xrightarrow{\alpha} a, baa \xrightarrow{\beta} aab \mid A, B \right).$$

Remark 2.2.6. For convenience, we mostly leave implicit the orientation of the 3-cells in the diagrams. We only label the corresponding area with the name of a 3-cell. The convention is that the source and the target of a 3-cell are always the upper and the lower paths, respectively, of the sphere bounding the area.

2.3. Homotopical reduction. A coherent presentation obtained by the homotopical completion procedure is not necessarily minimal, in the sense that it may contain superfluous cells. A homotopical reduction procedure aims to remove such superfluous cells by performing Tietze transformations (up to Nielsen transformations).

For a $(3, 1)$ -polygraph $X = (X_0, X_1, X_2, X_3)$, a **collapsible part** of X is a triple $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$, wherein $\Gamma_2, \Gamma_3, \Gamma_4$ respectively denote families of generating 2-cells

of X , generating 3-cells of X , 3-spheres of X_3^\top , such that the following requirements are met:

- every γ of every Γ_k is collapsible (possibly up to a Nielsen transformation);
- no γ of Γ_k is the target of an element of Γ_{k+1} ;
- there exist well-founded order relations on X_1 , X_2 and X_3 such that, for every γ in every Γ_k , the target of γ is strictly greater than every generating $(k-1)$ -cell that occurs in the source of γ .

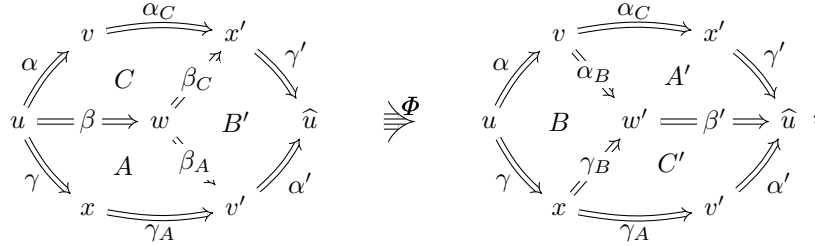
In that case, the recursive assignment

$$\pi_\Gamma(x) = \begin{cases} \pi_\Gamma(s(\gamma)) & \text{if } x = t(\gamma) \text{ for } \gamma \text{ in } \Gamma \\ 1_{\pi_\Gamma(s(x))} & \text{if } x \text{ in } \Gamma \\ x & \text{otherwise} \end{cases}$$

defines, by well-founded induction, a Tietze transformation. The result of this transformation is called the **homotopical reduction of X with respect to Γ** , and such a transformation is called a homotopical reduction procedure.

We have just recalled the definition of a generic collapsible part of a $(3,1)$ -polygraph X . For the applications considered here, however, it is practical to also recall a particular way to construct a collapsible part in the case when X is convergent and coherent. A **local triple branching** is an unordered triple $\{\alpha, \beta, \gamma\}$ of rewriting steps having a common source. A local triple branching is **trivial** if two of its components are equal or if one of its components forms branchings of the type $\{\alpha v, u\beta\}$, for $u = s(\alpha)$ and $v = s(\beta)$, with the other two. In a manner analogous to the case of local branchings, local triple branchings can be ordered by “inclusion”, and a minimal non-trivial local triple branching is called **critical**.

Let X be a convergent and coherent $(3,1)$ -polygraph. A **generating triple confluence** of X is a 3-sphere Φ , here formally presented as a 4-cell



where $\{\alpha, \beta, \gamma\}$ is a critical triple branching, and the other cells are obtained in the following way. Consider the branching $\{\alpha, \beta\}$: use confluence to get the 2-cells α_C and β_C ; then use coherence to get the 3-cell C . Similarly, treat the branchings $\{\beta, \gamma\}$ and $\{\alpha, \gamma\}$. In parallel, consider the branching $\{\alpha_C, \alpha_B\}$: use convergence to get the 2-cells γ' and β' which have the 1-cell \hat{u} as their common target; then use coherence to get the 3-cell A' . Similarly, treat the branching $\{\gamma_B, \beta', \gamma_A\}$. Finally, use coherence to get the 3-cell B' .

The source and target of the 3-sphere Φ consist of 3-cells of X in context: they have shape yXz , where X is a generating 3-cell and y and z are 1-cells. If one of those generating 3-cells occurs only once and in an empty context (meaning $y = z = 1$), then Φ provides a definition of X in terms of the other 3-cells. Thus, the 3-sphere Φ is collapsible, up to a Nielsen transformation (we Nielsen-transform Φ into a 3-sphere having X as target).

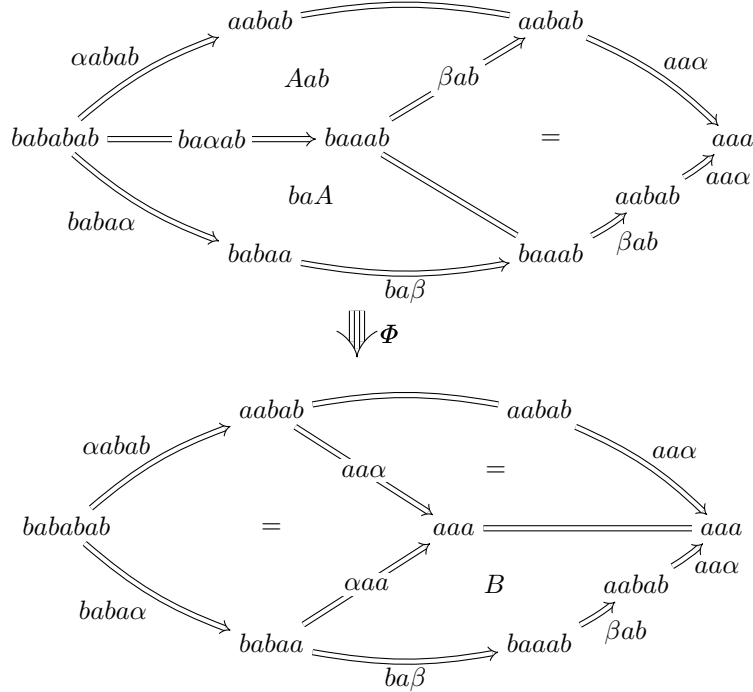
The component Γ_4 of a collapsible part can be constructed by iteratively examining all the critical triple branchings and taking the collapsible 3-spheres which are detected as described above. After constructing Γ_4 , proceed to construct Γ_3 in the following way. For every 3-cell A of X_3^\top , consider its source and target, which are

1-composites of rewriting steps $u\alpha v$, where α is a generating 2-cell and u and v are 1-cells. If any such α occurs only once and with $u = v = 1$, then A is collapsible (up to a Nielsen transformation). Iteratively examining all the 3-cells and taking the ones such as A , the component Γ_3 of a collapsible part can be constructed. Finally, Γ_2 is constructed by iteratively taking 2-cells whose source or target consists of a single generating 1-cell occurring only once.

Let X be a terminating 2-polygraph, with a termination order \leq . A **homotopical completion-reduction** of X is a $(3, 1)$ -polygraph, obtained as a homotopical reduction, with respect to a collapsible part, of a homotopical completion $\mathcal{H}(X)$ of X . Proposition 2.1.1 and Theorem 2.2.3 imply the following result.

Theorem 2.3.1. *Assume that X is a terminating 2-polygraph presenting a category \mathcal{C} . Then, every homotopical completion-reduction of X is a coherent presentation of \mathcal{C} .*

Example 2.3.2. Let us perform the homotopical reduction procedure on the homotopical completion $\mathcal{H}(K^+)$ of the Klein bottle monoid, computed in Example 2.2.5. We construct a collapsible part $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$. There is only one generating triple confluence (we display the 3-cells A and B differently now, to make the generating triple confluence more evident):



Hence the component Γ_4 of the collapsible part contains the 3-sphere Φ which has the 3-cell B as target (recall that we implicitly perform a higher Nielsen transformation when needed). By the definition of a collapsible part, we also need to provide a well-founded order relation on the set of generating 3-cells, such that, for every 3-sphere (X, Y) in Γ_4 , the target Y is strictly greater than every generating 3-cell that occurs in the source X . So, we put $B > A$. Proceeding as described in Subsection 2.3, we examine the remaining 3-cells and construct the component Γ_3 out of those 3-cells whose boundary contains a generating 2-cell occurring only once in the boundary. There is only one 3-cell left, namely A , and the 2-cell β appears only once in the boundary of A . So, Γ_3 contains A , and we order the set of generating 2-cells by setting $\beta > \alpha$. The component Γ_2 is empty because there

is no 2-cell whose source or target consists of a single generating 1-cell appearing only once.

Thus, after performing the homotopical reduction procedure with respect to the collapsible part $(\emptyset, \Gamma_3, \Gamma_4)$, we are left with the presentation

$$\left(a, b \mid bab \xrightarrow{\alpha} a \mid \emptyset \right)$$

which is thus coherent by Theorem 2.3.1. Note that having a coherent presentation X with the empty set of generating 3-cells means that any two parallel rewriting paths represent the same 2-cell in X_3^\top .

3. GARSIDE'S COHERENT PRESENTATIONS OF ARTIN-TITS AND GARSIDE MONOIDS

In this section, we recall the applications of the homotopical completion-reduction procedure for computing coherent presentations of Artin-Tits monoids and Garside monoids. Note that the homotopical completion-reduction procedure has also been applied to other examples (see [17] for plactic and Chinese monoids).

3.1. Divisibility in monoids. A monoid M is **left-cancellative** (resp. **right-cancellative**) if for all f, g and g' of M , the equality $fg = fg'$ (resp. $gf = g'f$) implies the equality $g = g'$. A monoid is **cancellative** if it is both left-cancellative and right-cancellative.

An element f of a monoid M is said to be a **left divisor** of $g \in M$, and g is said to be a **right multiple** of f , denoted by $f \preceq g$, if there is an element $f' \in M$ such that $ff' = g$. The set of all left divisors of g is denoted by $\text{Div}(g)$. We say that f is a proper left divisor of g , written as $f \prec g$, if $f \preceq g$ and $g \not\preceq f$. If M is left-cancellative, then the element f' is uniquely defined and called **the right complement of f in g** .

For an element h of a left-cancellative monoid M and a subfamily S of M , we say that h is a **left-gcd** (resp. **right-lcm**) of S if $h \preceq s$ (resp. $s \preceq h$) holds for all $s \in S$ and if every element of M which is a left divisor (resp. right multiple) of all $s \in S$ is also a left divisor (resp. right multiple) of h .

A (proper) right divisor, a left multiple, a left complement, a left-lcm and a right-gcd are defined similarly.

We say that a left-cancellative monoid M **admits conditional right-lcms** if any two elements having a common right multiple have a right-lcm.

3.2. Coxeter groups and Artin-Tits monoids. For elements s, t of a set S , the element $\langle st \rangle^m$ of the free monoid S^* is defined inductively by $\langle st \rangle^0 = 1$ and $\langle st \rangle^{m+1} = s \langle ts \rangle^m$. A **Coxeter group** W is a group generated by a finite S , and subject to relations

$$(3.1) \quad \{s^2 = 1, \langle st \rangle^{m_{st}} = \langle ts \rangle^{m_{st}} \mid s, t \in S\},$$

such that $(m_{st})_{s, t \in S}$ is a symmetric matrix whose entries are positive integers or ∞ (here, $m_{st} = \infty$ means that there is no given relation between s and t), and whose diagonal entries are all equal to 1. Note that a Coxeter group can admit several generating sets, but we always assume that such a set is fixed and totally ordered. The relations in (3.1), other than the involutions, are called **braid relations**. The **Artin-Tits monoid** $B^+(W)$ corresponding to a Coxeter group W is the monoid presented by

$$B^+(W) = \langle S \mid \{ \langle st \rangle^{m_{st}} = \langle ts \rangle^{m_{st}} \mid s, t \in S \} \rangle.$$

This presentation is called **Artin's presentation** of an Artin-Tits monoid. Since relations in $B^+(W)$ are homogeneous, one can define the length of an element in $B^+(W)$. By Matsumoto's theorem ([3, Corollary IX.1.11] or [13, Theorem 1.2.2]),

two minimal-length expressions (as products of elements of S) for an element in W can be obtained from one another by braid relations alone; this defines a length function $\ell : W \rightarrow \mathbb{N}$. Hence, the canonical projection map $\pi : B^+(W) \rightarrow W$ has a canonical set-theoretic section $\sigma : W \rightarrow B^+(W)$. The image of σ consists of all elements whose length is preserved by π . We often identify elements of $\text{im}(\sigma)$ with their images under π , to simplify notation. An Artin-Tits monoid is called **spherical** if its corresponding Coxeter group is finite.

Artin-Tits monoids are cancellative (see [1, Proposition 2.3]). Furthermore, if we regard $v \in W$ as an element of $B^+(W)$, by the canonical embedding provided by Matsumoto's theorem, then an equality $uu' = v$ in $B^+(W)$ implies that both u and u' are elements of W , and hence that the condition $\ell(uu') = \ell(u) + \ell(u')$ holds.

Artin-Tits monoids admit conditional right-lcms (see [1, Proposition 4.1]).

In a presentation of a monoid, a relation $u = v$ is called an ε -**relation** if one of the paths u, v is empty and the other is not. By [3, Lemma II.1.42], if a presentation of a monoid M contains no ε -relation, then M contains no nontrivial invertible elements. Note that a braid relation is not an ε -relation. Hence, an Artin-Tits monoid contains no nontrivial invertible element.

Example 3.2.1. Every free abelian monoid on a finite set is an Artin-Tits monoid because commutativity relations are braid relations; it is spherical because, if we impose involutions upon it, we get a finite Coxeter group.

The **braid monoid** B_n^+ is the monoid

$$(3.2) \quad B_n^+ \simeq \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, |i-j| \geq 2; \sigma_i \sigma_j = \sigma_j \sigma_i, |i-j| = 1 \rangle^+.$$

By definition, braid monoids are Artin-Tits monoids. The Coxeter group which corresponds to the braid monoid B_n^+ is the symmetric group S_n . So, braid monoids are spherical Artin-Tits monoids.

3.3. Garside's presentation of Artin-Tits monoids. The monoid $B^+(W)$ admits another presentation (see [13, Proposition 4.1.3]), where generators are elements of $W \setminus 1$, and relations are $u|v = uv$ whenever $\pi(u)\pi(v) = \pi(uv)$ and $\ell(\pi(u)) + \ell(\pi(v)) = \ell(\pi(uv))$ hold in W . Here, $u|v$ denotes product in W^* , the free monoid over W , whereas uv denotes product in W . This presentation is called **Garside's presentation** (see [22, Section 1]).

For all u and v in a Coxeter group W , the inequality $\ell(uv) \leq \ell(u) + \ell(v)$ holds. We recall the graphical notation which is introduced in [11, 3.1.1] to indicate whether the equality holds or not:

$$\begin{aligned} u \widehat{v} &\iff \ell(uv) = \ell(u) + \ell(v), \\ u \times v &\iff \ell(uv) < \ell(u) + \ell(v). \end{aligned}$$

The notation is generalised for a greater number of elements of W . For instance, in the case of three elements u, v and w of W , we write $u \widehat{v} \widehat{w}$ if both equalities $\ell(uv) = \ell(u) + \ell(v)$ and $\ell(vw) = \ell(v) + \ell(w)$ hold. This case splits into two disjoint subcases:

$$\begin{aligned} u \widehat{v} \widehat{w} &\iff \left(u \widehat{v} \widehat{w} \text{ and } \ell(uvw) = \ell(u) + \ell(v) + \ell(w) \right), \\ u \widehat{v} \times \widehat{w} &\iff \left(u \widehat{v} \widehat{w} \text{ and } \ell(uvw) < \ell(u) + \ell(v) + \ell(w) \right). \end{aligned}$$

Thus, Garside's presentation of the Artin-Tits monoid $B^+(W)$, seen as a 2-polygraph and denoted by $\text{Gar}_2(W)$, has a single generating 0-cell, elements of $W \setminus \{1\}$ as

generating 1-cells, and a generating 2-cell

$$\alpha_{u,v} : u|v \Rightarrow uv$$

for all $u, v \in W \setminus \{1\}$ such that $u \widehat{\ } v$ holds.

Note that the number of generators in Garside's presentation of an Artin-Tits monoid is finite if and only if the Artin-Tits monoid is spherical.

3.4. Garside's coherent presentation of Artin-Tits monoids. We fix a Coxeter group W , and we recall the particular application of the homotopical completion-reduction procedure, which is performed in [11, Section 3] and which results in a coherent presentation of the Artin-Tits monoid $B^+(W)$.

Let $\text{Gar}_3(W)$ denote the extended presentation of $B^+(W)$ obtained by adjoining to $\text{Gar}_2(W)$ a 3-cell

$$\begin{array}{ccccc} & & uv|w & & \\ & \nearrow \alpha_{u,v}|w & & \searrow \alpha_{uv,w} & \\ u|v|w & & A_{u,v,w} & & uvw \\ & \searrow \alpha_{u,v,w} & & \nearrow \alpha_{u,vw} & \\ u|\alpha_{v,w} & & u|vw & & \end{array}$$

for all u, v and w of $W \setminus \{1\}$ such that $u \widehat{\ } v \widehat{\ } w$.

Theorem 3.4.1 ([11, Theorem 3.1.3]). *For every Coxeter group W , the Artin-Tits monoid $B^+(W)$ admits $\text{Gar}_3(W)$ as a coherent presentation.*

Proof. Here is a brief overview of the proof. The main steps are: establishing termination, computing homotopical completion, computing homotopical reduction.

Firstly, one needs to establish that the 2-polygraph $\text{Gar}_2(W)$ is terminating. A strict order $<$ on W^* is obtained, thanks to favourable properties of Artin-Tits monoids, as follows: compare the lengths of elements of W^* , then the lengths of their components as elements of W starting from the right. By adding the diagonal to $<$, an order \leq is generated on $\text{Gar}_2(W)$, such that for every generating 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$, the strict inequality $s(\alpha_{u,v}) > t(\alpha_{u,v})$ holds. The relation \leq is well-founded and compatible with compositions. Therefore, the 2-polygraph $\text{Gar}_2(W)$ is terminating, so a homotopical completion-reduction of $\text{Gar}_2(W)$ can be computed following Remark 2.2.2.

We refer the reader to [11, Proposition 3.2.1] for elaboration, and we only give a sketch here. The 2-polygraph $\text{Gar}_2(W)$ has one critical branching for all u, v and w of $W \setminus \{1\}$ satisfying $u \widehat{\ } v \widehat{\ } w$. If $u \widehat{\ } v \widehat{\ } w$, the branching is already confluent, so only the generating 3-cell $A_{u,v,w}$ is adjoined. If $u \widehat{\ } v^x \widehat{\ } w$, on the other hand, a generating 2-cell is introduced to achieve confluence, and another generating 3-cell is adjoined for coherence. The added 2-cell causes new critical branchings, all of which are confluent. Seven families of generating 3-cells are adjoined. We do not give details here, as the cells also appear (in a more general context) in the proof of Proposition 5.3.1.

Lastly, the homotopical reduction procedure is applied to the computed homotopical completion of $\text{Gar}_2(W)$. It removes all the added families of generating cells, except for the generating 3-cells A , thus leaving precisely the $(3, 1)$ -polygraph $\text{Gar}_3(W)$. We refer the reader to [11, 3.2.2] for details.

By Theorem 2.3.1, the presentation $\text{Gar}_3(W)$ is then coherent. \square

The $(3, 1)$ -polygraph $\text{Gar}_3(W)$ is called the **Garside's coherent presentation of the Artin-Tits monoid $B^+(W)$** .

3.5. Garside monoids. Let \mathcal{C} be a left-cancellative category. Observe that a left-gcd of a subfamily S of \mathcal{C} is unique if, and only if, there are no nontrivial invertible elements in \mathcal{C} . For elements f and g of \mathcal{C} , we denote the left-gcd of $\{f, g\}$ by $f \wedge g$.

A **Garside monoid** (see [3, Definition I.2.1]) is a pair (M, Δ) such that the following conditions hold:

- M is a cancellative monoid;
- there is a map $\lambda : M \rightarrow \mathbb{N}$ such that $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $\lambda(f) = 0 \implies f = 1$;
- every two elements have a left-gcd and a right-gcd and a left-lcm and a right-lcm;
- $\Delta \in M$ is such that the left and the right divisors of Δ coincide, and they generate M ;
- the family of all divisors of Δ is finite.

Observe that the second condition implies that M has no nontrivial (here meaning different from 1) invertible element: $\lambda(1) = \lambda(1 \cdot 1) \geq \lambda(1) + \lambda(1)$ implies $\lambda(1) = 0$, whence $fg = 1$ implies $0 \geq \lambda(f) + \lambda(g)$ which further implies $f = g = 1$. Consequently, a left-gcd is unique in a Garside monoid.

The element Δ is called the **Garside element**. We write $\partial(u)$ for the right complement of an element u of $\text{Div}(\Delta)$ in Δ .

Example 3.5.1. The class of Garside monoids includes spherical Artin-Tits monoids (see [7]) and, consequently, braid monoids (see [1]). In the free abelian monoid $(\mathbb{N}^n, +)$, the Garside element is $\Delta_n = (1, \dots, 1)$. For the braid monoid B_n^+ , the Garside element Δ_n (called the half-turn braid) is defined inductively: $\Delta_1 = 1$, $\Delta_2 = \sigma_1$, $\Delta_n = \Delta_{n-1}\sigma_{n-1} \cdots \sigma_1$ for $n \geq 2$ (elements of $\text{Div}(\Delta)$ are called simple braids). For more examples, see [2, Example 2.3] (dual braid monoids) and [3, Examples I.2.7 and I.2.8].

In a Garside monoid (M, Δ) , a $\text{Div}(\Delta)$ -word $s_1 | \cdots | s_p$ is said to be **Δ -normal** if for every $i < p$,

$$(3.3) \quad \forall g \in M \setminus \{1\}, (g \preceq s_{i+1} \implies s_i g \not\preceq \Delta),$$

and it is said to be **strict** if additionally $s_p \neq 1$ holds. Note that the strict Δ -normal decomposition of 1 is the empty word. The condition (3.3) says that s_i is the maximal element of $\text{Div}(\Delta)$ which left divides $s_i s_{i+1} \cdots s_p$ (hence this decomposition is called **greedy**, a term we formally recall in Subsection 4.2), as illustrated by the following simple example.

Example 3.5.2. Let us compute the strict Δ_3 -normal decomposition of the element $g = (2, 1, 3)$ of the free abelian monoid \mathbb{N}^3 . The maximal element of $\text{Div}(\Delta)$ that divides g is Δ_3 . The division gives $(1, 0, 2)$. The maximal element of $\text{Div}(\Delta)$ that divides $(1, 0, 2)$ is $(1, 0, 1)$. The division gives $(0, 0, 1)$ which is in $\text{Div}(\Delta)$, thus the decomposition is complete. If we denote generators by a , b and c , we get $a^2bc^3 = abc|ac|c$.

The main idea is that the finite lattice $\text{Div}(\Delta)$ reveals the structure of the entire Garside monoid, as expressed in the following result.

Proposition 3.5.3 ([3, Proposition I.2.4], [2, Proposition 2.4]). *Let (M, Δ) be a Garside monoid. Every $f \in M$ admits a unique strict Δ -normal decomposition.*

We refer the reader to [2, Subsection 2.2] for an overview of the computational properties of the Δ -normal form.

A Δ -normal decomposition is considered in a broader context in Subsection 4.2. Here it suffices to examine the case where $f \in M$ is represented by a length-two $\text{Div}(\Delta)$ -word $u|v$ that is not Δ -normal. Observe that, in the length-two case, the

condition (3.3) translates into $\partial(u) \wedge v = 1$. Indeed, $u|v$ is Δ -normal if, and only if, there is no nontrivial left divisor g of v such that ug is in $\text{Div}(\Delta)$ (or, equivalently, $g \preceq \partial(u)$). Hence, a procedure can be established to find a Δ -normal decomposition of f , as follows. If there is a nontrivial left divisor g of v such that $g \preceq \partial(u)$, then $\partial(u) \wedge v$ exists by the definition of Garside monoid. We can obtain in a single step a Δ -normal form of f : replace u and v respectively by $u' = u(\partial(u) \wedge v)$ and the right complement v' of $\partial(u) \wedge v$ in v . Thus we have rewritten $u|v$ to $u'|v'$, which is Δ -normal. This step is called local sliding in [12, Subsection 2.2] as $(\partial(u) \wedge v)$ slides from v to u .

For the strict Δ -normal decomposition, one further distinguishes two cases:

- $\partial(u) \wedge v = v$, in which case $v' = 1$, and $u' = uv$ is returned as a length-one strict Δ -normal decomposition of f ;
- $\partial(u) \wedge v$ properly divides v , in which case $u'|v'$ is a strict Δ -normal decomposition of f .

3.6. Garside's coherent presentation of Garside monoids. For $u, v \in \text{Div}(\Delta)$, we set

$$\begin{aligned} u \widehat{\wedge} v &\iff \partial(u) \wedge v = v, \\ u \times v &\iff \partial(u) \wedge v \neq v. \end{aligned}$$

Note that there is a typo in [11, 3.3.2], where it is (wrongly) written $\partial(u) \wedge v = 1$ and $\partial(u) \wedge v \neq 1$, respectively.

Observe that the conditions $\partial(u) \wedge v = v$ and $\partial(u) \wedge v \neq v$ are equivalent to $uv \in \text{Div}(\Delta)$ and $uv \notin \text{Div}(\Delta)$, respectively. For three elements $u, v, w \in S$, the case when both conditions $uv \in \text{Div}(\Delta)$ and $vw \in \text{Div}(\Delta)$ hold is denoted by $u \widehat{\wedge} v \widehat{\wedge} w$. This case splits into two disjoint subcases:

$$\begin{aligned} u \widehat{\wedge} v \widehat{\wedge} w &\iff \left(u \widehat{\wedge} v \widehat{\wedge} w \text{ and } uvw \in \text{Div}(\Delta) \right), \\ u \widehat{\wedge} v \times \widehat{\wedge} w &\iff \left(u \widehat{\wedge} v \widehat{\wedge} w \text{ and } uvw \notin \text{Div}(\Delta) \right). \end{aligned}$$

Garside's presentation of a Garside monoid M is the 2-polygraph $\text{Gar}_2(M)$, having elements of $\text{Div}(\Delta) \setminus \{1\}$ as generating 1-cells and a generating 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$ whenever the condition $u \widehat{\wedge} v$ holds. It is easily verified that $\text{Gar}_2(M)$ is, indeed, a presentation of M , because uv is the strict (we leave out the 1 from $uv|1$) Δ -normal form of $u|v$ if, and only if, $u \widehat{\wedge} v$ holds. A (3, 1)-polygraph $\text{Gar}_3(M)$ is defined analogously to the one defined in Subsection 3.4.

The proof of Theorem 3.4.1 adapts in a straightforward manner to give the following result.

Theorem 3.6.1 ([11, Theorem 3.3.3]). *Every Garside monoid M admits $\text{Gar}_3(M)$ as a coherent presentation.*

4. GARSIDE FAMILIES

This section briefly recollects the basic notions and results concerning Garside families (for technical elaboration, see the book [3]).

4.1. Right-mcms. Let M be a left-cancellative monoid, and S a subfamily of M . The left divisibility relation \preceq is a preorder of elements; it is an order if, and only if, M has no nontrivial invertible element.

A subfamily S of a left-cancellative monoid M is **closed under right comultiple** if every common right multiple of two elements f and g of S (if there is any) is a right multiple of a common right multiple of f and g that lies in S .

For f and g in a monoid M , a minimal common right multiple, or **right-mcm**, of f and g is a right multiple h of f and g , such that no proper left divisor of h is a common right multiple of f and g . A monoid M **admits right-mcms** if, for all f and g of M , every common right multiple of f and g is a right multiple of some right-mcm of f and g . Observe that in a monoid admitting conditional right-lcms, the notions of a right-mcm and right-lcm coincide. Let us state a rather basic observation about right-mcm in a left-cancellative monoid, which we use in one step of the main proof in Subsection 5.3. The following lemma is similar to [18, Lemma 11.24], which deals with lcms whereas here it suffices to consider mcms (under weaker assumptions).

Lemma 4.1.1. *Assume that M is a left-cancellative monoid. If v' is a right-mcm of v_1 and v_2 in M , then uv' is a right-mcm of uv_1 and uv_2 for every u in M .*

Proof. Let x_1 and x_2 denote the right complements in v' of v_1 and v_2 , respectively. Note that uv' is a common right multiple of uv_1 and uv_2 , namely, $uv_1x_1 = uv' = uv_2x_2$. Assume that there is a common right multiple f of uv_1 and uv_2 such that f properly left divides uv' . Then there exist elements f_1, f_2 and a non-invertible element g in M satisfying $uv_1f_1 = f = uv_2f_2$ and $fg = uv'$. By the left cancellation property, the equations $uv_kf_kg = uv'$ yield $v_kf_kg = v'$ for $k \in \{1, 2\}$. Thus we obtain the common right multiple $v_1f_1 = v_2f_2$ of v_1 and v_2 , which properly left divides v' . This contradicts the fact that v' is a right-mcm of v_1 and v_2 . Therefore, uv' is a right-mcm of uv_1 and uv_2 . \square

Following [3, Propositions II.2.28 and II.2.29], a left-cancellative monoid M is said to be **left-noetherian** (resp. **right-noetherian**) if for every g in M , every strictly increasing sequence of right (resp. left) divisors of g with respect to the right divisibility (resp. left divisibility) is finite. A left-cancellative monoid M is **noetherian** if it is both left-noetherian and right-noetherian.

Example 4.1.2. Proper division, left or right, strictly reduces the length of an element of an Artin-Tits monoid. Therefore, no element admits an infinite number of divisors, so Artin-Tits monoids are noetherian.

Garside monoids are noetherian by definition (thanks to the map $\lambda : M \rightarrow \mathbb{N}$).

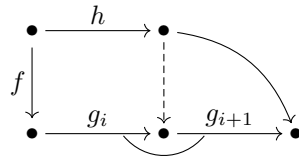
Proposition 4.1.3 ([3, Propositions II.2.40]). *Every left-cancellative left-noetherian monoid admits right-mcms.*

4.2. Notion of a Garside family. In this subsection, we recollect the definition and some basic properties of the all-important notion of a Garside family which provides a way of extending the notion of a greedy decomposition beyond Garside monoids.

Given a subfamily S of a left-cancellative monoid M , an M -word $g_1 | \cdots | g_q$ is said to be **S -greedy** if for all $i < q$,

$$\forall h \in S, \forall f \in M, (h \preceq fg_i g_{i+1} \implies h \preceq fg_i).$$

In other words, if the diagram



commutes without the dashed arrow, then there exists a dashed arrow making the square on the left commute. The arc joining g_i and g_{i+1} denotes greediness.

By definition, a word of length zero or one is S -greedy for any subfamily S . Given a subfamily S of M , an M -word $g_1 | \cdots | g_q$ is said to be **S -normal** if it is S -greedy and if, moreover, g_1, \dots, g_q all lie in S . An S -normal word $g_1 | \cdots | g_q$ is **strict** if $g_q \neq 1$. Observe that the existence of an S -normal form implies the existence of a strict one.

Note that, by the very definition of being greedy, a word is normal if, and only if, its length-two factors are. More is true: the procedure of transforming a word into its normal form consists of transforming its length-two factors (we refer the reader to [6] for elaboration).

In general, an **S -normal** decomposition of an element g of M is not unique. Nevertheless, the number of non-invertible letters in all S -normal decompositions of g is the same (see [4, Proposition 2.11] or [3, Proposition III.1.25] for exposition). If M has no nontrivial invertible element, then every g in M admits at most one strict S -normal decomposition. Given a subfamily S of a left-cancellative monoid M , and an element g of M admitting at least one S -normal decomposition, one defines the **S -length** of an element $g \in M$, written as $\|g\|_S$, to be the common number of non-invertible letters in all S -normal decompositions of g .

A subfamily S of a left-cancellative monoid M is called a **Garside family** in M if every element of M admits an S -normal decomposition. Observe that every left-cancellative monoid M is a Garside family in itself (for every g in M , simply take a length-one word g as a M -normal decomposition of g), so we are interested only in proper (meaning other than M itself) Garside families.

Example 4.2.1. Every Artin-Tits monoid admits a finite Garside family. In the case of a spherical Artin-Tits monoid, a finite Garside family is given by the corresponding Coxeter group. In the particular case of a braid monoid, the family of all simple braids is a Garside family.

The Coxeter group W which corresponds to a general Artin-Tits monoid $B^+(W)$ is a possibly infinite Garside family, but $B^+(W)$ admits a finite Garside family in any case (see [5]).

Any Garside monoid (M, Δ) has a finite Garside family given by $\text{Div}(\Delta)$ (see [4, Proposition 2.18] or [3, Proposition III.1.43]).

The following proposition gives a simple characterisation of a Garside family.

Proposition 4.2.2 ([4, Proposition 3.1] or [3, Proposition III.1.39]). *A subfamily S of a monoid M containing no nontrivial invertible element is a Garside family if, and only if, the following conjunction holds: S generates M and every element of S^2 admits an S -normal decomposition.*

Let us recall another characterisation of Garside family, one direction whereof we invoke in Subsection 5.3. More characterisations of Garside families can be found in [4, Subsection 3.2] or in [3, Subsection IV.1.2].

Proposition 4.2.3 ([4, Proposition 3.9]). *A family S of a left-cancellative monoid M containing no nontrivial invertible element is a Garside family if, and only if, the following conditions are satisfied: S generates M , it is closed under right comultiple and right divisor, and every non-invertible element of S^2 admits a \prec -maximal left divisor in S .*

We recall another result to be used in Subsection 5.3.

Lemma 4.2.4 ([3, Lemma IV.2.24]). *Assume that M is a left-cancellative monoid that contains no nontrivial invertible element and admits right-mcmts. Then for every subfamily S of M , the following are equivalent.*

- *The family S is closed under right comultiple.*

- The family S is **closed under right-mcm**, i.e. if f and g lie in S , then so does every right-mcm of f and g .

Given a Garside family S in a left-cancellative monoid with no nontrivial invertible element, the **normalisation map** $N^S : S^* \rightarrow S^*$ is the map which assigns to each element of $w \in S^* \setminus \{1\}$ the strict S -normal decomposition of the element represented by w ; and $N^S(1) = 1$. The following result provides an important property of S -normal decomposition. We use this property in Subsection 5.2.

Lemma 4.2.5 ([6, Lemma 6.9]). *Assume that M is a left-cancellative monoid having no nontrivial invertible element, and S is a Garside family in M . The normalisation map N^S derived from the Garside family S is **left-weighted**, i.e. for all s and t in S , the element s is a left divisor in M of the first letter of $N^S(s|t)$.*

A Garside family yields a presentation in the following sense.

Proposition 4.2.6 ([6, Proposition 6.17] or [14, Corollary 6.6.4]). *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family. Then M admits, as a convergent presentation, the 2-polygraph $(\{\bullet\}, S \setminus \{1\}, X_S)$, where $\{\bullet\}$ denotes a singleton and X_S is the set of generating 2-cells of the form*

$$(4.1) \quad s|t \Rightarrow N^S(s|t)$$

for all s and t in $S \setminus \{1\}$ such that $s|t$ is not S -normal. In particular, every Artin-Tits monoid admits a finite convergent presentation.

A Garside family also induces a “smaller” presentation, beside the one provided by Proposition 4.2.6, which will be instrumental in deriving our main result in the next section. The following proposition is an adaptation of [6, Proposition 6.15] using [6, Proposition 6.10].

Proposition 4.2.7. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family containing 1. Then M admits, as a presentation, the 2-polygraph $\text{Gar}_2(S)$ which contains a single generating 0-cell, one generating 1-cell for every element of $S \setminus \{1\}$, and one generating 2-cell of the form*

$$(4.2) \quad s|t \Rightarrow st,$$

for all s and t in $S \setminus \{1\}$ whose product st in M lies in S .

Proof. Proposition 4.2.6 grants a presentation of M in terms of S by the relations 4.1. Let us show that the relations (4.2) are included in the relations (4.1). If s and t in $S \setminus \{1\}$ are such that st lies in $S \setminus \{1\}$, then the strict S -decomposition of $s|t$ is st , hence $N^S(s|t) = st$. Otherwise, $st = 1$ holds and yields $N^S(s|t) = 1$. In both cases, the strict S -normal decomposition of $s|t$ is st . Hence, the relations (4.2) are included in the relations (4.1).

Conversely, let us show that each relation (4.1), with s and t in $S \setminus \{1\}$, follows from a finite number of relations (4.2). Assume that s and t lie in $S \setminus \{1\}$ and let $s'|t' := N^S(s|t)$. If $t' = 1$ holds, it implies $s' = st$, which is a (4.2) relation, so the result is true in this case. Otherwise, Lemma 4.2.5 implies that there exists r in M , satisfying $sr = s'$, which is a (4.2) relation. Being a right divisor of $s' \in S$, the element r also lies in S by Proposition 4.2.3. Multiplying the equality $sr = s'$ by t' on the right yields $srt' = s't' = st$. Then the left cancellation property of M implies $rt' = t$, which is a (4.2) relation. Since the relation $s|t = s|r|t' = s'|t'$ follows from the (4.2) relations $s|r = s'$ and $r|t' = t$, the result is true in this case, too. \square

We call the 2-polygraph $\text{Gar}_2(S)$ the **Garside's presentation** of M , with respect to Garside family S . We study it in the next section. Here, let us just observe that it extends the Garside's presentation of Artin-Tits monoids, recalled in Subsection 3.3.

Example 4.2.8. Garside's presentation $\text{Gar}_2(W)$ of an Artin-Tits monoid $B^+(W)$ is an instance of a Garside's presentation with respect to Garside family S . Indeed, the Artin-Tits monoid $B^+(W)$ is a cancellative monoid with no nontrivial invertible element, and the Coxeter group W is a Garside family containing 1; hence $B^+(W)$ meets all the requirements of Proposition 4.2.7 which, for this particular input, produces precisely Garside's presentation $\text{Gar}_2(W)$.

5. COHERENT PRESENTATIONS FROM GARSIDE FAMILIES

Having recalled necessary notions and results in previous sections, in this section we aim to state and prove Theorem 5.1.4 which provides a unifying generalisation of Theorems 3.4.1 and 3.6.1

5.1. Main statement and sketch of proof. In this subsection, we establish some notation and define a noetherianity condition. Then we state our main result.

Let M be a monoid generated by a set S containing 1. We redefine the notations $u \widehat{v}$ and $u^\times v$, as follows. Given two elements u and v of $S \setminus \{1\}$, we define:

$$\begin{aligned} u \widehat{v} &\iff uv \in S, \\ u^\times v &\iff uv \notin S. \end{aligned}$$

The notation extends to a greater number of elements, like in Subsection 3.6. So, for three elements $u, v, w \in S$, we write $u \widehat{v} \widehat{w}$ if both conditions $uv \in S$ and $vw \in S$ hold. The condition $u \widehat{v} \widehat{w}$ splits into two mutually exclusive subcases:

$$\begin{aligned} u \widehat{v} \widehat{w} &\iff (u \widehat{v} \widehat{w} \text{ and } uvw \in S), \\ u^\times \widehat{v} \widehat{w} &\iff (u \widehat{v} \widehat{w} \text{ and } uvw \notin S). \end{aligned}$$

We formally redefine symbols Gar_2 and Gar_3 in our general context as follows. The 2-polygraph $\text{Gar}_2(S)$ contains: a single generating 0-cell; one generating 1-cell for every element of $S \setminus \{1\}$; one generating 2-cell of the form

$$\alpha_{u,v} : u|v \Rightarrow uv,$$

for all u and v in $S \setminus \{1\}$ such that $u \widehat{v}$ holds. Here, $u|v$ denotes product in S^* , whereas uv denotes product in M . The (3,1)-polygraph $\text{Gar}_3(S)$ is consisting of the polygraph $\text{Gar}_2(S)$ and the generating 3-cells of the form

$$\begin{array}{ccccc} & & uv|w & & \\ & \nearrow \alpha_{u,v}|w & & \searrow \alpha_{uv,w} & \\ u|v|w & & A_{u,v,w} & & uvw \\ & \searrow u|\alpha_{v,w} & & \nearrow \alpha_{u,vw} & \\ & & u|vw & & \end{array}$$

for all u, v and w in $S \setminus \{1\}$ such that $u \widehat{v} \widehat{w}$.

Remark 5.1.1. Note that the 2-polygraph $\text{Gar}_2(S)$ is not a presentation of M , in general. Consequently, since $\text{Gar}_3(S)$ an extended presentation of a monoid presented by $\text{Gar}_2(S)$, it is not necessarily an extended presentation of M . Proposition 4.2.7 gives sufficient conditions for $\text{Gar}_2(S)$ to be a presentation of M , thus making $\text{Gar}_3(S)$ an extended presentation of M .

Definition 5.1.2. We say that a subfamily S of a monoid M is **locally right-noetherian** if there exists no $g \in S$ admitting an infinite sequence $(h_n)_{n=1}^\infty$ in $S \cap \text{Div}(g)$ such that for every n there exists a non-invertible f_n in S satisfying $h_n f_n = h_{n+1}$.

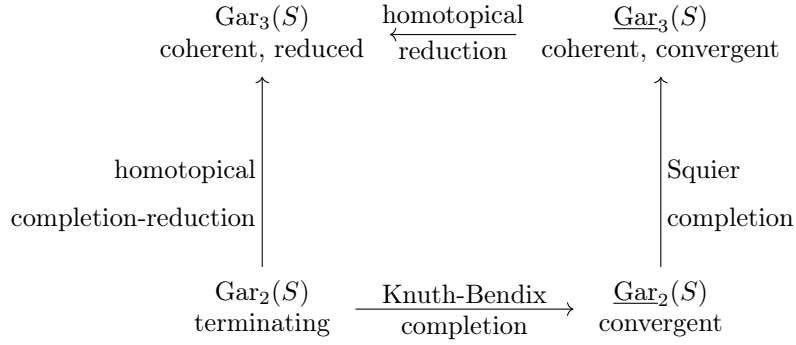
Note that we are using terminology from [3, Definition IV.2.16] for a slightly different definition.

Example 5.1.3. Every subfamily of a right-noetherian monoid is locally right-noetherian.

Now, we state the main result.

Theorem 5.1.4. *Assume that M is a left-cancellative monoid containing no non-trivial invertible element, and $S \subseteq M$ is a locally right-noetherian Garside family containing 1. If M admits right-mcms, then M admits the $(3, 1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation.*

The following diagram summarises key steps of the proof and thus motivates the next three subsections (which together contain the proof).



In Subsection 5.2, starting with the Garside's presentation $\text{Gar}_2(S)$ of M , we add the generating 2-cells β which results in a terminating presentation $\underline{\text{Gar}}_2(S)$. This is, in fact, a convergent presentation, namely a Knuth-Bendix completion of $\text{Gar}_2(S)$, but we do not prove it until Subsection 5.3. Nevertheless, this hindsight prompts us to begin Subsection 5.2 with a formal definition of the 2-polygraph $\underline{\text{Gar}}_2(S)$.

In Subsection 5.3, first we formally compute a Squier completion of the polygraph $\underline{\text{Gar}}_2(S)$, under certain assumptions on monoid. We denote the resulting $(3, 1)$ -polygraph by $\underline{\text{Gar}}_3(S)$. Then we show that this construction applies to a terminating presentation $\underline{\text{Gar}}_2(S)$ of M and produces a coherent convergent presentation $\underline{\text{Gar}}_3(S)$.

Finally, in Subsection 5.4, we compute a homotopical reduction of $\underline{\text{Gar}}_3(S)$ to obtain the $(3, 1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation of M .

Remark 5.1.5. Observe that the definition of the notations $u \widehat{\ } v$ and $u^\times v$ given in the beginning of the current subsection, generalises the corresponding definition of the length notation for Artin-Tits monoids recalled in Subsection 3.3, as well as the corresponding definition for Garside monoids recalled in Subsection 3.6.

Recall that Artin-Tits monoids and Garside monoids are cancellative and noetherian (Example 4.1.2), and that they contain no nontrivial invertible element.

Consequently, Theorem 5.1.4 specialises to Theorem 3.4.1 or Theorem 3.6.1 whenever a monoid considered is Artin-Tits or Garside, respectively. Therefore, Theorem 5.1.4 gives a common generalisation of the two distinct directions of extension, given in [11], of Deligne’s result [8, Theorem 1.5].

5.2. Attaining termination. In this subsection, we ensure that a certain presentation, denoted $\underline{\text{Gar}}_2(S)$, is terminating. This presentation arises naturally as a result of applying the Knuth-Bendix completion to the Garside’s presentation $\text{Gar}_2(S)$. Hence the motivation for the formal definition of the 2-polygraph $\underline{\text{Gar}}_2(S)$.

Let M be a monoid generated by a set S containing 1. Observe that the 2-polygraph $\text{Gar}_2(S)$ has exactly one critical branching for all u, v and w of $S \setminus \{1\}$ such that $u \frown v \frown w$ holds:

$$uv|w \xleftarrow{\alpha_{u,v}|w} u|v|w \xrightarrow{u|\alpha_{v,w}} u|vw.$$

If the subcase $u \frown v \frown w$ holds, then the branching is already confluent. Otherwise $u \overset{x}{\frown} v \frown w$ holds, and the branching requires a new generating 2-cell to reach confluence, so the generating 2-cell $\beta_{u,v,w} : u|vw \Rightarrow uv|w$ is adjoined. We write $\underline{\text{Gar}}_2(S)$ for the 2-polygraph which contains a single generating 0-cell, one generating 1-cell for every element of $S \setminus \{1\}$, the generating 2-cells

$$\begin{aligned} \alpha_{u,v} : u|v &\Rightarrow uv, & u, v &\in S \setminus \{1\}, \quad u \frown v, \\ \beta_{u,v,w} : u|vw &\Rightarrow uv|w, & u, v, w &\in S \setminus \{1\}, \quad u \overset{x}{\frown} v \frown w, \end{aligned}$$

To show that the 2-polygraph $\underline{\text{Gar}}_2(S)$, under certain conditions, is a Knuth-Bendix completion of the 2-polygraph $\text{Gar}_2(S)$, we need to ensure two things: a way to maintain a terminating presentation in the sense of Remark 2.2.2, and a demonstration that all new critical branchings caused by the generating 2-cells β are confluent. These are respectively given by Proposition 5.2.1, and the proof of Proposition 5.3.1.

For an element u of S^* , where S is a set, we use the following notations: $\ell(u)$, the S -length of u ; $h(u)$, the leftmost letter of u ; $t(u)$, the word obtained by removing the letter $h(u)$ from u .

Proposition 5.2.1. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a locally right-noetherian Garside family containing 1. Then the 2-polygraph $\underline{\text{Gar}}_2(S)$ is terminating.*

Proof. Let us first adopt some notations. For a generating 2-cell χ , a χ -step is a rewriting step in which the generating 2-cell involved is χ , and χ_i is a χ -step

$$\bullet \xrightarrow{w} \bullet \begin{array}{c} \xrightarrow{u} \\ \Downarrow \chi \\ \xrightarrow{v} \end{array} \bullet \xrightarrow{w'} \bullet,$$

where w has length $i-1$. If $u = i_1|i_2|\dots$ is an infinite sequence of positive integers, we denote the path $\dots \circ \chi_{i_2} \circ \chi_{i_1}$ by χ_u .

Suppose that there is an infinite rewriting path. Note that an α -step strictly reduces the $(S \setminus \{1\})$ -length of a word, so there can be only finitely many of the generating 2-cells α in any rewriting path. Hence, there is no loss in generality if we consider only β -steps. Namely, we can simply consider an infinite path after the last α -step is applied and we are left with an infinite path containing only β -steps.

So assume that there is an infinite rewriting path of β -steps. Let $\beta_{i_1|i_2|\dots}$ be such a path having source u of minimal $(S \setminus \{1\})$ -length. Note that $\ell(u)$ is at least two.

Note that the minimality assumption about $\ell(u)$ implies that the position 1 occurs infinitely many times in $i_1|i_2|\dots$. Namely, if the position 1 occurred only finitely many times in $i_1|i_2|\dots$, then $\beta_{i_{k+1}-1|i_{k+2}-1|\dots}$ would be an infinite path starting from $t(\beta_{i_1|i_2|\dots|i_k}(u))$ of $(S \setminus \{1\})$ -length $\ell(u) - 1$, where $i_k = 1$ is the last occurrence of 1 in the sequence $i_1|i_2|\dots$. That would contradict the minimality assumption about $\ell(u)$. We write $i_{c_1|i_{c_2}|\dots}$ for the constant subsequence of the sequence $i_1|i_2|\dots$ taking all the members whose value is 1. In other words, c_1 is the least j such that $i_j = 1$; and for all n , we have that c_{n+1} is the least j such that conditions $j > c_n$ and $i_j = 1$ hold.

Let $u^{(n)}$ denote the n th word in the path $\beta_{i_1|i_2|\dots}$, that is the source of the step β_{i_n} . Note that the leftmost letter of the word is modified by a step β_{i_n} if, and only if, i_n equals 1. In this case, the modification is such that the current leftmost letter $h(u^{(n)})$ is a proper left divisor of the next leftmost letter $h(u^{(n+1)})$, and the corresponding complement lies in S by the definition of the generating 2-cells β . In formal terms,

$$(5.1) \quad h(u^{(n+1)}) = \begin{cases} h(u^{(n)}) & \text{if } i_{n+1} \neq 1, \\ h(u^{(n)}) f_n \text{ for some } f_n \in S & \text{if } i_{n+1} = 1. \end{cases}$$

Let g denote the leftmost letter of the S -normal form of u . Since the normalisation map N^S is left-weighted by Lemma 4.2.5, we have that $h(u^{(n)})$ left divides g for all n .

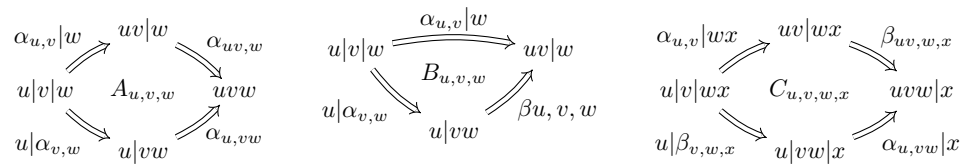
Consider the sequence

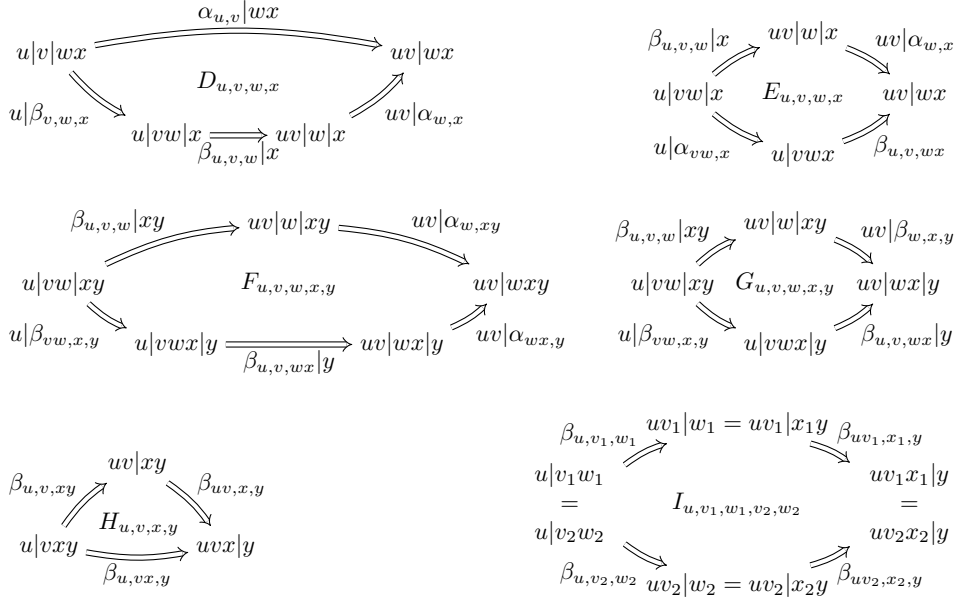
$$(5.2) \quad \left(h(u^{(c_n)}) \right)_{n=1}^{\infty}$$

in $S \cap \text{Div}(g)$. Observe that, by (5.1), we have $h(u^{(c_{n+1})}) = h(u^{(c_n)}) f_{c_n}$. The existence of the sequence (5.2) contradicts the fact that S is locally right-noetherian. We conclude that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is terminating. \square

5.3. Homotopical completion of Garside's presentation. In this subsection, we enrich Garside's presentation to reach a coherent convergent presentation. First (Proposition 5.3.1) we compute, purely formally, the homotopical completion of a terminating presentation of a monoid satisfying certain conditions, but not presumed to have a proper Garside family. Then we show, in Corollary 5.3.3, that this provides a coherent convergent presentation of a left-cancellative monoid containing no nontrivial invertible element, admitting right-mcms, and having a locally right-noetherian Garside family containing 1.

Proposition 5.3.1. *Assume that M is a left-cancellative monoid admitting right-mcms, and S is a subfamily of M closed under right-mcm and right divisor. Assume that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a terminating presentation of M . Then M admits, as a coherent convergent presentation, the $(3, 1)$ -polygraph $\underline{\text{Gar}}_3(S)$ which extends $\underline{\text{Gar}}_2(S)$ with the following nine families of generating 3-cells, indexed by all the possible elements of $S \setminus \{1\}$:*





The meanings of the 1-cells (i.e. words) x_1 , x_2 , y and x , y which appear respectively in the definitions of the generating 3-cells I and H , are as follows. Since v_1 and v_2 have the common right multiple $v_1w_1 = v_2w_2$, they also have a right-mcm. The words x_1 and x_2 are the right complements of v_1 and v_2 , respectively, in their right-mcm. The word y is the right complement of $v_1x_1 = v_2x_2$ in $v_1w_1 = v_2w_2$. If either x_1 or x_2 is equal to 1, then the other one is simply denoted by x (in the generating 3-cell H).

The structure of the following proof closely resembles that of the proof of [11, Proposition 3.2.1], but we need to devise more general arguments to assure favourable properties in a more general context.

Proof. Termination of the 2-polygraph $\text{Gar}_2(S)$ is assumed, so we can perform a relaxed version of the Knuth-Bendix completion procedure, as described in Remark 2.2.2, simultaneously with the Squier completion procedure. It will turn out that all critical branchings are confluent, and hence that only a Squier completion will be actually computed, i.e. no further 2-generating cells will be added.

Let us first consider critical branchings consisting only of the generating 2-cells α . There is only one such critical branching for all u , v and w of $S \setminus \{1\}$ such that $u \widehat{\vee} v \widehat{\wedge} w$ holds:

$$uv|w \xleftarrow{\alpha_{u,v}|w} u|v|w \xrightarrow{u|\alpha_{v,w}} u|vw.$$

If the subcase $u \widehat{\vee} v \widehat{\wedge} w$ holds, the branching is already confluent, so the homotopical completion procedure adjoins only the generating 3-cell $A_{u,v,w}$. If $u \widehat{\vee}^x v \widehat{\wedge} w$ holds, the branching is again confluent, so the generating 3-cell

$$\begin{array}{ccc} u|v|w & \xrightarrow{\alpha_{u,v}|w} & uv|w \\ & \searrow B_{u,v,w} \nearrow & \\ u|\alpha_{v,w} & & u|vw \end{array} \quad \beta_{u,v,w}$$

is adjoined.

Let us now consider critical branchings containing the generating 2-cell β . The sources of 2-cells forming such a branching can either overlap on one element of $S \setminus \{1\}$ or be equal, as the lengths in $(S \setminus \{1\})^*$ of the sources of the generating 2-cells α and β equal two. We consider the two cases accordingly.

For the first case, the proof of [11, Proposition 3.2.1] applies here verbatim. The source of a branching has length three, as a word in $(W \setminus \{1\})^*$. One of the 2-cells which form a branching, rewrites the leftmost two generating 1-cells of the source, and the other one rewrites the rightmost two. There are three distinct forms of such branchings:

$$\begin{aligned} uv|wx &\xleftarrow{\alpha_{u,v|wx}} u|v|wx \xrightarrow{u|\beta_{v,w,x}} u|vw|x, \\ uv|w|x &\xleftarrow{\beta_{u,v,w|x}} u|vw|x \xrightarrow{u|\alpha_{vw,x}} u|vwx, \\ vw|w|xy &\xleftarrow{\beta_{u,v,w|xy}} u|vw|xy \xrightarrow{u|\beta_{vw,x,y}} u|vwx|y. \end{aligned}$$

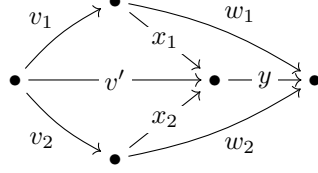
The first branching is defined under the condition $u \widehat{v} \overset{\times}{\widehat{w}} x$, which splits into two mutually exclusive possibilities $u \widehat{v} \overset{\times}{\widehat{w}} x$ and $u \overset{\times}{\widehat{v}} \widehat{w} x$, which respectively yield the generating 3-cells $C_{u,v,w,x}$ and $D_{u,v,w,x}$ by the homotopical completion procedure. The second branching is defined when $u \overset{\times}{\widehat{v}} \widehat{w} x$ holds, and it gives the generating 3-cell $E_{u,v,w,x}$. The third branching is defined under the conditions $u \overset{\times}{\widehat{v}} \widehat{w} \overset{\times}{\widehat{x}} y$ and $\ell(vwx) < \ell(vw) + \ell(xy)$. This situation splits into two mutually exclusive possibilities $u \overset{\times}{\widehat{v}} \widehat{w} \overset{\times}{\widehat{x}} y$ and $u \overset{\times}{\widehat{v}} \overset{\times}{\widehat{w}} \widehat{x} y$, which respectively yield the generating 3-cells $F_{u,v,w,x,y}$ and $G_{u,v,w,x,y}$.

We consider the second case in greater detail because this is where new justifications are needed. Assume that the two 2-cells which generate a critical branching, have the same source. One of those two 2-cells has to be a β (otherwise, the branching is trivial). Therefore, the source has to have a form $u|v_1w_1$ satisfying the condition $u \widehat{v_1} \overset{\times}{\widehat{w_1}}$. Since 2-cells α are not defined under this condition, the other 2-cell also has to be a β . The only way for the generating 2-cells β with the same source $u|v_1w_1$ to form a critical branching is for v_1w_1 to have another decomposition $v_1w_1 = v_2w_2$ such that $u \widehat{v_2} \overset{\times}{\widehat{w_2}}$. Then the branching is as follows:

$$uv_1|w_1 \xleftarrow{\beta_{u,v_1,w_1}} u|v_1w_1 = u|v_2w_2 \xrightarrow{\beta_{u,v_2,w_2}} uv_2|w_2.$$

Let us invoke the assumed property of M admitting right-mcms. Since v_1 and v_2 have a common right multiple, namely $v_1w_1 = v_2w_2$, they also have a right-mcm, say v' . Since S is closed under right-mcm by assumption, v' lies in S . By the left cancellation property which grants the uniqueness of right complements, we define x_1 and x_2 as the right complements in v' of v_1 and v_2 , respectively. Since S is closed under right divisor, x_1 and x_2 are elements of S . We also define y as the right complement of v' in $v_1w_1 = v_2w_2$. Note that y is in S as a right divisor of v_1w_1 which is in S . Uniqueness of the right complements of v_1 and v_2 in v_1w_1 and

v_2w_2 , respectively, yields $w_1 = x_1y$ and $w_2 = x_2y$. To sum up, the diagram



commutes, where \bullet denotes the unique 0-cell. Furthermore, the equality $w_k = x_ky$, the fact that v' lies in S , and the condition $v_k \widehat{w}_k$ together imply $v_k \widehat{x}_k \widehat{y}$ for $k \in \{1, 2\}$.

We have only showed that x_1 , x_2 and y are elements of S . Let us verify that all the generating 1-cells involved are, indeed, elements of $S \setminus \{1\}$. First we demonstrate that y cannot be equal to 1. Assume the opposite. Then the condition $u \widehat{v}_1^x \widehat{w}_1$ reduces to $u \widehat{v}_1^x x_1$. On the other hand, uv' is a right-mcm of uv_1 and uv_2 by Lemma 4.1.1. Since S is closed under right-mcm, uv' lies in S , which contradicts the condition $u \widehat{v}_1^x x_1$. Thus, we deduce that y is not equal to 1.

Note that if x_1 and x_2 were both equal to 1, the branching $\{\beta_{u,v_1,w_1}, \beta_{u,v_2,w_2}\}$ would be trivial. So, at most one of the 1-cells x_1 and x_2 can be equal to 1. If $x_2 = 1$, the generating 3-cell $H_{u,v,x,y}$ is constructed with $v := v_1$ and $x := x_1$. Similarly, for $x_1 = 1$, the generating 3-cell $H_{u,v,x,y}$ is constructed with $v := v_2$ and $x := x_2$. Finally, if neither x_1 nor x_2 is equal to 1, the generating 3-cell I_{u,v_1,w_1,v_2,w_2} is adjoined.

By Theorem 2.2.4, the constructed $(3, 1)$ -polygraph $\underline{\text{Gar}}_3(S)$ is a coherent convergent presentation of M . \square

We can now deduce that the 2-polygraph $\text{Gar}_2(S)$ is a Knuth-Bendix completion of the Garside's presentation $\text{Gar}_2(S)$, as hinted in Subsection 5.2.

Corollary 5.3.2. *Assume that M is a left-cancellative monoid containing no non-trivial invertible element, and $S \subseteq M$ is a locally right-noetherian Garside family containing 1. Then the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a convergent presentation of M .*

Proof. Proposition 4.2.7 grants that the 2-polygraph $\text{Gar}_2(S)$ is a presentation of M . Since the generating 2-cells α strictly decrease the S -length, the 2-polygraph $\text{Gar}_2(S)$ is terminating. Thanks to Proposition 5.2.1, we can compute its Knuth-Bendix completion in a manner described in Remark 2.2.2. As shown in Subsection 5.2, the generating 2-cells β are added.

Note that Proposition 4.2.3 and Lemma 4.2.4, together with the assumptions that S contains 1 and that M contains no nontrivial invertible element, yield the property of S being closed under right-mcm. By Proposition 4.2.3, S is closed under right divisor. With all these conditions satisfied, the proof of Proposition 5.3.1 applies in a straightforward fashion. In particular, it shows that all new critical branchings caused by the generating 2-cells β are confluent. Thus, the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a Knuth-Bendix completion of the Garside's presentation $\text{Gar}_2(S)$, which yields the desired conclusion by Theorem 2.2.1 and Remark 2.2.2. \square

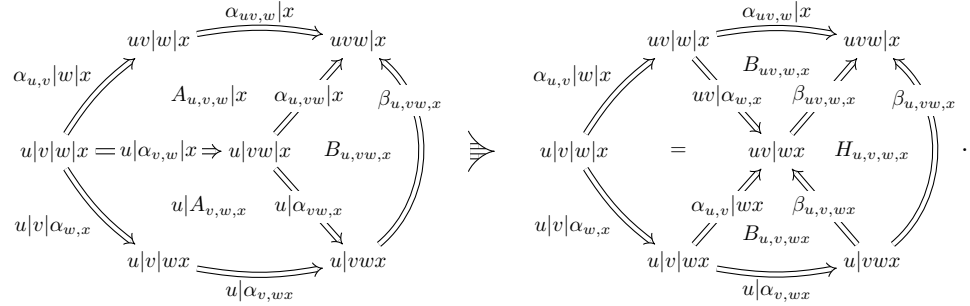
Observe that Proposition 5.2.1, together with Proposition 4.2.7, immediately implies that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a terminating presentation of M . On the other hand, the fact that $\underline{\text{Gar}}_2(S)$ is also a convergent presentation of M was reachable only after Proposition 5.3.1 when we made sure that no additional generating 2-cells were required to obtain confluence.

Corollary 5.3.3. *Assume that M is a left-cancellative monoid containing no non-trivial invertible element, and $S \subseteq M$ is a locally right-noetherian Garside family containing 1. If M admits right-mcms, then M admits the $(3,1)$ -polygraph $\text{Gar}_3(S)$, defined in Proposition 5.3.1, as a coherent convergent presentation.*

Proof. Corollary 5.3.2 grants that $\text{Gar}_2(S)$ is a terminating presentation of M . As shown in the proof of Corollary 5.3.2, all the requirements are met for applying Proposition 5.3.1, which completes the proof. \square

5.4. Homotopical reduction of Garside's presentation. The homotopical reduction procedure from [11, 3.2.2] applies verbatim to the coherent convergent presentation provided by Proposition 5.3.1 (and echoed by Corollary 5.3.3), with respect to a collapsible part Γ obtained as follows. The component Γ_4 of Γ contains seven generating triple confluences whose targets are the families C, \dots, I of generating 3-cells, with the order $I > H > \dots > C$. For the sake of illustration, we

recall one such generating triple confluence in the case $u \begin{array}{c} \diagup \\ v \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ w \\ \diagup \end{array} x$ (we refer the reader to [11, 3.2.2] for the other six generating triple confluences):



The target of this particular generating triple confluence is the generating 3-cell $H_{u,v,x,y}$.

The component Γ_3 of the collapsible part contains the family B of generating 3-cells having the generating 2-cells β as targets, with the order $\beta > \alpha$.

The homotopical reduction of the resulting $(3,1)$ -polygraph of Proposition 5.3.1, with respect to the collapsible part Γ , is precisely $\text{Gar}_3(S)$. By Theorem 2.3.1, we conclude that $\text{Gar}_3(S)$ is a coherent presentation of M . Through Corollary 5.3.3, the proof of Theorem 5.1.4 is hereby completed.

5.5. Noetherianity. Let us state an immediate corollary of Theorem 5.1.4, having somewhat simpler (although more restrictive) requirements.

Corollary 5.5.1. *Assume that M is a left-cancellative noetherian monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family containing 1. Then M admits the $(3,1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation.*

Proof. Since M is right-noetherian, S is locally right-noetherian. By Proposition 4.1.3, M admits right-mcms. So, all the conditions of Theorem 5.1.4 are satisfied. \square

The next section demonstrates advantages of using our results in applications. The following example, however, shows that taking a Garside family as a generating set is not always the most practical way to get a coherent presentation.

Example 5.5.2. We revisit the Klein bottle monoid K^+ from (Examples 2.2.5, 2.3.2, and). One of the infinitely many Garside families in K^+ , none of which is finite (see [3, Example IV.2.35]), is $S = \text{Div}(a^2)$. Let us check if the conditions of

Theorem 5.1.4 are satisfied. Note that K^+ is cancellative as it is embeddable in a group. The presentation (2.1) contains no ε -relation, hence K^+ has no nontrivial invertible element. Note that the left divisibility relation of K^+ is a linear order ([3, Figure I.6]), which is a lot more than necessary for admitting conditional right-lcms (consequently, right-mcms, too). However, the sequence $(ab^n)_{n=1}^\infty$ shows that S is not locally right-noetherian. Even worse, S contains an infinite path of the generating 2-cells β , as defined in Proposition 5.2.1:

$$b|a^2 \mapsto b^2|aba \mapsto b^3|ab^2a \mapsto \dots \mapsto b^q|ab^{q-1}a \mapsto \dots$$

Even if we took another Garside family, we would not be successful, as witnessed by [3, Example IV.2.35]. Therefore, neither Theorem 5.1.4 nor its proof is applicable to K^+ .

If one found a way to use a Garside family as a generating set, they would have an infinite number of 1-cells. On the other hand, by directly performing the homotopical completion-reduction procedure in Examples 2.2.5 and 2.3.2, we have demonstrated that the presentation (2.1), which has two generating 1-cells and one generating 2-cell, is coherent. Therefore, for this particular example, the direct application of the homotopical completion-reduction procedure is a preferable way of reaching a coherent presentation.

6. APPLICATIONS OF THEOREM 5.1.4

In this section, we consider applications of Theorem 5.1.4 to certain monoids. In Subsections 6.1 and 6.2, we apply it to monoids which are neither Artin-Tits nor Garside. We construct a coherent presentation of dual braid monoids in Subsection 6.3. Finally, in Subsection 6.4, we compute a finite coherent presentation of a non-spherical Artin-Tits monoid $B^+(W)$ with a finite Garside family F (hence, $F \neq W$).

6.1. The free abelian monoid over an infinite basis. Consider the free abelian monoid $\mathbb{N}^{(I)}$ of all I -indexed sequences of nonnegative integers with finite support. Note that $\mathbb{N}^{(I)}$ is not necessarily of finite type, hence it is neither Artin-Tits nor Garside. Define main steps are: establishing termination, computing homotopical completion, computing homotopical reduction.

$$S_I = \left\{ g \in \mathbb{N}^{(I)} \mid \forall k \in I, g(k) \in \{0, 1\} \right\}.$$

Observe that S_I is a Garside family in $\mathbb{N}^{(I)}$ (say, by apmain steps are: establishing termination, computing homotopical completion, computing homotopical reduction. plymain steps are: establishing termination, computing homotopical completion, computing homotopical reduction. ing Proposition 4.2.2). The following properties follow from the fact that the definition of the product on $\mathbb{N}^{(I)}$ is based on the pointwise addition of integers: $\mathbb{N}^{(I)}$ is a cancellative monoid, it has no nontrivial invertible elements, and it admits conditional right-lcms. Since every element of $\mathbb{N}^{(I)}$ has only finitely many divisors, $\mathbb{N}^{(I)}$ is noetherian. So, all the conditions of Theorem 5.1.4 are satisfied.

Let us describe the cells of the coherent presentation of $\mathbb{N}^{(I)}$ granted by Theorem 5.1.4. The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for $u, v \in S_I \setminus \{1\}$ with $uv \in S_I$, which in this particular context means that u and v have disjoint supports. A generating 3-cell $A_{u,v,w}$ is adjoined for any $u, v, w \in S_I \setminus \{1\}$ which have pairwise disjoint supports.

As expected, for $I = \{1, 2, \dots, n\}$, we recover Garside's presentation of the Artin-Tits monoid \mathbb{N}^n provided by Theorem 3.4.1, as well as Garside's presentation of the Garside monoid \mathbb{N}^n provided by Theorem 3.6.1.

6.2. Infinite braids. Denote by B_∞^+ the monoid of all positive braids on infinitely many strands indexed by positive integers, as defined in [3, Subsection I.3.1]. It is shown that B_∞^+ is not of finite type, therefore it is neither Artin-Tits nor Garside. Put

$$S_\infty = \bigcup_{n \geq 1} \text{Div}(\Delta_n),$$

where Δ_n denotes the half-turn braid. In other words, S_∞ consists of all simple braids for all $n \geq 1$. This is made precise in [3, Subsection I.3.1]. Basically, B_n^+ is identified with its image in B_{n+1}^+ under the homomorphism induced by the identity map on $\{\sigma_1, \dots, \sigma_n\}$. In that sense, B_∞^+ is seen as the union of all braid monoids B_n^+ . By Proposition 4.2.2, S_∞ is a Garside family in B_∞^+ . Cancellation properties, and having no nontrivial invertible elements are preserved from braid monoids because the respective definitions do not depend on n . The monoid is noetherian for the same reason as Artin-Tits monoids (Example 4.1.2). So, we can apply Theorem 5.1.4 to construct a coherent presentation.

The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for $u, v \in S_\infty \setminus \{1\}$ whenever $uv \in S_\infty$, which in this example means that uv is a simple braid. A generating 3-cell $A_{u,v,w}$ is adjoined for any $u, v, w \in S_\infty \setminus \{1\}$ with $uv \in S_\infty$, $vw \in S_\infty$, and $uvw \in S_\infty$, which here means that uv , vw and uvw are simple braids. So, formally, each cell is constructed exactly like in the coherent presentation provided by Theorem 3.4.1 for a (finite) braid monoid, regarded as an Artin-Tits monoid, which comes as no surprise because Theorem 5.1.4 is a formal generalisation of [11, Theorem 3.1.3].

6.3. Dual braid monoids. Let us consider the **dual braid monoid** B_n^{+*} , as defined in [3, Subsection I.1.3]. Its generators are $a_{i,j}$ with $1 \leq i < j < k \leq n$. They obey the following relations:

$$\begin{aligned} a_{i,j}a_{i',j'} &= a_{i',j'}a_{i,j} && \text{for } [i,j] \text{ and } [i',j'] \text{ disjoint or nested;} \\ a_{i,j}a_{j,k} &= a_{j,k}a_{i,k} = a_{i,k}a_{i,j} && \text{for } 1 \leq i < j < k \leq n. \end{aligned}$$

We refer the reader to [3, Subsection I.1.3] for elaboration.

The pair (B_n^{+*}, Δ_n^*) , with $\Delta_n^* = a_{1,2} \cdots a_{n-1,n}$, is a Garside monoid, as demonstrated in [3, Subsection I.1.3]. Therefore, computing a coherent presentation of a dual braid monoid does not really require Theorem 5.1.4, as Theorem 3.6.1 would suffice. To the best of our knowledge, however, it has not been done yet, so we do it here. Let us take B_4^{+*} for example. For convenience, we leave out the comma in the index of a .

The Garside family we consider is

$$\begin{aligned} \text{Div}(\Delta_4^*) &= \{1, a_{12}, a_{23}, a_{24}, a_{13}, a_{34}, a_{14}, \\ &\quad a_{12}a_{23}, a_{12}a_{34}, a_{23}a_{34}, a_{12}a_{24}, a_{23}a_{14}, a_{13}a_{34}, a_{12}a_{23}a_{34}\}. \end{aligned}$$

The lattice $(\text{Div}(\Delta_4^*), \preceq)$ can be seen in [3, Figure I.4].

The requirements of Theorem 5.1.4 are met, as noted in Remark 5.1.5. By Theorem 5.1.4, we take generating 2-cells of the form $u|v \Rightarrow uv$ for $u, v \in \text{Div}(\Delta_4^*) \setminus \{1\}$ such that $uv \in \text{Div}(\Delta_4^*)$. This yields the following family of 2-cells.

$$\begin{array}{ll} \alpha_{a_{12}, a_{23}} : a_{12}|a_{23} \Rightarrow a_{12}a_{23} & \alpha_{a_{23}, a_{34}} : a_{23}|a_{34} \Rightarrow a_{23}a_{34} \\ \alpha_{a_{23}, a_{13}} : a_{23}|a_{13} \Rightarrow a_{12}a_{23} & \alpha_{a_{34}, a_{24}} : a_{34}|a_{24} \Rightarrow a_{23}a_{34} \\ \alpha_{a_{13}, a_{12}} : a_{13}|a_{12} \Rightarrow a_{12}a_{23} & \alpha_{a_{24}, a_{23}} : a_{24}|a_{23} \Rightarrow a_{23}a_{34} \end{array}$$

$$\begin{array}{ll}
\alpha_{a_{12},a_{24}} : a_{12}|a_{24} \Rightarrow a_{12}a_{24} & \alpha_{a_{13},a_{34}} : a_{13}|a_{34} \Rightarrow a_{13}a_{34} \\
\alpha_{a_{24},a_{14}} : a_{24}|a_{14} \Rightarrow a_{12}a_{24} & \alpha_{a_{34},a_{14}} : a_{34}|a_{14} \Rightarrow a_{13}a_{34} \\
\alpha_{a_{14},a_{12}} : a_{14}|a_{12} \Rightarrow a_{12}a_{2,4} & \alpha_{a_{14},a_{13}} : a_{14}|a_{13} \Rightarrow a_{13}a_{34} \\
\\
\alpha_{a_{12},a_{34}} : a_{12}|a_{34} \Rightarrow a_{12}a_{34} & \alpha_{a_{23},a_{14}} : a_{23}|a_{14} \Rightarrow a_{23}a_{14} \\
\alpha_{a_{34},a_{12}} : a_{34}|a_{12} \Rightarrow a_{12}a_{34} & \alpha_{a_{14},a_{23}} : a_{14}|a_{23} \Rightarrow a_{23}a_{14}
\end{array}$$

$$\begin{array}{ll}
\alpha_{a_{12},a_{23}a_{34}} : a_{12}|a_{23}a_{34} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{12}a_{23},a_{34}} : a_{12}a_{23}|a_{34} \Rightarrow a_{12}a_{23}a_{34} \\
\alpha_{a_{23},a_{13}a_{34}} : a_{23}|a_{13}a_{34} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{12}a_{34},a_{24}} : a_{12}a_{34}|a_{24} \Rightarrow a_{12}a_{23}a_{34} \\
\alpha_{a_{24},a_{23}a_{14}} : a_{24}|a_{23}a_{14} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{23}a_{34},a_{14}} : a_{23}a_{34}|a_{14} \Rightarrow a_{12}a_{23}a_{34} \\
\alpha_{a_{13},a_{12}a_{34}} : a_{13}|a_{12}a_{34} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{12}a_{24},a_{23}} : a_{12}a_{2,4}|a_{23} \Rightarrow a_{12}a_{23}a_{34} \\
\alpha_{a_{34},a_{12}a_{24}} : a_{34}|a_{12}a_{24} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{23}a_{14},a_{13}} : a_{23}a_{14}|a_{13} \Rightarrow a_{12}a_{23}a_{34} \\
\alpha_{a_{14},a_{12}a_{23}} : a_{14}|a_{12}a_{23} \Rightarrow a_{12}a_{23}a_{34} & \alpha_{a_{13}a_{34},a_{12}} : a_{13}a_{34}|a_{12} \Rightarrow a_{12}a_{23}a_{34}
\end{array}$$

Now we construct the generating 3-cells $A_{u,v,w}$ for $u, v, w \in \text{Div}(\Delta_4^*) \setminus \{1\}$ with $uv \in \text{Div}(\Delta_4^*)$, $vw \in \text{Div}(\Delta_4^*)$, and $uvw \in \text{Div}(\Delta_4^*)$.

$$\begin{array}{ccc}
\alpha_{a_{12},a_{23}}|a_{34} & \xrightarrow{\alpha_{a_{12},a_{23},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{12}|a_{23}|a_{34} & \xrightarrow{A_{a_{12},a_{23},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{12}|a_{23}a_{34} & \xrightarrow{\alpha_{a_{12},a_{23}a_{34}}} & a_{12}|a_{23}a_{34}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{23},a_{34}}|a_{14} & \xrightarrow{\alpha_{a_{23},a_{34},a_{14}}} & a_{23}a_{34}|a_{14} \\
a_{23}|a_{34}|a_{14} & \xrightarrow{A_{a_{23},a_{34},a_{14}}} & a_{23}a_{34}|a_{14} \\
a_{23}|a_{34}a_{14} & \xrightarrow{\alpha_{a_{23},a_{34}a_{14}}} & a_{23}|a_{34}a_{14}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{23},a_{13}}|a_{34} & \xrightarrow{\alpha_{a_{23},a_{13},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{23}|a_{13}|a_{34} & \xrightarrow{A_{a_{23},a_{13},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{23}|a_{13}a_{34} & \xrightarrow{\alpha_{a_{23},a_{13}a_{34}}} & a_{23}|a_{13}a_{34}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{34},a_{24}}|a_{14} & \xrightarrow{\alpha_{a_{34},a_{24},a_{14}}} & a_{23}a_{34}|a_{14} \\
a_{34}|a_{24}|a_{14} & \xrightarrow{A_{a_{34},a_{24},a_{14}}} & a_{23}a_{34}|a_{14} \\
a_{34}|a_{24}a_{14} & \xrightarrow{\alpha_{a_{34},a_{24}a_{14}}} & a_{34}|a_{12}a_{24}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{13},a_{12}}|a_{34} & \xrightarrow{\alpha_{a_{13},a_{12},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{13}|a_{12}|a_{34} & \xrightarrow{A_{a_{13},a_{12},a_{34}}} & a_{12}a_{23}|a_{34} \\
a_{13}|a_{12}a_{34} & \xrightarrow{\alpha_{a_{13},a_{12}a_{34}}} & a_{13}|a_{12}a_{34}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{24},a_{23}}|a_{14} & \xrightarrow{\alpha_{a_{24},a_{23},a_{14}}} & a_{23}a_{34}|a_{14} \\
a_{24}|a_{23}|a_{14} & \xrightarrow{A_{a_{24},a_{23},a_{14}}} & a_{12}a_{23}|a_{34} \\
a_{24}|a_{23}a_{14} & \xrightarrow{\alpha_{a_{24},a_{23}a_{14}}} & a_{24}|a_{23}a_{14}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{12},a_{24}}|a_{23} & \xrightarrow{\alpha_{a_{12},a_{24},a_{23}}} & a_{12}a_{24}|a_{23} \\
a_{12}|a_{24}|a_{23} & \xrightarrow{A_{a_{12},a_{24},a_{23}}} & a_{12}a_{23}|a_{34} \\
a_{12}|a_{24}a_{23} & \xrightarrow{\alpha_{a_{12},a_{24}a_{23}}} & a_{12}|a_{23}a_{34}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{13},a_{34}}|a_{12} & \xrightarrow{\alpha_{a_{13},a_{34},a_{12}}} & a_{13}a_{34}|a_{12} \\
a_{13}|a_{34}|a_{12} & \xrightarrow{A_{a_{13},a_{34},a_{12}}} & a_{12}a_{23}|a_{34} \\
a_{13}|a_{34}a_{12} & \xrightarrow{\alpha_{a_{13},a_{34}a_{12}}} & a_{13}|a_{12}a_{34}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{24},a_{14}}|a_{23} & \xrightarrow{\alpha_{a_{24},a_{14},a_{23}}} & a_{12}a_{24}|a_{23} \\
a_{24}|a_{14}|a_{23} & \xrightarrow{A_{a_{24},a_{14},a_{23}}} & a_{12}a_{23}|a_{34} \\
a_{24}|a_{14}a_{23} & \xrightarrow{\alpha_{a_{24},a_{14}a_{23}}} & a_{24}|a_{23}a_{14}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{34},a_{14}}|a_{12} & \xrightarrow{\alpha_{a_{34},a_{14},a_{12}}} & a_{13}a_{34}|a_{12} \\
a_{34}|a_{14}|a_{12} & \xrightarrow{A_{a_{34},a_{14},a_{12}}} & a_{12}a_{23}|a_{34} \\
a_{34}|a_{14}a_{12} & \xrightarrow{\alpha_{a_{34},a_{14}a_{12}}} & a_{34}|a_{12}a_{24}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{14},a_{12}}|a_{23} & \xrightarrow{\quad} & a_{12}a_{24}|a_{23} \\
a_{14}|a_{12}|a_{23} & \xrightarrow{\quad} & A_{a_{14},a_{12},a_{23}} \\
a_{14}|\alpha_{a_{12},a_{23}} & \xrightarrow{\quad} & a_{14}|a_{12}a_{23}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{12}a_{24},a_{23}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{14},a_{12}a_{23}}
\end{array}
\qquad
\begin{array}{ccc}
\alpha_{a_{14},a_{13}}|a_{12} & \xrightarrow{\quad} & a_{13}a_{34}|a_{12} \\
a_{14}|a_{13}|a_{12} & \xrightarrow{\quad} & A_{a_{14},a_{13},a_{12}} \\
a_{14}|\alpha_{a_{13},a_{12}} & \xrightarrow{\quad} & a_{14}|a_{12}a_{23}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{13}a_{34},a_{12}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{14},a_{12}a_{23}}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{12},a_{34}}|a_{24} & \xrightarrow{\quad} & a_{12}a_{34}|a_{24} \\
a_{12}|a_{34}|a_{24} & \xrightarrow{\quad} & A_{a_{12},a_{34},a_{24}} \\
a_{12}|\alpha_{a_{34},a_{24}} & \xrightarrow{\quad} & a_{12}|a_{23}a_{34}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{12}a_{34},a_{24}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{12},a_{23}a_{34}}
\end{array}
\qquad
\begin{array}{ccc}
\alpha_{a_{23},a_{14}}|a_{13} & \xrightarrow{\quad} & a_{23}a_{14}|a_{13} \\
a_{23}|a_{14}|a_{13} & \xrightarrow{\quad} & A_{a_{23},a_{14},a_{13}} \\
a_{23}|\alpha_{a_{14},a_{13}} & \xrightarrow{\quad} & a_{23}|a_{13}a_{34}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{23}a_{14},a_{13}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{23},a_{13}a_{34}}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{a_{34},a_{12}}|a_{24} & \xrightarrow{\quad} & a_{12}a_{34}|a_{24} \\
a_{34}|a_{12}|a_{24} & \xrightarrow{\quad} & A_{a_{34},a_{12},a_{24}} \\
a_{34}|\alpha_{a_{12},a_{24}} & \xrightarrow{\quad} & a_{34}|a_{12}a_{24}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{12}a_{34},a_{24}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{34},a_{12}a_{24}}
\end{array}
\qquad
\begin{array}{ccc}
\alpha_{a_{14},a_{23}}|a_{13} & \xrightarrow{\quad} & a_{23}a_{14}|a_{13} \\
a_{14}|a_{23}|a_{13} & \xrightarrow{\quad} & A_{a_{14},a_{23},a_{13}} \\
a_{14}|\alpha_{a_{23},a_{13}} & \xrightarrow{\quad} & a_{14}|a_{12}a_{23}
\end{array}
\begin{array}{ccc}
& \xrightarrow{\quad} & \alpha_{a_{23}a_{14},a_{13}} \\
& \xrightarrow{\quad} & a_{12}a_{23}a_{34} \\
& \xrightarrow{\quad} & \alpha_{a_{14},a_{12}a_{23}}
\end{array}$$

We have thus computed the coherent presentation $\text{Gar}_3(\text{Div}(\Delta_4^*))$ of the dual braid monoid B_4^{+*} , which consists of thirteen generating 1-cells, twenty-eight generating 2-cells, and sixteen generating 3-cells.

Note that some of the generating 3-cells are superfluous, so let us further perform the homotopical reduction procedure as described in Subsection 2.3. A quick inspection reveals that there are no critical triple branchings, so we take the component Γ_4 of the collapsible part to be the empty set. For the component Γ_3 , we take the Nielsen transformations of the generating 3-cells

$$A_{a_{24},a_{23},a_{14}}, A_{a_{13},a_{34},a_{12}}, A_{a_{23},a_{14},a_{13}}, A_{a_{23},a_{34},a_{14}}, A_{a_{14},a_{12},a_{23}}, \\
A_{a_{34},a_{12},a_{24}}, A_{a_{13},a_{12},a_{34}}, A_{a_{23},a_{13},a_{34}}, A_{a_{12},a_{24},a_{23}}, A_{a_{12},a_{34},a_{24}}, A_{a_{12},a_{23},a_{34}}$$

whose respectively selected targets are the following generating 2-cells:

$$\alpha_{a_{24},a_{23}a_{14}}, \alpha_{a_{13}a_{34},a_{12}}, \alpha_{a_{23}a_{14},a_{13}}, \alpha_{a_{23}a_{34},a_{14}}, \alpha_{a_{14},a_{12}a_{23}}, \\
\alpha_{a_{34},a_{12}a_{24}}, \alpha_{a_{13},a_{12}a_{34}}, \alpha_{a_{23},a_{13}a_{34}}, \alpha_{a_{12}a_{24},a_{23}}, \alpha_{a_{12}a_{34},a_{24}}, \alpha_{a_{12}a_{23},a_{34}}.$$

We order the generating 2-cells as follows:

$$\begin{aligned}
\alpha_{a_{24},a_{23}a_{14}} &> \alpha_{a_{1,3}a_{3,4},a_{1,2}} > \alpha_{a_{23}a_{14},a_{13}} > \alpha_{a_{23}a_{34},a_{14}} > \\
&> \alpha_{a_{14},a_{12}a_{23}} > \alpha_{a_{34},a_{12}a_{24}} > \alpha_{a_{13},a_{12}a_{34}} > \alpha_{a_{23},a_{13}a_{34}} > \\
&> \alpha_{a_{12}a_{24},a_{23}} > \alpha_{a_{12}a_{34},a_{24}} > \alpha_{a_{12}a_{23},a_{34}} > \alpha_{a_{12},a_{23}a_{34}} > \\
&> \text{other generating 2-cells in any order (as they are not targets in } \Gamma_3).
\end{aligned}$$

For the component Γ_2 , we take the generating 2-cells

$$\alpha_{a_{12},a_{23}}, \alpha_{a_{23},a_{34}}, \alpha_{a_{12},a_{24}}, \alpha_{a_{13},a_{34}}, \alpha_{a_{12},a_{34}}, \alpha_{a_{23},a_{14}}, \alpha_{a_{12},a_{23}a_{34}}.$$

Any ordering of the generating 1-cells which respects the length in $\text{Div}(\Delta_4^*)^*$ will do.

Recall that, when cells are removed from a polygraph, the boundaries of the remaining cells are not necessarily defined anymore, in a manner of speaking. More precisely, if a boundary contained a removed cell, it is ‘‘updated’’ according to the recursive assignment given in Subsection 2.3. For convenience, we use the letter β (resp. B) to denote the new version of the cell α (resp. A) which shares the same index. Then, the homotopical reduction of $\text{Gar}_3(\text{Div}(\Delta_4^*))$ with respect to the collapsible part $(\Gamma_2, \Gamma_3, \emptyset)$ contains: a single generating 0-cell; the generating 1-cells $a_{12}, a_{23}, a_{24}, a_{13}, a_{34}, a_{14}$; the generating 2-cells

$$\begin{array}{ll}
\beta_{a_{23}, a_{13}} : a_{23}|a_{13} \Rightarrow a_{12}|a_{23}, & \beta_{a_{34}, a_{24}} : a_{34}|a_{24} \Rightarrow a_{23}|a_{34}, \\
\beta_{a_{13}, a_{12}} : a_{13}|a_{12} \Rightarrow a_{12}|a_{23}, & \beta_{a_{24}, a_{23}} : a_{24}|a_{23} \Rightarrow a_{23}|a_{34}, \\
\beta_{a_{24}, a_{14}} : a_{24}|a_{14} \Rightarrow a_{12}|a_{24}, & \beta_{a_{34}, a_{14}} : a_{34}|a_{14} \Rightarrow a_{13}|a_{34}, \\
\beta_{a_{14}, a_{1,2}} : a_{14}|a_{12} \Rightarrow a_{12}|a_{24}, & \beta_{a_{14}, a_{13}} : a_{14}|a_{13} \Rightarrow a_{13}|a_{34}, \\
\beta_{a_{34}, a_{12}} : a_{34}|a_{12} \Rightarrow a_{12}|a_{34}, & \beta_{a_{14}, a_{23}} : a_{14}|a_{23} \Rightarrow a_{23}|a_{14}.
\end{array}$$

and the generating 3-cells

$$\begin{array}{ccc}
\beta_{a_{34}, a_{24}}|a_{14} & \xrightarrow{a_{23}|\beta_{a_{34}, a_{14}}} & a_{23}|a_{13}|a_{34} \\
\Downarrow & & \Downarrow \beta_{a_{23}, a_{13}}|a_{34} \\
a_{34}|a_{24}|a_{14} & \xrightarrow{B_{a_{34}, a_{24}, a_{14}}} & a_{12}|a_{23}|a_{34} \\
\Downarrow & & \Downarrow a_{12}|\beta_{a_{34}, a_{2,4}} \\
a_{34}|\beta_{a_{24}, a_{14}} & \xrightarrow{\beta_{a_{34}, a_{12}}|a_{24}} & a_{12}|a_{34}|a_{24}
\end{array} \quad ,$$

$$\begin{array}{ccc}
\beta_{a_{24}, a_{14}}|a_{23} & \xrightarrow{a_{12}|\beta_{a_{24}, a_{23}}} & a_{12}|a_{23}|a_{34} \\
\Downarrow & & \Downarrow \beta_{a_{23}, a_{13}}|a_{34} \\
a_{24}|a_{14}|a_{23} & \xrightarrow{B_{a_{24}, a_{14}, a_{23}}} & a_{23}|a_{13}|a_{34} \\
\Downarrow & & \Downarrow a_{23}|\beta_{a_{34}, a_{14}} \\
a_{24}|\beta_{a_{14}, a_{23}} & \xrightarrow{\beta_{a_{24}, a_{23}}|a_{14}} & a_{23}|a_{34}|a_{14}
\end{array} \quad ,$$

$$\begin{array}{ccc}
\beta_{a_{34}, a_{14}}|a_{12} & \xrightarrow{a_{13}|\beta_{a_{34}, a_{12}}} & a_{13}|a_{12}|a_{34} \\
\Downarrow & & \Downarrow \beta_{a_{13}, a_{12}}|a_{34} \\
a_{34}|a_{14}|a_{12} & \xrightarrow{B_{a_{34}, a_{14}, a_{12}}} & a_{12}|a_{23}|a_{34} \\
\Downarrow & & \Downarrow a_{12}|\beta_{a_{34}, a_{2,4}} \\
a_{34}|\beta_{a_{14}, a_{12}} & \xrightarrow{\beta_{a_{34}, a_{12}}|a_{24}} & a_{12}|a_{34}|a_{24}
\end{array} \quad ,$$

$$\begin{array}{ccc}
\beta_{a_{14}, a_{13}}|a_{12} & \xrightarrow{a_{13}|\beta_{a_{34}, a_{12}}} & a_{13}|a_{12}|a_{34} \\
\Downarrow & & \Downarrow \beta_{a_{13}, a_{34}, a_{12}} \\
a_{14}|a_{13}|a_{12} & \xrightarrow{B_{a_{14}, a_{13}, a_{12}}} & a_{12}|a_{23}|a_{34} \\
\Downarrow & & \Downarrow a_{12}|\beta_{a_{24}, a_{23}} \\
a_{14}|\beta_{a_{13}, a_{12}} & \xrightarrow{\beta_{a_{14}, a_{12}}|a_{23}} & a_{12}|a_{24}|a_{23}
\end{array} \quad ,$$

$$\begin{array}{ccc}
\beta_{a_{14}, a_{23}}|a_{13} & \xrightarrow{a_{23}|\beta_{a_{14}, a_{13}}} & a_{23}|a_{13}|a_{34} \\
\Downarrow & & \Downarrow \beta_{a_{23}, a_{13}}|a_{34} \\
a_{14}|a_{23}|a_{13} & \xrightarrow{B_{a_{14}, a_{23}, a_{13}}} & a_{12}|a_{23}|a_{34} \\
\Downarrow & & \Downarrow a_{12}|\beta_{a_{24}, a_{23}} \\
a_{14}|\beta_{a_{23}, a_{13}} & \xrightarrow{\beta_{a_{14}, a_{12}}|a_{23}} & a_{12}|a_{24}|a_{23}
\end{array} \quad .$$

6.4. Non-spherical Artin-Tits monoids. For a spherical Artin-Tits monoid $B^+(W)$, Theorem 3.4.1 provides a finite coherent presentation having $W \setminus \{1\}$ as a generating set. On the other hand, if a Coxeter group W is infinite, Theorem 3.4.1 still provides a coherent presentation but an infinite one. Recall that every Artin-Tits monoid admits a finite Garside family (we refer the reader to [5] for elaboration), regardless of whether the monoid is spherical or not. An advantage of having Theorem 5.1.4 at our disposal is that we can take a finite Garside family for a generating set in computing a coherent presentation (whereas with Theorem 3.4.1, one has to take the corresponding Coxeter group).

Let us consider the Artin-Tits monoid of type \tilde{A}_2 , i.e. the monoid presented by

$$(6.1) \quad \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_3\sigma_1 \rangle^+.$$

By [5, Table 1 and Proposition 5.1], the smallest Garside family F in this monoid consists of the sixteen right divisors of the elements $\sigma_3\sigma_1\sigma_2\sigma_1$, $\sigma_1\sigma_2\sigma_3\sigma_2$, and $\sigma_2\sigma_3\sigma_1\sigma_3$. Namely,

$$F = \{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_2, \sigma_3\sigma_1, \sigma_1\sigma_3, \\ \sigma_1\sigma_2\sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3\sigma_1\sigma_3, \sigma_3\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_3\sigma_2, \sigma_2\sigma_3\sigma_1\sigma_3\}.$$

The Cayley graph of F can be seen in [5, Figure 1].

As noted in Remark 5.1.5, all the conditions of Theorem 5.1.4 are satisfied. Following Theorem 5.1.4, we construct a 2-cell $u|v \Rightarrow uv$ for $u, v \in F \setminus \{1\}$ with $uv \in F$. Thus we obtain the following family of 2-cells.

$$\begin{array}{ll} \alpha_{\sigma_1, \sigma_2} : \sigma_1 | \sigma_2 \Rightarrow \sigma_1 \sigma_2 & \alpha_{\sigma_2, \sigma_1} : \sigma_2 | \sigma_1 \Rightarrow \sigma_2 \sigma_1 \\ \alpha_{\sigma_2, \sigma_3} : \sigma_2 | \sigma_3 \Rightarrow \sigma_2 \sigma_3 & \alpha_{\sigma_3, \sigma_2} : \sigma_3 | \sigma_2 \Rightarrow \sigma_3 \sigma_2 \\ \alpha_{\sigma_3, \sigma_1} : \sigma_3 | \sigma_1 \Rightarrow \sigma_3 \sigma_1 & \alpha_{\sigma_1, \sigma_3} : \sigma_1 | \sigma_3 \Rightarrow \sigma_1 \sigma_3 \end{array}$$

$$\begin{array}{ll} \alpha_{\sigma_1, \sigma_2 \sigma_1} : \sigma_1 | \sigma_2 \sigma_1 \Rightarrow \sigma_1 \sigma_2 \sigma_1 & \alpha_{\sigma_2, \sigma_1 \sigma_2} : \sigma_2 | \sigma_1 \sigma_2 \Rightarrow \sigma_1 \sigma_2 \sigma_1 \\ \alpha_{\sigma_2, \sigma_3 \sigma_2} : \sigma_2 | \sigma_3 \sigma_2 \Rightarrow \sigma_2 \sigma_3 \sigma_2 & \alpha_{\sigma_3, \sigma_2 \sigma_3} : \sigma_3 | \sigma_2 \sigma_3 \Rightarrow \sigma_2 \sigma_3 \sigma_2 \\ \alpha_{\sigma_3, \sigma_1 \sigma_3} : \sigma_3 | \sigma_1 \sigma_3 \Rightarrow \sigma_3 \sigma_1 \sigma_3 & \alpha_{\sigma_1, \sigma_3 \sigma_1} : \sigma_1 | \sigma_3 \sigma_1 \Rightarrow \sigma_3 \sigma_1 \sigma_3 \\ \\ \alpha_{\sigma_1 \sigma_2, \sigma_1} : \sigma_1 \sigma_2 | \sigma_1 \Rightarrow \sigma_1 \sigma_2 \sigma_1 & \alpha_{\sigma_2 \sigma_1, \sigma_2} : \sigma_2 \sigma_1 | \sigma_2 \Rightarrow \sigma_1 \sigma_2 \sigma_1 \\ \alpha_{\sigma_2 \sigma_3, \sigma_2} : \sigma_2 \sigma_3 | \sigma_2 \Rightarrow \sigma_2 \sigma_3 \sigma_2 & \alpha_{\sigma_3 \sigma_2, \sigma_3} : \sigma_3 \sigma_2 | \sigma_3 \Rightarrow \sigma_2 \sigma_3 \sigma_2 \\ \alpha_{\sigma_3 \sigma_1, \sigma_3} : \sigma_3 \sigma_1 | \sigma_3 \Rightarrow \sigma_3 \sigma_1 \sigma_3 & \alpha_{\sigma_1 \sigma_3, \sigma_1} : \sigma_1 \sigma_3 | \sigma_1 \Rightarrow \sigma_3 \sigma_1 \sigma_3 \end{array}$$

$$\begin{array}{l} \alpha_{\sigma_3, \sigma_1 \sigma_2 \sigma_1} : \sigma_3 | \sigma_1 \sigma_2 \sigma_1 \Rightarrow \sigma_3 \sigma_1 \sigma_2 \sigma_1 \\ \alpha_{\sigma_1, \sigma_2 \sigma_3 \sigma_2} : \sigma_1 | \sigma_2 \sigma_3 \sigma_2 \Rightarrow \sigma_1 \sigma_2 \sigma_3 \sigma_2 \\ \alpha_{\sigma_2, \sigma_3 \sigma_1 \sigma_3} : \sigma_2 | \sigma_3 \sigma_1 \sigma_3 \Rightarrow \sigma_2 \sigma_3 \sigma_1 \sigma_3 \end{array}$$

$$\begin{array}{ll} \alpha_{\sigma_3 \sigma_1, \sigma_2 \sigma_1} : \sigma_3 \sigma_1 | \sigma_2 \sigma_1 \Rightarrow \sigma_3 \sigma_1 \sigma_2 \sigma_1 & \alpha_{\sigma_3 \sigma_2, \sigma_1 \sigma_2} : \sigma_3 \sigma_2 | \sigma_1 \sigma_2 \Rightarrow \sigma_3 \sigma_1 \sigma_2 \sigma_1 \\ \alpha_{\sigma_1 \sigma_2, \sigma_3 \sigma_2} : \sigma_1 \sigma_2 | \sigma_3 \sigma_2 \Rightarrow \sigma_1 \sigma_2 \sigma_3 \sigma_2 & \alpha_{\sigma_1 \sigma_3, \sigma_2 \sigma_3} : \sigma_1 \sigma_3 | \sigma_2 \sigma_3 \Rightarrow \sigma_1 \sigma_2 \sigma_3 \sigma_2 \\ \alpha_{\sigma_2 \sigma_3, \sigma_1 \sigma_3} : \sigma_2 \sigma_3 | \sigma_1 \sigma_3 \Rightarrow \sigma_2 \sigma_3 \sigma_1 \sigma_3 & \alpha_{\sigma_2 \sigma_1, \sigma_3 \sigma_1} : \sigma_2 \sigma_1 | \sigma_3 \sigma_1 \Rightarrow \sigma_2 \sigma_3 \sigma_1 \sigma_3 \end{array}$$

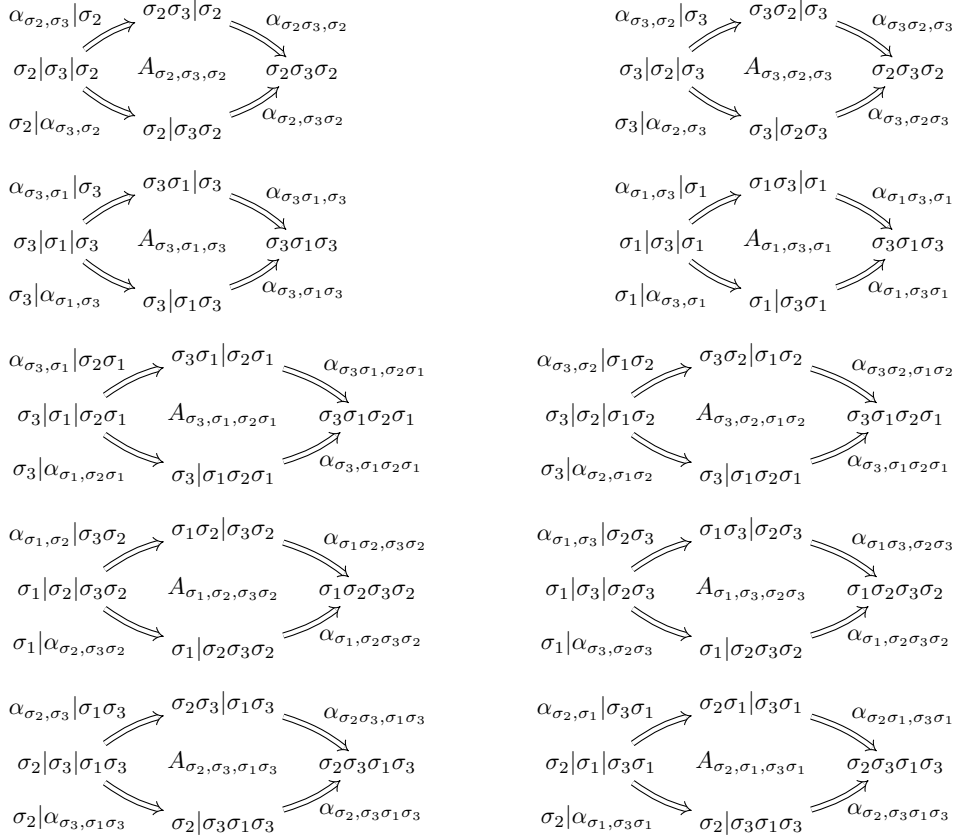
We proceed to construct the 3-cells $A_{u,v,w}$ for $u, v, w \in F \setminus \{1\}$ with $uv \in F$, $vw \in F$, and $uvw \in F$.

$$\begin{array}{ccc} \alpha_{\sigma_1, \sigma_2} | \sigma_1 & \xrightarrow{\quad} & \sigma_1 \sigma_2 | \sigma_1 \\ \sigma_1 | \sigma_2 | \sigma_1 & \xrightarrow{\quad} & \sigma_1 \sigma_2 \sigma_1 \\ \sigma_1 | \alpha_{\sigma_2, \sigma_1} & \xrightarrow{\quad} & \sigma_1 | \sigma_2 \sigma_1 \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \alpha_{\sigma_1, \sigma_2 \sigma_1} \\ & \xrightarrow{\quad} & \sigma_1 \sigma_2 \sigma_1 \\ & \xrightarrow{\quad} & \alpha_{\sigma_1, \sigma_2 \sigma_1} \end{array}$$

$A_{\sigma_1, \sigma_2, \sigma_1}$

$$\begin{array}{ccc} \alpha_{\sigma_2, \sigma_1} | \sigma_2 & \xrightarrow{\quad} & \sigma_2 \sigma_1 | \sigma_2 \\ \sigma_2 | \sigma_1 | \sigma_2 & \xrightarrow{\quad} & \sigma_1 \sigma_2 \sigma_1 \\ \sigma_2 | \alpha_{\sigma_1, \sigma_2} & \xrightarrow{\quad} & \sigma_2 | \sigma_1 \sigma_2 \end{array} \quad \begin{array}{ccc} & \xrightarrow{\quad} & \alpha_{\sigma_2 \sigma_1, \sigma_2} \\ & \xrightarrow{\quad} & \sigma_1 \sigma_2 \sigma_1 \\ & \xrightarrow{\quad} & \alpha_{\sigma_2, \sigma_1 \sigma_2} \end{array}$$

$A_{\sigma_2, \sigma_1, \sigma_2}$



We have thus computed the finite coherent presentation of the Artin-Tits monoid of type \tilde{A}_2 , which consists of fifteen generating 1-cells, twenty-seven generating 2-cells, and twelve generating 3-cells.

As in Subsection 6.3, we can further perform the homotopical reduction procedure. Let us construct a collapsible part. There are no critical triple branchings, so we set $\Gamma_4 := \emptyset$. For the component Γ_3 , we take the Nielsen transformations of the generating 3-cells

$$A_{\sigma_3, \sigma_1, \sigma_2 \sigma_1}, A_{\sigma_3, \sigma_2, \sigma_1 \sigma_2}, A_{\sigma_1, \sigma_2, \sigma_3 \sigma_2}, A_{\sigma_1, \sigma_3, \sigma_2 \sigma_3}, A_{\sigma_2, \sigma_3, \sigma_1 \sigma_3}, A_{\sigma_2, \sigma_1, \sigma_3 \sigma_1}$$

whose respective targets are the following generating 2-cells:

$$\alpha_{\sigma_3 \sigma_1, \sigma_2 \sigma_1}, \alpha_{\sigma_3 \sigma_2, \sigma_1 \sigma_2}, \alpha_{\sigma_1 \sigma_2, \sigma_3 \sigma_2}, \alpha_{\sigma_1 \sigma_3, \sigma_2 \sigma_3}, \alpha_{\sigma_2 \sigma_3, \sigma_1 \sigma_3}, \alpha_{\sigma_2 \sigma_1, \sigma_3 \sigma_1}.$$

We order the generating 2-cells as follows:

$$\alpha_{\sigma_3 \sigma_1, \sigma_2 \sigma_1} > \alpha_{\sigma_3 \sigma_2, \sigma_1 \sigma_2} > \alpha_{\sigma_1 \sigma_2, \sigma_3 \sigma_2} > \alpha_{\sigma_1 \sigma_3, \sigma_2 \sigma_3} > \alpha_{\sigma_2 \sigma_3, \sigma_1 \sigma_3} > \alpha_{\sigma_2 \sigma_1, \sigma_3 \sigma_1} > \\ > \text{other generating 2-cells in any order (as they are not targets in } \Gamma_3).$$

For the component Γ_2 , we take the generating 2-cells

$$\alpha_{\sigma_1, \sigma_2}, \alpha_{\sigma_2, \sigma_1}, \alpha_{\sigma_2, \sigma_3}, \alpha_{\sigma_3, \sigma_2}, \alpha_{\sigma_3, \sigma_1}, \alpha_{\sigma_1, \sigma_3}, \\ \alpha_{\sigma_1, \sigma_2 \sigma_1}, \alpha_{\sigma_2, \sigma_3 \sigma_2}, \alpha_{\sigma_3, \sigma_1 \sigma_3}, \alpha_{\sigma_3, \sigma_1 \sigma_2 \sigma_1}, \alpha_{\sigma_1, \sigma_2 \sigma_3 \sigma_2}, \alpha_{\sigma_2, \sigma_3 \sigma_1 \sigma_3}.$$

Any ordering of the generating 1-cells which respects the length in F^* will do.

The homotopical reduction of $\text{Gar}_3(F)$ with respect to the collapsible part $(\Gamma_2, \Gamma_3, \emptyset)$ contains: a single generating 0-cell; the generating 1-cells $\sigma_1, \sigma_2, \sigma_3$; the generating 2-cells $\alpha_{\sigma_2, \sigma_1 \sigma_2}, \alpha_{\sigma_3, \sigma_2 \sigma_3}, \alpha_{\sigma_1, \sigma_3 \sigma_1}$; and no generating 3-cells. As a

side result, we have thus shown that Artin’s presentation of the Artin-Tits monoid of type \tilde{A}_2 is coherent.

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