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# Analysis of the Topology of the Singularity Set of Planar 3-RPR Parallel Robots with Linear Platforms

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**Abstract.** In [4] most of the results for a general 3-RPR planar robot with linear platforms are proven, i.e. there is a singularity-free path between two symmetrical direct kinematics solutions and there are no singularity-free paths between non-symmetrical direct kinematics solutions. We provide alternative and easy theoretical proofs of these results. We describe the topology of the kinematic singularities of these robots. We prove that the singularity-free set of the domain of the kinematic map for such robots consists of three components. One of our main results is the detailed analysis of the special case, i.e. when the anchor points of the moving platform and the fixed platform have the same cross-ratios. For this case we show that, the singularity-free set consists of four connected components, no direct kinematics solutions can be connected without crossing the singularity and that the absolute value of the determinant of the Jacobian evaluates to the same number for any of the direct kinematics solutions. We provide some nice geometric properties of the direct kinematics solutions for this special case.

**Keywords:** direct kinematics, parallel planar robots, singularities

## 1 Introduction

There are surprisingly very little literatures on 3-RPR parallel planar robots with linear platforms. This is the mechanism we will deal with throughout this paper and for simplicity we will occasionally just call them 3-RPR robots. We could only guess that this structure is rather easy to study and the general 3-RPR parallel planar robots (with anchor points that are not necessarily collinear) has been widely studied. However, there is still little new knowledge to be extracted from this “simpler” planar robots. For instance, because of the simple structure, one is able to fully analyse the topology of the singularities, in  $SE(2)$ , of these mechanisms. Some results known for these robots are significantly simpler and perhaps somewhat surprisingly different at the same time. For instance, the number of real direct kinematics solutions for the linear platforms can be proven to be at most 4 while this is 6 for the triangular platforms (see Remark 2 or [4] or [5] Exercise 4.1). But more can be said about this simpler structure and in this paper we will give an analysis of the topology of the singularity surface of this mechanism, which to our knowledge has not been completely investigated.

We provide a proof of a known fact that for a sufficiently general 3-RPR robot there are two distinct direct kinematics solutions that can be connected via a path that does not cross singularity. Such questions were asked for other planar manipulators (e.g. [2]) and is of interest to experts in robotics and kinematics. In fact, more is known about our robot. It is known that there is a singularity-free path between two symmetrical solutions to the direct kinematics (i.e. pair of solutions for which the angles and heights of the moving platform have opposite signs). This was shown in [4]. We provide a topological proof of this as well.

The advantage of our proof is its simplicity, however there is also an advantage in the proof provided in [4], namely it is a constructive proof which can lead to an easy algorithm providing the path between symmetrical solutions. However, there is a subtle point in our proof, it provides geometric and topological insights to the kinematic singularity surface in  $SE(2)$ . We will indeed

show later a new result, namely that the singularity-free space of a sufficiently general 3-RPR robot consists of three connected components and then provide an alternative proof that non-symmetrical solutions cannot be connected without crossing singularity. We not only study these components but also show how the determinant of the Jacobian of the kinematic map evaluates at a point in any of these components. In fact, there is no explicit proof stating (which we prove) that the determinant of the Jacobian at non-symmetrical solutions evaluates to numbers with different signs (this is clear if the number of connected components of the singularity-free space is two, but we have three such components).

We study special 3-RPR robots as well, namely those for which the linear platforms have anchor points with the same cross-ratios. These robots differ from the general ones because no direct kinematics solutions can be connected without crossing singularity. These robots have easy but interesting topological structures. For instance, we show that the absolute value of the determinant of the Jacobian evaluates to the same number for any direct kinematics solution. Also we show that the singularity-free space of these robots, in  $SE(2)$ , consists of four connected components.

## 2 3-RPR Linear Planar Platforms

Consider a (non-trivial) 3-RPR planar parallel robot with linear platforms for which the anchor points are distinct. We can without loss of generality, (via scaling and change of zero-position) assume the following:

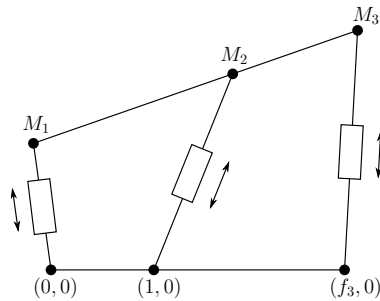
- The fixed platform lies on the  $x$ -axis
- The moving platform at zero-position, i.e. when the transformation of the moving platform with respect to the fixed platform is identity, lies along the  $x$ -axis.
- The anchor points of the fixed platform are  $(0,0), (1,0), (f_3,0)$  and at zero-position these anchor points are connected respectively, by an *RPR* leg, to the anchor points

$$M_1 = (0,0), \quad M_2 = (m_2,0), \quad M_3 = (m_3,0)$$

of the moving platform.

- We may assume  $1 < f_3$  and  $0 < m_2 < m_3$ .

The figure below illustrates such a robot.



We will now describe the kinematic map of this robot.

For simplicity, we write an element  $\sigma \in SE(2)$  as a triple  $(\theta, x, y)$  where  $\theta$  determines the orientation and  $(x, y)$  the translation vector. We will also often, write  $c$  instead of  $\cos \theta$  and  $s$  instead of  $\sin \theta$ .

Following the above notations, the kinematic map  $F : SE(2) \rightarrow \mathbb{R}^3$ , of the robot is given by

$$F(\theta, x, y) := (x^2 + y^2, (cm_2 - 1 + x)^2 + (sm_2 + y)^2, (cm_3 - f_3 + x)^2 + (sm_3 + y)^2)$$

This map takes the transformation of the moving platform (centred at  $M_1$ ) and maps it to the square of the length of each of the legs.

It is easy to compute the set of singularities of this map, it is given by:

$$\Sigma := \{(\theta, x, y) \in SE(2) : D(c, s, x, y) = 0\}$$

where  $D$  is the determinant of the Jacobian of  $F$  and it is a polynomial that is quadratic in  $y$  and in  $s$  and linear in  $x$  and in  $c$ . Namely

$$-c(f_3m_2 - m_3)y^2 + s(f_3m_2 - m_3)xy + s^2m_2m_3(f_3 - 1)x - s(cm_2m_3(f_3 - 1) + f_3(m_2 - m_3))y$$

Suppose now that the robot is sufficiently general (we will be more precise later) and that we are given squared leg length values  $\ell \in \mathbb{R}^3$ . We investigate the direct kinematics, i.e. the fiber  $F^{-1}(\ell)$ . For  $\ell = (l_1, l_2, l_3)$ , we have the following system of equations:

$$\begin{aligned} l_1 &= x^2 + y^2 \\ l_2 &= (cm_2 - 1 + x)^2 + (sm_2 + y)^2 \\ l_3 &= (cm_3 - f_3 + x)^2 + (sm_3 + y)^2 \end{aligned}$$

*Remark 1.* We note that if there is a solution for which the two platforms are parallel ( $\theta = 0$  or  $\theta = \pi$ ), then  $\ell$  must lie in the union of two hyperplanes in  $\mathbb{R}^3$ , say  $H_1 \cup H_2$ .

Suppose now that  $\ell$  is not in the  $H_1 \cup H_2$  defined in Remark 1. We can get rid of the squares  $x^2$  and  $y^2$  by substituting the first equation in the last two and obtain a linear system in  $x$  and  $y$ . This linear system is independent if we assume

$$f_3m_2 - m_3 \neq 0 \text{ and } s \neq 0$$

These inequalities are fulfilled because we assumed the robot is sufficiently general (to be precise, we mean that the cross ratios of the anchor points are not the same) and that  $\ell \notin H_1 \cup H_2$ .

In the above case, we can express  $x$  and  $y$  in terms of  $c$  and  $s$  and substitute this expression in  $l_1 = x^2 + y^2$  and, upon reduction modulo  $c^2 + s^2 - 1$ , obtain a univariate cubic polynomial  $h(c) \in \mathbb{R}[c]$  (for detail description of  $h$ , please see [1, 4]). The polynomial  $h(c)$  has the property that  $h(1)$  and  $h(-1)$  have the same signs.

*Remark 2.* We conclude that if  $\ell \in \mathbb{R}^3 \setminus \{H_1 \cup H_2\}$  (where  $H_1$  and  $H_2$  are defined in Remark 1), i.e. when the direct kinematics solutions do not involve the case where the platforms are parallel, then the solutions to the direct kinematics are obtained from finding the roots of the cubic univariate polynomial  $h(c)$ . However since  $h(c)$  has the property that  $h(1)$  and  $h(-1)$  have the same signs, there are maximum 2 distinct roots in the interval  $(-1, 1)$ . Since  $c = \cos(\theta) = \cos(-\theta)$ , there are at most 4 distinct real solutions to the direct kinematics problem. Thus, in terms of  $\theta$ , the direct kinematics solutions can be interpreted geometrically, as the intersection points of maximum two vertical lines and the unit circle in the  $cs$ -plane.

**Definition 1.** *The pair of direct kinematics solutions,  $(\theta, x, y)$  and  $(-\theta, x, -y)$ , will be called symmetrical solutions (this term is also used in [4]).*

We provide a short proof of a known result. (see [4])

**Proposition 1.** *Suppose our robot is a suitably general 3-RPR platform, to be precise we mean that the design parameters satisfy  $f_3m_2 - m_3 \neq 0$ . There are two distinct transformations that are not singular, say  $\sigma, \tau \in \text{SE}(2)$ , with the following properties:*

- $F(\sigma) = F(\tau)$
- There is a path  $\gamma : [0, 1] \rightarrow \text{SE}(2)$  from  $\sigma$  to  $\tau$  such that  $\gamma([0, 1]) \cap \Sigma = \emptyset$

*Proof.* We note that (see also [1])

$$D(\pm 1, 0, x, y) = \mp y^2(f_3m_2 - m_3)$$

So, for a general robot,  $(0, x, y) \in \Sigma$  implies  $y = 0$ .

Let  $\Delta_y(D)$  be the discriminant of  $D$  with respect to  $y$ . Then  $\Delta_y(D)$  factors as

$$\Delta_y(D)(c, s, x) = s^2 g(x, c)$$

where  $g(x, c)$  is a quadratic polynomial in  $x$  and in  $c$ . For a general robot, the coefficient of  $x^2$  in  $g(x, c)$  does not vanish and is the positive value  $(f_3 m_2 - m_3)^2$ .

Let  $\Delta_x(g)$  be the discriminant of  $g$  with respect to  $x$ . Then  $\Delta_x(g)$  factors as

$$\Delta_x(g)(c) = cp(m_2, m_3, f_3)$$

where  $p(m_2, m_3, f_3) = 16f_3 m_2 m_3 (m_3 - m_2)(f_3 - 1)(f_3 m_2 - m_3)^2 > 0$ , for a general robot.

Hence, for any  $\theta \in (-\pi/2, \pi/2)$  we get  $\Delta_x(g) > 0$ , i.e.  $g(x, \cos \theta)$  has two distinct roots in  $x$  for any  $\theta \in (-\pi/2, \pi/2)$  and since the coefficient of  $x^2$  is positive it attains negative values for any  $x$  between these two roots. Thus, we can find  $x = \alpha$  such that  $g(\alpha, 1) < 0$  and because  $g$  is continuous, there is a  $\delta \in (0, \pi/2)$  such that  $g(\alpha, \cos \theta) < 0$ , for all  $\theta \in (-\delta, \delta)$ .

Define

$$X := (-\delta, \delta) \times \{\alpha\} \times \mathbb{R} \setminus \{(0, \alpha, 0)\}.$$

Note that by the above analysis, any  $(\theta, x, y) \in X$  is a regular point of the kinematic map. But since  $F(\theta, x, y) = F(-\theta, x, -y)$ ,  $F$  restricted to  $X$  is not injective, i.e. there is a singularity-free path between two symmetrical direct kinematics solutions.

If we investigate the above proof we see that at the point  $(\theta, x, y) = (0, \alpha, 0)$ , where  $\alpha \neq 0$ , the set of real singularities is locally a line. This property of  $F$  was not directly revealed in [4], however [4] has shown a stronger result that implies our Proposition 1 (which would require a longer proof).

**Theorem 1.** *For a suitably general 3-RPR robot there are only three components of  $SE(2) \setminus \Sigma$ .*

*Proof.* We will abuse our notations and write  $D$  evaluated at  $\theta$  to mean  $D(\cos \theta, \sin \theta, x, y)$ . First we note that for a fixed  $\theta \in [0, 2\pi)$ , a section of  $\Sigma$  at this  $\theta$  is a hyperbola (possibly degenerate). This is because the determinant of the Hessian of  $D$  at any  $\theta$  is (a non-negative value)

$$(f_3 m_2 - m_3)^2 \sin^2 \theta$$

and this is the discriminant of the conic section described by  $D$  at  $\theta$ . Moreover, the determinant of the Hessian of the quadratic form defining this conic equation is

$$f_3 m_2 m_3 (m_3 - m_2)(f_3 - 1)(f_3 m_2 - m_3) \sin^4 \theta$$

This implies that the hyperbola at a  $\theta$ -section degenerates  $\Leftrightarrow \theta = 0$  or  $\theta = \pi$ . In fact, as our previous analysis shows, it degenerates into the double lines  $y^2 = 0$ , because we can check that

$$D(\pm 1, 0, x, y) = \mp y^2 (f_3 m_2 - m_3)$$

For a fixed  $\theta$ , we will call the  $\theta$ -section of  $\Sigma$  the hyperbola at  $\theta$ , as illustrated in Fig.1.

The slope of the principal axes of the hyperbola at  $\theta$  is also determined by the entries of the Hessian of  $D$ . One easily checks that the angle between the transverse axis of the hyperbola at  $\theta$  and the  $x$ -axis is  $\theta$  as well. So the hyperbola rotates counter-clockwise as we change  $\theta$  from 0 to  $2\pi$ . We may, without loss of generality, assume that at the  $\theta = 0$  section  $D$  evaluates to non-positive values while at the  $\theta = \pi$  section  $D$  evaluates to non-negative values. So as  $\theta$  continuously changes from 0 to  $\pi$  the following things happen:

1. The double line opens into two branches of a hyperbola.
2. The branches enclose connected components in the singularity free  $\theta \in (0, \pi)$  section, for which  $D$  evaluates to positive values.

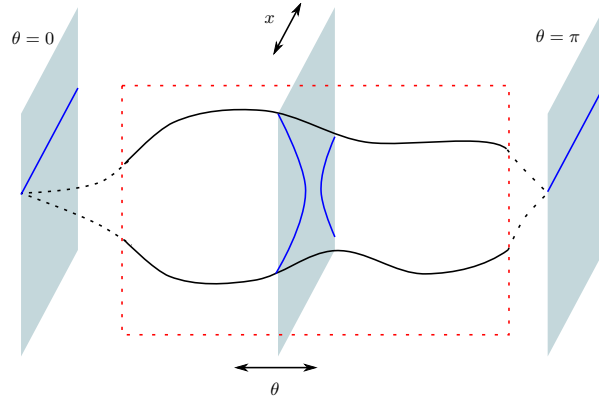


Fig. 1: An artistic illustration of the  $\theta$ -sections of  $\Sigma$ . At  $\theta = 0$  or  $\theta = \pi$  the hyperbola degenerates into double lines.

3. The connected component in the singularity free  $\theta \in (0, \pi)$  section between the two branches of the hyperbola at  $\theta$  has elements for which  $D$  evaluates to negative values.
4. When  $\theta$  is near  $\pi$  the branches begin to collapse so that at  $\theta = \pi$  we get a double line and at this section  $D$  evaluates any element outside this double line to positive values.

We can make similar observations when  $\theta$  changes from  $\pi$  to  $2\pi$ , where the hyperbola continues to rotate counter-clockwise but where the region for which  $D$  evaluates to positive values vanishes at  $\theta = 2\pi$  (which we identify with  $\theta = 0$  in  $S^1$ ). This behavior of the hyperbola is illustrated below.

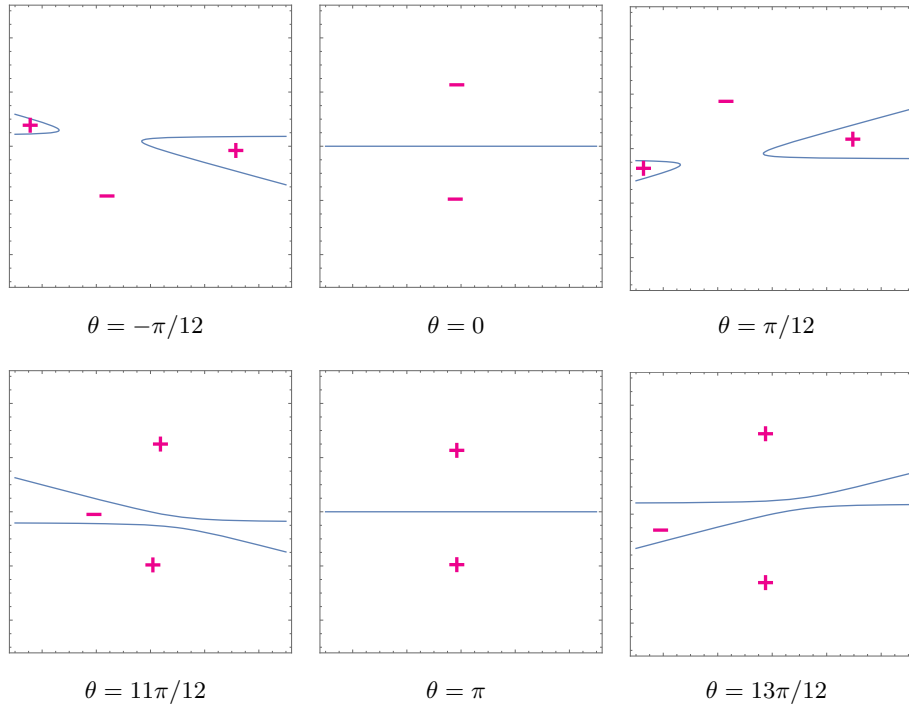


Fig. 2: How  $D$  evaluates for different  $\theta$ -sections

Consider now  $\sigma, \tau \in \text{SE}(2)$  such that  $F(\sigma) = F(\tau)$  and  $\text{sgn}(D(\sigma)) = \text{sgn}(D(\tau))$  is non-zero.

If  $D(\sigma) < 0$  then the orientations of  $\sigma$  and  $\tau$  are associated to a region between the two hyperbola branches in their respective  $\theta$ -sections, i.e. as we continuously follow the  $\theta$ -sections we can produce a singularity-free path from  $\sigma$  to  $\tau$ .

If  $D(\sigma) > 0$  then  $\sigma$  and  $\tau$  are located in the region enclosed by a branch of the hyperbola in their respective  $\theta$ -sections. Set  $\theta_\sigma$  and  $\theta_\tau$  be the angles associated to the orientations of  $\sigma$  and  $\tau$  respectively. Then there is a path from  $\sigma$  to  $\tau$  iff the branch of the hyperbola at  $\theta = \theta_\sigma$ , enclosing the region where  $\sigma$  lies, continuously moves to the branch of the hyperbola at  $\theta = \theta_\tau$  that encloses the region where  $\tau$  lies as we follow this path. In short, the two branches of the hyperbola at a  $\theta \in (0, 2\pi)$  section correspond to two distinct connected components of  $\text{SE}(2) \setminus \Sigma$ .

Thus, we conclude that there are three connected components of  $\text{SE}(2) \setminus \Sigma$ .

In the proof of the above Theorem more was shown. Starting from a  $\theta$ -section where one of the symmetrical solutions lies, and by continuously following the hyperbola branch, from the  $\theta$  to the  $-\theta$ -section, we see we end up in the hyperbola branch where the other symmetrical solution lies. So the symmetrical solutions belong to the same connected component of the singularity-free space. In fact, this was also proved in [4]. Thus, we can state the following:

**Corollary 1.** *For a suitably general 3-RPR platform, there is a singularity-free path between symmetrical direct kinematics solutions.*

Additionally, we are able to prove two more results, which are stated in the following Propositions.

**Proposition 2.** *For a suitably general 3-RPR platform, two non-symmetrical direct kinematics solutions evaluate to opposite signs in the determinant of the Jacobian.*

*Proof.* It is easy to check that symmetrical solutions evaluate to the same sign, since for any  $(\theta, x, y) \in \text{SE}(2)$ ,  $D(c, s, x, y) = D(c, -s, x, -y)$ . Let us assume that there are two non-symmetrical solutions that evaluate to the same sign. The all of the solutions would evaluate to the same sign and let it be  $+$ , without loss of generality. We know from Corollary 1 that symmetrical solutions can be connected by passing through  $\theta = 0$  or  $\pi$ . By assuming, without loss of generality, that  $f_3 m_2 - m_3 > 0$  we get  $D(1, 0, x, y) < 0$  and  $D(-1, 0, x, y) > 0$ . So, since we assumed them to be positive they should pass through  $\theta = \pi$ . Since symmetrical solutions can be connected singularity-free the line that is located furthest from  $c = -1$  would cross the other line while it connects the two solutions of that symmetrical pair, hence a singularity would occur, which is a contradiction.

**Proposition 3.** *For a suitably general 3-RPR platform, two non-symmetrical direct kinematics solutions cannot be connected without crossing singularities.*

*Proof.* Suppose we have two distinct non-symmetrical and non-singular solutions. Consider the vertical lines in Remark 2. Let us examine how these lines move while we are travelling in  $\text{SE}(2)$ . Switching between one solution of the direct kinematics to another would sparkle a parallel motion of the two lines along the circle which would end when the lines intersect the circle exactly at the same points as they initially did. Let us now distinguish between the following cases.

Firstly, consider the two lines moving towards opposite directions. Then, either they both move towards the boundary of the circle, or towards each other. If they move towards the boundary, they will bounce back when they reach  $c = 1$  and  $c = -1$  respectively and they will continue moving towards each other. If  $c = \pm 1$  results in a singular position then we are done. Otherwise, under this motion, at least one of them will return to its original position. But, we are not interested in that case because this would result in a change between symmetrical solutions. So, for a path between two non-symmetrical solutions to exist, the lines should maintain the trajectory of moving towards each other. Hence along this trajectory there should be a moment where the two lines merge because the lines in the end of the motion should intersect the circle precisely at the 4 initial points. Thus, the instance of the two lines merging, will result in a singular position.

On the other hand, let us consider them moving towards the same direction. Then, depending on the direction of the trajectory, the line fastest towards the boundary of the circle will bounce back first, resulting in a motion where they are both moving towards each other. Hence, it falls to the previous case and there will be an instance of them merging, before the lines intersect the circle at the initial points.

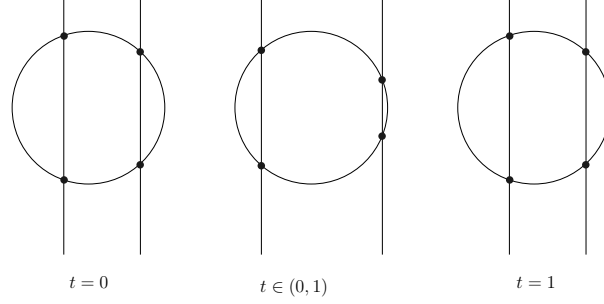


Fig. 3: An illustration of the two lines moving towards different directions. As  $t$  varies in  $[0, 1]$ , there is no way of eventually intersecting the circle in the same initial points, while we connect non-symmetrical solutions, without the two lines merging. Any of the 4 intersection points can be used as a reference point for this purpose. The target point could be any of the remaining points, but not the symmetric one.

### 3 Special Case $f_3 m_2 = m_3$

As discussed in [4], solving the direct kinematics in this case, amounts to solving a second degree polynomial equation in  $x$  for a specific  $c$  and  $y(x, c, s)$ . Hence the solutions will be of the form  $\sigma_1 = (\theta, x_1, y_1)$ ,  $\sigma_2 = (\theta, x_2, y_2)$  and  $\tau_1 = (-\theta, x_1, -y_1)$ ,  $\tau_2 = (-\theta, x_2, -y_2)$ , when  $\theta \neq 0$  or  $\pi$ .

*Remark 3.* By substituting  $m_3 = f_3 m_2$  in  $D$ , we get

$$D(c, s, x, y) = f_3 m_2 (f_3 - 1) ((1 - cm_2)y + sm_2 x)s.$$

Thus we observe that, for  $s = 0$ , we get  $D(\pm 1, 0, x, y) = 0, \forall x, y \in \mathbb{R}$ . Hence, for any  $x, y \in \mathbb{R}$ ,  $(0, x, y) \in \Sigma$  and  $(\pi, x, y) \in \Sigma$ . Also, it should be noted that, when  $\theta \neq 0$  and  $\theta \neq \pi$  there will always be 4 distinct direct kinematics solutions for a non-singular position.

**Proposition 4.** *If  $\sigma_1 = (\theta, x_1, y_1)$  and  $\sigma_2 = (\theta, x_2, y_2)$  are two solutions of the direct kinematics in the same  $\theta$ -section, then  $D(\sigma_1) = -D(\sigma_2)$ .*

*Proof.* By subtracting the first from the second equation of the direct kinematics we get,

$$2(cm_2 - 1)x + 2sm_2 y - 2cm_2 + m_2^2 + 1 + l_1 - l_2 = 0 \quad (1)$$

Since  $\theta$  is the same in  $\sigma_1$  and  $\sigma_2$ , they will belong in the same  $\theta$ -section of  $SE(2)$ , thus they will both satisfy equation (1), which can be rewritten, for some  $\kappa \in \mathbb{R}$ , as

$$(cm_2 - 1)x + sm_2 y + \kappa = 0 \quad (2)$$

Hence,  $(x_1, y_1)$  and  $(x_2, y_2)$  lie on a line in  $xy$ -plane with slope  $\frac{1 - cm_2}{sm_2}$ . Additionally, they will be points on a circle with origin  $(0, 0)$  and radius  $\sqrt{l_1}$ , since they both satisfy  $x^2 + y^2 = l_1$ .

On the other hand, the singularity surface on this  $\theta$ -section becomes the line

$$sm_2 x + (1 - cm_2)y = 0 \quad (3)$$

which passes through the origin of the  $xy$ -plane with slope  $\frac{sm_2}{cm_2 - 1}$ , hence it is a line perpendicular to the line defined by equation (2).

As a result,  $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$  will satisfy equation (3), which yields

$$(1 - cm_2)(y_1 + y_2) + sm_2(x_1 + x_2) = 0 \Leftrightarrow y_1(1 - cm_2) + x_1 sm_2 = -(y_2(1 - cm_2) + x_2 sm_2)$$

and thus  $D(c, s, x_1, y_1) = -D(c, s, x_2, y_2)$ .



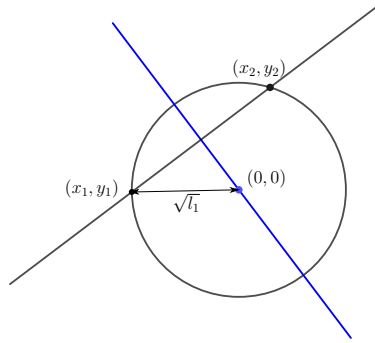


Fig. 4: An example of the singularity line crossing the middle point of the chord between the two direct kinematics solutions  $(x_1, y_1)$  and  $(x_2, y_2)$  on the same  $\theta$ -section.

**Corollary 2.** *If  $\sigma_1 = (\theta, x_1, y_1)$  and  $\sigma_2 = (\theta, x_2, y_2)$  are two solutions of the direct kinematics in the same  $\theta$ -section, then  $(\theta, \lambda(x_1 + x_2), \lambda(y_1 + y_2)) \in \Sigma$  for any  $\lambda \in \mathbb{R}$ .*

**Corollary 3.** *All paths between direct kinematics solutions must cross singularities.*

*Proof.* If  $\sigma_1, \sigma_2, \tau_1, \tau_2$  are as defined in the beginning of this section, then since  $c = \cos \theta = \cos(-\theta)$  and  $s = \sin \theta = -\sin(-\theta)$  we observe that, for  $i = 1, 2$ ,

$$D(\tau_i) = D(c, -s, x_i, -y_i) = D(c, s, x_i, y_i) = D(\sigma_i)$$

But since  $\sigma_i$  and  $\tau_i$ , for  $i = 1, 2$ , belong in opposite signed  $\theta$ -sections a path between them should involve crossing  $\theta = 0$  or  $\pi$  which is a singular position. By Proposition 4, solutions in the same  $\theta$ -section evaluate to different signs in the determinant of the Jacobian, hence they cannot be connected without crossing singularities.

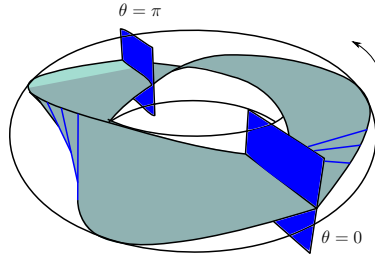


Fig. 5: An artistic illustration of the  $\theta$ -sections of the special case. At  $\theta = 0$  or  $\theta = \pi$  the singularities at these  $\theta$ -sections are the whole  $xy$ -plane. At other  $\theta$ -sections the singularity set is a (blue) line on the “twisted strip”. The singularity-free region can be thought of as the complement of the twisted strip and the two planes in the torus (there are 4 connected components!)

**Corollary 4.** *The absolute value of the determinant of the Jacobian evaluates to the same number for all direct kinematics solutions.*

*Proof.* From the previous results, it follows directly that

$$D(\tau_1) = -D(\tau_2) = -D(\sigma_2) = D(\sigma_1)$$

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