Minimal stretch maps between Euclidean triangles

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Abstract

Given two triangles whose angles are all acute, we find a homeomorphism with the smallest Lipschitz constant between them and we give a formula for the Lipschitz constant of this map. We show that on the set of pairs of acute triangles with fixed area, the function which assigns the logarithm of the smallest Lipschitz constant of Lipschitz maps between them is a symmetric metric. We show that this metric is Finsler, we give a necessary and sufficient condition for a path in this metric space to be geodesic and we determine the isometry group of this metric space.

This study is motivated by Thurston’s asymmetric metric on the Teichmüller space of a hyperbolic surface, and the results in this paper constitute an analysis of a basic Euclidean analogue of Thurston’s hyperbolic theory. Many interesting questions in the Euclidean setting deserve further attention.

Keywords: Thurston’s asymmetric metric, Teichmüller theory, space of Euclidean triangles, geodesics, Finsler structure

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1 Introduction

In this paper, we study a metric on a moduli space of Euclidean triangles which is an analogue of Thurston’s (asymmetric) metric on the Teichmüller space of a surface of finite type. Thurston introduced his metric in the 1985 preprint [16]. Since then, the metric has been studied from various viewpoints. A first survey appeared in 2007 [13], in which properties of this metric were compared to analogous properties of the Teichmüller metric. A more recent survey on this metric is in press [11]. The paper [13], published in 2015, contains a set of open problems on this metric. The paper [12] contains several results on the comparison of the Thurston metric with the Teichmüller metric. Several new techniques have been introduced recently in the study of Thurston’s metric; see in particular [1] [7], and the metric has been generalized to various settings, see [4] [6] for an analogous metric on spaces of geometrically finite hyperbolic manifolds and [8] for a generalization in the setting of higher Teichmüller theory. Euclidean analogues of this metric have also been studied, see [2] for an analogue on the Teichmüller space of Euclidean tori, [10] for a recent sequel, and the recent work [18] on an analogue on the moduli space of semi-translation surfaces.

There is a natural analogue of Thurston’s metric on a basic model space, namely, the moduli space of Euclidean triangles. This elementary setting has not been investigated yet. Our aim in this article is to settle the case of the moduli space of acute Euclidean triangles (that is, triangles whose three angles are acute), which turns out to be a natural space to study.

We now present the main results of this paper.

Consider a triangle in the Euclidean plane. Label its vertices by the set \{v_1, v_2, v_3\} so that this labeling induces a counter-clockwise orientation on the boundary of the triangle. We call such a triangle labeled.

Consider two labeled triangles \( T \) and \( T’ \) and let \( f : T \to T’ \) be a label-preserving homeomorphism. The Lipschitz constant of \( f \) is defined as

\[
L(f) = \sup_{x,y \in T, x \neq y} \frac{d_{\text{euc}}(f(x), f(y))}{d_{\text{euc}}(x, y)}
\]

where \( d_{\text{euc}} \) is the metric in the Euclidean plane. Now let

\[
L(T, T’) = \log\left(\inf\{L(f) : f \text{ is a label-preserving homeomorphism between } T \text{ and } T’\}\right).
\]

This formula induces a distance function on the space of Euclidean triangles which is an analogue of Thurston’s Lipschitz metric defined in the
hyperbolic setting (see [16, p. 4] where this distance function is also denoted by \( L \)).

We obtain the following:

- In the case where \( T \) and \( T' \) are acute triangles, we give another formula for the distance \( L(T, T') \), which we denote by \( m(T, T') \), in terms of the lengths of the edges and altitudes of the triangles \( T \) and \( T' \). The new formula is an analogue of Thurston’s definition of his distance in terms of lengths of simple closed geodesics (see [16, p. 4] where this distance function is denoted by \( K \)), and the equality \( L(T, T') = m(T, T') \) is an analogue of Thurston’s equality between his two distance functions \( K \) and \( L \) (see [16, p. 40]).

- For every \( A > 0 \), the metric induced by \( L \) on the space \( \mathcal{AT}_A \) of acute triangles having fixed area \( A \) is Finsler (this is an analogue of Thurston’s result in [16, p. 20]).

- We give a characterization of geodesics in \( \mathcal{AT}_A \): a path is geodesic if and only if the angle at each labeled vertex of a triangle in this family of triangles varies monotonically.

- The isometry group of \( \mathcal{AT}_A \) is isomorphic to \( S_3 \), the group of permutations of \( \{1, 2, 3\} \).

The paper is organized as follows. Section 2 is the technical heart of our work. For any two labeled triangles \( T \) and \( T' \), we define their Lipschitz distance, and then we define the distance \( m(T, T') \) and show that \( m(T, T') = L(T, T') \) for any acute triangles \( T \) and \( T' \). In Section 3 we introduce several spaces of triangles and study some of their topological and metric properties. Among these spaces, the space of acute triangles and the space of non-obtuse triangles having area \( A \) will play central roles in this paper. We denote these spaces by \( \mathcal{AT}_A \) and \( \mathcal{AT}_A \) respectively. In Section 4 we give a necessary and sufficient condition for a path in \( \mathcal{AT}_A \) to be a geodesic. In Section 5 we prove that the metric \( L \) on \( \mathcal{AT}_A \) is Finsler. We determine the isometry group of \( \mathcal{AT}_A \) in Section 6.

We introduce some notation used throughout the paper. Let \( T \) be a labeled triangle, with vertices \( v_1, v_2, v_3 \). For \( i \in \{1, 2, 3\} \), we denote the edge opposite to the vertex \( v_i \) by \( e_i \). We denote the angle at the vertex \( v_i \) by \( \theta_i \) and the altitude from the vertex \( v_i \) by \( h_i \). We let \( ||e_i|| \) and \( ||h_i|| \) be the lengths of \( e_i \) and \( h_i \), respectively. We denote the intersection of the altitude \( h_i \) with the line which contains the edge \( e_i \) by \( p_i \). A triangle with vertices
$v_1, v_2, v_3$ is denoted by $\Delta v_1 v_2 v_3$. The line segment between two points $x$ and $y$ in the Euclidean plane is denoted by $[x, y]$, and its length by $|[x, y]|$. Finally, $\text{Area}(T)$ will denote the area of a triangle $T$.

2 Acute triangles

Assume that $T$ and $T'$ are two acute triangles. We use the notation introduced above for the triangle $T$. Similarly, for $T'$, we denote the vertices by $v'_1, v'_2, v'_3$, the edge opposite to the end $v'_i$ by $e'_i$, the angle at the vertex $e'_i$ by $\theta'_i$, etc.

**Proposition 1.** For any two acute triangles $T$ and $T'$, we have:

$$\exp(L(T, T')) \geq \max \left\{ \frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|}, \frac{|h'_1|}{|h_1|}, \frac{|h'_2|}{|h_2|}, \frac{|h'_3|}{|h_3|} \right\}. $$

**Proof.** Since any label-preserving homeomorphism from $T$ to $T'$ sends $e_1$ to $e'_1$, $e_2$ to $e'_2$ and $e_3$ to $e'_3$, it is clear that

$$\exp(L(T, T')) \geq \max \left\{ \frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|} \right\}. $$

Let $f : T \to T'$ be a label-preserving homeomorphism. Since $T$ is acute it follows that each $p_i$ lies in the interior of the edge $e_i$. Therefore we have

$$d_{\text{euc}}(f(p_i), f(v_i)) = d_{\text{euc}}(f(p_i), v'_i) \geq |h'_i|.$$ 

Since

$$d_{\text{euc}}(p_i, v_i) = |h_i|,$$

it follows that $L(f) \geq \frac{|h'_i|}{|h_i|}$. Therefore for any two acute triangles $T$ and $T'$, we have:

$$\exp(L(T, T')) \geq \max \left\{ \frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|}, \frac{|h'_1|}{|h_1|}, \frac{|h'_2|}{|h_2|}, \frac{|h'_3|}{|h_3|} \right\},$$

which is the inequality we need.

For two arbitrary labeled triangles $T$ and $T'$ we define

$$m(T, T') = \log(\max \left\{ \frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|}, \frac{|h'_1|}{|h_1|}, \frac{|h'_2|}{|h_2|}, \frac{|h'_3|}{|h_3|} \right\}).$$

(1)
Assume that we scale the triangle $T$ by a factor $\lambda$ and the triangle $T'$ by a factor $\lambda'$, where $\lambda, \lambda' > 0$. Let us denote the scaled triangles by $\lambda T$ and $\lambda' T'$. This means that the triangle $\lambda T$ has edge lengths $\lambda ||e_1||$, $\lambda ||e_2||$ and $\lambda ||e_3||$ and that the triangle $\lambda' T'$ has side lengths $\lambda' ||e'_1||$, $\lambda' ||e'_2||$ and $\lambda' ||e'_3||$.

The following formulae are clear:

$$\exp(L(\lambda T, \lambda' T')) = \frac{\lambda'}{\lambda} \exp(L(T, T'))$$  \hspace{1cm} (2)

$$\exp(m(\lambda T, \lambda' T')) = \frac{\lambda'}{\lambda} \exp(m(T, T'))$$  \hspace{1cm} (3)

**Remark 1.** We shall prove that for any two acute triangles $T$ and $T'$, $L(T, T') = m(T, T')$. See Theorem 1. We shall use the following fact in the proof. If $\lambda, \lambda' > 0$, $L(T, T') = m(T, T')$ if and only if $L(\lambda T, \lambda' T') = m(\lambda T, \lambda' T')$. This follows from Inequalities (2) and (3).

### 2.1 Right triangles

Right triangles appear naturally as sitting on the boundary of the moduli space of acute triangle.

Assume that $T$ and $T'$ are two right triangles where $\theta_1$ and $\theta'_1$ are equal to $\frac{\pi}{2}$. See Figure 1. In this subsection we calculate $L(T, T')$.

**Proposition 2.** Let $T$ and $T'$ be two right triangles so that $\theta_1 = \theta'_1 = \frac{\pi}{2}$. Then

$$\exp(L(T, T')) = \max\{\frac{||e'_3||}{||e_3||}, \frac{||e'_2||}{||e_2||}\}.$$  \hspace{1cm} (4)

**Proof.** It is clear that

$$\exp(L(T, T')) \geq \max\{\frac{||e'_3||}{||e_3||}, \frac{||e'_2||}{||e_2||}\}. \hspace{1cm} (4)$$

Let us show that the other inequality is true as well.

We consider a Euclidean system of coordinates $(x, y)$ in $\mathbb{R}^2$.

By performing some isometries to $T$ and $T'$, we may assume that the vertices $v_1$ and $v'_1$ are at the origin, the sides $e_3$ and $e'_3$ are on the $x$-axis, and the sides $e_2$ and $e'_2$ are on the $y$-axis. Consider the following homeomorphism from $T$ to $T'$:

$$f : (x, y) \rightarrow \left(\frac{||e'_3||}{||e_3||} x, \frac{||e'_2||}{||e_2||} y\right).$$

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Figure 1: A best Lipschitz map between the two triangles is the affine map.
It is clear that

\[ L(f) \geq \max\{ \|e'_3\|, \|e'_2\| \}. \]

Let \( q_1 = (x_1, y_1) \) and \( q_2 = (x_2, y_2) \) be two points in \( T \). Then

\[
d_{\text{euc}}(f(q_1), f(q_2)) = d_{\text{euc}}((\|e'_2\|, \|e'_1\|, x_1, y_1), (\|e'_2\|, \|e'_1\|, x_2, y_2))
\]

\[
= \sqrt{(\|e'_3\|)^2(x_2 - x_1)^2 + (\|e'_2\|)^2(y_2 - y_1)^2}
\]

\[
\leq \max\{ \|e'_3\|, \|e'_2\| \} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

\[
= \max\{ \|e'_3\|, \|e'_2\| \} d(q_1, q_2).
\]

Therefore \( L(f) \leq \max\{ \|e'_3\|, \|e'_2\| \} \). Combined with (4), this gives \( L(f) = \max\{ \|e'_3\|, \|e'_2\| \} \). From this we conclude that \( \exp(L(T, T')) = \max\{ \|e'_3\|, \|e'_2\| \} \).

Note that \( \exp(L(T, T')) \) is given by an infimum and this infimum is attained by the map we just constructed. Therefore the map we defined above is a best Lipschitz map.

2.2 A problem related to right triangles

**Proposition 3.** Let \( T \) and \( T' \) be two acute triangles so that \( \theta_2 \leq \theta'_2 \) and \( \theta_3 \leq \theta'_3 \). Assume that \( \|h_1\| = \|h'_1\| \). Then \( \exp(L(T, T')) = 1 \) and there is a label-preserving homeomorphism \( f : T \to T' \) so that \( L(f) = 1 \).

**Proof.** We can move the triangle \( T \) into \( T' \) so that their altitudes coincide. See Figure 2. Therefore we have two triangles \( T = \Delta v_1v_2v_3 \) and \( T' = \Delta v'_1v'_2v'_3 \) so that \( v_1 = v'_1 \) and \( h_1 = h'_1 \). We want to find \( L(T, T') \). Since \( T \) and \( T' \) have the same altitude from \( v_1 = v'_1 \), we readily see that for any label-preserving homeomorphism \( f : T \to T' \), \( L(f) \geq 1 \). Now we construct a label-preserving homeomorphism \( f : T \to T' \) so that \( L(f) = 1 \); this will complete the proofs of the two statements in the proposition. Consider the homeomorphism \( f \) as in Section 2.1 which sends the triangle \( \Delta v_1v_2p_1 \) to
Figure 2: If the triangle $T'$ is contained in the triangle $T'$ and $T$ and $T'$ have the same altitudes, then $\exp(L(T, T')) = 1$. 
the triangle $\triangle v_1v_2p_1$. Also consider the homeomorphism $h$ as in Section 2.1 which sends the triangle $\triangle v_1p_1v_3$ to the triangle $\triangle v_1p_1v_3'$. Clearly $L(g) = L(h) = 1$ and since $g$ and $h$ agree on the altitude $h_1 = h'_1 = |v_1p_1|$, they induce a homeomorphism $f : T \to T'$. Furthermore it is clear that $L(f) = 1$.

\[\Box\]

### 2.3 Acute triangles

We are now ready to prove the main result on the space of acute triangles. It is an analogue, in the elementary case we are discussing, of a result of Thurston in [16] (see Corollary 8.5).

**Theorem 1.** If $T$ and $T'$ are two acute triangles, then

$$L(T,T') = m(T,T').$$

Furthermore, there exists a minimal Lipschitz map between the two acute triangles.

**Proof.** As already noted, we have $\exp(L(T,T')) \geq \exp(m(T,T'))$. See Proposition [1]. We will show that $\exp(L(T,T')) \leq \exp(m(T,T'))$. There exist distinct $i,j \in \{1,2,3\}$ such that either

1. $\theta_i \leq \theta'_i$ and $\theta_j \leq \theta'_j$, or
2. $\theta_i \geq \theta'_i$ and $\theta_j \geq \theta'_j$.

Without loss of generality we suppose that $\{i,j\} = \{2,3\}$.

Assume that the first case holds, that is, $\theta_2 \leq \theta'_2$ and $\theta_3 \leq \theta'_3$. We may scale the triangles $T$ and $T'$ so that they have altitudes of the same length from the vertex labeled by 1. That is, we may suppose that $|h_1| = |h'_1|$. In that case the triangle $T'$ can be moved inside the triangle $T$ so that their altitudes coincide. See Figure 2. Therefore there exists a homeomorphism $f : T \to T'$ such that $L(f) = 1$. Hence we have

$$\exp(L(T,T')) \leq L(f) = 1 \leq \exp(m(T,T')).$$

This completes the proof of the claim for the first case.

Assume now that the second case holds, that is, $\theta_2 \geq \theta'_2$ and $\theta_3 \geq \theta'_3$. After scaling the triangle $T$ by $\frac{1}{|v_1|}$ and the triangle $T'$ by $\frac{1}{|v'_1|}$, we may suppose that the lengths of $e_1$ and $e'_1$ are equal to 1. Also, we may assume that $T$ and $T'$ share an edge and two vertices, that is, $e_1 = e'_1$, $v_2 = v'_2$ and $v_3 = v'_3$. It follows that $T$ contains $T'$. See Figure 3. Consider the triangle
Figure 3: The case where $\theta_2 \geq \theta'_2$ and $\theta_3 \geq \theta'_3$
\[ \Delta q_1v_2v_3 \] where \( q_1 \) is the intersection of the line passing through the points \( v'_1 \) and \( v_3 \) with the line segment \([v_1, v_2]\). Consider the homeomorphism \( g \) sending \( \Delta v_1v_2v_3 \) to \( \Delta q_1v_2v_3 \) which is the identity on the triangle \( \Delta v_1p_1v_3 \) and which maps the \( \Delta v_1p_1v_3 \) to the triangle \( \Delta q_1p_1v_3 \) as in the Section 2.1. Clearly \( L(g) = 1 \).

Now consider the homeomorphism \( h \) from \( \Delta q_1v_2v_3 \) to \( \Delta v'_1v_2v_3 \) which is the identity on the triangle \( \Delta p'_1v_2v_3 \) and which sends \( \Delta q_1v_2p'_1 \) to \( \Delta v'_1v_2p'_2 \) as in Section 2.1. Clearly \( L(h) = 1 \).

Therefore \( h \circ g \) is a homeomorphism between \( T \) and \( T' \) satisfying \( L(h \circ g) = 1 \). We have

\[
\exp(L(T, T')) \leq L(h \circ g) = 1 \leq \exp(m(T, T')).
\]

In particular \( L(T, T') \) is given by an infimum which is attained by the map constructed in the proof of Theorem 1 when \( T \) and \( T' \) are acute triangles. Thus, there exists a best Lipschitz map between two acute triangles.

Remark 2. Assume that \( T \) and \( T' \) have the same area. Since

\[
||e_i|| \cdot ||h_i|| = ||e'_i|| \cdot ||h'_i||,
\]

it follows that

\[
\frac{||e_i||}{||e'_i||} = \frac{||h'_i||}{||h_i||} \text{ for any } i \in \{1, 2, 3\}.
\]

Therefore

\[
m(T, T') = \log(\max\{\frac{||e'_1||}{||e_1||}, \frac{||e'_2||}{||e_2||}, \frac{||e'_3||}{||e_3||}\})
\]

\[
= \max\{\log(||e'_i||) - \log(||e_i||)\}.
\]

Remark 3. (The case of obtuse triangles.) Consider Figure 4. Let \( T \) and \( T' \) be the triangles \( \Delta v_1v_2v_3 \) and \( \Delta v'_1v'_2v'_3 \), respectively. Note that \( v_2 = v'_2 \) and \( v_3 = v'_3 \). We will show that \( L(T, T') > m(T, T') \). Indeed it is easy to see that

\[
\exp(m(T, T')) = \frac{||h'_1||}{||h_1||} = \frac{3}{1} = 3.
\]

Consider the point \( p_1 \) which is the intersection of the altitude from the vertex \( v_1 \) and the edge \( e_1 \). If \( f \) is any label-preserving homeomorphism from \( T \) to \( T' \). Then \( f(p_1) \) is on the interior of the edge \( e_1 \). It follows that
Figure 4: Two triangles $T$ and $T'$ so that $L(T, T') > m(T, T')$.

\[ d_{\text{euc}}(f(p_1), f(v_1)) = d_{\text{euc}}(f(p_1), v_1') > d_{\text{euc}}(v_2, v_1) = 2\sqrt{3}. \]

Hence
\[ L(f) > \frac{d_{\text{euc}}(f(p_1), f(v_1))}{d_{\text{euc}}(p_1, v_1)} = 2\sqrt{3}. \]

Thus it follows that $\exp(L(T, T')) \geq 2\sqrt{3} > 3 = \exp(m(T, T'))$. Note that one can scale the triangles $T$ and $T'$ to get $\lambda T$ and $\lambda T'$ so that they have same area. In that case we have $L(\lambda T, \lambda T') > m(\lambda T, \lambda T')$.

2.4 Some Facts about Two Non-obtuse Triangles Sharing an Edge

Let $T$ be a triangle with vertices $v_1, v_2, v_3$ and $T'$ be a triangle with vertices $v_1', v_2', v_3'$ such that $v_1 = v_1'$ and $v_2 = v_2'$. Suppose that $T$ and $T'$ are non-obtuse and $T$ is contained in $T'$, as in Figure 5. We claim that

\[ \exp(m(T, T')) = \frac{||h_3'||}{||h_3||}. \]

Consider the line which passes through $v_3$ and which is parallel to the line passing through $v_1$ and $v_2$. Let $y$ be the point of intersection of this line with the edge $e_1'$. Then we have
Similarly

\[ \frac{||h'_3||}{||h_3||} = \frac{||e'_1||}{||(v_2, y)||} \geq \frac{||e'_1||}{||e_1||}. \]

Now we show that

\[ \frac{||h'_3||}{||h_3||} \geq \frac{||h'_1||}{||h_1||}. \]

We have

\[ \frac{||h'_3||}{||h_3||} = \frac{\text{Area}(T')}{\text{Area}(T)} = \frac{||h'_1||}{||h_1||}. \]

Since \( \frac{||e'_1||}{||e_1||} \geq 1 \), we have \( \frac{||h'_3||}{||h_3||} \geq \frac{||h'_1||}{||h_1||} \). Similarly, \( \frac{||h'_3||}{||h_3||} \geq \frac{||h'_2||}{||h_2||} \). Since the other arguments of \( \exp(m(T, T')) \) are less than or equal to 1, we have

\[ \exp(m(T, T')) = \frac{||h'_3||}{||h_3||}. \]
Now we want to scale $T$ so that the new triangle and $T'$ have the same area. Clearly we need to scale $T$ by the factor $\lambda = \frac{||h_3'||}{||h_3||}$. Let $T'' = \lambda T$. Then Equality 3 implies that
\[
\exp(m(T'', T')) = \lambda = \frac{||e_3'||}{||e_3||} = \frac{||e_3'||}{||e_3||},
\]
where $e_3'$ is the edge of $T''$ which is opposite the vertex with label 3, $v_3''$. Let us summarize the above discussion as a lemma.

**Lemma 1.** Let $T$ and $T'$ be labeled non-obtuse triangles having the same area. There exists $i \in \{1, 2, 3\}$ such that

1. $\theta_j \leq \theta_j'$ and $\theta_k \leq \theta_k'$, or
2. $\theta_j \geq \theta_j'$ and $\theta_k \geq \theta_k'$,

where $j, k \in \{1, 2, 3\}$, $j, k \neq i$. Then, we have
\[
\exp(m(T, T')) = \max\{ ||e_i|| : ||e_i'|| \}
\]
or equivalently,
\[
m(T, T') = |\log(||e_i'||) - \log(||e_i||)|.
\]
More precisely,

1. If $\theta_j \leq \theta_j'$ and $\theta_k \leq \theta_k'$
\[
\exp(m(T, T')) = \frac{||e_i||}{||e_i'||},
\]
2. and if $\theta_j \geq \theta_j'$ and $\theta_k \geq \theta_k'$
\[
\exp(m(T, T')) = \frac{||e_i'||}{||e_i||}.
\]

**Lemma 2.** Let $T$ and $T'$ be two labeled triangles of the same area with angles $(\theta_1, \theta_2, \theta_3)$ and $(\theta_1', \theta_2', \theta_3')$. Let $\{i, j, k\} = \{1, 2, 3\}$.

1. If $\theta_i > \theta_i'$, $\theta_j < \theta_j'$ and $\theta_k \leq \theta_k'$, then
\[
\exp(m(T, T')) = \frac{||e_i||}{||e_i'||} > \max\{ ||e_k|| : ||e_k'|| \}
\]

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2. If \( \theta_i > \theta'_i, \theta_j < \theta'_j \text{ and } \theta_k > \theta'_k \), then
\[
\exp(m(T, T')) = \frac{||e'_j||}{||e_j||} > \max\{\frac{||e_k||}{||e'_k||}, \frac{||e'_k||}{||e_k||}\}.
\]

**Proof.** Without loss of generality we will assume that \( i = 1, j = 2, k = 3 \).

1. By Lemma 1, we know that \( \exp(m(T, T')) = \frac{||e_1||}{||e'_1||} \), and clearly \( \exp(m(T, T')) > 1 \). This means that
\[
\frac{||e'_1||}{||e_1||} \cdot ||e'_2|| \sin \theta'_2 > ||e_1|| \cdot ||e_3|| \sin \theta_2,
\]
since \( \theta < \theta'_2 < \frac{\pi}{2} \). This is a contradiction since \( T \) and \( T' \) have the same area.

2. We know that
\[
1 < \exp(m(T, T')) = \frac{||e'_2||}{||e_2||} \geq \max\{\frac{||e_3||}{||e'_3||}, \frac{||e'_3||}{||e_3||}\}.
\]
If \( \frac{||e'_2||}{||e_2||} = \frac{||e_3||}{||e'_3||} \), then
\[
||e_2|| \cdot ||e_3|| \sin \theta_1 > ||e'_2|| \cdot ||e'_3|| \sin \theta'_1,
\]
which is a contradiction. If \( \frac{||e'_2||}{||e_2||} = \frac{||e'_3||}{||e_3||} \), then we have
\[
||e'_1|| ||e'_3|| \sin \theta'_2 = ||e_1|| ||e_3|| \sin \theta_2.
\]
Thus
\[
\frac{||e_1||}{||e'_1||} = \frac{||e'_3||}{||e_3||} \sin \theta'_2 > \frac{||e'_3||}{||e_3||} = \frac{||e'_2||}{||e_2||} = \exp(m(T, T')),
\]
which is impossible. This completes the proof of the claim.
Figure 6: Two triangles sharing an edge but one is not contained in the other.
Remark 4. Let $T$ and $T'$ be two labeled triangles of equal area so that $\theta_i > \theta_i'$ and $\theta_j < \theta_j'$. Scale $T$ to get a new triangle $T''$ so that $||e''_k|| = ||e'_k||$. See Figure 6. Then it follows from the discussion before Lemma 1 that

$$\max\left\{\frac{||e_k||}{||e'_k||}, \frac{||e'_k||}{||e_k||}\right\} = \max\left\{\frac{\sqrt{||h''_3||}}{||h'_3||}, \frac{\sqrt{||h'_3||}}{||h''_3||}\right\}$$

Thus Lemma 2 implies that

$$\exp(m(T, T')) > \max\left\{\frac{\sqrt{||h''_3||}}{||h'_3||}, \frac{\sqrt{||h'_3||}}{||h''_3||}\right\}.$$

3 The space of triangles

In this section we will consider the set of isometry classes of triangles. First we define the notion of a metric. The definition we give is different from the usual definition of a metric since we drop the symmetry axiom.

Definition 1. A metric on a set $X$ is a function $\eta : X \times X \to \mathbb{R}$ such that

- $\eta(x, x) = 0$ for all $x \in X$,
- $\eta(x, y) > 0$ if $x \neq y$,
- $\eta(x, y) + \eta(y, z) \geq \eta(x, z)$ for all $x, y, z \in X$.

The pair $(X, d)$ or the set $X$ is called a metric space. If $\eta(x, y) = \eta(y, x)$ for all $x, y \in X$, then the metric is called symmetric. Otherwise, the metric is called asymmetric.

For a fixed $A > 0$, let $\mathcal{T}_A$ be the set of equivalence classes of labeled triangles with area $A$, where two triangles are equivalent if there is an isometry between them which respects the labeling.

We show that $L$ defines a metric on $\mathcal{T}_A$. If $T$ is a labeled triangle with area $A$, then we denote its equivalence class in $\mathcal{T}_A$ as $[T]$. It is clear that $L$ gives a well-defined function on $\mathcal{T}_A \times \mathcal{T}_A$. We denote this function by $\mathcal{L}$ as well.

The proof of the following lemma is similar to that of Proposition 2.1 of [16].

Lemma 3. Let $T$ and $T'$ be two labeled triangles of equal area. If $L(T, T') \leq 0$, then $L(T, T') = 0$ and $T$ and $T'$ are isometric.
Proof. For any $\lambda \geq 0$ the set of $\lambda$-Lipschitz maps from $T$ to $T'$ which respect the labeling is equicontinuous. Suppose that $L(T, T') \leq 0$. Then we can pick a map $f : T \to T'$ which has minimum global Lipschitz constant $e^{L(T, T')} \leq 1$.

If $L(T, T') < 0$, we can assume that $f$ is a homeomorphism and that its Lipschitz constant (which may be larger than $e^{L(T, T')}$) is $< 1$. But then we can cover the surface $T$ by a countable family of discs of different radii such that the interiors of these discs are disjoint and their complement has measure zero, such that $f$ maps one of these discs to a disc of strictly smaller radius. This is impossible since $T$ and $T'$ have the same area. Thus, $L(T, T') = 0$.

Now taking a covering of the surface $T$ by a countable family of discs of different radii such that the interiors of these discs are disjoint and their complement has measure zero, we conclude, since the Lipschitz constant of this map is 1, that each such disc is mapped by $f$ surjectively onto a disc of the same radius. Repeating the same argument with a covering of $T$ by discs whose radii tend uniformly to zero, we see that $f$ is an isometry. \(\square\)

Note that Lemma 3 implies that for any $[T], [T'] \in \mathcal{T}_A$ we have $L([T], [T']) \geq 0$ and $L([T], [T']) = 0$ if and only if $[T] = [T']$.

Lemma 4. If $T, T', T''$ are three labeled triangles of the same area, then

$$L(T, T') + L(T', T'') \geq L(T, T'').$$

Proof. Let $f : T \to T'$ and $g : T' \to T''$ be label-preserving homeomorphisms. The assertion follows from the fact that

$$L(g \circ f) \leq L(g) L(f).$$

\(\square\)

The following theorem is immediate from Lemma 3 and Lemma 4.

**Theorem 2.** The function $L$ is a metric on $\mathcal{T}_A$.

Clearly the function $m$ defined in Equation (1) induces a function on $\mathcal{T}_A \times \mathcal{T}_A$. We denote this function by $m$ as well. Our next objective is to prove that $m$ is a symmetric metric on $\mathcal{T}_A$.

**Theorem 3.** $m$ is a symmetric metric on $\mathcal{T}_A$. 18
Proof. Let \([T], [T'] \in \mathcal{X}_A\). Then \(T\) has edges of length \(|e_1|, |e_2|, |e_3|\) and \(T'\) has edges of length \(|e'_1|, |e'_2|, |e'_3|\). Since \(T\) and \(T'\) have same area, by Remark 2 we have

\[
m(T, T') = \log(\max\left\{\frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|}, \frac{|e_1|}{|e'_1|}, \frac{|e_2|}{|e'_2|}, \frac{|e_3|}{|e'_3|}\right\}).
\]

It is clear from this formula that \(m\) separates points and \(m\) is symmetric. Let \(T''\) be another triangle with area \(A\) having edges with length \(|e''_1|, |e''_2|, |e''_3|\). It follows from the following inequality that \(m\) satisfies the triangle inequality:

\[
\max\left\{\frac{|e'_1|}{|e_1|}, \frac{|e'_2|}{|e_2|}, \frac{|e'_3|}{|e_3|}, \frac{|e_1|}{|e'_1|}, \frac{|e_2|}{|e'_2|}, \frac{|e_3|}{|e'_3|}\right\} \\
\times \max\left\{\frac{|e''_1|}{|e'_1|}, \frac{|e''_2|}{|e'_2|}, \frac{|e''_3|}{|e'_3|}, \frac{|e'_1|}{|e''_1|}, \frac{|e'_2|}{|e''_2|}, \frac{|e'_3|}{|e''_3|}\right\} \\
\geq \max\left\{\frac{|e''_1|}{|e_1|}, \frac{|e''_2|}{|e_2|}, \frac{|e''_3|}{|e_3|}, \frac{|e_1|}{|e''_1|}, \frac{|e_2|}{|e''_2|}, \frac{|e_3|}{|e''_3|}\right\}
\]

If there is no risk of confusion, we will denote the equivalence class of a labeled triangle \(T\) in \(\mathcal{X}_A\) by \(T\) as well.

### 3.1 The space of acute triangles

We denote the set of equivalence classes of acute triangles by \(\mathcal{T}\). For each fixed \(A > 0\), let \(\mathcal{T}_A\) be the set of equivalence classes of acute triangles having area \(A\). By Theorem 1 the restrictions of \(L\) and \(m\) on \(\mathcal{T}_A\) give the same metric.

### 3.2 Two models for \(\mathcal{T}_A\)

In this section we introduce two models of \(\mathcal{T}_A\). First, since each labeled triangle is determined by the length of its edges, there is an injection \(\mathcal{T}_A \to \mathbb{R}^3_+\) sending the class of a labeled triangle \(T\) to \((|e_1|, |e_2|, |e_3|))\), where \(e_1, e_2, e_3\) are the edges of \(T\). The image of this map is a 2-dimensional submanifold of \(\mathbb{R}^3_+\). Furthermore \(m\) induces a metric in this manifold and we denote this metric by \(m\) as well. Clearly, if \(a = (a_1, a_2, a_3)\) and \(a' = (a'_1, a'_2, a'_3)\) are in this manifold, then
\[ m(a, a') = \log(\max\left\{ \frac{a_1}{a'_1}, \frac{a_1}{a'_1}, \frac{a_2}{a'_2}, \frac{a_3}{a'_3}, \frac{a_1}{a'_1}, \frac{a'_1}{a}_2, \frac{a_3}{a'_3}, \frac{a'_3}{a}} \right\}). \]

If there is no risk of confusion we will denote this submanifold by \( \mathfrak{AT} \) as well. We will call this model as the edge model.

Now we introduce the other model. We first note that each labelled triangle having area \( A \) is determined by its angles. That is, the map \( \mathfrak{AT} \to \mathbb{R}^+ \times \mathbb{R}^+ \) sending a triangle \( T \) to \( (\theta_1, \theta_2, \theta_3) \) is injective, where \( \theta_1, \theta_2, \theta_3 \) are angles at the vertices of \( T \). Its image is the interior of a Euclidean equilateral triangle in a hyperplane in \( \mathbb{R}^3 \). If there is no risk of confusion, we will also denote this image by \( \mathfrak{AT} \). We will denote by \( (\theta_1, \theta_2, \theta_3) \) the triangle having angles \( \theta_1, \theta_2, \theta_3 \). We will call this model the angle model.

### 3.3 Topology of \( \mathfrak{T}_A \)

We consider the edge model for \( \mathfrak{T}_A \). It is equipped with two topologies: the one which comes from the Euclidean metric (the one induced from the ambient space) and the one which comes from \( m \). We denote these topologies by \( T_{\text{euc}} \) and \( T_m \), respectively.

**Proposition 4.** \( T_{\text{euc}} \) and \( T_m \) coincide.

*Proof.* It is enough to show that the identity map of \( \mathfrak{T}_A \) is a homeomorphism. This can be done by proving that a sequence \( T_n = (a_n, b_n, c_n) \) with respect to \( m \) if and only if \( T_n \to (a, b, c) \) with respect to \( d_{\text{euc}} \). If \( (a_n, b_n, c_n) \to (a, b, c) \) with respect to \( d_{\text{euc}} \), then \( \log a_n \to \log a \), \( \log b_n \to \log b \) and \( \log c_n \to \log c \). Thus \( (a_n, b_n, c_n) \to (a, b, c) \) with respect to \( m \) as \( n \to \infty \). If \( (a_n, b_n, c_n) \to (a, b, c) \) with respect to \( m \), then \( \log a_n - \log a \to 0 \), \( \log b_n - \log b \to 0 \) and \( \log c_n - \log c \to 0 \) as \( n \to \infty \). Then \( a_n \to a \), \( b_n \to b \) and \( c_n \to c \) as \( n \to \infty \), but this means that \( (a_n, b_n, c_n) \to (a, b, c) \) with respect to \( d_{\text{euc}} \) as \( n \to \infty \). \( \square \)

### 3.4 The Space of non-obtuse triangles

We will prove that \( \mathfrak{T}_A \) is complete and introduce the space of non-obtuse triangles as the closure of \( \mathfrak{AT} \) in \( \mathfrak{T}_A \).

**Proposition 5.** \( (\mathfrak{T}_A, m) \) is complete.

*Proof.* We use the edge model of \( \mathfrak{T}_A \). Let \( T_n = (a_n, b_n, c_n) \) be a Cauchy sequence of triangles with edge lengths \( a_n, b_n, c_n \). It follows that \( \log a_n, \log b_n, \log c_n \) are Cauchy sequences. Then we have
\[ \log a_n \to \log a, \log b_n \to \log b, \log c_n \to \log c. \]

\((a_n, b_n, c_n) \to (a, b, c)\) (with respect to \(m\)). \((a, b, c)\) is a triangle with area \(A\) since area is a continuous function of the length of edges of triangles, and \((a, b, c)\) satisfies strict triangle inequalities since the corresponding “triangle” is non-degenerate. Indeed any degenerate triangle has zero area.

Furthermore, the set of non-obtuse triangles is a closed subset of \(T_A\). This follows from the fact that the angle functions are continuous. The set of acute triangles \(AT_A\) is an open subset of \(T_A\) and it is easy to see that its closure in \(T_A\) is the set of non-obtuse triangles of area \(A\), which is denoted by \(AT_A\).

### 3.5 \(T_A\) and \(T_A'\) are isometric

For any two triangles \(T\) and \(T'\), we have

\[ m(\lambda T, \lambda T') = m(T, T'), \]

It follows that \(T_A\) and \(T_{A'}\) are isometric by the map sending the class of a triangle \(T\) to the class of the triangle \(\lambda T\). Similarly, for any \(A, A' > 0\), \(AT_A\) and \(AT_A'\) are isometric.

### 4 Geodesics in \(AT_A\)

Let \((X, d)\) be a metric space where \(d\) is not necessarily symmetric. We use the following version of the notion of geodesic:

**Definition 2.** A geodesic in \(X\) is a map \(h : I \to X\) where \(I\) is an interval of \(\mathbb{R}\) such that for any triple \(x_1, x_2, x_3 \in I\) satisfying \(x_1 \leq x_2 \leq x_3\), we have

\[ d(h(x_1), h(x_3)) = d(h(x_1), h(x_2)) + d(h(x_2), h(x_3)). \]

We also require that \(h\) is not constant on any subinterval of \(I\).

Note that if \(d\) is symmetric, a map obtained from \(h\) by reversing the direction of a geodesic is also a geodesic. But this is not necessarily true for asymmetric metrics. Note also that if \(d\) is asymmetric, one needs, in the above definition, to be careful about the order of the arguments.

For \(A > 0\), let \(T = (a_1, a_2, a_3)\) and \(T' = (a'_1, a'_2, a'_3)\) be two different elements in \(AT_A\), where we use the edge model. In this section we prove
that $T$ and $T'$ can be joined by a geodesic. In other words, we show that the metric space $(\mathbb{F}A, m)$ is geodesic.

As before, there is unique $i \in 1, 2, 3$ such that one of the following holds:

1. $\theta_j \leq \theta'_j$ and $\theta_k \leq \theta'_k$,
2. $\theta_j \geq \theta'_j$ and $\theta_k \geq \theta'_k$,

where $j, k \in \{1, 2, 3\}$ and $j, k \neq i$. Without loss of generality, we assume that $i = 3$ and

$$\theta_1 \geq \theta'_1 \text{ and } \theta_2 \geq \theta'_2,$$

and we shall find a geodesic from $T$ to $T'$. Since geodesics are reversible in a symmetric metric space, this argument should cover the case

$$\theta_1 \leq \theta'_1 \text{ and } \theta_2 \leq \theta'_2.$$

Consider the scaled triangles $\bar{T} = \frac{1}{a_3} T = (a_1, \frac{a_2}{a_3}, 1)$ and $\bar{T}' = \frac{1}{a_3'} (\frac{a'_1}{a_3'}, \frac{a'_2}{a_3'}, 1)$. The fact that they have an edge of equal length and the conditions on the angles imply that $\bar{T}'$ can be drawn inside $\bar{T}$. See Figure 7. Let the set of vertices of $\bar{T}'$ be $\{\bar{v}'_1, \bar{v}'_2, \bar{v}'_3\}$ and the set of vertices of $\bar{T}$ be $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$. Consider a family of triangles $\{\bar{T}_t\}, t \in [0, 1], \text{ with angles } (\theta_1(t), \theta_2(t), \theta_3(t))$ having the following properties:

1. For each $t$, $\bar{T}_t$ is a triangle so that two of its vertices are $\bar{v}_1$ and $\bar{v}_2$.
2. $\bar{T}_0 = \bar{T}$ and $\bar{T}_1 = \bar{T}'$.
3. Each $\theta_i(t)$ is either a decreasing or increasing function.
4. Each $\theta_i(t)$ is a continuous function.
5. $(\theta_1(t), \theta_2(t), \theta_3(t))$ is not constant on any subinterval of $[0, 1]$.

See Figure 7 for an example of such a family.

If we scale each $\bar{T}_t$ by a factor $\lambda(t)$ so that the area of $\lambda(t)\bar{T}_t$ is $A$, then we get a family of triangles $T_t = (a_1(t), a_2(t), a_3(t))$ so that $T_0 = (a_1, a_2, a_3)$ and $T_1 = (a'_1, a'_2, a'_3)$. By Lemma 1 this family has the following property:

$$\exp(m(T_{t_1}, T_{t_2})) = \frac{a_3(t_2)}{a_3(t_1)}$$

(8)

for any $t_1 \leq t_2, t_1, t_2 \in [0, 1]$. It follows that if $t_1 \leq t_2 \leq t_3$ then
Figure 7: $\alpha(t)$ is a parametrization of the line segment between $\bar{v}_3$ and $\bar{v}'_3$. The family of triangles $T_t = \Delta\bar{v}_1\bar{v}_2\alpha(t)$ is a geodesic in $\mathfrak{A}_A$.

Thus, $m(T_{t_1}, T_{t_3}) = m(T_{t_1}, T_{t_2}) + m(T_{t_2}, T_{t_3})$.

This means that the 1-parameter family of triangles $T_t$ forms a geodesic. Therefore we proved the following.

**Theorem 4.** The space $(\mathfrak{A}_A, m)$ is geodesic, that is, any two points of $\mathfrak{A}_A$ can be joined by a geodesic.

In fact, we also proved that any two distinct point in $\mathfrak{A}_A$ can be joined by a special geodesic which is a straight line segment in the angle model of $\mathfrak{A}_A$. To be more precise let $T = (\theta_1, \theta_2, \theta_3)$ and $T' = (\theta'_1, \theta'_2, \theta'_3)$. Let

$$\theta_i(t) = (1 - t)\theta_i(t) + t\theta'_i(t), t \in [0, 1], i \in \{1, 2, 3\}.$$ 

Then clearly $T_t = (\theta_1(t), \theta_2(t), \theta_3(t))$ is a geodesic joining $T$ and $T'$.  

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Remark 5. If $T$ and $T'$ are two elements in $\mathfrak{F}_A$ that have different angles at each vertex, then there are uncountably many geodesics joining them up to parametrization. But if $T$ and $T'$ have equal angle at a vertex, then up to parametrization there is a unique geodesic joining them.

Now we determine when a path in $\mathfrak{F}_A$ is a geodesic.

Theorem 5. Let $T_t = (\theta_1(t), \theta_2(t), \theta_3(t))$, $t \in [a, b]$ be a continuous family of triangles which is not constant on any subinterval of $[a, b]$. Then $T_t$ is a geodesic if and only if each $\theta_i(t)$ is increasing or decreasing.

Proof. The above argument shows that if each $\theta_i(t)$ is decreasing or increasing, then $T_t$ is a geodesic. To prove the converse, we assume that there is a family $T_t$ which is a geodesic but not all $\theta_i(t)$ are monotone. We also assume that $\theta_1(a) \leq \theta_1(b)$, $\theta_2(a) \leq \theta_2(b)$ and $\theta_3(a) > \theta_3(b)$. It follows that one of the $\theta_i(t)$ is not monotone. Since it is not possible that both of $\theta_1(t)$ and $\theta_2(t)$ are increasing, but $\theta_3(t)$ is not monotone, we should only consider the cases where $\theta_1(t)$ or $\theta_2(t)$ are not increasing. By symmetry, we only need to assume that $\theta_1$ is not increasing. Since $\theta_1(a) \leq \theta_1(b)$, there exists $t_1, t_2, t_3 \in [a, b]$ such that $t_1 < t_2 < t_3$ and $\theta_1(t_1) < \theta_1(t_2) > \theta_1(t_3)$.

Consider the triangles $T_{t_1}, T_{t_2}$ ans $T_{t_3}$. Scale these triangles so that the edges opposite to the vertex with label 3 have the same length. See Figure 8 Let $\bar{T}_{t_i}$ be the scaled triangles. Let $h_3(t_i)$ be the altitude of $\bar{T}_{t_i}$ from the vertex labeled by 3. We have

$$m(T_{t_1}, T_{t_3}) = \log\left(\frac{\sqrt{||h_3(t_3)||}}{\sqrt{||h_3(t_1)||}}\right),$$

$$m(T_{t_1}, T_{t_2}) \geq \log\left(\frac{\sqrt{||h_3(t_2)||}}{\sqrt{||h_3(t_1)||}}\right),$$

$$m(T_{t_2}, T_{t_3}) > \log\left(\frac{\sqrt{||h_3(t_3)||}}{\sqrt{||h_3(t_2)||}}\right),$$

(see Section 2.4). Thus, $m(T_{t_1}, T_{t_3}) < m(T_{t_1}, T_{t_2}) + m(T_{t_2}, T_{t_3})$. Hence the family $\{T_t\}$ does not form a geodesic. \qed

5 The Finsler Structure on $\mathfrak{F}_A$

Let us first recall the notion of a Finsler structure on a differentiable manifold. We work in the global Finsler setting adopted e.g. in [14] (that
Figure 8
is, without the tensor apparatus). We shall prove that the metric space 
\((\mathfrak{A}_A, m)\), or, equivalently, \((\mathfrak{A}_A, L)\), is Finsler, that is, it is a length metric
associated with a Finsler structure. We start with the definition of a weak norm.

**Definition 3.** Let \(V\) be a real vector space. A weak norm on \(V\) is a function \(V \to [0, \infty)\) such that for every \(v \in V\) the following properties hold for every \(v\) and \(w\) in \(V\):

1. \(||v|| = 0\) if and only if \(v = 0\),
2. \(||tv|| = t||v||\) for every \(t > 0\),
3. \(||tv + (1 - t)w|| \leq t||v|| + (1 - t)||w||\) for every \(t \in [0, 1]\).

Let \(M\) be a differentiable manifold and \(TM\) be the tangent bundle of \(M\).

**Definition 4.** A Finsler structure on \(M\) is a function \(F : TM \to [0, \infty)\) such that

1. \(F\) is continuous,
2. for each \(x \in M\), \(F|_{T_xM}\) is a weak norm.

Let \(F\) be a Finsler structure on a manifold \(M\). For each \(C^1\) curve \(c : [a, b] \to M\), we define

\[ l(c) = l_F(c) = \int_a^b F(\dot{c}(t))dt. \]

**Definition 5.** A metric \(d\) on a differentiable manifold \(M\) is called Finsler if it is the length metric associated to a Finsler structure, that is, if there exist a Finsler structure \(F\) on \(M\) such that for every \(x, y \in M\) we have

\[ d(x, y) = \inf\{l_F(c)\} \]

where \(c\) ranges over all piecewise \(C^1\) curves such that \(c(0) = x\) and \(c(1) = y\).

Now we show that the metric \(m\) on \(\mathfrak{A}_A\) is Finsler. Recall that we identified \(\mathfrak{A}_A\) with the submanifold in \(\mathbb{R}^3_+\):

\[ \mathfrak{A}_A = \{(a_1, a_2, a_3) \in \mathbb{R}^3_+ : a_1 + a_2 - a_3 > 0, a_2 + a_3 - a_1 > 0, a_3 + a_1 - a_2 > 0, \text{Area}(a_1, a_2, a_3) = A\}. \]
Let $F$ be the following function on $\mathfrak{A}_A$:

$$(a_1, a_2, a_3, v_1, v_2, v_3) \mapsto \max_i \left\{ \frac{|v_i|}{a_i} \right\}.$$  

Here, we have identified the tangent space of the manifold $\mathfrak{A}_A$ at a point $(a_1, a_2, a_3)$ with the Euclidean subspace of $\mathbb{R}^{3+}$ spanned by the set $\mathfrak{A}_A$ itself, and $(v_1, v_2, v_3)$ are the coordinates of a tangent vector in the tangent space at the point $(a_1, a_2, a_3)$.

It is not difficult to see that $F$ is a Finsler structure on $\mathfrak{A}_A$.

**Theorem 6.** The metric $m$ on $\mathfrak{A}_A$ is Finsler. More precisely, the metric $m$ is the length metric associated with the Finsler structure $F$.

**Proof.** Let $a' = (a_1', a_2', a_3')$ and $a'' = (a_1'', a_2'', a_3'')$ be in $\mathfrak{A}_A$. As in Section 4, we may assume that

$$\exp(m(a', a'')) = \frac{a''_3}{a'_3}$$

and therefore there exists a $C^1$ function $g : [0, 1] \to \mathfrak{A}_A$, $g(t) = (a_1(t), a_2(t), a_3(t))$ such that $g(0) = a'$, $g(1) = a''$ and

$$\exp(m(c(t_1), c(t_2))) = \frac{a_3(t_2)}{a_3(t_1)}$$

for all $t_1, t_2 \in [0, 1], t_1 \leq t_2$. Note that this implies that $a_3(t)$ is increasing.

We claim that

$$F(\dot{g}(t)) = \left| \frac{d(a_3(t))}{a_3(t)} \right| = \frac{\dot{a}_3(t)}{a_3(t)}$$

for each $t_0 \in [0, 1]$. Otherwise there exists $t_1 \in [0, 1]$ and $i \in \{1, 2\}$ such that

$$\frac{|\dot{a}_i(t_1)|}{|a_i(t_1)|} > \frac{\dot{a}_3(t_1)}{a_3(t_1)}.$$

Assume that

$$\frac{\dot{a}_i(t_1)}{a_i(t_1)} > \frac{\dot{a}_3(t_1)}{a_3(t_1)}.$$

It follows that $\frac{d}{dt}(\log(\frac{a_i(t)}{a_3(t)})) > 0$ at $t = t_1$ and we may assume that $t_1 < 1$. Therefore there is $t_2 > t_1$ such that

$$\log\left(\frac{a_i(t_2)}{a_3(t_2)}\right) > \log\left(\frac{a_i(t_1)}{a_3(t_1)}\right).$$
Hence we have

\[
\frac{a_i(t_2)}{a_3(t_2)} > \frac{a_i(t_1)}{a_3(t_1)}, \text{ or equivalently } \frac{a_i(t_2)}{a_i(t_1)} > \frac{a_3(t_2)}{a_3(t_1)},
\]

This implies that \( \exp(m(g(t_1), g(t_2))) \geq \frac{a_i(t_2)}{a_i(t_1)} > \frac{a_3(t_2)}{a_3(t_1)}, \) which is a contradiction. In a similar manner we can show that

\[
\frac{-\dot{a}_i(t)}{a_i(t)} \leq \frac{\dot{a}_3(t)}{a_3(t)}.
\]

Equation (9) implies that

\[
l_F(g(t)) = \int_0^1 F(\dot{g}(t)) = \int_0^1 \frac{\dot{a}_3(t)}{a_3(t)} dt = \log a_3(1) - \log a_3(0) = \log a''_3 - \log a'_3 = m(a', a'').
\]

Now we show that for any \( C^1 \) curve \( c : [0, 1] \to \mathcal{A}_A \), \( c(t) = (a_1(t), a_2(t), a_3(t)) \), we have

\[
l_F(c) \geq m(a', a'')
\]

We have

\[
L_f(c) \geq \int_0^1 \left| \frac{\dot{a}_i(t)}{a_i(t)} \right| dt \geq |\log a_i(1) - \log a_i(0)| = |\log a''_i - \log a'_i|.
\]

Hence we have

\[
L_f(c) \geq \max_i \{ |\log a''_i - \log a'_i| \} = m(a, a').
\]

We conclude that

\[
m(a', a'') = \inf l_F(c),
\]

where \( c : [0, 1] \to \mathcal{A}_A \) is a \( C^1 \) curve such that \( c(0) = a' \) and \( c(1) = a'' \). \( \square \)
6 Symmetries of the space $\mathfrak{AT}$

In this section we study the isometry group $\text{Isom}(\mathfrak{AT})$ of $\mathfrak{AT}$ with respect to the metric $m$.

The space $\mathfrak{AT}$ has three boundary components, each of them corresponding to one angle of the triangle becoming right. If we consider the angle model and fix $i \in \{1, 2, 3\}$, then a boundary component is given as follows:

\[\{ (\theta_1, \theta_2, \theta_3) : 0 < \theta_1, \theta_2, \theta_3 \leq \frac{\pi}{2}, \theta_1 + \theta_2 + \theta_3 = \pi, \theta_i = \frac{\pi}{2} \}.\]

This shows first that topologically $\mathfrak{AT}$ is a disc with three punctures on its boundary. A puncture may be regarded as a “triangle” with angles $\frac{\pi}{2}, \frac{\pi}{2}, 0$ and area $A$. Furthermore these boundary components are geodesics since any injective continuous map from $[0, 1]$ to a boundary component is a geodesic. This can easily be deduced from Theorem 5. We represent the boundary component for which $\theta_i = \frac{\pi}{2}$ by $B_i$. Note that in the angle model, $\mathfrak{AT}$ is just a Euclidean triangle with three punctures at its vertices.

It is not difficult to see that $\mathfrak{AT}$ is unbounded. Let us denote the equilateral triangle having area $\frac{1}{2}$ by $T_e$. Consider a sequence of isosceles triangles $T_n = (\theta_1(n), \theta_2(n), \theta_3(n))$ so that $\theta_2(n) = \theta_3(n)$ and $\lim_{n \to \infty} \theta_3(n) = \frac{\pi}{2}$. It requires a simple calculation to show that $m(T_e, T_n) \to \infty$ as $n \to \infty$. Now consider a sequence of triangles $T'_n = (\theta_1(n), \frac{\pi}{2}, \frac{\pi}{2})$, where $\theta_1(n) \to \frac{\pi}{2}$ as $n \to \infty$. Let us also consider the sequence $T''_n = (\frac{\pi}{2}, \theta_1(n), \theta_3(n))$. It can be shown that $m(T'_n, T''_n) \to 0$ as $n \to \infty$. See Remark 6. This means that $\mathfrak{AT}$ resembles an ideal triangle in the hyperbolic plane: it has three geodesic boundary components and any two of these components is a line which converges from each side to a puncture.

Now we consider the isometry group $\text{Isom}(\mathfrak{AT})$. The symmetric group $S_3 = \text{Sym}\{1, 2, 3\}$ can be regarded as a subgroup of $\text{Isom}(\mathfrak{AT})$ as follows. Let $\sigma \in S_3$ and consider the length model for $\mathfrak{AT}$. We identify $\sigma$ with the map sending $(a_1, a_2, a_3)$ to $(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$. It is not difficult to see that this map is an isometry of $\mathfrak{AT}$. Thus we may consider $S_3$ as a subgroup of $\text{Isom}(\mathfrak{AT})$. We wish to prove now that there are no other isometries, that is, $S_3 = \mathfrak{AT}$.

Let $\{i, j, k\} = \{1, 2, 3\}$. The 2-uple $\{T, T'\}$ is called an $i$-pair if $T \in B_j$, $T' \in B_k$ and $\theta_k = \theta'_j$. Note that since $\theta_i = \theta'_i$, there is a unique geodesic between $T$ and $T'$ up to parametrization.
**Lemma 5.** If \( \{T, T'\} \) and \( \{T_1, T'_1\} \) are different \( i \)-pairs, then \( m(T, T') \neq m(T_1, T'_1) \).

**Proof.** Without loss of generality assume that \( i = 1, j = 2, k = 3 \). Since \( A_T A \) and \( A_T A'_1 \) are isometric, we may suppose that \( A = \frac{1}{2} \). This will make the calculations easier. Then in the edge model

\[
T = (a, \sqrt{a^2 + \frac{1}{a^2}}, \frac{1}{a}), T' = (a, \frac{1}{a}, \sqrt{a^2 + \frac{1}{a^2}}).
\]

It follows that

\[
m(T, T') = \log \sqrt{a^2 + \frac{1}{a^2}} - \log \frac{1}{a} = \frac{1}{2} \log(1 + a^4),
\]

which is an injective function of \( a \). This proves the claim.

**Remark 6.** As \( a \to 0 \), \( m(T, T') = \frac{1}{2} \log(1 + a^4) \to 0 \). This means that the distance between two boundary components is arbitrarily small near a puncture.

**Theorem 7.** \( \text{Isom}(A_T A) = S_3 \).

**Proof.** It suffices to prove that an isometry that sends \( B_i \) to itself for all \( i \in \{1, 2, 3\} \) is the identity. Let \( \sigma \) be such an isometry. Consider an \( i \)-pair \( \{T, T'\} \). First of all observe that \( \{\sigma(T), \sigma(T')\} \) is an \( i \)-pair. This is true since two elements in different boundary components are \( l \)-pairs for some \( l \in \{1, 2, 3\} \) if and only if there is, up to parametrization, a unique geodesic between them. But to be joined by a unique geodesic is an isometry invariant. It follows that \( m(T, T') = m(\sigma(T), \sigma(T')) \). Since \( \sigma \) fixes boundary components, Lemma 5 implies that \( T = \sigma(T) \) and \( T' = \sigma(T') \). Therefore \( \sigma \) gives us an isometry of the geodesic between \( T \) and \( T' \). But if we restrict \( m \) to this geodesic, we get the usual metric on some interval in the real line. Since an isometry of an interval fixing its endpoint is trivial, it follows that the restriction of \( \sigma \) to such a geodesic is the identity. Since any point in \( A_T A \) lies in such a geodesic, it follows that \( \sigma \) is the identity map.

In Figure 9, we give the angle model together with some 1-pairs and geodesics between them. Note that the geodesic between two \( i \)-pairs is a straight line segment which is parallel to the boundary component \( B_i \). Consider the function from \( A_T A \to \mathbb{R}_+^* \) which sends a triangle to its angle at the \( i \)-th vertex. Then it follows that the inverse image of any point in \( \mathbb{R}_+^* \) is a geodesic segment between two \( i \)-pairs.
Figure 9: The angle model
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References


[7] Y. Huang and A. Papadopoulos, Optimal Lipschitz maps on one-holed tori and the Thurston metric theory of Teichmüller space, Geometriae Dedicata Published online, April 2021, 24 p.


